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Composition Series of Tensor Product

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Abstract: Given a quantized enveloping algebra $U_q(\mathfrak{g})$ and a pair of dominant weights $(\lambda,\ \mu)$, we extend a conjecture of Lusztig's in [13] to a more general form and then prove this extended version of the conjecture. Namely we prove that for any symmetrizable Kac-Moody algebra $\mathfrak{g}$, there is a composition series of the $U_q(\mathfrak{g})$-module $V(\lambda)\otimes V(\mu)$ compatible with the canonical basis. As a byproduct, the celebrated Littlewood-Richardson rule is derived and we also construct, in the same manner, a composition series of $V(\lambda)\otimes V(-\mu)$ compatible with the canonical basis when $\mathfrak{g}$ is of affine type and the level of $\lambda-\mu$ is nonzero.
Keywords: Canonical basis, crystal basis, composition series.

1. Introduction

Let $U_q(\mathfrak{g})$ be a quantized enveloping algebra associated to an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{g}$. In [13], for dominant integral weights λ and μ, Lusztig constructed a canonical basis for the $U_q(\mathfrak{g})$-module $V(\lambda)\otimes V(-\mu)$, where $V(\lambda)$ is an irreducible highest weight integrable $U_q(\mathfrak{g})$-module of highest weight λ and $V(-\mu)$ is an irreducible lowest weight integrable $U_q(\mathfrak{g})$-module of lowest weight $-\mu$. This basis has many remarkable properties and can be lifted to a basis of the modified quantized enveloping algebra $\widetilde{U}$. Since then the canonical basis as well as the corresponding crystal basis of both this tensor product and $\widetilde{U}$ are widely investigated by many mathematicians e.g. [1, 8, 14, 15].

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Due to the stable property of the basis, there are quite a few submodules of $V(\lambda)\otimes V(-\mu)$ compatible with the canonical basis, that is, every such submodule is spanned by parts of the basis. Lusztig conjectured further in [13] that in the case $\mathfrak{g}$ is of finite type there is a composition series of $V(\lambda)\otimes V(-\mu)$ compatible with the canonical basis and he proved the conjecture in the case of type A_1 by a direct computation. Later in chapter 27 of [14] concerning about the based module, Lusztig proved that for any integrable $U_q(\mathfrak{g})$-module $M=\bigoplus_{\xi\in P_+} M[\xi]$ in category $\mathcal{O}_{int}$ where $M[\xi]$ is the sum of all submodules of M isomorphic to $V(\xi)$, $M[\lambda]$ is compatible with the canonical basis of M if λ is maximal among those ξ such that $M[\xi]$ is nonzero. Though not pointing out, Lusztig's proof of this result implies the conjecture and provided an inductive construction for the composition series since, in particular, $V(\lambda)\otimes V(-\mu)$ is in category $\mathcal{O}_{int}$ when $\mathfrak{g}$ is of finite type. The crystal structures of both $V(\lambda)\otimes V(-\mu)$ and $\widetilde{U}$ are extensively investigated by Kashiwara in [8]. In [15] Lusztig investigated the two-sided cells in the canonical basis of $\widetilde{U}$ for $\mathfrak{g}$ of finite type and he raised some conjectures in affine type case which were finally solved by Beck and Nakajima in [1].

In [2], a filtration of $V(\Lambda_i)\otimes V(-\Lambda_j)$ of $U_q(\mathfrak{g})$ was constructed, for $\mathfrak{g}$ which is of affine type and where Λ_i and Λ_j are fundamental weights. Each $U_q(\mathfrak{g})$-submodules appeared in this filtration is generated by the tensor product of u_{Λ_i} with an extremal vector of $V(-\Lambda_j)$. It turns out that all of the $U_q(\mathfrak{g})$-submodules appeared in this filtration are compatible with the canonical basis which can be proved using an important lemma due to Kashiwara and some results for Demazure modules. Motivated by the construction of the filtration in [2], we construct the composition series of $V(\lambda)\otimes V(\mu)$ directly for $\mathfrak{g}$ of any type in the same fashion. The conjecture by Lusztig is then a special case since $V(\mu)$ is also a lowest weight module for $\mathfrak{g}$ of finite type. This is quite different from the argument in Chapter 27 in Lusztig's book [14] and one can derive from our proof the Littlewood-Richardson rule for decomposing the tensor product $V(\lambda)\otimes V(\mu)$ into a direct sum of irreducible modules, which is also known by the work of Littelmann [9].

On geometric aspects, quiver varieties were introduced by Nakajima in order to get integrable highest weight representations of symmetric Kac-Moody algebra $\mathfrak{g}$. Furthermore, there is also a geometric construction of tensor product $V(\lambda_1)\otimes\cdots\otimes V(\lambda_r)$ using quiver varieties [17]. To realize this tensor product, Malkin also introduced in [16] the tensor product variety. Though both constructions are in classical case ($q = 1$), it would be interesting to consider the geometric construction of the composition series using Nakajima's quiver variety or Malkin's tensor product variety. We will study this topic in the forth coming publications.

The arrangement of the paper is the following: in section 2, we recall some basics of the theory of crystal basis and canonical basis. In particular, we recall the construction of the canonical basis of $V(\lambda)\otimes V(-\mu)$ due to Lusztig. Next in section 3, the extended Lusztig's conjecture is proved by building up the required composition series explicitly using the theory of crystal basis due to Kashiwara. Then we reintroduce the Littlewood-Richardson rule and compare this composition series with Lusztig's inductive construction. Finally in the last section we study the tensor product $V(\lambda)\otimes V(-\mu)$ for any symmetrizable Kac-Moody algebra $\mathfrak{g}$. In particular, the connected components of the crystal graph of $U_q(g)a_{\lambda-\mu}$ are completely determined and a composition series of $V(\lambda)\otimes V(-\mu)$ is constructed compatible with the canonical basis when $\mathfrak{g}$ is of affine type and the level of $\lambda-\mu$ is nonzero.

2. Lusztig's Construction of Canonical Basis

2.1. **Notations.** Let $\mathfrak{g} = \mathfrak{g}(A)$ be an arbitrary symmetrizable Kac-Moody algebra over $\mathbb{Q}$ where A is the $n\times n$ generalized Cartan matrix and let $\mathfrak{h}$ be the Cartan subalgebra which is of dimension $2n - rank(A)$. We denote by $I = \{1,\cdots,n\}$ the index set. Let $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ be the root lattice and set $Q_+ = \bigoplus_{i\in I}\mathbb{Z}_+\alpha_i$ where α_i are the simple roots. Denote by $\{h_i \in \mathfrak{h} \mid i \in I\}$ the set of simple coroots. $P^\vee$ is defined to be a free $\mathbb{Z}$-module with a basis

$$\{h_i \mid i\in I\}\bigcup\{d_j\in\mathfrak{h} \mid 1\leqslant j\leqslant n-rank(A)\},$$

called the dual weight lattice. We also define $P = \{\lambda\in\mathfrak{h}^* \mid \langle h,\lambda\rangle\in\mathbb{Z}\ \forall h\in P^\vee\}$ to be the weight lattice. Note that there is a symmetric bilinear form on P such that

$$\frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)} = \langle h_i,\lambda\rangle,$$

for $i\in I$, $\lambda\in P$. Let $P_+ = \{\lambda\in\mathfrak{h}^*|\langle h_i,\lambda\rangle\in\mathbb{Z}_+\ \ \forall i\in I\ \}$ be the set of dominant weights. Denote by Λ_i the fundamental weight, i.e. $\langle h_i,\Lambda_j\rangle = \delta_{ij}\ \ \forall i,j\in I$. The partial order on P is defined as $\xi\geqslant\varphi$ if $\xi-\varphi\in Q_+$.

The quantized enveloping algebra $U_q(\mathfrak{g})$ is defined as a k-algebra with generators E_i, F_i and q^h for all $i\in I$ and $h\in P^\vee$, where $k=\mathbb{Q}(q)$. The relations are as in [8]. Let $U_q(\mathfrak{g})^+$ (*resp.* $U_q(\mathfrak{g})^-$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the E_i (*resp.* F_i) for all $i \in I$. Note that irreducible integrable highest and lowest weight $U_q(\mathfrak{g})$-modules can be indexed by P_+ and $-P_+$ respectively. Namely, for $\lambda \in P_+$ (*resp.* $\lambda \in -P_+$), we denote by $V(\lambda)$ the irreducible highest (*resp.* lowest) weight $U_q(\mathfrak{g})$-module of highest (*resp.* lowest) weight λ and let u_λ be the highest (*resp.* lowest) weight vector. Let $\mathcal{O}_{int}$ denote the category of integrable $U_q(\mathfrak{g})$-modules M which are direct sums of irreducible integrable highest weight modules.

