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# Infinitesimal Isospectral Deformations of Symmetric Spaces, II: Quotients of the Special Unitary Group of Rank Two

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**Abstract:** We study the space $I(X)$ of infinitesimal isospectral deformations of an irreducible and reduced symmetric space $X$ of compact type when $X$ is a quotient of the special unitary group $G = SU(n)$, with $n \geq 3$. If $X$ is the reduced space of the special unitary group $SU(n)$ or of the special Lagrangian Grassmannian $SU(n)/SO(n)$, the non-zero $G$-invariant symmetric 3-form on $X$ gives rise to a linear mapping $\Phi_0 : C^\infty_{\mathbb{R}}(X) \to I(X)$, where $C^\infty_{\mathbb{R}}(X)$ is the space of real-valued functions on $X$. Previously, we constructed a subspace $\mathcal{F}_X$ of $C^\infty_{\mathbb{R}}(X)$ of finite-codimension and showed that the restriction $\Phi : \mathcal{F}_X \to I(X)$ of $\Phi_0$ is a monomorphism. Here we prove that, when $n = 3$, the mapping $\Phi$ is an isomorphism and thus obtain in this case an explicit description of the deformation space $I(X)$.
**Keywords:** symmetric space, special unitary group, special Lagrangian Grassmannian, reduced Lagrangian Grassmannian, Radon transform, infinitesimal isospectral deformation, symmetric form, Guillemin condition.

## Introduction

Motivated by a result of Guillemin, in [3] we introduced the space $I(X)$ of infinitesimal isospectral deformations of a Riemannian symmetric space $(X, g)$ of compact type. We are interested in determining the space $I(X)$ when $X$ is irreducible and reduced. The reduced space of $X$ constructed in [3] is a symmetric

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space covered by $X$ and which is not the cover of another symmetric space; we say that $X$ is reduced if it is equal to its reduced space. If $I(X)$ vanishes, we say that $(X, g)$ is rigid in the sense of Guillemin; in this case, we know that every isospectral deformation of the metric $g$ is trivial to first-order, and so the space $X$ is spectrally rigid to first-order.

The only irreducible and reduced spaces for which it is known that the space of infinitesimal isospectral deformations is non-trivial are quotients of the special unitary group. In [4] and [5], by means of the homogeneous polynomials of degree 3 on the Lie algebra of the special unitary group, we produced non-trivial infinitesimal isospectral deformations of a symmetric space belonging to one of the following families of irreducible symmetric spaces, where the integer $n$ is $\geq 3$:

(i) the reduced space of the symmetric space $SU(n)/SO(n)$;

(ii) the reduced space of the special unitary group $SU(n)$;

(iii) the reduced space of the symmetric space $SU(2n)/Sp(n)$.

In fact, if $X$ is one of these spaces, we constructed an explicit infinite-dimensional space $\mathcal{F}_X$ of real-valued functions on $X$ and an injective mapping

$$\Phi : \mathcal{F}_X \to I(X).$$

The symmetric space $SU(n)/SO(n)$ is the special Lagrangian Grassmannian; its reduced space, which we call the reduced Lagrangian Grassmannian, is the quotient of $SU(n)/SO(n)$ by the action of a cyclic group of order $n$ consisting of isometries. The reduced space of the special unitary group $G = SU(n)$ viewed as a symmetric space is the quotient group $G/S$, where $S$ is the center of $G$; the latter group is isomorphic to the adjoint group of $\mathfrak{su}(n)$ and is called the reduced unitary group.

In this paper, we describe explicitly the deformation spaces of two of these reduced spaces, which are quotients of the special unitary group $SU(3)$ and which are of rank 2, namely: the reduced Lagrangian Grassmannian, quotient of the symmetric space $SU(3)/SO(3)$, and the reduced unitary group $SU(3)/S$. Here we show that the mapping $\Phi$ is an isomorphism for these two symmetric spaces.

As in [2], we say that a symmetric $p$-form $u$ on a symmetric space $(X, g)$ satisfies the Guillemin condition if, for every maximal flat totally geodesic torus $Z$

contained in $X$ and for all parallel vector fields $\zeta$ on $Z$, the integral

$$\int_Z u(\zeta, \zeta, \dots, \zeta)\, dZ$$

vanishes, where $dZ$ is the Riemannian measure of $Z$. The kernel $\mathcal{N}_p$ of the Radon transform for $p$-forms consists precisely of those forms satisfying the Guillemin condition.

Let $\{g_t\}$ be a family of Riemannian metrics on $X$, with $g_0 = g$; assume that $\{g_t\}$ is an isospectral deformation of $g$ (i.e., that the spectrum of the Laplacian of the metric $g_t$ is independent of $t$). Guillemin proved, using the methods he introduced in [7], that the corresponding infinitesimal deformation $h = \frac{d}{dt} g_{t|t=0}$ of the metric $g$ belongs to the kernel $\mathcal{N}_2$. If $\varphi_t$ is a one-parameter family of diffeomorphisms of $X$, the family $\{\varphi_t^* g\}$ is a trivial isospectral deformation; in fact, the space $\mathcal{L}_2$ of Lie derivatives of the metric $g$ is a subspace of $\mathcal{N}_2$. This leads us to define the space $I(X)$ of infinitesimal isospectral deformations as the orthogonal complement of $\mathcal{L}_2$ in $\mathcal{N}_2$. Thus we have the orthogonal decomposition

$$\mathcal{N}_2 = \mathcal{L}_2 \oplus I(X),$$

and we denote by $P$ the orthogonal projection of $\mathcal{N}_2$ onto $I(X)$. If $I(X)$ vanishes, the infinitesimal deformation $h$ is a Lie derivative of the metric and the deformation $\{g_t\}$ is trivial to first-order.

