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Non-simply Laced McKay Correspondence and Triality

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Abstract: The classical McKay correspondence establishes a one-to-one correspondence between finite subgroups of $SU(2)$ and simply-laced root systems, namely root systems of ADE type. In this article, we extend the McKay correspondence to all root systems, simply-laced or not, and relate this correspondence to triality of quaternions.
Keywords: McKay correspondence, quaternion, triality, crepant resolution.

1. Introduction

1.1. The McKay correspondence says that, conjugacy classes of finite subgroups $\widetilde{\Gamma}$ of $SU(2)$, are in one-to-one correspondence with simply-laced simple Lie algebras, or equivalently, Dynkin diagrams of ADE type.

In this article, we extend this correspondence to all simple Lie algebras, simply-laced or not, using the triality of the quaternions $\mathbb{H}$. This correspondence for non-simply laced Dynkin diagrams is more or less known. We give an explicit and unified description, and relate it to the triality of the quaternions $\mathbb{H}$.

Note that $SU(2) = Sp(1)$ acts on $\mathbb{H}$ by left multiplications and the adjoint map $Ad : SU(2) \to SO(3)$ is the universal covering of $SO(3)$, which is the automorphism group of $\mathbb{H}$ as a normed division algebra, i.e. $SO(3) = Aut(\mathbb{H})$ and $SU(2) = \widetilde{Aut}(\mathbb{H})$. Recall that there are only four normed division algebras $\mathbb{A}$ and they are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. Such an algebraic structure is closely related to the notion of a normed triality:

$$t : V_1 \times V_2 \times V_3 \to \mathbb{R}.$$

Indeed given any triality triple $\vec{v} = (v_1, v_2, v_3)$, we obtain canonical identifications $V_1 \cong V_2 \cong V_3 \cong V_3^*$ and t determines a product structure on V_i. This makes V_i

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into a normed division algebra $\mathbb{A}$. We write $t_{\mathbb{A}}$ instead of t and we denote the set of all triality triples as $Tri_{\mathbb{A}}$.

The symmetry group $Aut(t_{\mathbb{A}}) \subset O(V_1)\times O(V_2)\times O(V_3)$ of triality is isomorphic to $SU(2)^3/\mathbb{Z}_2$ (resp. $Spin(8)$) when $\mathbb{A}$ equals $\mathbb{H}$ (resp. $\mathbb{O}$). Also we denote the universal cover of $Aut(t_{\mathbb{A}})$ by $\widetilde{Aut}(t_{\mathbb{A}})$. Then $\widetilde{Aut}(t_{\mathbb{H}}) = SU(2)^3$. The projection to each V_i induces a homomorphism $p_i : \widetilde{Aut}(t_{\mathbb{A}}) \to O(V_i)$. In general, $Aut(t_{\mathbb{A}})$ acts transitively on $Tri_{\mathbb{A}}$ and the isotropy group is isomorphic to $Aut(\mathbb{A})$, which equals $SO(3)$ (resp. G_2) when $\mathbb{A}$ equals $\mathbb{H}$ (resp. $\mathbb{O}$). As a result, subgroups of $Aut(\mathbb{A})$ correspond to subgroups of $Aut(t_{\mathbb{A}})$ which fix some point in $Tri_{\mathbb{A}}$.

From the above discussions, the McKay correspondence can be rephrased as a correspondence between simply laced root systems and finite subgroups of $\widetilde{Aut}(t_{\mathbb{H}})$ fixing some element in $Tri_{\mathbb{H}}$, and we call such a subgroup *an isotropic subgroup*. Our main result says that if we consider pairs of finite isotropic subgroups of $\widetilde{Aut}(t_{\mathbb{H}})$ inducing the same symmetries on each V_i, then the McKay correspondence can be extended to *all* root systems.

A pair $(\mathfrak{g}, \tau)$ with $\mathfrak{g}$ a (complex) simple Lie algebra and τ an outer automorphism determines a Lie algebra $\mathfrak{g}^\tau$ by taking the fixed part. Equivalently, a simply laced Dynkin diagram D and a diagram automorphism τ determines a Dynkin diagram D_τ by taking the folding. All non-simply laced simple Lie algebras (or equivalently, Dynkin diagrams) can be obtained from simply laced ones in such a way. Namely, $\mathfrak{g}^\tau = B_n, C_n, F_4, G_2$ when $\mathfrak{g} = D_{n+1}, A_{2n-1}, E_6, D_4$ respectively with τ being of order 2 except for D_4, and 3 for D_4.

Theorem 1. *There is a one-to-one correspondence between the pairs $(\mathfrak{g}, \tau)$ and the equivalence classes of pairs of finite isotropic subgroups $\widetilde{\Gamma}, \widetilde{\Gamma}' \subset \widetilde{Aut}(t_{\mathbb{H}})$ of the same order satisfying $p_i(\widetilde{\Gamma}) = p_i(\widetilde{\Gamma}') \subset O(V_i) \cong O(4)$ for all i.*

To prove this result, we first establish the following non-simply laced McKay correspondence. Our construction is different from the classical restricted-induced construction in [14] (see also [10]).

Theorem 2. *The equivalence classes of pairs $(\widetilde{\Gamma}, O_v)$ with $\widetilde{\Gamma}$ a finite subgroup of $SU(2)$ and O_v an element of the outer automorphism group of $\widetilde{\Gamma}$ induced by $Ad(v)$ where $v \in \mathbb{H}$, are in one-to-one correspondence with the pairs $(\mathfrak{g}, \tau)$ as above. In particular, we obtain all non-simply laced root systems from the pairs $(\widetilde{\Gamma}, O_v)$.*

The Dynkin diagram associated with a pair $(\widetilde{\Gamma}, O_v)$ is obtained as follows. Let $W_0, W_1, \cdots, W_n$ be all irreducible representations of $(\widetilde{\Gamma}, O_v)$. W_0 is the trivial one. Let W be the standard representation of $SU(2)$. Assign a node p_l to each W_l. Assume $W_l \otimes W \cong \oplus_m a_{lm} W_m$. When $a_{lm} = a_{ml} \neq 0$ (both are equal to 1), we draw a_{lm} edges to connect the nodes p_l and p_m. When $a_{lm} \neq a_{ml}$, one of them,

$(\widetilde{\Gamma}, O_v)$	$D_{(\widetilde{\Gamma}, O_v)}$
$\mathcal{C}_{n+1}$, $O_v = 1$	A_n :
$B\mathcal{D}_{n-2}$, $O_v = 1$	D_n :
$B\mathcal{T}$, $O_v = 1$	E_6 :
$B\mathcal{O}$, $O_v = 1$	E_7 :
$B\mathcal{I}$, $O_v = 1$	E_8 :
$\mathcal{C}_{2n}$, $\lvert O_v \rvert = 2$	C_n :
$\mathcal{C}_{2n+1}$, $\lvert O_v \rvert = 2$	A_n :
$B\mathcal{D}_{n-1}$, $\lvert O_v \rvert = 2$	B_n :
$B\mathcal{T}$, $\lvert O_v \rvert = 2$	F_4 :
$B\mathcal{D}_2$, $\lvert O_v \rvert = 3$	G_2 :