As is well known, if $\mathfrak{g}$ is of finite type, the Weyl group W of the Lie algebra $\mathfrak{g}$ is a finite group and there is a unique longest element $w_0 \in W$. In this case, the irreducible module $V(\lambda)$ is finite dimensional and hence it is also a lowest weight module of lowest weight $w_0\lambda$.

Note that $U_q(\mathfrak{g})$ is a Hopf algebra and thus the tensor product of $U_q(\mathfrak{g})$-modules has a structure of $U_q(\mathfrak{g})$-module through the coproduct on $U_q(\mathfrak{g})$. There is a $\mathbb{Q}$-automorphism of $U_q(\mathfrak{g})$, denoted by $^-$, such that

$$\overline{q} = q^{-1},\ \overline{q^h} = q^{-h},\ \overline{E_i} = E_i,\ \overline{F_i} = F_i.$$

Let $\widetilde{U}_q(\mathfrak{g})$ or simply $\widetilde{U}$ be the modified quantized enveloping algebra [8] generated by $U_q(\mathfrak{g})a_\lambda$ for $\lambda \in P$ subject to the relations:

$$q^h a_\lambda = q^{\langle h,\lambda\rangle} a_\lambda,\ a_\lambda a_\mu = \delta_{\lambda,\mu} a_\lambda,\ u a_\lambda = a_{\lambda+\xi} u \text{ for } u \in U_q(\mathfrak{g})_\xi$$

where $U_q(\mathfrak{g})_\xi = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\langle h,\lambda\rangle} u\ \forall h \in P^\vee\}$. Note that

$$\widetilde{U} = \bigoplus_{\lambda \in P} U_q(\mathfrak{g}) a_\lambda.$$

2.2. **Canonical Basis.** Canonical bases are constructed by Lusztig for both $U_q(\mathfrak{g})^\pm$ and some classes of $U_q(\mathfrak{g})$-modules [10, 11, 12, 13]. This basis was subsequently studied by Kashiwara [4, 5, 7, 8] who called it the global crystal basis. Hereafter we will follow Lusztig's terminology of *canonical basis* while using the notations of global crystal basis due to Kashiwara.

For details on definition of (abstract) crystal, one can refer to [4, 5, 6]. We only mention here that for $\lambda \in P_+$, $V(\lambda)$ admits a crystal basis $(L(\lambda), B(\lambda))$ where $B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda) \mid r \geqslant 0, i_k \in I\ \} \setminus \{0\}$ and there is a similar result for lowest weight module $V(-\lambda)$ [4, 5]. We denote also by u_λ its image in $L(\lambda)/qL(\lambda)$ if this causes no confusion. For a $U_q(\mathfrak{g})$-module M, there is an involution $^-$ on M such that

$$\overline{u \cdot m} = \overline{u} \cdot \overline{m} \quad \forall u \in U_q(\mathfrak{g}),\ m \in M,$$

which will be called bar involution hereafter. Assume that M has a crystal basis $(L(M), B(M))$ and $M_{\mathbb{Q}}$ is a $\mathbb{Q}[q, q^{-1}]$-lattice of M. $(L(M), \overline{L(M)}, M_{\mathbb{Q}})$ is said to be a balanced triple if

$$L(M) \cap \overline{L(M)} \cap M_{\mathbb{Q}} \cong L(M)/qL(M).$$

Suppose that the balanced triple does exist for M, then we have a basis consisting of bar-invariant elements in $L(M) \cap \overline{L(M)} \cap M_{\mathbb{Q}}$, called *canonical basis* in this paper (see [5] for details). We denote it by $\{G(b)|b \in B(M)\}$.

Definition 2.1. Let M and N be $U_q(\mathfrak{g})$-modules with canonical bases,

(i) a $U_q(\mathfrak{g})$ (or $U_q(\mathfrak{g})^{\pm}$)-submodule M' of M is said to be nice (or compatible with the canonical basis of M) if M' is spanned as a k-vector space by parts of the canonical basis of M.

(ii) a $U_q(\mathfrak{g})$-morphism $f:\ M \longrightarrow N$ is said to be nice (or compatible with canonical bases) if f maps any canonical basis element of M to either zero or a canonical basis element of N and if $ker f$ is nice.

(iii) a filtration or a composition series of a $U_q(\mathfrak{g})$-module M is said to be nice (or compatible with the canonical basis) if any submodule in the filtration or composition series is nice.

For $\lambda \in \pm P_+$, we define the bar involution on $V(\lambda)$ by

$$\overline{x \cdot u_\lambda} = \overline{x} \cdot u_\lambda$$

for all $x \in U_q(\mathfrak{g})$. As is well known, $V(\lambda)$ has a canonical basis $\{G(b)|b \in B(\lambda)\}$. Note that $U_q(\mathfrak{g})^{\mp}$ also has a canonical basis $\{G(b)|b \in B(\pm\infty)\}$ such that $\{G(b)u_\lambda|b \in B(\pm\infty)\}\backslash\{0\}$ coincides with the above set.

2.3. **Canonical Bases in Tensor Product.** For $U_q(\mathfrak{g})$-modules M and N with bar involutions where $M \in \mathcal{O}_{int}$, the $U_q(\mathfrak{g})$-module $M \otimes N$ can be endowed with a bar involution as

$$\overline{u \otimes v} = \Theta(\overline{u} \otimes \overline{v})$$

for all $u \in M, v \in N$, where Θ is the quasi R-matrix [3].

We focus our attention on $V(\lambda)\otimes V(\mu)$, where $\lambda,\ \mu\in P_+$. Since both $V(\lambda)$ and $V(\mu)$ have canonical bases, $V(\lambda)\otimes V(\mu)$ has a natural basis $\{G(b_1)\otimes G(b_2)|b_1\in B(\lambda), b_2\in B(\mu)\}$. The bar involution acts on this basis as

$$\overline{G(b_1)\otimes G(b_2)}\in G(b_1)\otimes G(b_2)+\sum_{wtb_1'>wtb_1, wtb_2'<wtb_2}\mathbb{Z}[q,q^{-1}]G(b_1')\otimes G(b_2').$$

If a partial order is fixed on the natural basis according to the lexicographical order on $\{(wt(b_1),wt(b_2))\mid b_1\in B(\lambda),b_2\in B(\mu)\}$, then one gets a new basis of $V(\lambda)\otimes V(\mu)$ that is bar-invariant with upper triangular relations with the above natural one.

Proposition 2.2. *([13]) For $b_1\otimes b_2\in B(\lambda)\otimes B(\mu)$ there exists a unique element*

$$(b_1\diamond b_2)_{\lambda,\mu}\in G(b_1)\otimes G(b_2)+\sum_{wtb_1'>wtb_1, wtb_2'<wtb_2} q\mathbb{Z}[q]G(b_1')\otimes G(b_2')$$

satisfying $\overline{(b_1\diamond b_2)_{\lambda,\mu}}=(b_1\diamond b_2)_{\lambda,\mu}$. Hence $\{(b_1\diamond b_2)_{\lambda,\mu}|b_1\in B(\lambda),b_2\in B(\mu)\}$ forms a new basis of $V(\lambda)\otimes V(\mu)$.

Note that $V(\lambda)\otimes V(\mu)$ has a crystal basis $(L(\lambda)\otimes L(\mu),\ B(\lambda)\otimes B(\mu))$ and for $b_1\otimes b_2\in B(\lambda)\otimes B(\mu)$, the corresponding canonical basis element

$$G(b_1\otimes b_2)=(b_1\diamond b_2)_{\lambda,\mu}.$$

In particular, $G(b_1\otimes b_2)=G(b_1)\otimes G(b_2)$ if $b_1=u_\lambda$. This basis is constructed in the same fashion as that of Lusztig's canonical basis of $V(\lambda)\otimes V(-\mu)$ [13]. When $\mathfrak{g}$ is of finite type, our basis coincides with Lusztig's basis for $V(\lambda)\otimes V(w_0\mu)$ since the $U_q(\mathfrak{g})$-morphism $f:V(\mu)\longrightarrow V(w_0\mu)$ which takes u_μ to the canonical basis element of hight weight in $V(w_0\mu)$ is easily seen to be a nice isomorphism. Therefore $V(\lambda)\otimes V(-\mu)$ is a special case in our consideration for $\mathfrak{g}$ of finite type but things are quite different in affine or indefinite types since this tensor product is not in category $\mathcal{O}_{int}$ any more. As is known $V(\lambda)\otimes V(-\mu)$ is a cyclic $U_q(\mathfrak{g})$-module generated by $u_\lambda\otimes u_{-\mu}$. We mention here a result of Lusztig's (Theorem 2 in [13]) on the stability property for the canonical basis of this tensor product, which is actually true for $\mathfrak{g}$ of any type.