Let $(X, g)$ be a reduced symmetric space belonging to one of the above three families of reduced spaces. The universal cover $\tilde{X}$ of $X$ is an irreducible symmetric space corresponding to a Riemannian symmetric pair $(\tilde{G}, K)$, which is in fact one of the following pairs

$$(G, SO(n)), \qquad (G \times G, G^*), \qquad (G, Sp(n)),$$

where $G = SU(n)$ for the first two pairs and $G = SU(2n)$ for the latter pair, with $n \geq 3$, and where $G^*$ is the diagonal of $G \times G$. We view the symmetric space $X$ as a homogeneous space of the group $\tilde{G}$. The symmetric space $X$ carries a unique (up to a constant) $\tilde{G}$-invariant symmetric 3-form $\sigma$, which is induced by the $G$-invariant homogeneous polynomial $Q$ on the Lie algebra $\mathfrak{g}_0$ of $G$ defined by

$$Q(A) = i \operatorname{Tr} A^3,$$

for all $A \in \mathfrak{g}_0$ (see [3, §2]). The form $\sigma$ induces an injective mapping $\tilde{\sigma}$ from the space of 1-forms on $X$ to the space of symmetric 2-forms on $X$. According to [4], a 1-form $\theta$ on $X$ satisfies the Guillemin condition if and only if the symmetric 2-form $\tilde{\sigma}(\theta)$ satisfies the Guillemin condition. We consider the $\tilde{G}$-module $C^\infty_{\mathbb{R}}(X)$ of real-valued functions on $X$; if $f$ is an element of $C^\infty_{\mathbb{R}}(X)$, the symmetric 2-form $\tilde{\sigma}(df)$ satisfies the Guillemin condition. In [4] and [5], we proved that the space $\mathcal{F}'_X$ of functions $f \in C^\infty_{\mathbb{R}}(X)$ for which the symmetric 2-form $\tilde{\sigma}(df)$ is a Lie derivative of the metric $g$ is the direct sum of two irreducible $\tilde{G}$-submodules $\mathcal{B}_{\mathbb{R}}$ and $\mathbb{R}(X)$ of $C^\infty_{\mathbb{R}}(X)$, where $\mathbb{R}(X)$ is the space of constant functions on $X$. Thus if $\mathcal{F}_X$ is the orthogonal complement of $\mathcal{F}'_X$ in $C^\infty_{\mathbb{R}}(X)$, the sum

$$\mathcal{L}_2 \oplus \tilde{\sigma}(d\mathcal{F}_X)$$

is direct; thus we know that the mapping

$$\Phi = P\cdot\tilde{\sigma}\cdot d : \mathcal{F}_X \to I(X)$$

is injective and $\Phi(\mathcal{F}_X)$ is an infinite-dimensional subspace of $I(X)$. The main results of this paper imply that the mapping $\Phi$ is also surjective when $n = 3$.

Henceforth, we suppose that $G = SU(3)$, and that $\tilde{X}$ is equal either to the space $SU(3)/SO(3)$, with $\tilde{G} = G$, or to the group $SU(3)$, with $\tilde{G} = G \times G$. In both cases, the space $X$ is the quotient of $\tilde{X}$ by the action of a cyclic group $\Sigma$ of order 3 consisting of isometries which commute with the action of $\tilde{G}$. We consider the $\tilde{G}$-module $C^\infty(\tilde{X})$ of complex-valued functions and the $\tilde{G}$-module $C^\infty(S^pT^*_{\mathbb{C}})$ of complex symmetric $p$-forms on the space $\tilde{X}$. To the form $\sigma$, we associate an elliptic homogeneous differential operator $D_\sigma$ on $\tilde{X}$ with values in the space of symmetric 2-forms. In order to demonstrate that the equality

$$\Phi(\mathcal{F}_X) = I(X)$$

holds on $X$, it suffices to show that a $\Sigma$-invariant 2-form on $\tilde{X}$ satisfying the Guillemin condition belongs to the image of $D_\sigma$. The ellipticity of $D_\sigma$ allows us to exploit the harmonic analysis on the homogenous space $\tilde{X}$ of the group $\tilde{G}$.

In §§3, 4 and 12, using the Littlewood-Richardson rule we compute the multiplicity of an arbitrary isotypic component of the $\tilde{G}$-module $C^\infty(S^pT^*_{\mathbb{C}})$, with $p = 1, 2$. We then determine in §§5 and 12 all the highest weight vectors of such an isotypic component and express them in terms of a family of functions and a finite set of 1-forms on $\tilde{X}$. We wish to point out that the descriptions given there are remarkably simple. More precisely, we construct two explicit functions $f_1$ and $f_2$ on $\tilde{X}$ and consider the family $\mathcal{U}$ of functions $f_{r,s} = f_1^r \cdot f_2^s$ on $\tilde{X}$, where

$r, s$ are integers $\geq 0$. If $\tilde{X} = SU(3)/SO(3)$, the highest weight vector of an irreducible $\tilde{G}$-module of $C^\infty(S^pT^*_{\mathbb{C}})$, with $p = 1, 2$, can be expressed in terms of functions belonging to the family $\mathcal{U}$ and of two explicit 1-forms. On $\tilde{X} = SU(3)$, we construct explicit 1-forms $\vartheta_j$, with $1 \leq j \leq 8$, which are highest weight vectors of irreducible $\tilde{G}$-submodules of $C^\infty(T^*_{\mathbb{C}})$; then any highest weight vector of an irreducible $\tilde{G}$-submodule of $C^\infty(T^*_{\mathbb{C}})$ can be expressed as a linear combination of forms of the type $f\vartheta_j$, with $f \in \mathcal{U}$ and $1 \leq j \leq 8$, and any highest weight vector of an irreducible $\tilde{G}$-submodule of $C^\infty(S^2T^*_{\mathbb{C}})$ can be expressed as a linear combination of the forms bel! onging to the family

$$\{u_1 g, \tilde{\sigma}(u_2\vartheta_j), u_3\vartheta_k \cdot \vartheta_l\},$$

with $u_1, u_2, u_3 \in \mathcal{U}$ and $1 \leq j, k, l \leq 8$. These descriptions allow us to tell which of these highest weight vectors are $\Sigma$-invariant. Also if $W$ is an isotypic component of the $\tilde{G}$-module $C^\infty(S^pT^*_{\mathbb{C}})$, with $p = 1, 2$, we are able to see that its $\tilde{G}$-submodule $W^\Sigma$ consisting of its $\Sigma$-invariant forms either is equal to $W$ or vanishes.

