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Local GW Invariants of Elliptic Multiple Fibers

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Abstract: We use simple geometric arguments to calculate the dimension zero local Gromov-Witten invariants of elliptic multiple fibers. This completes the calculation of all dimension zero GW invariants of elliptic surfaces with $p_g > 0$.
Keywords: Local GW invariants, elliptic surfaces, spectral flow.

Let X be a Kähler surface with $p_g > 0$. By the Enriques-Kodaira classification (cf. [BHPV]), its minimal model is a $K3$ or Abelian surface, a surface of general type or an elliptic surface. Each holomorphic 2-form α on X defines an almost complex structure

$$J_\alpha = (Id + JK_\alpha)\, J\, (Id + JK_\alpha). \tag{0.1}$$

Here, J is the complex structure on X and the endomorphism K_α of TX is defined by the formula $\langle u, K_\alpha v\rangle = \alpha(u,v)$ where $\langle\ ,\ \rangle$ is the Kähler metric. This J_α satisfies:

Lemma 0.1 ([L]). *If f is a J_α-holomorphic map that represents a nontrivial (1,1) class then its image lies in the support of the zero divisor D_α of α and f is, in fact, holomorphic.*

The Gromov-Witten invariant $GW_{g,n}(X,A)$ is a (virtual) count of holomorphic maps representing the class A. In particular, the invariant $GW_{g,n}(X,A)$ vanishes unless A is a (1,1) class since every holomorphic map represents (1,1) class. Note

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that each canonical divisor D of X is a zero divisor of a holomorphic 2-form. Lemma 0.1 thus shows that the GW invariant is a sum

$$GW_{g,n}(X,A) \;=\; \sum GW^{loc}_{g,n}(D_k,A_k)$$

over the connected components D_k of the canonical divisor D of local invariants that counts the contribution of maps whose image lies in D_k (cf. [LP], [KL]). It follows that the GW invariants of minimal $K3$ or Abelian surfaces are trivial except possibly for the trivial homology class because their canonical divisors are trivial.

The local GW invariants have a universal property. If X is a minimal surface of general type with a smooth canonical divisor D then the local invariants associated with D, and hence GW invariants, are determined by the normal bundle of D — in fact, there exists a universal function of c_1^2 and c_2 that gives the GW invariants of X (cf. Section 7 of [LP]).

If $\pi : X \to C$ is a minimal elliptic surface with $p_g > 0$, after suitable deformation, we can assume X has a canonical divisor of the form

$$\sum_i n_i F^i \;+\; \sum_k (m_k - 1) F_{m_k}$$

where F^i is a regular fiber and F_{m_k} is a smooth multiple fiber of multiplicity m_k (cf. Proposition 6.1 of [LP]). In this case, the GW invariants of X are sums of universal functions, and are completely determined by the multiplicities m_k and the number

$$c_\pi \;=\; \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C)$$

(cf. Section 6 of [LP]). In particular, the generating function for the set of all *dimension zero* GW invariants of X is given by

$$GW^0_X \;=\; c_\pi \sum_{d>0} GW^{loc}_1(F,d)\, t^d \;+\; \sum_k \sum_{d>0} GW^{loc}_1(F_{m_k},d)\, t^d_{m_k} \tag{0.2}$$

where the formal variables t and t_{m_k} are for the fiber class $[F]$ and the multiple fiber classes $[F_{m_k}]$ respectively; these satisfy $t_{m_k}^{m_k} = t$. The local invariants in (0.2) are counts of multiple covers of elliptic curves together with signs determined by the GW theory of 4-manifolds.

Some of the generating functions in (0.2) are known. In cases of the regular fiber F and the multiple fiber F_2, it was proved in Section 10 of [LP] that

$$(0.3)\quad GW_1^{loc}(F,d) \;=\; -\frac{1}{d}\,\sigma(d) \quad\text{ and }\quad GW_1^{loc}(F_2,d) \;=\; \frac{1}{d}\left(\sigma(d) \,-\, 2\,\sigma\Big(\frac{d}{2}\Big)\right)$$

where $\sigma(d) = \sum_{k|d} k$ if d is a positive integer and $\sigma(d) = 0$ otherwise. In this note we use geometric arguments to obtain the terms in (0.2) associated with fibers of higher multiplicity. Our main theorem is the following formula for the local invariants $GW_1^{loc}(F_m, d)$ for $m > 2$. This completes the calculation of all dimension zero GW invariants of all minimal elliptic surfaces with $p_g > 0$.