Proposition 2.3. *For any $\lambda,\mu,\theta\in P_+$, the $U_q(\mathfrak{g})$-morphism*

$$\phi: V(\lambda+\theta)\otimes V(-\theta-\mu)\longrightarrow V(\lambda)\otimes V(-\mu)$$

which takes $u_{\lambda+\theta}\otimes u_{-\theta-\mu}$ to $u_\lambda\otimes u_{-\mu}$ is a surjective nice $U_q(\mathfrak{g})$-morphism.

We can get some submodules of $V(\lambda) \otimes V(-\mu)$ compatible with the canonical basis of $V(\lambda) \otimes V(-\mu)$ by means of the above maps, but usually one cannot get a composition series consisting of the nice submodules obtained above.

Example 2.4. *In A_2 case, consider $V(\Lambda_1) \otimes V(-\Lambda_1 - \Lambda_2)$. Since we have*

$$V(\Lambda_1) \otimes V(-\Lambda_1 - \Lambda_2) \xrightarrow{\phi} V(0) \otimes V(-\Lambda_2) \cong V(-\Lambda_2)$$

then $V(\Lambda_1) \otimes V(-\Lambda_1 - \Lambda_2) \supseteq ker\phi \supseteq 0$ is a filtration compatible with the canonical basis, but $ker\phi$ is far from being an irreducible module.

We denote by $B(\lambda, -\mu)$ the crystal basis of $V(\lambda) \otimes V(-\mu)$. It can be seen from Proposition 2.3 that there is an embedding of crystals

$$B(\lambda, -\mu) \hookrightarrow B(\lambda + \theta, -\theta - \mu)$$

and note that it is strict, i.e. the embedding map commutes with all the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$. For $\lambda, \mu \in P_+$, let $\Phi : U_q(\mathfrak{g})a_{\lambda-\mu} \longrightarrow V(\lambda, -\mu)$ be the $U_q(\mathfrak{g})$-map taking $a_{\lambda-\mu}$ to $u_\lambda \otimes u_{-\mu}$. It is known that $\widetilde{U}$ as well as each $U_q(\mathfrak{g})a_\lambda$ have canonical bases and Φ is a nice surjective $U_q(\mathfrak{g})$-map [8, 13]. We denote the crystal basis of $\widetilde{U}$ (*resp.* $U_q(\mathfrak{g})a_\lambda$) by $\widetilde{B}$ (*resp.* $B(U_q(\mathfrak{g})a_\lambda)$). Hence we have an embedding of crystals $B(\lambda, -\mu) \hookrightarrow B(U_q(\mathfrak{g})a_{\lambda-\mu})$. It can be viewed as

$$B(\lambda, -\mu) \subseteq B(\lambda + \theta, -\theta - \mu) \subseteq B(U_q(\mathfrak{g})a_{\lambda-\mu}) \subseteq \widetilde{B}.$$

Note that

$$B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$$

where T_λ is a crystal consisting of a single element t_λ with $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ for all $i \in I$. For $b \in B(\lambda, -\mu) \subseteq \widetilde{B}$, we denote the corresponding canonical basis element in $V(\lambda, -\mu)$ or $\tilde{U}$ by the same $G(b)$ if there is no confusion.

3. Composition Series of $V(\lambda) \otimes V(\mu)$

3.1. **Kashiwara's Lemma.** We fix $\lambda, \mu \in P_+$ hereafter. In [13], Lusztig conjectured that there exists a nice composition series of $V(\lambda) \otimes V(-\mu)$ if $\mathfrak{g}$ is of finite type. One may extend this conjecture by changing $V(-\mu)$ to $V(\mu)$ and omitting the assumption that $\mathfrak{g}$ is of finite type. This section is devoted to the proof of this extended Lusztig's conjecture. In order to do that, we need the following lemma due to Kashiwara [6] who proved the lemma in case of $g = sl_2$ and claimed that it is true in general.

Lemma 3.1. *([6]) Let M be an integrable $U_q(\mathfrak{g})$-module with a canonical basis. If N is a nice $U_q(\mathfrak{g})^+$-submodule of M, then $U_q(\mathfrak{g})N$ is a nice $U_q(\mathfrak{g})$-submodule of M, i.e. $U_q(\mathfrak{g})N = \bigoplus_{b\in B(U_q(\mathfrak{g})N)\subseteq B(M)} kG(b)$. Moreover,*

$$B(U_q(\mathfrak{g})N) = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_m}b \mid m \geqslant 0, i_1,\cdots,i_m \in I, b\in B(N)\}\setminus\{0\}.$$

For completeness, we give a full proof of Kashiwara's lemma. First assume that M is a finite dimensional $U_q(sl_2)$-module with canonical basis and we denote by $B(M)$ or B for simplicity the crystal basis of M. As is defined by M. Kashiwara in [6], $I^l(M)$ is the sum of all $l+1$-dimensional irreducible submodules of M. Hence $M = \bigoplus_l I^l(M)$. Set $I^l(B) = \{b\in B|\varepsilon(b)+\varphi(b) = l\}$ and one can see that

$$B = \bigoplus_l I^l(B),$$

where $\bigoplus$ here simply means a union. Note that the decomposition of M into isotypical components $I^l(M)$'s is compatible with the decomposition of crystal basis B into $I^l(B)$'s, but it is usually not compatible with the canonical basis. Set $W^l(M) = \bigoplus_{l'\geqslant l} I^{l'}(M)$ and $W^l(B) = \{b\in B|\varepsilon(b)+\varphi(b)\geqslant l\}$. We know from [6] that $W^l(M)$ is a nice $U_q(sl_2)$-submodule of M, i.e.

$$W^l(M) = \bigoplus_{b\in W^l(B)} kG(b).$$

Moreover, if $b\in I^l(B)$, then

$$F_i^{(k)}G(b) = \begin{bmatrix}\varepsilon_i(b)+k\\ k\end{bmatrix}_i G(\tilde{f}_i^k b) \quad (mod\ W^{l+1}(M)),$$
$$E_i^{(k)}G(b) = \begin{bmatrix}\varphi_i(b)+k\\ k\end{bmatrix}_i G(\tilde{e}_i^k b) \quad (mod\ W^{l+1}(M)).$$

Let N be a nice $U_q(sl_2)^+$-submodule of M, i.e.

$$N = \bigoplus_{b\in B(N)\subseteq B(M)} kG(b).$$

Set $\widetilde{N} = U_q(sl_2)N$, $I^l(B(N)) = B(N)\bigcap I^l(B)$, $W^l(B(N)) = \bigcup_{k\geqslant l} I^k(B(N))$, $W^l(N) = W^l(M)\bigcap N$ and $B(\widetilde{N}) = \bigcup_{m\geqslant 0}\tilde{f}^m B(N)\setminus\{0\}$. We have the following lemma.

Lemma 3.2. *([6]) For N, $W^l(N)$, $\widetilde{N}$, $B(\widetilde{N})$ defined as above,*

(i) $\tilde{e}_i B(N)\subseteq B(N)\bigcup\{0\}$.
(ii) $W^l(N) = \bigoplus_{b\in W^l(B(N))} kG(b)$.

(iii) $W^l(\widetilde{N}) = U_q(sl_2)W^l(N)$.
(iv) $\widetilde{N} = \bigoplus_{b\in B(\widetilde{N})\subseteq B(M)} kG(b)$.

Definition 3.3. An integrable $U_q(sl_2)$-module M is said to be truncated if $M = \bigoplus_{j\geqslant 0} I^j(M)$ where there exists an $l \geqslant 0$ such that $I^j(M) = 0$ for all $j \geqslant l$.