Let $\widetilde{\mathcal{N}}_{2,\mathbb{C}}$ denote the $\tilde{G}$-submodule of $C^\infty(S^2T^*_{\mathbb{C}})$ consisting of complex symmetric 2-forms satisfying the Guillemin condition. In order to prove the desired equality, i.e., that the mapping $\Phi$ is surjective, we simply need to show the following: if $W$ is a non-zero isotypic component of the $G$-submodule $C^\infty(S^2T^*_{\mathbb{C}})$ satisfying $W = W^\Sigma$, the space $W \cap \widetilde{\mathcal{N}}_{2,\mathbb{C}}$ belongs to the image of $D_\sigma$. This last fact follows from an appropriate bound for the multiplicity of the $\tilde{G}$-submodule $W \cap \widetilde{\mathcal{N}}_{2,\mathbb{C}}$, or equivalently for the dimension of the vector space $W^\natural \cap \widetilde{\mathcal{N}}_{2,\mathbb{C}}$, where $W^\natural$ is the subspace of $W$ generated by its highest weight vectors. In fact, the dimension of the space $W^\natural$ is always $\leq 6$, and we achieve this bound by constructing a mapping $W^\natural \to \mathbb{R}^q$ whose kernel contains $W^\natural \cap \widetilde{\mathcal{N}}_{2,\mathbb{C}}$ and showing that it is surjective, where the integer $q$ depends on $W^\natural$ and is equal to 1, 2 or 3. The equality

$$\mathcal{L}_2 \cap \tilde{\sigma} d C^\infty_{\mathbb{R}}(X) = \tilde{\sigma} d\mathcal{B}_{\mathbb{R}},$$

which implies that the mapping $\Phi$ is surjective, enters into defining these bounds. In §§10 and 14, we obtain the required bounds by computing specific integrals over a suitably chosen family of maximal flat totally geodesic tori of $\tilde{X}$. We need to evaluate various coefficients of polynomials arising from certain trigonometric integrals obtained by means of the identities of §9. We give a complete explanation for only two such computations in Proposition 10.1; all the others are obtained by similar methods. Here as in [3], we require the WZ theory, as described in [8], to prove the combinatorial identities of §8, which are of independent interest.

The proof of our main results allows us to show that the maximal flat Radon transform for functions on $X$ is injective and that a 1-form on $X$ satisfying the Guillemin condition is exact (see §7).

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## Contents

# CHAPTER I: The Lagrangian Grassmannians

## 1. Riemannian manifolds

Let $X$ be a differentiable manifold, whose tangent and cotangent bundles we denote by $T = T_X$ and $T^* = T^*_X$, respectively. We consider the space of complex-valued functions $C^\infty(X)$ (resp. real-valued functions $C^\infty_{\mathbb{R}}(X)$) on $X$. Let $E$ be a vector bundle over $X$; we denote by $E_{\mathbb{C}}$ its complexification, by $\mathcal{E}$ the sheaf of sections of $E$ over $X$ and by $C^\infty(E)$ the space of global sections of $E$ over $X$. By $S^kE$, we shall mean the $k$-th symmetric product of the vector bundle $E$. We shall identify $S^kT^*$ with a sub-bundle of the $k$-th tensor product $\bigotimes^k T^*$ of $T^*$ as in §1, Chapter I of [2]; in particular, if $\alpha, \beta \in T^*$, the symmetric product $\alpha \cdot \beta$ is identified with the element $\alpha \otimes \beta + \beta \otimes \alpha$ of $\bigotimes^2 T^*$. If $u$ is a section of $S^pT^*$ over $X$, with $p \geq 1$, we consider the morphism of vector bundles

$$u^\flat : T \to S^{p-1}T^*,$$

defined by

$$(u^\flat \xi)(\eta_1, \dots, \eta_{p-1}) = u(\xi, \eta_1, \dots, \eta_{p-1}),$$

for $\xi, \eta_1, \dots, \eta_{p-1} \in T$.

Let $g$ be a Riemannian metric on $X$. We denote by $g^\sharp : T^* \to T$ the inverse of the isomorphism $g^\flat : T \to T^*$. If $u$ is a section of $S^pT^*$ over $X$, we consider the morphism of vector bundles

$$\tilde{u} = u^\flat \cdot g^\sharp : T^* \to S^{p-1}T^*.$$

We also consider the scalar products on the vector spaces $C^\infty(X)$, $C^\infty(T)$ and $C^\infty(S^2T^*)$, defined in terms of the Riemannian measure of $X$ and the scalar products on the vector bundles $T$ and $S^2T^*$ induced by the metric $g$.

We denote by $\operatorname{Hess} f$ the Hessian of a real-valued function $f$ on $X$. The Killing operator

$$D_0 : \mathcal{T} \to S^2\mathcal{T}^*$$

of $(X, g)$, sends a vector field $\xi$ into the Lie derivative $\mathcal{L}_\xi g$ of $g$ along $\xi$. We also consider the divergence operator

$$\operatorname{div} : S^2\mathcal{T}^* \to \mathcal{T}^*,$$

as defined in §1, Chapter I of [2]; we recall that the formal adjoint of $D_0$ is equal to $2g^\sharp \cdot \operatorname{div} : S^2\mathcal{T}^* \to \mathcal{T}$. When $X$ is compact, since the operator $D_0$ is elliptic, we therefore have the orthogonal decomposition

$$C^\infty(S^2T^*) = D_0C^\infty(T) \oplus \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0 \,\} \tag{1.1}$$

given by the relation (1.11) of [2]; we denote by

$$P : C^\infty(S^2T^*) \to \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0 \,\}$$

the projection determined by the decomposition (1.1).