Main Theorem. *Let $m \geq 3$. Then*

$$GW_1^{loc}(F_m,d) \;=\; \frac{1}{d}\left(\sigma(d) \,-\, m\,\sigma\Big(\frac{d}{m}\Big)\right).$$

The contribution of each degree d cover f of elliptic curve F_m is, as a map into a 4-manifold, determined by the normal bundle N_m of F_m. In cases of $F_1 = F$ and F_2, the almost complex structure J_α on X is generic in the sense that the linearized operator L_f (see (1.7) below) is invertible and hence the contribution of f is $(-1)^{h^0(N_m)}/|\mathrm{Aut}(f)|$ (cf. Section 10 of [LP]). When $m \geq 3$, J_α is, in general, no longer generic. We need to perturb J_α to generic J. In Section 2, using the universal property of local invariants (see (1.3) below), we choose a local model that is convenient for our calculation. In Section 3, when L_f is not invertible, we use a lifting property of covering space to calculate the contribution of f that proves the Main Theorem. The information for dimension zero GW invariants of elliptic surfaces with $p_g > 0$ is the same as for its Seiberg-Witten invariants. We spell out the specific connection in Remark 3.4.

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1. Dimension Zero Genus One Local GW Invariants

Let X be a (not necessarily compact) elliptic Kähler surface with a holomorphic 2-form α. The 2-form α defines an almost complex structure J_α on X by the formula (0.1). Suppose that the zero divisor D_α of α has a smooth reduction $D_\alpha = (m-1)D$ where D is a regular fiber or a multiple fiber of multiplicity m

for some integer $m > 1$. The adjunction formula then shows $c_1(TX)([D]) = 0$ and $c_1(N) = 0$ where N is the normal bundle of D. The moduli space

$$\overline{\mathcal{M}}_1^{\alpha}(X, d[D]) \tag{1.1}$$

of stable J_α-holomorphic maps from curves of genus one representing the class $d[D]$ ($d \neq 0$) carries a (virtual) fundamental class

$$[\overline{\mathcal{M}}_1^{\alpha}(X, d[D])]^{vir} \tag{1.2}$$

that is defined by the GW theory of 4-manifolds (cf. Section 4 of [LP]). This (virtual) fundamental class (1.2) has dimension zero since $c_1(TX)([D]) = 0$. *The dimension zero genus one local Gromov-Witten invariant of X associated with the zero divisor D_α* is then

$$GW_1^{loc}(X, D_\alpha, d) \,:=\, [\overline{\mathcal{M}}_1^{\alpha}(X, d[D])]^{vir}.$$

This local GW invariant has the following universal property. Let X' be another elliptic Kähler surface with a holomorphic 2-form α' whose zero divisor $D_{\alpha'} = (m'-1)D'$ where D' is a regular fiber or a multiple fiber of multiplicity m'. Let N' be the normal bundle of D'. If $m = m'$ and $h^0(N) = h^0(N')$ then

$$GW_1^{loc}(X, D_\alpha, d) \,=\, GW_1^{loc}(X', D_{\alpha'}, d) \tag{1.3}$$

(cf. Section 6 of [LP]). We set

(1.4)

$$GW_1^{loc}(X, D_\alpha, d) \,=\, \begin{cases} GW_1^{loc}((m-1)F, d) & \text{if } \ D \text{ is a regular fiber} \\ GW_1^{loc}(F_m, d) & \text{if } \ D \text{ is a } m\text{-multiple fiber} \end{cases}$$

It was proved in Example 4.4 of [LP] that

$$GW_1^{loc}(mF, d) \,=\, m\, GW_1^{loc}(F, d). \tag{1.5}$$

As given in (0.2), all dimension zero GW invariants of minimal elliptic surfaces with $p_g > 0$ are sums of local invariants in (1.4).