Recall that Lemma 3.2 (iv) is proved by showing

$$W^l(\widetilde{N}) = \bigoplus_{b\in W^l(B(\widetilde{N}))} kG(b)$$

through a descending induction on l since both sides equal zero when l is sufficiently large. Thus the above results also hold when we modify M to be a truncated integrable $U_q(sl_2)$-module, that is,

Lemma 3.4. *Let M be a truncated integrable $U_q(sl_2)$-module with a canonical basis. If N is a nice $U_q(sl_2)^+$-submodule of M, then $U_q(sl_2)N$ is a nice $U_q(sl_2)$-submodule of M, i.e.*

$$U_q(sl_2)N = \bigoplus_{b\in B(U_q(sl_2)N)\subseteq B(M)} kG(b).$$

Moreover, $B(U_q(sl_2)N) = \bigcup_{m\geqslant 0} \tilde{f}^m B(N) \setminus \{0\}$.

Furthermore, we can prove the following lemma.

Lemma 3.5. *Let M be an (possibly infinite dimensional) integrable $U_q(sl_2)$-module with a canonical basis. If N is a nice $U_q(sl_2)^+$-submodule of M, then $U_q(sl_2)N = U_q(sl_2)^-N$ is a nice $U_q(sl_2)$-submodule of M. Moreover,*

$$B(U_q(sl_2)N) = \bigcup_{m\geqslant 0} \tilde{f}^m B(N) \setminus \{0\}.$$

Proof. One can define a nice $U_q(sl_2)$-submodule $W^l(M)$ of M for any $l \geqslant 0$ as before. Hence $M/W^l(M)$ is a truncated module with a canonical basis $\{G(b) + W^l(M) | b \in I^j(B), j < l\}$ and $(N+W^l(M))/W^l(M)$ is a nice $U_q(sl_2)^+$-submodule. Applying Lemma 3.4, we have

$$U_q(sl_2)((N + W^l(M))/W^l(M)) = \bigoplus_{b\in\bigoplus_{j<l} I^j(B(N))} k(G(\tilde{f}^m b) + W^l(M)).$$

It follows that

$$U_q(sl_2)(N+W^l(M)) = (\bigoplus_{b\in\bigoplus_{j<l} I^j(B(N))} kG(\tilde{f}^m b))\bigoplus(\bigoplus_{b\in\bigoplus_{j\geqslant l} I^j(B)} kG(b)).$$

Set $\widetilde{N} = U_q(sl_2)N$. We have $U_q(sl_2)(N+W^l(M)) = \widetilde{N}+W^l(M)$. Hence

$$\widetilde{N} = \bigcap_{l\geqslant 0}(\widetilde{N}+W^l(M)) = \bigcap_{l\geqslant 0}(\bigoplus_{b\in\bigoplus_{j<l} I^j(B(N))} kG(\tilde{f}^m b)\bigoplus\bigoplus_{b\in\bigoplus_{j\geqslant l} I^j(B)} kG(b))$$

which is easily seen to be a nice $U_q(sl_2)$-submodule of M. We denote by B^l the crystal basis of $\widetilde{N}+W^l(M)$, i.e.

$$B^l = \{\tilde{f}^m b \mid b\in I^j(B(N)),\ j<l,\ m\geqslant 0,\ \tilde{f}^m b\neq 0\}\cup W^l(B).$$

Since $\tilde{f}^m b\in I^j(B)$ for $b\in I^j(B(N))$ and $m\geqslant 0$ such that $\tilde{f}^m b\neq 0$, we have for $l<k$, $B^l\supseteq B^k$ and $B^k\bigcap I^l(B) = \bigcup_{m\geqslant 0}\tilde{f}^m I^l(B(N))\setminus\{0\}$. It follows that

$$B(\widetilde{N})\bigcap I^l(B) = (\bigcap_{k\geqslant 0} B^k)\bigcap I^l(B) = \bigcup_{m\geqslant 0}\tilde{f}^m I^l(B(N))\setminus\{0\}$$

and hence we have $B(\widetilde{N}) = \bigcup_{l\geqslant 0}(B(\widetilde{N})\bigcap I^l(B)) = \bigcup_{m\geqslant 0}\tilde{f}^m B(N)\setminus\{0\}$. □

We define $U_q(sl_2(i))$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by E_i, F_i and $q^{\frac{(\alpha_i,\alpha_i)}{2}h_i}$ for some $i\in I$. Since N is a nice $U_q(\mathfrak{g})^+$-submodule of M, it is also a nice $U_q(sl_2(i))^+$-submodule. Hence $U_q(sl_2(i))N$ is a nice $U_q(sl_2(i))$-submodule of M by Lemma 3.5. It is easy to see that $U_q(\mathfrak{g})^+U_q(sl_2(i)) = U_q(sl_2(i))U_q(\mathfrak{g})^+$. Hence

$$U_q(sl_2(i))N = U_q(sl_2(i))U_q(\mathfrak{g})^+N = U_q(\mathfrak{g})^+U_q(sl_2(i))N$$

is still a $U_q(\mathfrak{g})^+$-module. Repeating this, one can see that

$$U_q(sl_2(i_1))\cdots U_q(sl_2(i_m))N$$

is a nice $U_q(\mathfrak{g})^+$-submodule of M which admits a crystal basis

$$\{\tilde{f}_{i_1}^{r_1}\cdots\tilde{f}_{i_m}^{r_m}b \mid r_1,\ \cdots,r_m\in\mathbb{Z}_+,\ b\in B(N)\}\setminus\{0\}.$$

This proves Lemma 3.1 since

$$U_q(\mathfrak{g})N = \sum_{i_1,\cdots,i_m\in I} U(sl_2(i_1))\cdots U_q(sl_2(i_m))N.$$

3.2. **Composition Series.** The following construction of composition series is inspired by [2]. For $b \in B(\mu)$ with $wtb = \mu - \sum_{i\in I} m_i\alpha_i$ where $m_i \geqslant 0$, set $l(b) = \sum_{i\in I} m_i$. Since $B(\mu) = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_l}u_\mu \mid i_1,\cdots,i_l \in I, l \geqslant 0\} \setminus \{0\}$, b is of the form $\tilde{f}_{i_1}\cdots\tilde{f}_{i_l}u_\mu$ for some $i_1,\cdots,i_l \in I$, $l \geqslant 0$. Hence $wtb = \mu - \sum_{j=1}^{l}\alpha_{i_j}$, which implies $l = l(b)$. One can define $|b|$ to be the $l(b)$-tuple $(i_1,\cdots,i_{l(b)})$ such that $(i_1,\cdots,i_{l(b)})$ is minimal in lexicographic order among tuples $(j_1,\cdots,j_{l(b)})$ such that $\tilde{f}_{j_1}\cdots\tilde{f}_{j_{l(b)}}u_\mu = b$, i.e.

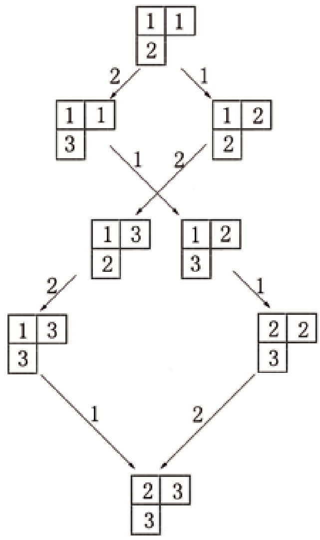
$$|b| = min\{(j_1,\cdots,j_{l(b)}) \mid b = \tilde{f}_{j_1}\cdots\tilde{f}_{j_{l(b)}}u_\mu\}.$$

Set $|u_\mu| = 0$. Note that the order on I is given as $1 < 2 < \cdots < n-1 < n$. If $|b_1| = |b_2| = (i_1,\cdots,i_l)$, we have $b_1 = b_2 = \tilde{f}_{i_1}\cdots\tilde{f}_{i_l}u_\mu$ which implies that there is a one to one correspondence between $B(\mu)$ and $\{|b| \mid b \in B(\mu)\}$. Thus we have a total order on $B(\mu)$ as the following,

$$b_1 \leqslant b_2 \ \text{ iff } \ l(b_1) > l(b_2) \ \text{ or } \ l(b_1) = l(b_2) \ \text{ but } \ |b_1| \geqslant |b_2|.$$

Obviously $b_1 < b_2$ if $wtb_1 < wtb_2$.