We now suppose that $X$ is a symmetric space of compact type. We know that there is a Riemannian symmetric pair $(G, K)$ of compact type, where $G$ is a compact, semi-simple Lie group and $K$ is a closed subgroup of $G$, such that the space $X$ is isometric to the homogeneous space $G/K$ endowed with a $G$-invariant metric. We shall identify $X$ with $G/K$. We shall denote by $\mathfrak{g}_0$ the Lie algebra of $G$. The pair $(G, K)$ is associated with an orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$ of compact type, where $\theta$ is an involutive automorphism of $\mathfrak{g}_0$. The spaces $C^\infty(X)$, $C^\infty(T)$ and $C^\infty(T_{\mathbb{C}})$ and the spaces $C^\infty(S^pT^*)$ and $C^\infty(S^pT^*_{\mathbb{C}})$ of symmetric $p$-forms on $X$ inherit structures of $G$-modules from the action of $G$ on $X$.

We consider the $G$-submodule $\mathcal{N}_p = \mathcal{N}_{p,X}$ of $C^\infty(S^pT^*)$ consisting of all symmetric $p$-forms satisfying the Guillemin condition; the complexification $\mathcal{N}_{p,\mathbb{C}}$ of $\mathcal{N}_p$ shall be viewed as the $G$-submodule of $C^\infty(S^pT^*_{\mathbb{C}})$ consisting of all complex symmetric $p$-forms satisfying the Guillemin condition. We recall that $D_0C^\infty(T)$ is a $G$-submodule of $\mathcal{N}_2$ and that $dC^\infty_{\mathbb{R}}(X)$ is a $G$-submodule of $\mathcal{N}_1$ (see Lemma 2.10 of [2]). We consider the space of infinitesimal isospectral deformations of $g$ defined by

$$I(X) = \{\, h \in \mathcal{N}_2 \mid \operatorname{div} h = 0 \,\}.$$

From the decomposition (1.1), we obtain the orthogonal decomposition

$$\mathcal{N}_2 = D_0C^\infty(T) \oplus I(X); \tag{1.2}$$

moreover, the orthogonal projection of $\mathcal{N}_2$ onto $I(X)$ is equal to the restriction of the projection $P$ to $\mathcal{N}_2$. Thus the vanishing of the space $I(X)$ is equivalent to the fact that the space $X$ is rigid in the sense of Guillemin (see [3, §1]).

Let $\Gamma$ be the dual of the group $G$, that is, the set of equivalence classes of irreducible $G$-modules over $\mathbb{C}$. Let $F$ be a $G$-invariant complex sub-bundle of $T_{\mathbb{C}}$ or $S^pT^*_{\mathbb{C}}$. If $\gamma$ is an element of $\Gamma$, we denote by $C^\infty_\gamma(X)$ and $C^\infty_\gamma(F)$ the isotypic components of the $G$-modules $C^\infty(X)$ and $C^\infty(F)$, respectively, corresponding to $\gamma$. Let $\Gamma_0$ be the subset of $\Gamma$ consisting of those elements $\gamma$ of $\Gamma$ for which the $G$-module $C^\infty_\gamma(X)$ is non-zero; for $\gamma \in \Gamma_0$, we know that the $G$-module $C^\infty_\gamma(X)$ is irreducible. Let $\gamma_0$ be the element of $\Gamma_0$ corresponding to the trivial irreducible $G$-module $\mathbb{C}$. We know that the space $C^\infty_{\gamma_0}(T_{\mathbb{C}})$ always vanishes.

Let $\gamma$ be an element of $\Gamma$ and $E_\gamma$ be an irreducible $G$-module corresponding to $\gamma$. A $G$-submodule $W$ of $C^\infty_\gamma(F)$, with $\gamma \in \Gamma$, is isomorphic to the direct sum of $k$ copies of $E_\gamma$; this integer $k$ is called the multiplicity of the $G$-module $W$ and denoted by $\operatorname{Mult} W$. If we choose a Cartan subalgebra of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ and fix a system of positive roots of $\mathfrak{g}$, we recall that the dimension of the weight subspace of $W$, corresponding to the highest weight of $E_\gamma$, is equal to the multiplicity of $W$ (see [2, Chapter II]).

Let $\sigma$ be a $G$-invariant symmetric 3-form on $X$; we consider the $G$-equivariant morphism of vector bundles

$$\tilde{\sigma} : T^* \to S^2T^*$$

induced by $\sigma$. If the space $X$ is irreducible and the form $\sigma$ is non-zero, we know that $\tilde{\sigma}$ is a monomorphism. If $F$ denotes the trivial real line bundle over $X$, we associate with $\sigma$ the first-order differential operator

$$D_\sigma : \mathcal{T} \oplus \mathcal{F} \to S^2\mathcal{T}^*$$

defined by

$$D_\sigma(\xi, f) = D_0\xi + \tilde{\sigma} df,$$

for $\xi \in C^\infty(T)$ and $f \in C^\infty_{\mathbb{R}}(X)$.

Let $\Sigma$ be a finite group of isometries of $X$ of order $m$; suppose that the elements of $\Sigma$ commute with the action of $G$ on $X$ and that the group $\Sigma$ acts without fixed points. Then the quotient $Y = X/\Sigma$ is a manifold and the natural projection $\varpi : X \to Y$ is an $m$-fold covering. Thus the metric $g$ induces a Riemannian metric $g_Y$ on $Y$ such that $\varpi^* g_Y = g$. Let $(G, K')$ be another Riemannian symmetric pair associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$. Assume that $K$ is a subgroup of $K'$ and that there exists a $G$-equivariant diffeomorphism $\varphi : Y \to G/K'$ which has the following property: when we identify $X$ with $G/K$, the projection $\varphi \circ \varpi$ is equal to the natural projection $G/K \to G/K'$. Under these conditions, the space $(Y, g_Y)$ is isometric to the symmetric space $G/K'$ of compact type endowed with a $G$-invariant metric. In [3, §1], we saw that the reduced space of $X$ can be written as the quotient $X/\Sigma$, where $\Sigma$ is an appropriate group of isometries of $X$, and that it also satisfies all of the above assumptions.