In the below, we will give a precise description on the (virtual) fundamental class (1.2) which will be used for our calculation in Section 3. The point in the moduli space (1.1) is an equivalence class $[f, C]$ of stable maps (f, C) where two stable maps (f, C) and (f', C') are equivalent if there is a biholomorphic map $\sigma : C \to C'$ with $f' \circ \sigma = f$. By Lemma 0.1, if $d \neq 0$ every representative (f, C) of $[f, C]$ is a holomorphic d-fold covering map from C to D. Thus, if D is given by a lattice Λ in the complex plane then $[f, C]$ is determined by an index d sublattice of Λ. In particular, the moduli space (1.1) consists of $\sigma(d)$ points.

On the other hand, since the (virtual) fundamental class (1.2) is defined by the GW theory of 4-manifolds, as described in Section 3 of [IP], it is a finite sum

$$[\overline{\mathcal{M}}_1^{\alpha}(X, d[D])]^{vir} = \sum c([f, C]) \tag{1.6}$$

over $[f, C] \in \overline{\mathcal{M}}_1^{\alpha}(X, d[D])$ of the contributions $c([f, C])$ that are defined as follows. Choose a $p \in D$ and a small disk B in X with $B \cap D = \{p\}$ and, once and for all, fix a map (f, C, x) with $f(x) = p$ such that (f, C) represents $[f, C]$. Then for a generic almost complex structure J on X that is sufficiently close to J_α and tamed by the Kähler form on X, there are finitely many J-holomorphic maps (f_i, C_i, x_i) from smooth genus one curves with one marked point such that (i) $f_i(x_i) \in B$ (ii) each (f_i, C_i, x_i) is C^0-close to (f, C, x) (in a suitable space of maps) and (iii) the index zero operator

$$L_{f_i} : \Omega^0(f_i^* N_i) \to \Omega^{0,1}(f_i^* N_i) \tag{1.7}$$

has trivial kernel (or equivalently L_{f_i} is invertible) where the operator L_{f_i} is obtained by linearizing J-holomorphic map equation (see Remark 1.1 below) and restricting to the normal bundle N_i of the image of f_i. Denote by

$$\mathcal{M}_{(f,C,x),B,J}$$

the set of such J-holomorphic maps (f_i, C_i, x_i). Notice that for each (f_i, C_i, x_i) the preimage $f_i^{-1}(B)$ consists of $d = |\mathrm{Aut}(f)|$ distinct points x_{ij}. Since the automorphism group of C_i acts transitively, for each x_{ij} there exists an automorphism σ_j of C_i with $\sigma_j(x_i) = x_{ij}$ such that $(f_i \circ \sigma_j, C_i, x_i)$ is also contained in the set $\mathcal{M}_{(f,C,x),B,J}$. The contribution $c([f, C])$ is thus the (weighted) sum

$$c([f, C]) = \frac{1}{d} \sum (-1)^{SF(L_{f_i})}$$

over f_i in $\mathcal{M}_{(f,C,x),B,J}$ where the sign of each f_i is given by the mod 2 spectral flow $SF(L_{f_i})$ of the invertible operator L_{f_i}. In particular, $SF(L_{f_i}) = 0$ if L_{f_i} is complex linear, namely J-linear.

Remark 1.1. The operator $D_{f_i} : \Omega^0(f_i^* TX) \to \Omega^{0,1}(f_i^* TX)$ obtained by linearizing J-holomorphic map equation at f_i is given by

$$D_{f_i}(\xi)(v) = \nabla_v \xi + J \nabla_{jv} \xi + \frac{1}{2}\big[(\nabla_\xi J)(df_i(jv)) - J(\nabla_\xi J)(v)\big] \tag{1.8}$$

where $\xi \in \Omega^0(f_i^*TX)$, $v \in TC_i$ and j is the complex structure on C_i. Here ∇ is the pull-back connection on f_i^*TX of the Levi-Civita connection of the metric on X that is defined by the Käler form and J (cf. Lemma 6.3 of [RT]).