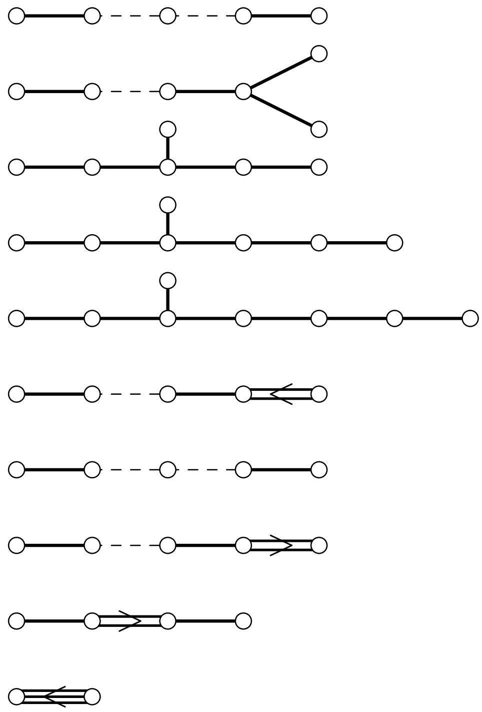

FIGURE 1. Dynkin diagrams associated with the pairs $(\widetilde{\Gamma}, O_v)$

say a_{lm}, must be 2 or 3, and another is 1. In this case, we draw a_{lm} directed edges from p_m to p_l. Thus, we obtain a diagram $D^{aff}_{(\widetilde{\Gamma}, O_v)}$. Removing the node p_0 and all the edges with p_0 as an endpoint from $D^{aff}_{(\widetilde{\Gamma}, O_v)}$, we obtain a diagram $D_{(\widetilde{\Gamma}, O_v)}$, and it is a Dynkin diagram. Conversely, each Dynkin diagram can be obtained in such a way. When $O_v = id$, we obtain a simply laced Dynkin diagram. We illustrate this correspondence in Figure 1, where the groups $\mathcal{C}_n$, $B\mathcal{D}_n$, $B\mathcal{T}$, $B\mathcal{O}$, and $B\mathcal{I}$ are respectively the cyclic, binary dihedron, binary tetrahedron, binary octahedron, and binary icosahedron groups.

1.2. Remark. When $\mathbb{A} = \mathbb{R}$ the similar correspondence is rather trivial, as $Aut(\mathbb{R}) = \{1, -1\}$. When $\mathbb{A} = \mathbb{C}$ we have $U(1) = Aut(\mathbb{C})$. Finite subgroups of $U(1) = S^1$ are finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ ($n \in \mathbb{N}$). These finite groups are also subgroups of $SO(3) = Aut(\mathbb{H})$, since $Aut(\mathbb{C}) \subset Aut(\mathbb{H})$.

When $\mathbb{A} = \mathbb{O}$, the octonions or the Cayley numbers, we have $Aut(\mathbb{O}) = G_2$ and $Aut(t_{\mathbb{O}}) = Spin(8)$. Finite subgroups Γ of $SU(3) \subset G_2$ were studied by many mathematicians, as well as physicists, for example, see [8][12][13][15][16] and the references therein, and the McKay correspondence relates the geometry of the Calabi-Yau threefolds which are the crepant resolutions of $\mathbb{C}^3/\Gamma$ and the orbifold (or stingy) geometry of $\mathbb{C}^3/\Gamma$ ([12][13]). There is also a version at the level of derived categories, see [3]. It is natural to ask a similar question for $\mathbb{R}^7/\Gamma$ and its G_2-resolution for finite subgroup $\Gamma \subseteq G_2$. It is also interesting to understand the octonions analogue of our result here.

1.3. In this short article, we first recall in §2 the classification of finite subgroups of $SU(2)$, and study their (special) outer automorphisms (i.e. automorphisms induced by adjoint actions).

In §3, we present an explicit description for non-simply laced McKay correspondence. Our result is Theorem 16 and Corollary 17 (see also Theorem 2).

In §4, we explain the relation between non-simply laced McKay correspondence and triality. The result is Theorem 1 (see also Theorem 25 and Corollary 26).

2. Finite Subgroups of $SU(2)$

2.1. Finite Subgroups of $SO(3)$ and $SU(2)$, and the Quaternions. Recall, finite subgroups of $SO(3)$ are classified completely, up to conjugacy (see [5][7]).

Proposition 3. *The classification of finite subgroups Γ of $SO(3)$, up to conjugacy, is listed in the following table. Here A, B, C are the generators of Γ. The last column lists the regular polyhedrons whose symmetry groups are the respective finite groups.*

Group, Γ	*Order*	*Generating Relations*	*(Regular) Polyhedrons*
Cyclic, $\mathcal{C}_n$	n	$A^n = B^n = C^1 = ABC = 1$	*n-gon's in $\mathbb{R}^2$*
Dihedron, $\mathcal{D}_n$	$2n$	$A^n = B^2 = C^2 = ABC = 1$	*n-gon's in $\mathbb{R}^3$*
Tetrahedron, $\mathcal{T}$	12	$A^3 = B^3 = C^2 = ABC = 1$	*Tetrahedrons*
Octahedron, $\mathcal{O}$	24	$A^4 = B^3 = C^2 = ABC = 1$	*Octahedrons*
Icosahedron, $\mathcal{I}$	60	$A^5 = B^3 = C^2 = ABC = 1$	*Icosahedron*

Let $\mathbb{H}$ be the algebra of quaternions, with the basis $1, \mathbb{i}, \mathbb{j}, \mathbb{k}$. For $a = a_0 + a_1\mathbb{i} + a_2\mathbb{j} + a_3\mathbb{k} \in \mathbb{H}$ with $a_i \in \mathbb{R}$, we denote $Re(a) = a_0 \in Re(\mathbb{H})$, $Im(a) = a_1\mathbb{i} + a_2\mathbb{j} + a_3\mathbb{k} \in Im(\mathbb{H})$, and the conjugate $\bar{a} = Re(a) - Im(a)$. When $a_0 = 0$, a is called a pure quaternion. If a is a unit quaternion, then a can be written as the form

$$a = \cos\alpha + y\sin\alpha = e^{y\alpha},$$

where $\cos\alpha = Re(a)$ and y is a pure unit quaternion.