We consider the $G$-submodule $C^\infty(X)^\Sigma$ of $C^\infty(X)$ consisting of all $\Sigma$-invariant functions on $X$. If $F$ is a sub-bundle of $T_{\mathbb{C}}$ or $S^pT^*_{\mathbb{C}}$ which is invariant under both the groups $G$ and $\Sigma$, we consider the $G$-submodule $C^\infty(F)^\Sigma$ of $C^\infty(F)$ consisting of all $\Sigma$-invariant sections of $F$ over $X$. If $F$ is the vector bundle $T_{\mathbb{C}}$ or $S^pT^*_{\mathbb{C}}$, we consider the $G$-submodules

$$C^\infty_\gamma(X)^\Sigma = C^\infty(X)^\Sigma \cap C^\infty_\gamma(X), \qquad C^\infty_\gamma(F)^\Sigma = C^\infty(F)^\Sigma \cap C^\infty_\gamma(F)$$

of $C^\infty_\gamma(X)$ and $C^\infty_\gamma(F)$, respectively. Let $\Gamma_1$ be the subset of $\Gamma_0$ consisting of all elements $\gamma$ of $\Gamma_0$ such that

$$C^\infty_\gamma(X)^\Sigma = C^\infty_\gamma(X).$$

If the symmetric 3-form $\sigma$ is $\Sigma$-invariant, the symmetric form $\sigma$ induces a symmetric 3-form $\sigma_Y$ on $Y$ such that

$$\sigma = \varpi^*\sigma_Y.$$

We consider the morphism of vector bundles

$$\tilde\sigma_Y : T^*_Y \to S^2T^*_Y$$

induced by the symmetric 3-form $\sigma_Y$. If $\varphi$ is a 1-form on $Y$, we have

$$\varpi^*\tilde\sigma_Y(\varphi) = \tilde\sigma(\varpi^*\varphi). \tag{1.3}$$

Suppose that, if $\varphi$ is an arbitrary 1-form on $X$ satisfying the Guillemin condition, the symmetric 2-form $\tilde\sigma(\varphi)$ on $X$ also satisfies the Guillemin condition. Then for all $\gamma \in \Gamma$, since the differential operators $D_0$ and $\tilde\sigma d$ are homogeneous, we have the inclusions

$$\begin{gathered} D_0C^\infty(T)^\Sigma + \tilde\sigma dC^\infty_{\mathbb{R}}(X)^\Sigma \subset \mathcal{N}_2 \cap C^\infty(S^2T^*)^\Sigma, \\ D_0C^\infty_\gamma(T_{\mathbb{C}})^\Sigma + \tilde\sigma dC^\infty_\gamma(X)^\Sigma \subset \mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}})^\Sigma, \end{gathered} \tag{1.4}$$

for all $\gamma \in \Gamma$.

We now further assume that $X$ is irreducible and is not equal to a simple Lie group. We may suppose that the Lie group $G$ is simple; then the complexification $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}_0$ is simple. Let $\gamma_1$ be the element of $\Gamma$ which is the equivalence class of the irreducible $G$-module $\mathfrak{g}$. The space $\mathcal{K}$ of all Killing vector fields on $X$, i.e., the space of all solutions $\xi \in C^\infty(T)$ of the equation $D_0\xi = 0$, is isomorphic to $\mathfrak{g}_0$; thus we may view its complexification $\mathcal{K}_{\mathbb{C}}$ as a $G$-submodule of $C^\infty_{\gamma_1}(T_{\mathbb{C}})$. If $X$ is not Hermitian, according to the equality (2.27) of [2] we know that $\gamma_1$ does not belong to $\Gamma_0$ and that

$$C^\infty_{\gamma_1}(T_{\mathbb{C}}) = \mathcal{K}_{\mathbb{C}}. \tag{1.5}$$

**Proposition 1.1.** *Suppose that the symmetric space $X$ of compact type is irreducible; assume that it is neither Hermitian nor equal to a simple Lie group. Let $\sigma$ be a non-zero symmetric 3-form on $X$ which is both $G$-invariant and $\Sigma$-invariant and let $\sigma_Y$ be the symmetric 3-form on the symmetric space $Y$ satisfying $\varpi^*\sigma_Y = \sigma$. Suppose that the following hypotheses hold:*

(a) *If a 1-form $\varphi$ on $X$ satisfies the Guillemin condition, the symmetric 2-form $\tilde{\sigma}(\varphi)$ on $X$ also satisfies the Guillemin condition.*

(b) *There exists an element $\gamma_2$ of $\Gamma_1$, not equal to $\gamma_0$ or $\gamma_1$, such that*

$$D_0 C^\infty(T_{\mathbb{C}}) \cap \tilde{\sigma} d C^\infty(X) = \tilde{\sigma} d C^\infty_{\gamma_2}(X).$$

(c) *The differential operator $D_\sigma$ is elliptic.*

*Then the following assertions are equivalent:*

(i) *The equality*

$$\mathcal{N}_{2,Y} = D_0 C^\infty(T_Y) + \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y)$$

*holds.*

(ii) *The equality*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty(S^2 T^*_{\mathbb{C}})^\Sigma = D_0 C^\infty(T_{\mathbb{C}})^\Sigma + \tilde{\sigma} d C^\infty(X)^\Sigma$$

*holds.*

(iii) *We have*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{\gamma_0}(S^2 T^*_{\mathbb{C}})^\Sigma = \mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{\gamma_1}(S^2 T^*_{\mathbb{C}})^\Sigma = \{0\}, \tag{1.6}$$