Example 3.6. *In the case of type A, there is a combinatorial realization of the crystal* $B(\lambda)$ *for* $\lambda \in P_+$. *If* $U_q(\mathfrak{g}) = U_q(sl_3)$, $B(\Lambda_1+\Lambda_2) \cong B(\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix})$ *and the crystal graph is given as the following,*



We have $|\begin{smallmatrix}1&1\\2&\end{smallmatrix}| = 0$, $|\begin{smallmatrix}1&1\\3&\end{smallmatrix}| = (2)$, $|\begin{smallmatrix}1&2\\2&\end{smallmatrix}| = (1)$, $|\begin{smallmatrix}1&3\\2&\end{smallmatrix}| = (2,1)$, $|\begin{smallmatrix}1&2\\3&\end{smallmatrix}| = (1,2)$, $|\begin{smallmatrix}1&3\\3&\end{smallmatrix}| = (2,2,1)$, $|\begin{smallmatrix}2&2\\3&\end{smallmatrix}| = (1,1,2)$, $|\begin{smallmatrix}2&3\\3&\end{smallmatrix}| = (1,2,2,1)$. *Hence the order* *on $B(\Lambda_1+\Lambda_2)$ is given as the following,*

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} > \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}.$$

For $b\in B(\mu)$, we define a k-subspace $V_b(\mu)$ of $V(\mu)$ spanned by all canonical basis elements $G(c)$ such that $c\geqslant b$, i.e. $V_b(\mu):=\sum_{c\geqslant b}kG(c)$.

Lemma 3.7. *For $\mu\in P_+$ and $b\in B(\mu)$, $V_b(\mu)$ is a nice $U_q(\mathfrak{g})^+$-submodule of $V(\mu)$ and $B(V_b(\mu))=\{c\in B(\mu)\mid c\geqslant b\}$.*

Proof. We only need to show that $V_b(\mu)$ is a $U_q(\mathfrak{g})^+$-submodule of $V(\mu)$. For any $c\in B(\mu)$ where $c\geqslant b$, one can see that $V_c(\mu)\subseteq V_b(\mu)$ and

$$U_q(\mathfrak{g})^+G(c)=\bigoplus_{\xi\in Q_+}U_q(\mathfrak{g})^+_\xi G(c)=kG(c)\bigoplus\bigoplus_{\xi\in Q_+\setminus\{0\}}U_q(\mathfrak{g})^+_\xi G(c).$$

For $\xi\in Q_+\setminus\{0\}$,

$$\begin{aligned}U_q(\mathfrak{g})^+_\xi G(c)\subseteq V(\mu)_{wtc+\xi}&=\sum_{wtd=wtc+\xi}kG(d)\\&\subseteq\sum_{wtd>wtc}kG(d)\subseteq\sum_{d\geqslant c}kG(d)=V_c(\mu).\end{aligned}$$

Hence $\bigoplus_{\xi\in Q_+\setminus\{0\}}U_q(\mathfrak{g})^+_\xi G(c)\subseteq V_c(\mu)$ and furthermore,

$$U_q(\mathfrak{g})^+G(c)\subseteq V_c(\mu)\subseteq V_b(\mu).$$

It follows that $U_q(\mathfrak{g})^+V_b(\mu)=\sum_{c\geqslant b}U_q(\mathfrak{g})^+G(c)\subseteq V_b(\mu)$. Thus $V_b(\mu)$ is a nice $U_q(\mathfrak{g})^+$-submodule of $V(\mu)$. □

Clearly, the above proof is independent of the order on

$$B(\mu)_l=\{b\in B(\mu)\mid l(b)=l\}.$$

More generally, we can choose any total order on $B(\mu)$ such that $b_1<b_2$ if $wtb_1<wtb_2$.

For $b\in B(\mu)$, we define a $U_q(\mathfrak{g})$-submodule $F_\lambda(b)$ of $V(\lambda)\otimes V(\mu)$ generated by $u_\lambda\otimes V_b(\mu)$, i.e.

$$F_\lambda(b):=U_q(\mathfrak{g})(u_\lambda\otimes V_b(\mu)).$$

Since it follows from the coproduct formula that

$$U_q(\mathfrak{g})^+(u_\lambda\otimes V_b(\mu))=u_\lambda\otimes U_q(\mathfrak{g})^+V_b(\mu)=u_\lambda\otimes V_b(\mu)$$

and

$$u_\lambda \otimes V_b(\mu) = \sum_{c \geqslant b} k u_\lambda \otimes G(c) = \sum_{c \geqslant b} k G(u_\lambda \otimes c),$$

$u_\lambda \otimes V_b(\mu)$ is a nice $U_q(\mathfrak{g})^+$-submodule of $V(\lambda) \otimes V(\mu)$. We have the following proposition according to Lemma 3.1.

Proposition 3.8. *For λ, $\mu \in P_+$ and $b \in B(\mu)$, $F_\lambda(b)$ is a nice $U_q(\mathfrak{g})$-submodule of $V(\lambda) \otimes V(\mu)$. Moreover,*

$$B(F_\lambda(b)) = \{\ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes c) \mid i_1, \cdots, i_l \in I, \quad l \geqslant 0, \quad c \geqslant b\} \setminus \{0\}.$$

Theorem 3.9. *For λ, $\mu \in P_+$, $\{\ F_\lambda(b) \mid b \in B(\mu)\}$ forms a nice ascending filtration of $V(\lambda) \otimes V(\mu)$ as the following,*

$$0 \subseteq F_\lambda(b_1) \subseteq F_\lambda(b_2) \subseteq F_\lambda(b_3) \subseteq \cdots \tag{1}$$

where $u_\mu = b_1 > b_2 > b_3 > \cdots$ is a complete list of $B(\mu)$. Moreover, for two neighbors $c > b$ in $B(\mu)$, $F_\lambda(b)/F_\lambda(c) \cong V(\lambda + wtb)$ if $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$, otherwise $F_\lambda(b) = F_\lambda(c)$.

Proof. It suffices to show the second half. We have $B(F_\lambda(b)) \supseteq B(F_\lambda(c))$ if $c > b$ are two neighbors in $B(\mu)$. Claim that

$$B(F_\lambda(b)) \setminus B(F_\lambda(c)) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes b) \mid i_1, \cdots, i_l \in I, l \geqslant 0\} \setminus \{0\}$$

if $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$, otherwise $B(F_\lambda(b)) = B(F_\lambda(c))$. Indeed, if $B(F_\lambda(b)) \setminus B(F_\lambda(c))$ is non-empty, it follows from Proposition 3.8 that any element in $B(F_\lambda(b)) \setminus B(F_\lambda(c))$ is of the form $\tilde{f}_{j_1} \cdots \tilde{f}_{j_k}(u_\lambda \otimes d)$ for some $j_1, \cdots, j_k \in I$, $k \geqslant 0$ and $d \in B(\mu)$ where $c > d \geqslant b$ and it implies $d = b$. Hence if $u_\lambda \otimes b \in B(F_\lambda(b)) \setminus B(F_\lambda(c))$, we have

$$B(F_\lambda(b)) \setminus B(F_\lambda(c)) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(u_\lambda \otimes b) \mid i_1, \cdots, i_l \in I, l \geqslant 0\} \setminus \{0\},$$

otherwise if $u_\lambda \otimes b \in B(F_\lambda(c))$, $B(F_\lambda(b)) = B(F_\lambda(c))$. If $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$, assume that $u_\lambda \otimes b \notin B(F_\lambda(b)) \setminus B(F_\lambda(c))$. We have $u_\lambda \otimes b \in B(F_\lambda(c))$ and it is of the form $\tilde{f}_{l_1} \cdots \tilde{f}_{l_t}(u_\lambda \otimes d)$ for some $l_1, \cdots, l_t \in I$, $t \geqslant 0$ and $d \in B(\mu)$ where $d \geqslant c > b$. Since $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$, it implies $t = 0$ and $u_\lambda \otimes b = u_\lambda \otimes d$ which is a contradiction. Thus $u_\lambda \otimes b \in B(F_\lambda(b)) \setminus B(F_\lambda(b_1))$. Conversely, if $\tilde{e}_i(u_\lambda \otimes b) \neq 0$ for some $i \in I$, $\tilde{e}_i(u_\lambda \otimes b) = u_\lambda \otimes \tilde{e}_i b \neq 0$ where $wt\tilde{e}_i b = wtb + \alpha_i$. It follows that $\tilde{e}_i b > b$ and furthermore, $\tilde{e}_i b \geqslant c$. Hence

$$u_\lambda \otimes b = \tilde{f}_i \tilde{e}_i(u_\lambda \otimes b) = \tilde{f}_i(u_\lambda \otimes \tilde{e}_i b) \in B(F_\lambda(c)).$$

We have proved the claim which implies the theorem. □

By deleting superfluous terms in the filtration (1), we have a nice composition series of $V(\lambda)\otimes V(\mu)$.