The set of all unit quaternions forms a multiplicative group, which is isomorphic to $SU(2)$:

$$a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k} \mapsto \begin{pmatrix} a + b\mathbb{i} & c + d\mathbb{i} \\ -c + d\mathbb{i} & a - b\mathbb{i} \end{pmatrix}.$$

Identifying $\mathbb{H}$ with $\mathbb{C}^2$ by the map

$$x = x_0 + x_1\mathbb{i} + x_2\mathbb{j} + x_3\mathbb{k} \mapsto (x_0 + x_1\mathbb{i}, -x_2 + x_3\mathbb{i}),$$

we see that $g(x) = g \cdot x$, for any $x \in \mathbb{H}, g \in SU(2)$.

In $\mathbb{R}^3 = Im(\mathbb{H})$, the reflections and rotations can be represented by quaternion multiplications.

Remark 4. *(See* [6]*) Let y be a unit pure quaternion.*

(1) *Denote S_y the reflection of $Im(\mathbb{H})$ in the plane perpendicular to y. Then S_y is represented by the quaternion transformation*

$$S_y(x) = yxy \text{ for } x \in Im(\mathbb{H}).$$

(2) *Denote $R_{(y,\alpha)}$ the rotation of $Im(\mathbb{H})$ about y through 2α. Then $R_{(y,\alpha)}$ is represented by the quaternion transformation*

$$R_{(y,\alpha)}(x) = e^{-\alpha y} x e^{\alpha y} \text{ for } x \in Im(\mathbb{H}).$$

(3) *The group of all unit quaternions is $2:1$ homomorphic to the group of all rotations that leave the origin fixed, that is, the group $SO(3)$. The kernel of this homomorphism is $\{\pm 1\}$.*

Let $\widetilde{\Gamma} \neq \{0\}$ be a finite subgroup of unit quaternions, or equivalently $\widetilde{\Gamma} \subset SU(2)$. There are two different classes.

The first class: If $\widetilde{\Gamma}$ contains -1, then by Remark 4, $\widetilde{\Gamma}$ is $2:1$ homomorphic to a finite subgroup Γ of $SO(3)$. By Proposition 3, there are 5 cases. Moreover, in terms of generators and generating relations, $\widetilde{\Gamma}$ can be written as the form $\widetilde{\Gamma} = < A, B, C >$ where $A^p = B^q = C^r = ABC = -1$.

Proposition 5. *Let $-1 \in \widetilde{\Gamma}$. Then the classification is listed in the following table. Here (A, B, C) in the third, forth and fifth row are respectively* $(1/2(1 + \mathbb{i} + \mathbb{j} + \mathbb{k}), 1/2(1 + \mathbb{i} - \mathbb{j} + \mathbb{k}), \mathbb{i})$, $(1/2(1 + \mathbb{i} + \mathbb{j} + \mathbb{k}), 1/2(1 + \mathbb{i} - \mathbb{j} + \mathbb{k}), \mathbb{i})$ *and* $(\cos\pi/5 + \mathbb{k}\cos\pi/3 + \mathbb{i}\sin\pi/10, \cos\pi/3 + \mathbb{k}\cos\pi/5 + \mathbb{j}\sin\pi/10, \mathbb{k})$.

Group, $\widetilde{\Gamma}$	Γ	Order	(p,q,r)	(A,B,C)
Cyclic, $B\mathcal{C}_n$	$\mathcal{C}_n$	$2n$	$(n,n,1)$	$(e^{\pi\mathbb{i}/n}, e^{-\pi\mathbb{i}/n}, -1)$
Binary Dihedron, $B\mathcal{D}_n$	$\mathcal{D}_n$	$4n$	$(n,2,2)$	$(e^{\pi\mathbb{i}/n}, \mathbb{j}, e^{\pi\mathbb{i}/n}\mathbb{j})$
Binary Tetrahedron, $B\mathcal{T}$	$\mathcal{T}$	24	$(3,3,2)$	(A,B,C)
Binary Octahedron, $B\mathcal{O}$	$\mathcal{O}$	48	$(4,3,2)$	(A,B,C)
Binary Icosahedron, $B\mathcal{I}$	$\mathcal{I}$	160	$(5,3,2)$	(A,B,C)

The second class: If $\widetilde{\Gamma}$ does not contain -1, each of its elements $e^{y\alpha}$ represents uniquely the rotation through 2α about y, so $\widetilde{\Gamma}$ is isomorphic to a subgroup of $SO(3)$. In this case, none of the rotations can be a half-turn, because then the corresponding quaternion would be pure, and its square would be -1. Looking through the list of finite subgroups of $SO(3)$, we see that the only kind not containing a half-turn is the cyclic group $\mathcal{C}_n$, where n is odd. Hence we have

Proposition 6. *The only finite subgroups of quaternion not containing -1 are the cyclic groups of odd order.*

Altogether, we obtain the complete classification of the finite subgroups of $SU(2)$.

Theorem 7. *(see [6]) Let $\widetilde{\Gamma}$ be a finite subgroup of quaternion.*

(1) *If $\widetilde{\Gamma}$ does not contain -1, then $\widetilde{\Gamma}$ is a cyclic subgroup of odd order.*
(2) *Otherwise, $\widetilde{\Gamma}$ is one of the five types listed in Proposition 5.*

2.2. The Outer Automorphisms. By definition,

$$Inn(\widetilde{\Gamma}) = \{Ad(x)|x \in \widetilde{\Gamma}, Ad(x)(y) := x^{-1}yx\}.$$

Slightly different from the standard notations, we denote

$$Out(\widetilde{\Gamma}) = (Aut(\widetilde{\Gamma}) \cap Ad(SU(2)))/Inn(\widetilde{\Gamma})$$

and call it the *outer automorphism group* of $\widetilde{\Gamma}$ by abuse of ambiguity. For $v \in SU(2) \subset \mathbb{H}$, we shall use O_v to denote the image of $Ad(v)$ in $Out(\widetilde{\Gamma})$ if $Ad(v) \in Aut(\widetilde{\Gamma})$. So when $v \in \widetilde{\Gamma}$, $O_v = id_{\widetilde{\Gamma}}$.