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{\gamma_2}(S^2 T^*_{\mathbb{C}})^\Sigma = D_0 C^\infty_{\gamma_2}(T_{\mathbb{C}})^\Sigma, \tag{1.7}$$

*and the equality*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{\gamma}(S^2 T^*_{\mathbb{C}})^\Sigma = D_0 C^\infty_{\gamma}(T_{\mathbb{C}})^\Sigma + \tilde{\sigma} d C^\infty_{\gamma}(X)^\Sigma \tag{1.8}$$

*holds for all $\gamma \in \Gamma$, with $\gamma \neq \gamma_0, \gamma_1, \gamma_2$.*

(iv) *The equalities* (1.6) *hold; moreover, if* $\gamma$ *is an arbitrary element of* $\Gamma$ *not equal to* $\gamma_0$ *or* $\gamma_1$*, the inequality*

$$\mathrm{Mult}\,\big(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}})^\Sigma\big) \leq 1 + \mathrm{Mult}\, C^\infty_\gamma(T_{\mathbb{C}})^\Sigma$$

*holds whenever* $\gamma$ *belongs to* $\Gamma_1$ *and is not equal to* $\gamma_2$*, and the inequality*

$$\mathrm{Mult}\,\big(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}})^\Sigma\big) \leq \mathrm{Mult}\, C^\infty_\gamma(T_{\mathbb{C}})^\Sigma$$

*holds whenever* $\gamma$ *is equal to* $\gamma_2$ *or does not belong to* $\Gamma_1$*.*

*Proof.* Lemma 2.17 of [2], together with the relations (2.6) of [2] and (1.3), gives us the equivalence of (i) and (ii). Since $D_\sigma$ is an elliptic homogeneous differential operator, from the inclusions (2.12) and Propositions 2.2,(iii) of [2], by (1.4) we infer that assertion (ii) is equivalent to the fact that the equality (1.8) holds for all $\gamma \in \Gamma$. According to our hypothesis (b), the equality (1.8), with $\gamma = \gamma_2$, is equivalent to (1.7). When $\gamma = \gamma_0$, we know that the spaces $dC^\infty_\gamma(X)$ and $C^\infty_\gamma(T_{\mathbb{C}})$ vanish. On the other hand, since $X$ is not Hermitian, we saw that the spaces $C^\infty_{\gamma_1}(X)$ and $D_0C^\infty_{\gamma_1}(T_{\mathbb{C}})$ vanish. Hence the equalities (1.8), with $\gamma$ equal to $\gamma_0$ or $\gamma_1$, are equivalent to the relations (1.6). Thus the assertions (ii) and (iii) are equivalent. Since $\mathrm{Mult}\, C^\infty_\gamma(X)^\Sigma$ is equal to 1 when the element $\gamma$ of $\Gamma$ belongs to $\Gamma_1$ and vanishes otherwise, the equivalence of (iii) and (iv) follows from the inclusions (1.4) and the hypothesis (b). □

We no longer assume that the space $X$ is irreducible and also allow it to be equal to a Lie group. The following result is a direct consequence of Lemma 2.17 and Proposition 2.32 of [2].

**Proposition 1.2.** *Let* $X$ *be a symmetric space of compact type and* $Y$ *be the symmetric space equal to the quotient of* $X$ *by the finite group* $\Sigma$ *of isometries of* $X$*. Then the following assertions are equivalent:*

(i) *A* 1*-form on* $Y$ *satisfies the Guillemin condition if and only if it is exact.*

(ii) *The equality*

$$\mathcal{N}_{1,\mathbb{C}} \cap C^\infty(T^*_{\mathbb{C}})^\Sigma = dC^\infty(X)^\Sigma$$

*holds.*

(iii) *The equality*

$$\mathcal{N}_{1,\mathbb{C}} \cap C^\infty_\gamma(T^*_{\mathbb{C}})^\Sigma = dC^\infty_\gamma(X)^\Sigma$$

*holds for all* $\gamma \in \Gamma$*.*

(iv) *The inequality*

$$\mathrm{Mult}\left(\mathcal{N}_{1,\mathbb{C}} \cap C^\infty_\gamma(T^*_\mathbb{C})^\Sigma\right) \leq 1$$

*holds whenever* $\gamma$ *belongs to* $\Gamma_1$, *and the equality*

$$\mathcal{N}_{1,\mathbb{C}} \cap C^\infty_\gamma(T^*_\mathbb{C})^\Sigma = \{0\}$$

*holds whenever* $\gamma$ *does not belong to* $\Gamma_1$.

## 2. The special unitary group

Let $n \geq 3$ be a given integer. Let $X = G$ be the special unitary group $SU(n)$. If $B$ denotes the Killing form of the Lie algebra $\mathfrak{g}_0 = \mathfrak{su}(n)$, we endow $X$ with the bi-invariant Riemannian metric $g_0$ induced by $-B$. As usual, we identify the $G$-module $\mathfrak{g}_0$ with the tangent space of $X$ at the identity element $e_0 = I_n$ of $G$.

We consider the space $M_n$ of all $n \times n$ complex matrices. For $1 \leq j, k \leq n$, let $E_{jk} = (c_{lr})$ be the element of $M_n$ determined by $c_{jk} = 1$ and $c_{lr} = 0$ whenever $(l, r) \neq (j, k)$. If $1 \leq j, k \leq n$ and $1 \leq l \leq n-1$ are integers, with $j \neq k$, the matrices

$$A_{jk} = E_{jk} - E_{kj}, \qquad B_{jk} = i(E_{jk} + E_{kj}), \qquad C_l = i(E_{ll} - E_{l+1,l+1})$$

of $M_n$ belong to $\mathfrak{g}_0$; in fact, the set of all these matrices $\{A_{jk}, B_{jk}, C_l\}$, with $1 \leq j < k \leq n$ and $1 \leq l \leq n-1$, form a basis of $\mathfrak{g}_0$.