2. Local Model

Once and for all, fix an integer $m \geq 2$ and let D denote the elliptic curve given as the complex plane (with coordinate z) modulo the lattice $\mathbb{Z} + (mi)\mathbb{Z}$. Then $S = D \times \mathbb{C}$ has an automorphism φ of order m defined by

$$\varphi(z, w) \;=\; \big(z + i, e^{2\pi i/m} \cdot w\big)$$

such that all powers φ^i are fixed-point free where w is a coordinate on $\mathbb{C}$. Let S_m be the quotient of S by the group $\{\varphi^i\}$ and $q : S \to S_m$ the quotient map. The map $S \to \mathbb{C} : (z, w) \to w^m$ then factors through S_m to give an elliptic fibration $S_m \to \mathbb{C}$ whose central fiber is a m-multiple fiber D_m given by the lattice $\mathbb{Z} + i\mathbb{Z}$ with torsion normal bundle N_m of order m :

$$\begin{array}{ccc} S = D \times \mathbb{C} & \xrightarrow{\quad q \quad} & S_m \\ & {\scriptstyle (z,w)\to w^m} \searrow \quad \swarrow & \\ & \mathbb{C} & \end{array}$$

The following simple observation is a key fact for our subsequent discussions. Let $f : C \to D_m$ be a holomorphic map of degree d from an elliptic curve C that is given by a sublattice of $\mathbb{Z} + i\mathbb{Z}$ of the form

$$a\mathbb{Z} \,+\, (bi + k)\mathbb{Z} \qquad \text{with} \qquad d \,=\, ab, \;\; 0 \,\leq\, k \,\leq\, a - 1.$$

Write $D \times \{0\} \subset S$ simply as D.

Lemma 2.1. *Let D, N_m and $f : C \to D_m$ be as above. Then,*

$$f \text{ factors through } D \quad\Longleftrightarrow\quad a \,|\, \tfrac{d}{m} \quad\Longleftrightarrow\quad f^*N_m \,=\, \mathcal{O}_C$$

Proof. f factors through $D \Longleftrightarrow a\mathbb{Z} + (bi + k)\mathbb{Z}$ is a sublattice of $\mathbb{Z} + (mi)\mathbb{Z} \Longleftrightarrow m|b \Longleftrightarrow a|\frac{d}{m}$. This shows the first assertion. Observe that for the restriction map $g_m = q_{|D} : D \to D_m$,

$$g_m^*(N_m) \,=\, g_m^*([D_m]_{|D_m}) \,=\, q^*([D_m])_{|D} \,=\, [q^*D_m]_{|D} \,=\, [D]_{|D} \,=\, \mathcal{O}_D \tag{2.1}$$

where $[D_m]$ is the line bundle associated to the divisor D_m, $N_m = [D_m]_{|D_m}$ by adjunction, the pullback divisor $q^*D_m = D$ and again by adjunction $[D]_{|D}$ is the normal bundle of D that is trivial. Write as $N_m = \mathcal{O}_{D_m}(p-q)$ where

$$\int_q^p dz \;=\; \frac{k_1}{m} + i\,\frac{k_2}{m} \qquad \text{for some} \qquad 0 \le k_1, k_2 \le m-1.$$

Then, by (2.1) and the Abel's Theorem, $g_m^* N_m = \mathcal{O}_D(\,\sum_j (p_j - q_j)\,)$ for some p_j, q_j such that

$$\sum_j \int_{q_j}^{p_j} dz \;=\; k_1 + ik_2 \;\equiv\; 0 \mod \mathbb{Z} + (mi)\mathbb{Z}.$$

Consequently, $k_2 = 0$ and $\gcd(m, k_1) = 1$ since N_m is torsion of order m. Now, again by the Abel's Theorem, $f^* N_m = \mathcal{O}_C(\,\sum_\ell (t_\ell - s_\ell)\,)$ for some s_ℓ, t_ℓ such that

$$\sum_\ell \int_{s_\ell}^{t_\ell} dz \;=\; \frac{dk_1}{m} \;\equiv 0 \mod a\mathbb{Z} + (bi+k)\mathbb{Z} \quad \Longleftrightarrow \quad f^*N_m = \mathcal{O}_C$$

Therefore, $a|\frac{d}{m} \iff m|b \iff a \,|\, \frac{dk_1}{m} \iff f^*N_m = \mathcal{O}_C$. This shows the second assertion. $\square$

Remark 2.2. Since $q : S \to S_m$ is a covering map, Lemma 2.1 shows $f : C \to D_m \subset S_m$ lifts to $\tilde{f} : C \to D \subset S$ if and only if $f^*N_m = \mathcal{O}_C$. On the other hand, the Kähler form $-\frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$ on $\mathbb{C}^2$ descends to a Kähler form $\tilde{\omega}$ on S that is φ-invariant, so $\tilde{\omega}$ also descends to a Kähler form ω on S_m such that $q^*\omega = \tilde{\omega}$.