Corollary 3.10. *For λ, $\mu \in P_+$, there is a nice ascending composition series of $U_q(\mathfrak{g})$-module $V(\lambda)\otimes V(\mu)$ by listing the elements in $\{F_\lambda(b) \mid b\in B(\mu), \tilde{e}_i(u_\lambda\otimes b) = 0 \ \ \forall i\in I\}$ according to the descending order on $B(\mu)$.*

Lusztig's conjecture for $\mathfrak{g}$ of finite type is then an immediate consequence of the Corollary 3.10.

Corollary 3.11. *For λ, $\mu\in P_+$ and $\mathfrak{g}$ of finite type, there is a nice composition series of $U_q(\mathfrak{g})$-module $V(\lambda)\otimes V(-\mu)$ by listing the elements in $\{F_\lambda(b) \mid b\in B(-\mu), \tilde{e}_i(u_\lambda\otimes b)=0 \ \ \forall i\in I\}$ according to the descending order on $B(-\mu)$.*

Example 3.12. *For $\mathfrak{g}=sl_3$, consider the $U_q(\mathfrak{g})$-mod $V(\Lambda_1)\otimes V(-\Lambda_1-\Lambda_2)$ as in Example 2.4. Since $V(-\Lambda_1-\Lambda_2)\cong V(\Lambda_1+\Lambda_2)$ where the total order on the crystal basis $B(\Lambda_1+\Lambda_2)$ of $V(\Lambda_1+\Lambda_2)$ is given as in Example 3.6, there exists a nice filtration of the tensor product*

$$\begin{aligned}0 \subseteq & F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\Big)\subseteq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\Big)\subseteq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\Big)\subseteq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\Big)\subseteq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\Big)\\ \subseteq & F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\Big)\subseteq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\Big)\subseteq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}\Big)=V(\Lambda_1)\otimes V(-\Lambda_1-\Lambda_2).\end{aligned}$$

One can check that $u_{\Lambda_1}\otimes \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$, $u_{\Lambda_1}\otimes \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$, $u_{\Lambda_1}\otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ are maximal vectors while the others are not. Hence

$$0\subsetneq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\Big)\subsetneq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\Big)\subsetneq F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\Big)=V(\Lambda_1)\otimes V(-\Lambda_1-\Lambda_2)$$

is the nice composition series of $V(\Lambda_1)\otimes V(-\Lambda_1-\Lambda_2)$ where $F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\Big)\cong V(2\Lambda_1+\Lambda_2)$, $F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\Big)/F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\Big)\cong V(2\Lambda_2)$, $F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\Big)/F_{\Lambda_1}\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\Big)\cong V(\Lambda_1)$.

From the proof of Theorem 3.9 one can derive the generalized Littlewood-Richardson rule for symmetrizable Kac-Moody algebra $\mathfrak{g}$, that is,

$$V(\lambda)\otimes V(\mu)\cong \bigoplus_{b\in B(\mu),\ \tilde{e}_i(u_\lambda\otimes b)=0\ \forall i\in I} V(\lambda+wtb).$$

This generalized Littlewood-Richardson rule was proved by Littelmann using path model [9], see also [4]. One can see from the tensor rule of crystal bases that $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$ is equivalent to

$$\tilde{e}_i^{\langle h_i, \lambda\rangle + 1} b = 0 \text{ for all } i \in I$$

and such a crystal basis element b is called λ-*dominant* in [9].

3.3. **Comparison With Lusztig's Composition Series.** As stated in the introduction, one can also construct a composition series of $V(\lambda) \otimes V(\mu)$ in an inductive way due to Lusztig. To be precise, for any $M \in \mathcal{O}_{int}$ with a canonical basis, we write M as a direct sum of isotypical components $M = \bigoplus_{\xi \in P_+} M[\xi]$. Let λ_1 be a maximal weight in the set $\{\xi \in P_+ \mid M[\xi] \neq 0\}$. We can see from the proof of Proposition 27.1.7 in [14] that there exists a nice submodule $V_1 \cong V(\lambda_1)$ of M. Go on this procedure by changing M to $M_2 := M/V_1$ and so on. Thus we have a nice $U_q(\mathfrak{g})$-submodule $V_i \cong V(\lambda_i)$ of M_i for some $\lambda_i \in P_+$ maximal in the weights of M_i where $M_1 = M$ and $M_{i+1} = M_i/V_i$. Let π_i be the canonical map $\pi_i : M_i \longrightarrow M_{i+1}$. We obtain then a sequence consisting of nice surjective $U_q(\mathfrak{g})$-maps

$$M = M_1 \xrightarrow{\pi_1} M_2 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{i-1}} M_i \xrightarrow{\pi_i} M_{i+1} \xrightarrow{\pi_{i+1}} \cdots .$$

We define $F_i(M)$ to be the kernel of $\pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_1$ for $i \geqslant 1$ and set $F_0(M) = 0$. One can see easily from the construction that

$$0 = F_0(M) \subseteq F_1(M) \subseteq \cdots \subseteq F_i(M) \subseteq F_{i+1}(M) \subseteq \cdots \tag{2}$$

is a nice composition series of M where $F_i(M)/F_{i-1}(M) \cong V(\lambda_i)$. Furthermore, it is clear to see that $\lambda_i \geqslant \lambda_j$ for $i < j$ if they are comparable. In particular, for $\lambda, \mu \in P_+$, there is a nice composition series of $V(\lambda) \otimes V(\mu)$. We denote by F_i the $U_q(\mathfrak{g})$-submodule $F_i(V(\lambda) \otimes V(\mu))$ of $V(\lambda) \otimes V(\mu)$ defined above for simplicity.

Let b'_j be the unique highest weight element in $B(F_j) \setminus B(F_{j-1})$. We know from the previous subsection that $b'_j \in B(\lambda) \otimes B(\mu)$ is of the form $u_\lambda \otimes c_j$ for some $c_j \in B(\mu)$ such that $\tilde{e}_i(u_\lambda \otimes c_j) = 0$ for all $i \in I$. One can see that $\lambda_j = \lambda + wtc_j$ and $\{c_j \mid j = 1, 2, \cdots\}$ is a complete set of elements b such that $u_\lambda \otimes b$ is maximal. One can arrange a total order on $B(\mu)$ satisfying the following two conditions,

(i) for $b, c \in B(\mu)$, $b < c$ if $wtb < wtc$.
(ii) $c_1 > c_2 > c_3 > \cdots > c_j > c_{j+1} > \cdots$.

Indeed we can define u_μ to be the maximum in $B(\mu)$ (one can see $u_\mu = c_1$), then choose an element in $B(\mu) \setminus \{u_\mu\}$ maximal in weight to be the second and so on only to ensure that $c_1 > c_2 > c_3 > \cdots > c_j > c_{j+1} > \cdots$. It is feasible since one can see from the inductive construction of composition series that $wtc_i \geqslant wtc_j$ for $i < j$ if they are comparable. Once such a total order on $B(\mu)$ is fixed, we immediately obtain, by Corollary 3.10, a nice composition series of $V(\lambda) \otimes V(\mu)$

$$0 \subseteq F_\lambda(c_1) \subseteq F_\lambda(c_2) \subseteq \cdots \subseteq F_\lambda(c_i) \subseteq F_\lambda(c_{i+1}) \subseteq \cdots \tag{3}$$

It is clear that (3) coincides with (2) when $M = V(\lambda) \otimes V(\mu)$, i.e. $F_i = F_\lambda(c_i)$.

Conversely, if we construct the nice composition series of $V(\lambda) \otimes V(\mu)$

$$0 := F_\lambda(b_0) \subseteq F_\lambda(b_1) \subseteq F_\lambda(b_2) \subseteq \cdots \subseteq F_\lambda(b_i) \subseteq F_\lambda(b_{i+1}) \subseteq \cdots \tag{4}$$

as in the previous subsection, it can be seen from the choice of total order that $\lambda_i \geqslant \lambda_j$ for $i < j$ if they are comparable where $\lambda_i \in P_+$ is such that $F_\lambda(b_i)/F_\lambda(b_{i-1}) \cong V(\lambda_i)$. Hence for $M = V(\lambda) \otimes V(\mu) = M_1$, we define $M_i = M/F_\lambda(b_{i-1})$, $V_i = F_\lambda(b_i)/F_\lambda(b_{i-1})$ and π_i as stated above. It follows easily that the composition series inductively constructed is exactly (4), i.e. $F_i(M) := ker(\pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_1) = F_\lambda(b_i)$. Hence we get the same nice composition series of the tensor product in two different approaches.