Theorem 8. *Up to obvious equivalences, $(\widetilde{\Gamma}, O_v)$ are classified into the following types.*

(1) *When $O_v = id$, the classification of $(\widetilde{\Gamma}, O_v)$ is the same as that of $\widetilde{\Gamma}$.*
(2) *Otherwise, there are the following four types.*
    (i) *$\widetilde{\Gamma} = \mathcal{C}_n$, and O_v is of order 2. When $n = 2m$ is even, that is $\widetilde{\Gamma} = B\mathcal{C}_m$, we call $(\widetilde{\Gamma}, O_v)$ of C_n-type.*
    (ii) *$\widetilde{\Gamma} = B\mathcal{D}_n$, and O_v is of order 2. $(\widetilde{\Gamma}, O_v)$ is called of B_{n+1}-type.*

(iii) $\widetilde{\Gamma} = B\mathcal{T}$, *and* O_v *is of order* 2. $(\widetilde{\Gamma}, O_v)$ *is called of* F_4*-type.*
(iv) $\widetilde{\Gamma} = B\mathcal{D}_2$, *and* O_v *is of order* 3. $(\widetilde{\Gamma}, O_v)$ *is called of* G_2*-type.*

Proof. This is a direct consequence of the following Proposition 9. □

Proposition 9. *The outer automorphism group (defined as above) of* $\widetilde{\Gamma}$ *is* $\mathbb{Z}_2$ *for* $\widetilde{\Gamma} = \mathcal{C}_n, B\mathcal{D}_n (n \neq 2)$ *or* $B\mathcal{T}$. *For* $\widetilde{\Gamma} = B\mathcal{D}_2$, *it is* S_3, *the symmetry group of* 3 *letters. In all other cases, the outer automorphism group is trivial.*

Proof. Let $g = e^{\alpha y} = \cos\alpha + y \sin\alpha \in \widetilde{\Gamma}$ and $v = e^{x\beta} \in SU(2)$. Then $Ad(v)(g) = v^{-1}e^{\alpha y}v = v^{-1}(\cos\alpha)v + v^{-1}(y\sin\alpha)v = \cos\alpha + (v^{-1}yv)\sin\alpha = e^{\alpha(v^{-1}yv)}$. Thus we can see that $Ad(v)$ transforms the rotation by 2α about the vector $y \in Im(\mathbb{H})$ into the rotation about $v^{-1}yv \in Im(\mathbb{H})$ by the same angle. But the action $y \mapsto v^{-1}yv$ itself is the rotation by 2β with the vector x, since $v = e^{x\beta}$. Then according to the classification of finite subgroups of $SO(3)$ (or $SU(2)$), it suffices to show the existence of the non-trivial outer automorphisms O_v with $v \in SU(2)$, in the case where $-1 \in \widetilde{\Gamma}$. For convenience, we suppose the generators of $\widetilde{\Gamma}$ are taken as in Remark 5. For $\widetilde{\Gamma} = \mathcal{C}_n$ ($B\mathcal{D}_n(n \neq 2)$ or $B\mathcal{T}$, respectively), we take $v = \mathbb{j}$ (respectively, $e^{-\pi\mathbb{i}/2n}$, $\mathbb{i}e^{\pi\mathbb{j}/4}$). One can check directly that v satisfies the condition. For $\widetilde{\Gamma} = B\mathcal{D}_2$, one can check that O_{v_1} and O_{v_2} generate the outer automorphism group S_3, where $v_1 = e^{(\pi/3)(\mathbb{i}+\mathbb{j}+\mathbb{k})/\sqrt{3}}$ and $v_2 = (\mathbb{i}+\mathbb{j})/\sqrt{2}$. For the case where $\widetilde{\Gamma}$ does not contain -1, the result comes from the following lemma. □

Lemma 10. *Take* $(\widetilde{\Gamma}, O_v)$ *as above. Let* $g \in \widetilde{\Gamma}$ *be an element of order* $d > 1$. *If* $O_v(g) = g^l$ *then* $l \equiv \pm 1$ *mod* (d). *If* $g = e^{2\pi\mathbb{i}/d}$, *we can take* $v = \mathbb{k}$ *such that* $O_v(g) = g^{-1}$.

Proof. Note that $\widetilde{\Gamma}$ must be conjugate with $e^{2\pi\mathbb{i}/d}$ in $SU(2)$. Let $g = ue^{2\pi\mathbb{i}/d}u^{-1}$ with $u \in SU(2)$. Then $hgh^{-1} = g^l$ implies that $hue^{2\pi\mathbb{i}/d}u^{-1}h^{-1} = ue^{2\pi\mathbb{i}/d}u^{-1}$. Thus $(u^{-1}hu)e^{2\pi\mathbb{i}/d}(u^{-1}hu)^{-1} = e^{2\pi\mathbb{i}k/d}$. Therefore we can assume $g = e^{2\pi\mathbb{i}/d}$. As elements of $SU(2)$,

$$g = \begin{pmatrix} e^{2\pi\mathbb{i}/d} & 0 \\ 0 & e^{-2\pi\mathbb{i}/d} \end{pmatrix} \quad \text{and} \quad \mathrm{h} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. One can check that there are only two solutions for (a, b, l): $(1, 0, 1)$ and $(0, 1, -1)$. □

Example 11. *In the case that* $\widetilde{\Gamma} = B\mathcal{D}_2$, *we can take* $\widetilde{\Gamma} = \{\pm 1, \pm\mathbb{i}, \pm\mathbb{j}, \pm\mathbb{k}\}$, $v = e^{(2\pi/3)(\mathbb{i}+\mathbb{j}+\mathbb{k})/\sqrt{3}}$ *and* $Ad(v)$ *acts just as the permutation* $(\mathbb{i}\mathbb{j}\mathbb{k})$. *This is the classical triality.*

3. McKay Correspondence and Dynkin Diagrams

In this section we first recall the classical McKay correspondence between subgroups of $SU(2)$ and ADE diagrams [10]. To obtain non-simply laced Dynkin diagrams, one just takes the "folding". In [14], Slodowy considered the relation between finite subgroups of $SU(2)$, Dynkin diagrams and simple singularities, and he realized the diagram automorphisms as the automorphisms of the desingularizations.

By considering the irreducible representations and the conjugacy classes associated with a finite subgroup $\widetilde{\Gamma}$ with an outer automorphism, we give an explicit and unified description for McKay correspondence in all cases.

Let us first recall how to obtain ADE-Dynkin diagrams from finite subgroups of $SU(2)$. Let $\widetilde{\Gamma}$ be a finite subgroup of $SU(2)$. Let $W = \mathbb{C}^2$ be the standard representation of $SU(2)$. And let $W_l, l = 0, 1, \cdots, s$ be all of the irreducible representations of $\widetilde{\Gamma}$, where W_0 is the trivial representation, and $s+1$ is the number of the conjugacy classes of $\widetilde{\Gamma}$ (recall that the number of irreducible representations equals to the number of conjugacy classes, for a finite group). Then we have

$$W \otimes W_l \cong \bigoplus_m a_{lm} W_m,$$

where a_{lm} is the multiplicity of W_m in this decomposition. Assigning a node p_l to each W_l and drawing a_{lm} edges to connect p_l and p_m, we obtain the so-called *affine McKay quiver* of $\widetilde{\Gamma}$, denoted by $D^{aff}(\widetilde{\Gamma})$. McKay found that $D^{aff}(\widetilde{\Gamma})$ was an affine Dynkin diagram of type A_n (respectively D_n, E_6, E_7, E_8) for $\widetilde{\Gamma} = \mathcal{C}_{n+1}$ (respectively $B\mathcal{D}_{n-2}, B\mathcal{T}, B\mathcal{O}, B\mathcal{I}$). If we remove the node p_0 and all the edges with p_0 as an endpoint, we obtain the corresponding Dynkin diagrams (of finite type), denoted as $D(\widetilde{\Gamma})$.