For $p \geq 2$, the homogeneous polynomial $Q_p$ on $\mathfrak{g}_0$ defined by

$$Q_p(\xi) = (-i)^p \operatorname{Tr} \xi^p,$$

for all $\xi \in \mathfrak{g}_0$, is $G$-invariant, non-zero and real-valued; therefore it gives rise to a non-zero bi-invariant symmetric $p$-form $\sigma'_p$ on $X$. We know that the metric $g_0$ is equal to the symmetric 2-form $2n \cdot \sigma'_2$ and that $\sigma'_3$ is up to a constant the unique bi-invariant symmetric 3-form on $X$ (see [3, §2]).

We shall always consider the space $X = SU(n)$, with $n \geq 3$, endowed with the Riemannian metric $g' = \sigma'_2$. We easily verify that the product of matrices $C_j \cdot C_k$ is equal to 0, for all $1 \leq j, k \leq n-1$, with $|j - k| \geq 2$, and hence that

$$(2.1) \qquad g'(C_j, C_j) = 2, \qquad g'(C_l, C_{l+1}) = -1, \qquad g'(C_k, C_q) = 0,$$

for all $1 \leq j, k, q \leq n-1$ and $1 \leq l \leq n-2$, with $q \geq k+2$. Moreover, we verify that

$$g'(C_l, B_{jk}) = 0, \qquad g'(B_{jk}, B_{rs}) = 2\delta_{jr}\delta_{ls}, \tag{2.2}$$

for all $1 \leq l \leq n-1$ and $1 \leq j, k, r, s \leq n$ and, with $j < k$ and $r < s$.

We now consider the mapping

$$\iota' : \mathbb{R}^{n-1} \to G,$$

which sends $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathbb{R}^{n-1}$ into the diagonal matrix

$$\iota'(\theta) = \operatorname{diag}\left(e^{ix_1}, \dots, e^{ix_n}\right)$$

of $G$, where

$$x_1 = \theta_1, \qquad x_j = \theta_j - \theta_{j-1}, \qquad x_n = -\theta_{n-1},$$

for $2 \leq j \leq n-1$. If $\{e'_1, \dots, e'_{n-1}\}$ is the standard basis of $\mathbb{R}^{n-1}$ and $\Lambda'$ is the lattice of $\mathbb{R}^{n-1}$ generated by the basis $\{2\pi e'_j\}_{1 \leq j \leq n-1}$ of $\mathbb{R}^{n-1}$, the mapping $\iota'$ induces by passage to the quotient an imbedding

$$\iota' : \mathbb{R}^{n-1}/\Lambda' \to G.$$

The image of the mappings $\iota'$ is the maximal torus $H$ of the group $G$ which consists of all diagonal matrices of $G$ and is therefore a maximal flat totally geodesic torus of $G$ viewed as a symmetric space. Clearly we have $\iota'(0) = e_0$. We previously considered this maximal torus $H$ of $G$ in [4, §3] and [5, §2].

The complexification $\mathfrak{h}$ of the Lie algebra $\mathfrak{h}_0$ of $H$ is a Cartan algebra of the complexification $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of the simple Lie algebra $\mathfrak{g}_0$, and consists of all diagonal matrices of $\mathfrak{g}$. In fact, the matrices $\{C_1, \dots, C_{n-1}\}$ form a basis of $\mathfrak{h}_0$. For $1 \leq j \leq n$, the linear form $\lambda_j : \mathfrak{h} \to \mathbb{C}$, sending the diagonal matrix with $a_1, \dots, a_n \in \mathbb{C}$ as its diagonal entries into $a_j$, is purely imaginary on $\mathfrak{h}_0$. We write $\alpha_j = \lambda_j - \lambda_{j+1}$, for $1 \leq j \leq n-1$. Then

$$\{\, \lambda_j - \lambda_k \mid 1 \leq j, k \leq n \text{ and } j \neq k \,\}$$

is the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. As in [4, §5], we take

$$\{\alpha_1, \dots, \alpha_{n-1}\}$$

as a system of simple roots of $\mathfrak{g}$; the corresponding system of positive roots is

$$\Delta^+ = \{\, \lambda_j - \lambda_k \mid 1 \leq j < k \leq n \,\}.$$

If $\alpha$ is the root $\lambda_j - \lambda_k$, with $1 \leq j, k \leq n$ and $j \neq k$, the root subspace $\mathfrak{g}_\alpha$ corresponding to $\alpha$ is generated by $E_{jk}$ (over $\mathbb{C}$). We have the decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha};$$

in fact, $\mathfrak{n}^+$ is spanned by the elements $E_{jk}$ of $\mathfrak{g}$, with $1 \leq j < k \leq n$. The corresponding fundamental weights are

$$\varpi_j = \lambda_1 + \cdots + \lambda_j,$$

with $1 \leq j \leq n-1$; in fact, $\varpi_j$ is the highest weight of the irreducible $G$-module $\bigwedge^j \mathbb{C}^n$, and we have

$$\varpi_k(C_j) = i\delta_{jk},$$

for $1 \leq j, k \leq n-1$. The unique element $w_0$ of the Weyl group of $\mathfrak{g}$ determined

$$w_0(\Delta^+) = -\Delta^+$$

is the involutive automorphism satisfying

$$w_0(\varpi_j) = -\varpi_{n-j}, \tag{2.3}$$

for $1 \leq j \leq n-1$ (see [4, §5] or [5, §4]). A dominant integral form $\lambda$ for $G$ may be written in a unique way

$$\lambda = \gamma_{r_1,\dots,r_{n-1}} = r_1\varpi_1 + \cdots + r_{n-1}\varpi_{n-1}, \tag{2.4}$$

where $r_1, \dots, r_{n-1}$ are non-negative integers. Thus the highest weight of an irreducible (complex) $G$-module has a unique expression of this form and so is equal to

$$c_1\lambda_1 + \cdots + c_{n-1}\lambda_{n-1}, \tag{2.5}$$

where $c_1, \dots, c_{n-1}$ are integers satisfying $c_1 \geq \cdots \geq c_{n-1} \geq 0$; hence we may identify the dual $\Gamma$ of $G$ with the set of all linear forms on $\mathfrak{h}$ which can be written in the form (2.4) (or equivalently in the form (2.5)).