3. Calculation

Let $q : (S, D) \to (S_m, D_m)$ be as in Section 2. Fix a holomorphic 2-from

$$\alpha \;=\; w^{m-1} dw \wedge dz$$

on S whose zero divisor is $(m-1)D$ and let J_α denote the almost complex structure on S defined by the formula (0.1). The 2-form α is φ-invariant, so it descends to a holomorphic 2-form α_m on S_m whose zero divisor is $(m-1)D_m$. We denote by $J_m = J_{\alpha_m}$ the almost complex structure on S_m defined by the 2-form α_m. Since D_m is a multiple fiber of multiplicity m,

$$GW_1^{loc}(F_m, d) \;=\; [\,\overline{\mathcal{M}}_1^{\alpha_m}(S_m, d[D_m])\,]^{vir}$$

where the right-hand side is given by the sum of contributions as in (1.6). In order to calculate them, we decompose the moduli space $\overline{\mathcal{M}}_1^{\alpha_m}(S_m, d[D_m])$ as a disjoint union

$$\overline{\mathcal{M}}_1^{\alpha_m}(S_m, d[D_m]) \;=\; \mathcal{M}^+_{m,d} \coprod \mathcal{M}^-_{m,d}$$

where

$$\mathcal{M}^+_{m,d} = \{\,[f,C] \,:\, h^0(f^*N_m) = 0\,\} \;\text{ and }\; \mathcal{M}^-_{m,d} = \{\,[f,C] \,:\, h^0(f^*N_m) = 1\,\}.$$

It then follows from Lemma 2.1 that

$$\#\,\mathcal{M}^+_{m,d} \;=\; \sigma(d) \,-\, \sigma\Big(\frac{d}{m}\Big) \qquad \text{and} \qquad \#\,\mathcal{M}^-_{m,d} \;=\; \sigma\Big(\frac{d}{m}\Big) \tag{3.1}$$

where $\#A$ is the cardinality of a set A.

We first calculate the contribution $c([f,C])$ of $[f,C]$ in $\mathcal{M}^+_{m,d}$. In the below, we always assume $m \geq 3$ and $m|d$.

Lemma 3.1. *If* $[f,C] \in \mathcal{M}^+_{m,d}$ *then* $c([f,C]) = \frac{1}{d}$.

Proof. The linearized operator L_f has the form $L_f = \overline{\partial}_f + R_m$ where $\overline{\partial}_f$ is the usual $\overline{\partial}$-operator on f^*N_m and the zeroth order term R_m is given by

$$R_m(\xi) \;=\; -\nabla_\xi K_{\alpha_m} \circ J_{\alpha_m} \circ df \qquad \text{for} \qquad \xi \in \Omega^0(f^*N_m)$$

(cf. Section 8 of [LP]). But, $R_m \equiv 0$ since α_m (and hence K_{α_m}) vanishes of order $m-1 \geq 2$ along D_m. Consequently, $\dim\ker L_f = 2h^0(f^*N_m) = 0$, so L_f is invertible with $SF(L_f) = 0$. Now, the proof follows from the fact $f : C \to D_m$ has degree d. $\Box$

Let $[f,C] \in \mathcal{M}^-_{m,d}$. The proof of Lemma 3.1 shows $L_f = \overline{\partial}_f$ is not invertible. In this case, we will uses the m-fold covering map $q : S \to S_m$ to calculate the contribution $c([f,C])$. Observe that by Lemma 2.1 the map

$$\mathcal{M}^-_{m,d} \to \overline{\mathcal{M}}_1^{\alpha}(S, \tfrac{d}{m}[D]) \qquad \text{defined by} \qquad [f,C] \to [\tilde{f},C]$$

is one-to-one and onto where $\tilde{f}$ is a lift of f.