In the following we consider the pair $(\widetilde{\Gamma}, O_v)$, where v is a unit quaternion.

Definition 12. *An irreducible representation of the pair $(\widetilde{\Gamma}, O_v)$ is a representation W of Γ, which satisfies the following two conditions.*

(a) *W is invariant under $Ad(v)$, that is, $Ad(v)$ preserves the characters of W.*
(b) *W can not be written as a direct sum $W_1 \oplus W_2$ with W_i satisfying the condition (a).*

Definition 13. *An O_v-conjugacy class of $(\widetilde{\Gamma}, O_v)$ is an equivalence class of conjugacy classes of $\widetilde{\Gamma}$, where two conjugacy classes $\bar{g}_1$ and $\bar{g}_2$ are called equivalent to each other if $Ad(v)(\bar{g}_1) = \bar{g}_2$ or $Ad(v)(\bar{g}_2) = \bar{g}_1$.*

The outer automorphism group $Out(\widetilde{\Gamma})$ acts on the set of the representations of $\widetilde{\Gamma}$ as follows. Let $O_v \in Out(\widetilde{\Gamma})$, and V be a representation of $\widetilde{\Gamma}$. We obtain a new representation V' by composing the action of $\widetilde{\Gamma}$ on V with $O_v : \widetilde{\Gamma} \to \widetilde{\Gamma}$.

Theorem 14. *Given any finite subgroup $\widetilde{\Gamma} \subseteq SU(2)$ invariant under $O_v \in Out(\widetilde{\Gamma})$, we have*

(i) *O_v induces a diagram automorphism on $D(\widetilde{\Gamma})$;*
(ii) *the number of irreducible representations of $(\widetilde{\Gamma}, O_v)$ is equal to the number of O_v-conjugacy classes of $(\widetilde{\Gamma}, O_v)$.*

Proof. (ii) is a direct consequence of (i). We check (i) directly in each case.

For $\widetilde{\Gamma} = \mathcal{C}_n$, $\widetilde{\Gamma}$ is generated by $g = e^{2\pi \mathbb{i}/n}$. For $l = 0, \cdots, n-1$, let $W_l \cong \mathbb{C}$ be the l-th irreducible representation, namely, it is one-dimensional with basis e_l satisfying $g(e_l) = g^l \cdot e_l$. Note that W_0 is the trivial irreducible representation. Take $v = \mathbb{j}$. Then $O_v \neq id_G$, and $O_v(g) = g^{-1}$. So O_v transforms the character of W_l to the one of W_{n-l}. That is, O_v induces a diagram automorphism of $D(\widetilde{\Gamma})$.

For $\widetilde{\Gamma} = B\mathcal{D}_n$ and O_v of order 2. As a subgroup of $SU(2)$, we can take $\widetilde{\Gamma}$ to be generated by two elements

$$g = \begin{pmatrix} e^{\mathbb{i}\pi/n} & 0 \\ 0 & e^{-\mathbb{i}\pi/n} \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $W_l \cong \mathbb{C}^2$ be the l-th 2-dimensional representation, with the action given by $g(e_l, f_l) = (e_l, f_l)g^l, h(e_l, f_l) = ((-1)^l f_l, e_l)$, where e_l, f_l is a basis of W_l. Note that W_l is irreducible for $l \neq 0, n$. W_0 splits into two irreducible representations W_{01} and W_{02}, where W_{02} is the trivial one and $g|_{W_{01}} = id, h|_{W_{01}} = -id$. Also W_n splits into $W_n = W_{n1} \oplus W_{n2}$. When n is even, $W_{n1} = \mathbb{C} < e_n + f_n >$, $W_{n2} = \mathbb{C} < e_n - f_n >$. When n is odd, $W_{n1} = \mathbb{C} < e_n + \mathbb{i} f_n >$, $W_{n2} = \mathbb{C} < e_n - \mathbb{i} f_n >$. Take $v = \mathbb{k}$, then O_v just interchanges the characters of W_{n1} and W_{n2}.

The situation for $B\mathcal{T}$ is similar but more complicated. The character table for $B\mathcal{T}$ is

char	1	C	-1	A^2	B^2	A	B
1	1	1	1	1	1	1	1
2	2	0	-2	-1	-1	1	1
3	3	-1	3	0	0	0	0
$2'$	2	0	-2	$-\rho$	$-\rho^2$	ρ^2	ρ
$2''$	2	0	-2	$-\rho^2$	$-\rho$	ρ	ρ^2
$1'$	1	1	1	ρ	ρ^2	ρ^2	ρ
$1''$	1	1	1	ρ^2	ρ	ρ	ρ^2

where $\rho = e^{2\pi i/3}$, and the entries in the first column are all irreducible representations indexed by the affine Dynkin diagram of E_6. Let O_v interchange the generators A and B. Then O_v interchanges pairwise the 5-th and the 6-th columns, the 7-th and the 8-th columns in this table. Hence it also interchanges pairwise the 5-th and the 6-th rows, the 7-th and the 8-th rows. That is, O_v induces a diagram automorphism of $D(\widetilde{\Gamma})$.

The proof for $G = B\mathcal{D}_2$ is very easy, according to Remark 11. □

Similar to the classical case, we define the McKay quiver for a pair $(\widetilde{\Gamma}, O_v)$ as follows.