4. Nice Filtration of $V(\lambda) \otimes V(-\mu)$

4.1. **Filtration.** In the previous section we have proved, by Corollary 3.11, Lusztig's conjecture that the $U_q(\mathfrak{g})$-module $V(\lambda) \otimes V(-\mu)$ has a nice composition series for $\mathfrak{g}$ of finite type and $\lambda, \mu \in P_+$. For an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{g}$, the $U_q(\mathfrak{g})$-module $V(\lambda) \otimes V(-\mu)$ also admits a canonical basis as mentioned previously. But the tensor product may have infinite dimensional weight spaces (when λ and μ are both nontrivial) and have no maximal weights. Therefore it does not belong to category $\mathcal{O}_{int}$ and Lusztig's approach to construct nice submodules of $V(\lambda) \otimes V(-\mu)$ fails while our method still works in this case. To be precise, though we cannot obtain a composition series of the tensor product in general, we find a nice filtration of it instead which helps us to understand the structure of this module.

Indeed, we can define a total order on $B(-\mu)$ similarly. For $b \in B(-\mu)$ which is of the form $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l} u_{-\mu}$, set $l(b) = l$ and define $|b|$ to be the $l(b)$-tuple

$(i_1, \cdots, i_{l(b)})$ such that $(i_1, \cdots, i_{l(b)})$ is minimal in lexicographic order among tuples $(j_1, \cdots, j_{l(b)})$ such that $\tilde{e}_{j_1} \cdots \tilde{e}_{j_{l(b)}} u_{-\mu} = b$, i.e.

$$|b| = min\{(j_1, \cdots, j_{l(b)}) \mid b = \tilde{e}_{j_1} \cdots \tilde{e}_{j_{l(b)}} u_{-\mu}\}.$$

Set $|u_{-\mu}| = 0$. A total order on $B(-\mu)$ is defined as

$$b_1 \leqslant b_2 \text{ iff } l(b_1) < l(b_2) \text{ or } l(b_1) = l(b_2) \text{ but } |b_1| \leqslant |b_2|.$$

As in section 3, for $b \in B(-\mu)$, $V_b(-\mu)$ is defined as a k-subspace of $V(-\mu)$ spanned by all $G(c)$ such that $c \geqslant b$ and let $F_\lambda(b)$ be the $U_q(\mathfrak{g})$-submodule of $V(\lambda) \otimes V(-\mu)$ generated by $u_\lambda \otimes V_b(-\mu)$, i.e.

$$F_\lambda(b) := U_q(\mathfrak{g})(u_\lambda \otimes V_b(-\mu)).$$

We have the following theorem by Lemma 3.1, which can be similarly proved as Theorem 3.9.

Theorem 4.1. *For λ, $\mu \in P_+$, $\{ F_\lambda(b) \mid b \in B(-\mu)\}$ forms a nice descending filtration of $V(\lambda) \otimes V(-\mu)$ as the following*

$$V(\lambda) \otimes V(-\mu) = F_\lambda(b_1) \supseteq F_\lambda(b_2) \supseteq F_\lambda(b_3) \supseteq \cdots \tag{5}$$

where $u_{-\mu} = b_1 < b_2 < b_3 < \cdots$ is a complete list of $B(-\mu)$. Moreover, for two neighbors $b < c$ in $B(-\mu)$, $F_\lambda(b)/F_\lambda(c) \cong V(\lambda + wtb)$ if $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$, otherwise $F_\lambda(b) = F_\lambda(c)$.

Actually the order on $B(-\mu)$ can be chosen only to satisfy the property that $b_1 < b_2$ if $wtb_1 < wtb_2$. In contrast to Corollary 3.11, usually we cannot get a nice composition series of $V(\lambda) \otimes V(-\mu)$ by deleting superfluous terms in (5). More precisely, the intersection of all submodules in (5) might be nonzero. For example, when $\mathfrak{g}$ is of affine type and $\lambda - \mu$ is of a negative level, $F_\lambda(b) = V(\lambda) \otimes V(-\mu)$ for all $b \in B(-\mu)$.

Similarly, with the order on $B(\lambda)$ defined in section 3, we can construct another nice filtration of $V(\lambda) \otimes V(-\mu)$. For $b \in B(\lambda)$, define $F_{-\mu}(b)$ to be the $U_q(\mathfrak{g})$-submodule of $V(\lambda) \otimes V(-\mu)$ generated by $G(c) \otimes u_{-\mu}$ for all $c \leqslant b$. Note that when we change $U_q(\mathfrak{g})^+$ to $U_q(\mathfrak{g})^-$, Lemma 3.1 is also true which implies the following theorem.

Theorem 4.2. *For λ, $\mu \in P_+$, $\{ F_{-\mu}(b) \mid b \in B(\lambda)\}$ forms a nice descending filtration of $V(\lambda) \otimes V(-\mu)$ as the following,*

$$V(\lambda) \otimes V(-\mu) = F_{-\mu}(b_1) \supseteq F_{-\mu}(b_2) \supseteq F_{-\mu}(b_3) \supseteq \cdots \tag{6}$$

where $u_\lambda = b_1 > b_2 > b_3 > \cdots$ *is a complete list of* $B(\lambda)$*. Moreover, for two neighbors* $b > c$ *in* $B(\lambda)$*,* $F_{-\mu}(b)/F_{-\mu}(c) \cong V(-\mu + wtb)$ *if* $\tilde{f}_i(b \otimes u_{-\mu}) = 0$ *for all* $i \in I$*, otherwise,* $F_{-\mu}(b) = F_{-\mu}(c)$*.*

4.2. **Affine Type Case.** For $\lambda \in P$, note that there is a subcrystal $B^{max}(\lambda)$ of $B(U_q(\mathfrak{g})a_\lambda)$ consisting of some $*$-extremal elements which is exactly the crystal basis of extremal weight module $V^{max}(\lambda)$ (see [8] for details). It is proved in [8] that

$$V^{max}(\lambda) \cong V^{max}(w\lambda)$$

for any $w \in W$ and $V^{max}(\lambda) \cong V(\lambda)$ for $\lambda \in \pm P_+$.

Proposition 4.3. *(*[8]*) For any connected component* B *of* $\widetilde{B}$*, there is an* $l > 0$ *such that* $(wtb, wtb) \leqslant l$ *for all* $b \in B$*. Moreover,* B *contains an extremal vector and can be embedded into* $B^{max}(\lambda)$ *for some* $\lambda \in P$*.*

For $\mathfrak{g}$ of affine type, let $c \in \mathfrak{h}$ be the canonical central element of $\mathfrak{g}$. Given $\lambda \in P$, we define $\langle c, \lambda \rangle$ to be the level of λ, denoted by *level*(λ). The corollary below follows immediately from Proposition 4.3.

Corollary 4.4.
(i) *For* λ *with level*$(\lambda) > 0$*,* $B(U_q(\mathfrak{g})a_\lambda)$ *is a union of highest weight crystals.*
(ii) *For* λ *with level*$(\lambda) < 0$*,* $B(U_q(\mathfrak{g})a_\lambda)$ *is a union of lowest weight crystals.*

It follows from the corollary that for $\lambda, \mu \in P_+$, $B(\lambda, -\mu)$ is a union of highest (*resp.* lowest) weight crystals if *level*$(\lambda - \mu) > 0$ (*resp.* *level*$(\lambda - \mu) < 0$). We define $W(\lambda, -\mu)$ (*resp.* $U(\lambda, -\mu)$) to be a k-subspace $\bigcap_{b \in B(-\mu)} F_\lambda(b)$ (*resp.* $\bigcap_{b \in B(\lambda)} F_{-\mu}(b)$) of $V(\lambda) \otimes V(-\mu)$ and set

$$M(\lambda, -\mu) = (V(\lambda) \otimes V(-\mu))/W(\lambda, -\mu)$$
$$(\textit{resp.} \quad N(\lambda, -\mu) = (V(\lambda) \otimes V(-\mu))/U(\lambda, -\mu)).$$

Denote by $B^+(\lambda, -\mu)$ (*resp.* $B^-(\lambda, -\mu)$) the subcrystal of $B(\lambda, -\mu)$ which is the union of all connect components of $B(\lambda, -\mu)$ that are not highest (*resp.* lowest) weight crystals.