Lemma 3.2. *If* $[f,C] \in \mathcal{M}^-_{m,d}$ *then* $c([f,C]) = \frac{1}{m}\, c([\tilde{f},C])$.

Proof. Let $B = \{0\} \times \Delta \subset S$ where Δ is a small disk around 0 in $\mathbb{C}$ and $B_m = q(B)$ and fix a map (f,C,x) with $f(x) \in B_m$ such that (f,C) represents $[f,C]$. Since the restriction map $q_{|B} : B \to B_m$ is one-to-one, Lemma 2.1 shows that (f, C, x) uniquely lifts to a J_α-holomorphic map $(\tilde{f}, C, x)$ with $\tilde{f}(x) \in B$ such that $(\tilde{f}, C)$ represents $[\tilde{f}, C]$ in $\overline{\mathcal{M}}_1^\alpha(S, \frac{d}{m}[D])$.

Let ω and $\tilde{\omega}$ be the Kähler forms as in Remark 2.2 and choose a generic ω-tamed almost complex structure J on S_m that is close to J_m. Then, we have

- J lifts to an $\tilde{\omega}$-tamed almost complex structure $\tilde{J}$ on S close to J_α such that $dq \circ \tilde{J} = J \circ dq$,
- each f_i in $\mathcal{M}_{(f,C,x),B_m,J}$ is homotopic to f since f_i is C^0-close to f, so (f_i, C_i, x_i) also uniquely lifts to $\tilde{J}$-holomorphic maps $(\tilde{f}_i, C_i, x_i)$ with $\tilde{f}(x_i) \in B$ such that $(\tilde{f}_i, C_i, x_i)$ is C^0-close to $(\tilde{f}, C, x)$.

The pair (ω, J) defines a metric g on S_m whose lift $\tilde{g} = q^* g$ is the same metric defined by the pair $(\tilde{\omega}, \tilde{J})$. Let ∇ and $\tilde{\nabla}$ respectively denote the pull-back connections on $f_i^* TS_m$ and $\tilde{f}_i^* TS$ of the Levi-Civita connection of g and $\tilde{g}$. The differential dq then induces a bundle isomorphism $dq : \tilde{f}_i^* TS \to \tilde{f}_i^* q^* TS_m = f_i^* TS_m$ such that $dq \circ \tilde{\nabla} = \nabla \circ dq$ (see [W] page 138) and hence by the formula (1.8) we have

$$dq \circ D_{\tilde{f}_i} = D_{f_i} \circ dq \tag{3.2}$$

The differential dq also induces a bundle isomorphism $dq_i : \tilde{f}_i^* \tilde{N}_i \to f_i^* N_i$ and restricting the equation (3.2) to $\tilde{f}_i^* \tilde{N}_i$ and $f_i^* N_i$ gives

$$dq_i \circ L_{\tilde{f}_i} = L_{f_i} \circ dq_i$$

where $\tilde{N}_i$ and N_i are normal bundles of $\mathrm{Im}(\tilde{f}_i)$ and $\mathrm{Im}(f_i)$ respectively. Therefore, $L_{\tilde{f}_i}$ is also invertible and hence there is one-to-one correspondence

$$\mathcal{M}_{(f,C,x),B_m,J} \to \mathcal{M}_{(\tilde{f},C,x),B,\tilde{J}} \qquad \text{given by} \qquad (f_i, C_i, x_i) \to (\tilde{f}_i, C_i, x_i).$$

Let $\tilde{L}_t$ be a path of first order elliptic operators from an invertible $\tilde{J}$-linear operator $\tilde{L}_0$ to $\tilde{L}_1 = L_{\tilde{f}_i}$ with all $\tilde{L}_t$ invertible except at finitely many t_k. Then, $dq_i \circ \tilde{L}_t \circ (dq_i)^{-1}$ is also a path from invertible J-linear operator to L_{f_i} such that

$$SF(L_{\tilde{f}_i}) \equiv \sum_k \dim\ker \tilde{L}_{t_k} = \sum_k \dim\ker dq_i \circ \tilde{L}_{t_k} \circ (dq_i)^{-1} \equiv SF(L_{f_i}) \pmod 2.$$