Definition 15. *Given a pair $(\widetilde{\Gamma}, O_v)$, let $W_l, l = 0, \cdots, n$ be all irreducible representations of $(\widetilde{\Gamma}, O_v)$, where W_0 is the trivial one, and $n+1$ is the number of the O_v-conjugacy classes. Assign a node p_l to each irreducible representation W_l. Let $W = \mathbb{C}^2$ be the standard representation of $SU(2)$. Assume that $W_l \otimes W \cong \oplus_m a_{lm} W_m$. When $a_{lm} = a_{ml} \neq 0$ (it must be equal to 1 in this situation), we draw a_{lm} edges to connect the nodes p_l and p_m. When $a_{lm} \neq a_{ml}$, one of them, say a_{lm}, must be 2 or 3, and another is 1. In this case, we draw a_{lm} directed edges from p_m to p_l. Thus, we obtain a diagram, called the affine McKay quiver of $(\widetilde{\Gamma}, O_v)$, and denoted by $D^{aff}_{(\widetilde{\Gamma}, O_v)}$. Removing the node p_0 and all the edges with p_0 as an endpoint from $D^{aff}_{(\widetilde{\Gamma}, O_v)}$, we obtain a diagram, denoted by $D_{(\widetilde{\Gamma}, O_v)}$, called the (finite) McKay quiver of $(\widetilde{\Gamma}, O_v)$.*

Theorem 16. *Given any finite subgroup $\widetilde{\Gamma} \subseteq SU(2)$ and $O_v \in Out(\widetilde{\Gamma})$, $D_{(\widetilde{\Gamma}, O_v)}$ must be one of the following types.*

(1) *For $O_v = id_{\widetilde{\Gamma}}$, $D_{(\widetilde{\Gamma}, O_v)}$ is a Dynkin diagram of ADE-type.*
(2) *Suppose O_v is non-trivial.*
   (i) *For $\widetilde{\Gamma} = \mathcal{C}_{2n+1}$, $D_{(\widetilde{\Gamma}, O_v)}$ is of A_n-type.*
   (ii) *For $\widetilde{\Gamma} = B\mathcal{C}_n = \mathcal{C}_{2n}$, $D_{(\widetilde{\Gamma}, O_v)}$ is of C_n-type.*
   (iii) *For $\widetilde{\Gamma} = B\mathcal{D}_n$ and $(O_v)^2 = id_{\widetilde{\Gamma}}$, $D_{(\widetilde{\Gamma}, O_v)}$ is of B_{n+1}-type.*
   (iv) *For $\widetilde{\Gamma} = B\mathcal{T}$, $D_{(\widetilde{\Gamma}, O_v)}$ is of F_4-type.*
   (v) *For $\widetilde{\Gamma} = B\mathcal{D}_2$ and $(O_v)^3 = id_{\widetilde{\Gamma}}$, $D_{(\widetilde{\Gamma}, O_v)}$ is of G_2-type.*

Proof. (1) is the classical McKay correspondence, see [10]. By Theorem 14, we obtain non-simply laced Dynkin diagrams from simply laced ones. Let $\{W_l | l = 0, 1, \cdots, s\}$ is the set of all irreducible representations of $(\widetilde{\Gamma}, O_v)$, where $s+1$ is the number of the O_v-conjugacy classes of $(\widetilde{\Gamma}, O_v)$, and W_0 is the trivial one. Let $a_{lm}, l, m = 0, \cdots, s$ be the non-negative numbers determined by the following decomposition

$$W_l \otimes W \cong \bigoplus_m a_{lm} W_m.$$

We only need to verify the number of undirected edges connecting the nodes p_l and p_m is equal to a_{lm} where $a_{lm} = a_{ml}$; and the number of directed edges connecting from p_m to p_l is equal to a_{lm} where $a_{lm} > a_{ml} = 1$.

This follows from a direct checking. For example, we compute $D_{(\widetilde{\Gamma}, O_v)}$ for $\widetilde{\Gamma} = \mathcal{C}_{2k}$ and $O_v \neq id$. Just as in the proof of Theorem 14, suppose $\widetilde{\Gamma}$ is generated by $g = e^{2\pi \mathbb{i}/(2k)}$. Let $W_l \cong \mathbb{C}$ is the l-th irreducible representation of $\widetilde{\Gamma}$, which is one-dimensional with basis e_l, where $g(e_l) = g^l \cdot e_l$, for $l = 0, \cdots, 2k-1$. Note that W_0 is the trivial irreducible representation. Take $v = \mathbb{j}$. Then $O_v \neq id_{\widetilde{\Gamma}}$. And $O_v(g) = g^{-1}$. So the irreducible representations for $(\widetilde{\Gamma}, O_v)$ are $U_0 = W_0, U_l = W_l \oplus W_{2k-l}, 0 < l < k, U_k = W_k$. Let $U_l = \bigoplus_m a_{lm} U_m$. Then we can see that a_{lm} satisfies the constrains, since $U_0 \otimes W = U_1$, $U_1 \otimes W = 2U_0 \oplus U_2$, $U_l \otimes W = U_{l-1} \oplus U_{l+1}$, for $1 < l < k-2$ and $U_{k-1} \otimes W = U_{k-2} \oplus 2U_k$, $U_k \otimes W = U_{k-1}$. □

Corollary 17. *Except the case $(\widetilde{\Gamma}, O_v)$ where $\widetilde{\Gamma} = \mathcal{C}_{2k+1}$ and $O_v \neq id_{\widetilde{\Gamma}}$, $D_{(\widetilde{\Gamma}, O_v)}$ is just one of the affine Dynkin diagrams of untwisted type, and the matrix $2I - (a_{ij})$ is just the Cartan matrix of the corresponding affine Dynkin diagram.*

Remark 18. *In fact, this outer automorphism O_v induces an outer automorphism of a regular polyhedron $\mathcal{P}$ (that is, interchanging the vertices and the faces of $\mathcal{P}$). For example, the tetrahedron has such a non-trivial outer automorphism. Accordingly the binary tetrahedron group as well as its McKay quiver also has such an outer automorphism.*

4. Triality and Non-simply Laced McKay Correspondence

We have seen that the McKay correspondence in non-simply laced cases is induced by that one in simply laced case with a symmetry of $\mathbb{H}$. Essentially, all symmetries of $\mathbb{H}$ come from the triality property on $\mathbb{H}$. In this section, we formulate this correspondence in terms of triality.

4.1. Triality. In the following, we recall the theory on triality. For references, see [1] and [2]. Let $V_i, i = 1, 2, 3$ be three real vector spaces of finite dimension.