Proposition 4.5. *For* $\lambda, \mu \in P_+$*,*

(i) *both* $W(\lambda, -\mu)$ *and* $U(\lambda, -\mu)$ *are nice* $U_q(\mathfrak{g})$*-submodules of* $V(\lambda) \otimes V(-\mu)$*. Moreover,* $B(W(\lambda, -\mu)) = B^+(\lambda, -\mu)$ *and* $B(U(\lambda, -\mu)) = B^-(\lambda, -\mu)$*.*

(ii) *both* $M(\lambda, -\mu)$ *and* $N(\lambda, -\mu)$ *admit canonical bases and* $B(M(\lambda, -\mu)) = B(\lambda, -\mu) \setminus B^{+}(\lambda, -\mu)$, $B(N(\lambda, -\mu)) = B(\lambda, -\mu) \setminus B^{-}(\lambda, -\mu)$.

Proof. $W(\lambda, -\mu)$ admits a $U_q(\mathfrak{g})$-action since every $F_\lambda(b)$ does. The conclusion for $W(\lambda, -\mu)$ in (i) follows from Theorem 3.9 and that any maximal vector in $B(\lambda, -\mu)$ is of the form $u_\lambda \otimes b$ with $b \in B(-\mu)$ and $\varepsilon_i(b) \leqslant \langle h_i, \lambda\rangle$ for all $i \in I$. It is similar for $U(\lambda, -\mu)$ and (ii) is implied by (i). □

When $\mathfrak{g}$ is of finite type, one can see that $W(\lambda, -\mu) = U(\lambda, -\mu) = 0$ and both (1) and (6) provide composition series of $V(\lambda) \otimes V(-\mu)$ by deleting superfluous terms.

For two crystals B_1 and B_2 where B_1 is connected, let $[B_2 : B_1]$ be the cardinality of the set which consists of all connected components of B_2 isomorphic to B_1, i.e. $[B_2 : B_1] = \{B \subset B_2 \mid B \cong B_1\}^{\#}$.

Theorem 4.6. *For* $\lambda \in P_+$ *and* $\mu \in P$, $[B(U_q(\mathfrak{g})a_\mu) : B(\lambda)] = dim V(\lambda)_\mu$.

Proof. We only need to find out all maximal vectors in $B(U_q(\mathfrak{g})a_\mu)$. Note that $B(U_q(\mathfrak{g})a_\mu) = B(\infty) \otimes T_\mu \otimes B(-\infty)$ and $\tilde{e}_i$ acts on it as

$$\tilde{e}_i(b_1 \otimes t_\mu \otimes b_2) = \begin{cases} (\tilde{e}_i b_1) \otimes t_\mu \otimes b_2 \text{ if } \varphi_i(b_1) + \langle h_i, \mu\rangle \geqslant \varepsilon_i(b_2) \\ b_1 \otimes t_\mu \otimes (\tilde{e}_i b_2) \text{ if } \varphi_i(b_1) + \langle h_i, \mu\rangle < \varepsilon_i(b_2). \end{cases}$$

Assume that $b_1 \otimes t_\mu \otimes b_2$ is maximal, since $\tilde{e}_i b_2 \neq 0$ for all $b_2 \in B(-\infty)$, we have $\tilde{e}_i b_1 = 0$ and

$$\varphi_i(b_1) + \langle h_i, \mu\rangle \geqslant \varepsilon_i(b_2) \tag{7}$$

for all $i \in I$. Hence $b_1 = u_\infty$ which is the image of 1.

Now, we claim that $u_\infty \otimes t_\mu \otimes b_2$ is a maximal vector of weight λ iff $wtb_2 = \lambda - \mu$ and $\varphi_i(b_2) \leqslant \langle h_i, \lambda\rangle$ for all $i \in I$. Indeed, if $u_\infty \otimes t_\mu \otimes b_2$ is maximal and $wt(u_\infty \otimes t_\mu \otimes b_2) = \mu + wtb_2 = \lambda$, then $wtb_2 = \lambda - \mu$ and (7) holds which can be rewritten as $\langle h_i, \mu\rangle \geqslant \varepsilon_i(b_2)$ since $\varphi_i(u_\infty) = 0$. It follows from $\varphi_i(b_2) - \varepsilon_i(b_2) = \langle h_i, wtb_2\rangle$ that $\langle h_i, \mu\rangle \geqslant \varphi_i(b_2) - \langle h_i, wtb_2\rangle$ which implies $\varphi_i(b_2) \leqslant \langle h_i, \lambda\rangle$. The other side of the claim is easy to prove.

It has been shown by Kashiwara in [5] that for $\xi \in P_+$ there is an embedding of crystals

$$\tau : \ B(-\xi) \longrightarrow T_{-\xi} \otimes B(-\infty)$$

whose image is $Im\tau = \{t_{-\xi} \otimes b \mid \varphi_i^*(b) \leqslant \langle h_i, \xi\rangle \ \forall\ i \in I\}$. Hence for $\eta \in P$,

$$\{b \in B(-\infty)_{\xi-\eta} | \ \varphi_i^*(b) \leqslant \langle h_i, \xi\rangle \ \forall\ i \in I\}^{\#} = dimV(-\xi)_{-\eta} = dimV(\xi)_\eta \tag{8}$$

Recall that $*$ acts bijectively on $B(-\infty)$. By restricting the $*$-action on

$$\{b \in B(-\infty) \mid \varphi_i(b) \leqslant \langle h_i, \lambda\rangle \ \forall\ i \in I\},$$

we get a bijection between $\{b \in B(-\infty) \mid \varphi_i(b) \leqslant \langle h_i, \lambda\rangle \ \forall\ i \in I\}$ and $\{b \in B(-\infty) \mid \varphi_i^*(b) \leqslant \langle h_i, \lambda\rangle \ \forall\ i \in I\}$. Hence there is a bijection between

$$\{b \in B(-\infty)_{\lambda-\mu} \mid \varphi_i(b) \leqslant \langle h_i, \lambda\rangle \ \forall\ i \in I\}$$

and

$$\{b \in B(-\infty)_{\lambda-\mu} \mid \varphi_i^*(b) \leqslant \langle h_i, \lambda\rangle \ \forall\ i \in I\}.$$

From (8) and the claim above we know that the number of maximal vectors in $B(U_q(\mathfrak{g})a_\mu)$ of weight λ equals

$$\{b \in B(-\infty)_{\lambda-\mu} | \varphi_i(b) \leqslant \langle h_i, \lambda\rangle \ \forall\ i \in I\}^{\#} = dimV(\lambda)_\mu.$$

□

Let P_0 be the subset of P_+ consisting of weights λ such that $\langle h_i, \lambda\rangle = 0$ for all $i \in I$. We have the following corollary.

Corollary 4.7. (i) *$W(\lambda, -\mu) = N(\lambda, -\mu) = 0$ and $M(\lambda, -\mu) = U(\lambda, -\mu) = V(\lambda) \otimes V(-\mu)$ if $level(\lambda - \mu) > 0$.*

(ii) *$W(\lambda, -\mu) = N(\lambda, -\mu) = V(\lambda) \otimes V(-\mu)$ and $M(\lambda, -\mu) = U(\lambda, -\mu) = 0$ if $level(\lambda - \mu) < 0$.*

(iii) *$M(\lambda, -\mu) = N(\lambda, -\mu)$ is a 1-dimensional trivial module if $\lambda - \mu \in P_0$, otherwise if $\lambda - \mu \notin P_0$ is of level 0, $W(\lambda, -\mu) = U(\lambda, -\mu) = V(\lambda) \otimes V(-\mu)$ and $M(\lambda, -\mu) = N(\lambda, -\mu) = 0$.*

Proof. (*i*), (*ii*) come from Corollary 4.4. (*iii*) holds since there is no highest or lowest weight subcrystal in $B(\lambda, -\mu)$ if $\lambda - \mu \notin P_0$ is of level 0 while there is only one trivial subcrystal for $\lambda - \mu \in P_0$ by Theorem 4.6. □

We can see from this corollary that for $\mathfrak{g}$ of affine type, (5) (*resp.* (6)) provides a nice composition series of $V(\lambda) \otimes V(-\mu)$ by deleting superfluous terms when $\lambda - \mu$ is of a positive (*resp.* negative) level.

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