Now, noting $\deg(f) = d$ and $\deg(\tilde{f}) = \frac{d}{m}$, we have

$$c([f, C]) = \frac{1}{d} \sum_{f_i} (-1)^{SF(L_{f_i})} = \frac{1}{d} \sum_{\tilde{f}_i} (-1)^{SF(L_{\tilde{f}_i})} = \frac{1}{m}\, c([\tilde{f}, C]). \quad \square$$

We are now ready to prove the Main Theorem in the introduction.

Proof of the Main Theorem : It follows from Lemma 3.1, Lemma 3.2 and (3.1) that

$$GW_1^{loc}(F_m, d) = \sum_{[f,C]\in\mathcal{M}^+_{m,d}} c([f,C]) + \sum_{[f,C]\in\mathcal{M}^-_{m,d}} c([f,C]) \tag{3.3}$$
$$= \frac{1}{d}\left(\sigma(d) - \sigma\Big(\frac{d}{m}\Big)\right) + \frac{1}{m}\,[\,\overline{\mathcal{M}}{}_1^{\alpha}(S, \tfrac{d}{m}[D])\,]^{vir}.$$

Since the 2-form α on S has the zero divisor $(m-1)D$, so by (1.5) and (0.3) we have

$$[\,\overline{\mathcal{M}}{}_1^{\alpha}(S, \tfrac{d}{m}[D])\,]^{vir} = GW_1^{loc}((m-1)F, \tfrac{d}{m}) = -(m-1)\,\frac{m}{d}\,\sigma\Big(\frac{d}{m}\Big). \tag{3.4}$$

Now, the proof follows from (3.3) and (3.4). □

Remark 3.3. One can also use the above argument to compute $GW_1^{loc}(F_2, d)$, replacing the "Taubes type" argument used in [LP]. Specifically, for each f in $\overline{\mathcal{M}}{}_1^{\alpha_2}(S_2, d[D_2])$ the linearized operator L_f is invertible with $SF(L_f) \equiv h^0(f^*N_2)$ (mod 2) (cf. Proposition 9.2 of [LP]). Thus, by (3.1) we have

$$GW_1^{loc}(F_2, d) = [\,\overline{\mathcal{M}}{}_1^{\alpha_2}(S_2, d[D_2])\,]^{vir} = \frac{1}{d}\left(\sigma(d) - \sigma\Big(\frac{d}{2}\Big)\right) - \frac{1}{d}\,\sigma\Big(\frac{d}{2}\Big)$$
$$= \frac{1}{d}\left(\sigma(d) - 2\sigma\Big(\frac{d}{2}\Big)\right).$$

Remark 3.4. Ionel and Parker [IP] showed how GW invariants for the class A of a symplectic 4-manifold X are related with the Taubes' Gromov invariants $Gr_X(A)$ [T] that count embedded (not necessarily connected) curves in X representing the class A. They used a particular function $F(t)$ that satisfies

$$\prod_d F\big(t^d\big)^{-\frac{1}{d}\sigma(d)} = (1-t)$$

to relate Taubes' counting of multiple covers of embedded tori with the dimension zero genus one GW invariants. Let X be a minimal elliptic surface with $p_g > 0$. In this case, any GW invariant constrained to pass through generic points vanishes (cf. Corollary 3.4 of [LP]). So, by (0.2), (0.3) and the Main Theorem, the relation between two set of invariants (Theorem 4.5 of [IP]) yields

$$\sum_A Gr_X(A)\, t_A = \prod_{d,k} F(t^d)^{c_\pi GW_1^{loc}(F,d)}\, F(t_{m_k}^d)^{GW_1^{loc}(F_{m_k},d)}$$
$$= (1-t)^{c_\pi} \prod_k (1 + t_{m_k} + \cdots + t_{m_k}^{m_k-1}).$$

This also gives the well-known Seiberg-Witten invariants SW of X (cf. [FM], [B], [FS]) due to the famous Taubes' theorem $SW = Gr$.

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