Definition 19. *A triality is a trilinear map*

$$t : V_1 \otimes V_2 \otimes V_3 \to \mathbb{R}$$

such that for any non-zero $v_1 \in V_1, v_2 \in V_2$ there exists a $v_3 \in V_3$ such that $t(v_1 \otimes v_2 \otimes v_3) \neq 0$ (and similarly for $v_1, v_3 \neq 0, v_2, v_3 \neq 0$). If each V_i has a norm, we say that f is a normed triality if $|t(v_1 \otimes v_2 \otimes v_3)| \leq ||v_1|| \cdot ||v_2|| \cdot ||v_3||$, *and for all* $v_1, v_2 \neq 0$*, there is a* $v_3 \neq 0$ *for which the bound is attained (and similarly for the other two cases).*

Example 20. $V_1 = V_2 = V_3 = \mathbb{R}, \mathbb{C}$*, or* $\mathbb{H}$ *respectively and take* $t(x \otimes y \otimes z) = Re(xyz)$*. Then* t *is a normed triality.*

Theorem 21. *A triality exists only if* $dimV_1 = dimV_2 = dimV_3 = 1, 2, 4$ *or* 8*.*

Proof. See [1] for reference. For later uses, we sketch the proof. Suppose we are given a triality t. Then for any $e_1 \neq 0$ we get a duality of V_2, V_3, so we must have $dimV_2 = dimV_3$ and similarly $dimV_1 = dimV_2 = dimV_3$. We can transpose V_3 to get $t' : V_1 \otimes V_2 \to V_3^*$. Choose $e_1 \neq 0$ in V_1 and use it to identify V_2 with V_3^* by $v_2 \mapsto t'(e_1 \otimes v_2)$. Similarly choose $e_2 \neq 0$ in V_2 to identify V_1 with V_3^*. We now have $t'' : V_3^* \otimes V_3^* \to V_3^*$ for which $t''(e_1 \otimes e_2)$ acts as a two-sided unit and t'' is non-singular in that if $x, y \neq 0 \in V_3^*$, then $t''(x \otimes y) \neq 0$. So we have a division algebra $\mathbb{A}$ over $\mathbb{R}$ and consequently $dimV_i = 1, 2, 4$ or 8. If we start with a normed triality, we then obtain a normed algebra $\mathbb{A}$, and in this case, we should take unit vectors e_1, e_2 and e_3. □

From the above proof we see that for a normed triality $t : V_1 \times V_2 \times V_3 \to \mathbb{R}$, given a triple $\vec{v} = (v_1, v_2, v_3)$ where $v_i \in V_i, i = 1, 2, 3$ such that $||v_i|| = 1$ and $t(v_1, v_2, v_3) = 1$, we obtain a normed division algebra $\mathbb{A}$. Such a triple $\vec{v} = (v_1, v_2, v_3)$ is called a *triality triple*. We will write $t_{\mathbb{A}}$ instead of t and denote the set of all triality triples as $Tri_{\mathbb{A}}$, that is

$$Tri_{\mathbb{A}} = \{(v_1, v_2, v_3) \in V_1 \times V_2 \times V_3 \mid t(v_1, v_2, v_3) = 1, ||v_i|| = 1, i = 1, 2, 3\}.$$

It is well-known that there are only four normed (finite) $\mathbb{R}$-algebras and they are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

An automorphism of the normed triality $t : V_1 \times V_2 \times V_3 \to \mathbb{R}$ is a triple of norm-preserving maps $f_i : V_i \to V_i$ such that

$$t(f(\vec{v})) = t(\vec{v})$$

for all $\vec{v} \in V_1 \times V_2 \times V_3$, where $f = (f_1, f_2, f_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $f(\vec{v}) = (f_1(v_1), f_2(v_2), f_3(v_3))$. These automorphisms form a group we call $Aut(t_{\mathbb{A}})$. Take a triality triple $\vec{v} = (v_1, v_2, v_3) \in Tri_{\mathbb{A}}$, and denote

$$Aut(t_{\mathbb{A}}, \vec{v}) = \{f \in Aut(t_{\mathbb{A}}) : f(\vec{v}) = \vec{v}\}.$$

Lemma 22. $Aut(t_{\mathbb{A}})$ *acts transitively on the set* $Tri_{\mathbb{A}}$ *of triality triples, and there is a canonical isomorphism* $\psi_{\vec{v}} : Aut(\mathbb{A}) \xrightarrow{\sim} Aut(t_{\mathbb{A}}, \vec{v})$ *for any triality triple* $\vec{v}$*.*

Proof. The first part is trivial since given two unit vectors u_i, v_i in V_i, there is a norm-preserving map $f_i \in Aut(V_i)$, such that $f_i(u_i) = v_i$. To prove the second part, we identify V_i to $\mathbb{A}$ by identifying v_i to the identity element 1 of $\mathbb{A}$. Then for any $f \in Aut(t_{\mathbb{A}}, \vec{v})$, under above identifications, f preserves the algebra structure of $\mathbb{A}$, therefore $f \in Aut(\mathbb{A})$. The converse is also true. □

Corollary 23. *For all $\vec{u}, \vec{v} \in Tri_{\mathbb{A}}$, there exists a $\phi_{\vec{u}\vec{v}} \in Aut(t_{\mathbb{A}})$, such that $Aut(t_{\mathbb{A}}, \vec{u}) = \phi_{\vec{u}\vec{v}}^{-1}(Aut(t_{\mathbb{A}}, \vec{v}))\phi_{\vec{u}\vec{v}}$.*

Let $p_i : Aut(t_{\mathbb{A}}) \to O(V_i)$ be the homomorphism induced by the projection to the i^{th} component: $V_1 \times V_2 \times V_3 \to V_i, i = 1, 2, 3$. Then by construction, for any $\vec{v} \in Tri_{\mathbb{A}}$, the map $p_i : Aut(t_{\mathbb{A}}, \vec{v}) \xrightarrow{\sim} p_i(Aut(t_{\mathbb{A}}, \vec{v})) \subset O(V_i)$ is an isomorphism onto its image, for all i.

4.2. McKay Correspondence via Triality of $\mathbb{H}$. From now on, we assume $\mathbb{A} = \mathbb{H}$. The universal cover (i.e. double cover) $\widetilde{Aut}(t_{\mathbb{H}})$ of $Aut(t_{\mathbb{H}})$ is isomorphic to $SU(2)^3$. Let $\widetilde{\Gamma}$ be a finite subgroup of $\widetilde{Aut}(t_{\mathbb{H}})$ with the image $\Gamma \subset Aut(t_{\mathbb{H}})$. A subgroup of $\widetilde{Aut}(t_{\mathbb{H}})$ is called an *isotropic subgroup* if it fixes some element in $Tri_{\mathbb{H}}$. By Lemma 22, a finite isotropic subgroup of $\widetilde{Aut}(t_{\mathbb{H}})$ is in fact a subgroup of $\widetilde{Aut}(\mathbb{H}) \cong SU(2)$. Thus finite subgroups of $SO(3)$ (or $SU(2)$) are identified with finite isotropic subgroups of $Aut(t_{\mathbb{H}})$ (or $\widetilde{Aut}(t_{\mathbb{H}})$).

Considering the pairs of finite isotropic subgroups $\widetilde{\Gamma}, \widetilde{\Gamma}'$ of $\widetilde{Aut}(t_{\mathbb{H}})$, of the same order, with $p_1(\Gamma) = p_1(\Gamma')$ (or equivalently, for all p_i). In the following we show that the classification of such pairs is equivalent to the classification of the pairs $(\widetilde{\Gamma}, O_v)$, where $\widetilde{\Gamma} \subset SU(2), v \in \mathbb{H}$.

Assume $\vec{u}, \vec{v} \in Tri_{\mathbb{H}}$. According to [1], we have

$SO(3) = Aut(\mathbb{H}) \cong Aut(t_{\mathbb{H}}, \vec{u}) \subset Aut(t_{\mathbb{H}}) \cong (Sp(1) \times Sp(1) \times Sp(1))/\{\pm 1\}$.

Without loss of generality, we can take $\vec{u} = \vec{e} = (1, 1, 1)$ and $\vec{v} = (v_1, v_2, v_3)$. In this case, we can take $V_i = \mathbb{H}, i = 1, 2, 3$, then the map $t_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ given by $t(u, v, w) = Re(uvw)$ defines the normed triality of $\mathbb{H}$. And this triality implies that any triality triple (v_1, v_2, v_3) determines unique isomorphisms $V_1 \cong V_2 \cong V_3 = \mathbb{H}$ (every isomorphism is in fact an automorphism of $\mathbb{H}$). And $SO(3) = Aut(t_{\mathbb{H}}, \vec{u}) \cong Aut(t_{\mathbb{H}}, \vec{v})$, where the isomorphism, is just $\phi_{\vec{v}} := \phi_{\vec{u}\vec{v}}$ as above, is given by $g \mapsto (\phi_{\vec{v},1}(g), \phi_{\vec{v},2}(g), g)$ with $\phi_{\vec{v},i}$ being an isomorphism of groups induced by $\vec{v} = (v_1, v_2, v_3)$. Note that the dual space $\mathbb{H}^*$ is naturally identified with $\overline{\mathbb{H}} = \mathbb{H}$, where $\bar{x} = Re(x) - Im(x)$, since the inner product on $\mathbb{H}$ is defined as

$$< x, y >:= Re(\bar{x}y) = Re(x\bar{y}).$$

This implies for $v_1 \in V_1$, t induces an isomorphism from $V_2 = \mathbb{H}$ to $\mathbb{H}$ defined by $t_1 : y \mapsto v_1 y$ for $y \in V_2$. Thus $\phi_{\vec{v},1}(g) = t_1^{-1} g t_1$. Similarly $\phi_{\vec{v},2}(g) = t_2^{-1} g t_2$ with t_2 defined by $t_2(x) = x v_2$ for $x \in V_1$. Since v_1, v_2 are unit quaternions, $\phi_{\vec{v},1}, \phi_{\vec{v},2}$ are elements of $SU(2)$. Pulled back to $SU(2)$, $\phi_{\vec{v},1}, \phi_{\vec{v},2}$ induces automorphisms of $SU(2)$, which are just the conjugations $Ad(v_1) : g \mapsto v_1^{-1} g v_1$ and the identity.

The reason is the following. Let $g \in SU(2)$, then $g(x) = g \cdot x$, since we consider $SU(2)$ as the group of unit quaternions. We have the following commutative diagram:

$$\begin{array}{ccc} x & \xrightarrow{v_1 \cdot} & v_1 x \\ {\scriptstyle \phi_{\vec{v},1}(g)}\downarrow & & \downarrow{\scriptstyle g\cdot} \\ \phi_1(g)(x) & \xrightarrow{v_1 \cdot} & y. \end{array}$$

Thus $\phi_{\vec{v},1}(g) = Ad(v_1)(g) = v_1^{-1} g v_1$. Similarly, we have $\phi_{\vec{v},2}(g) = g$. Hence the isomorphism $\phi_{\vec{v}} = (Ad(v_1), id, id)$. Thus we have proved that

Proposition 24. *Let $\widetilde{\Gamma}, \widetilde{\Gamma}'$ be two finite subgroups of $\widetilde{Aut}(t_{\mathbb{H}})$ of the same order, fixing respectively $\vec{u}, \vec{v} \in Tri_{\mathbb{H}}$, and let $p_1(\widetilde{\Gamma}) = p_1(\widetilde{\Gamma}')$. Then $\phi_{\vec{u}\vec{v}}$ induces an (outer) automorphism on $p_1(\widetilde{\Gamma}) = p_1(\widetilde{\Gamma}')$.*

Let $\widetilde{\Gamma} \subset SU(2)$, and $\widetilde{\Gamma}', \widetilde{\Gamma''}$ be two finite isotropic subgroups of $\widetilde{Aut}(t_{\mathbb{H}})$, of the same order, and assume $p_1(\widetilde{\Gamma}') = p_1(\widetilde{\Gamma''})$. Let $v \in \mathbb{H}$. Consider the equivalence classes of the pairs $(\widetilde{\Gamma}', \widetilde{\Gamma''})$ in the obvious sense.

Theorem 25. *The equivalence classes of the pairs $(\widetilde{\Gamma}', \widetilde{\Gamma''})$ as above are in one-to-one correspondence with the equivalence classes of the pairs $(\widetilde{\Gamma}, O_v)$.*

Proof. Given a pair $(\widetilde{\Gamma}', \widetilde{\Gamma''})$ as above, up to conjugation, we can take $\vec{u} = (1, 1, 1)$ and $\vec{v} = (v_1, v_2, v_3)$. We take $\widetilde{\Gamma} \subset SU(2)$ to be the finite subgroup of the same order such that $\Gamma = p_1(\widetilde{\Gamma}') = p_1(\widetilde{\Gamma''})$, and we take $x = v_1$. Conversely, given $(\widetilde{\Gamma}, O_x)$, we take $\vec{u} = (1, 1, 1)$ and $\vec{v} = (x, x^{-1}, 1)$. Then we can take $\widetilde{\Gamma''}$ to be group consisting of the diagonal elements of $\widetilde{\Gamma} \times \widetilde{\Gamma} \times \widetilde{\Gamma}$, and $\widetilde{\Gamma}' = (Ad(x), id, id)(\widetilde{\Gamma''})$. Then by construction and by Proposition 24, one can easily see the above correspondence is one-to-one. ☐

Corollary 26. *The equivalence classes of the pairs $(\widetilde{\Gamma}', \widetilde{\Gamma''})$ as in Proposition 24, are in one-to-one correspondence with the pairs (D, τ) with D a simply laced Dynkin diagram and τ a diagram automorphism (see Figure 1).*

Remark 27. *The same arguments as the proof of Proposition 24 and Theorem 25 work in fact for all normed division algebras.*

Much less is known about general finite subgroups of $G_2 = Aut(\mathbb{O})$. For finite subgroups of $SU(3) \subset G_2$, there is a geometric McKay correspondence ([12][13] etc). But the McKay quiver is usually more complicated. We hope to have a detailed study of the octonion case in the future.

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