If $\gamma = \gamma_{r_1,r_2,\dots,r_{n-1}}$ is an element of $\Gamma$, where $r_1, r_2, \dots, r_{n-1}$ are non-negative integers, by (2.3), the unique element $\bar{\gamma}$ of $\Gamma$ determined by

$$w_0(\gamma) = -\bar{\gamma}$$

is equal to $\gamma_{r_{n-1},\dots,r_2,r_1}$; in particular, if $\gamma$ is the element $\varpi_k$ of $\Gamma$, we have the equality $\bar\gamma = \varpi_{n-k}$.

## 3. Branching laws and multiplicities

If $V$ is a real or complex vector space, we denote by $S^kV$ and $\bigwedge^l V$ the $k$-th symmetric product and the $l$-th exterior product of $V$, respectively. Let $n \geq 3$ be a given integer and let $U$ be the real vector space $\mathbb{R}^n$ endowed with its standard Euclidean scalar product. If $k \geq 2$, we consider the kernel $S^k_0U^*$ of the trace mapping $S^kU^* \to S^{k-2}U^*$ defined in [4, §2].

We consider the groups $G = SU(n)$ and $K = SO(n)$. The complexification $U_{\mathbb{C}}$ of $U$ is a $G$-module, and so the $k$-th symmetric product $S^kU^*_{\mathbb{C}}$ of $U^*_{\mathbb{C}}$ inherits a $G$-module structure. In fact, the space $S^kU^*_{\mathbb{C}}$ is an irreducible $G$-module and, for $k \geq 2$, the complexification $S^k_0U^*_{\mathbb{C}}$ of $S^k_0U^*$ is an irreducible $K$-module.

A partition $\pi = (\pi_1, \dots, \pi_{n-1})$ is an $(n-1)$-tuple of integers satisfying

$$\pi_1 \geq \pi_2 \geq \cdots \geq \pi_{n-1} \geq 0.$$

We say that a partition $\pi = (\pi_1, \dots, \pi_{n-1})$ is even if all its integers $\pi_j$ are even. Let

$$\gamma = a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}$$

be an element of $\Gamma$, where $a_1, \dots, a_{n-1}$ are non-negative integers. We associate with the element $\gamma$ the partition $\pi(\gamma) = (\pi_1, \dots, \pi_{n-1})$, where

$$\pi_j = a_1 + \cdots + a_{n-j},$$

for $1 \leq j \leq n-1$; in fact, this partition uniquely determines the element $\gamma$ of $\Gamma$. Let $N_0(\gamma)$ be the integer which is equal to 1 if the partition $\pi(\gamma)$ is even and 0 otherwise. We consider an irreducible $G$-module $E_\gamma$ corresponding to $\gamma$. Let $k \geq 1$ be a given integer and consider the set $\Sigma(\gamma, k)$ of all partitions $\eta = (\eta_1, \dots, \eta_{n-1})$ defined as follows: a partition $\eta = (\eta_1, \dots, \eta_{n-1})$ belongs to $\Sigma(\gamma, k)$ if and only if there exist integers $\nu_1, \dots, \nu_n \geq 0$ satisfying the relations

$$\eta_j = \nu_j - \nu_n, \qquad \nu_j \geq \pi_j \geq \nu_{j+1},$$
$$\nu_1 + \cdots + \nu_n = \pi_1 + \cdots + \pi_{n-1} + k,$$

for $1 \leq j \leq n-1$. We denote by $N_k(\gamma)$ the cardinality of the set $\Sigma'(\gamma, k)$ consisting of all even partitions of $\Sigma(\gamma, k)$.

Since $G$ is a real form of the group $SL(n, \mathbb{C})$ and the subgroup $K$ of $G$ is equal to $G \cap SO(n, \mathbb{C})$, from Pieri's formula (see Proposition 15.25,(i) and formula (A.7) of [1] and [3, §10]) and the relation (10.4) of [3], for $k \geq 2$, we deduce that the multiplicity $M_k(\gamma)$ of the irreducible $K$-module $S^k_0 U^*_{\mathbb{C}}$ in the decomposition of $E_\gamma$ viewed as a $K$-module is equal to

$$\dim \operatorname{Hom}_K(S^k_0 U^*_{\mathbb{C}}, E_\gamma) = N_k(\gamma) - N_{k-2}(\gamma). \tag{3.1}$$

By (3.1), we know that $M_2(\gamma)$ is equal to $N_2(\gamma) - 1$ if the partition $\pi(\gamma)$ is even (i.e., if the integers $a_j$ are even) and to $N_2(\gamma)$ otherwise.

If $p, q$ are given integers, let $\varepsilon^p_q$ be the integer equal to 1 if $p \geq q$ and 0 otherwise.

**Lemma 3.1.** *Suppose that $n = 3$. Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma_{r_1,r_2}$ of $\Gamma$. Then the integers $N_2(\gamma)$ and $M_4(\gamma)$ are given by the following table:*

| Conditions on $r_1$ and $r_2$ | $N_2(\gamma)$ | $M_4(\gamma)$ |
|---|---|---|
| $r_1, r_2$ even | $1 + \varepsilon^{r_1}_2 + \varepsilon^{r_2}_2$ | $\varepsilon^{r_1}_2 \varepsilon^{r_2}_2 + \varepsilon^{r_1}_4 + \varepsilon^{r_2}_4$ |
| $r_1, r_2$ odd | $1$ | $\varepsilon^{r_1}_3 + \varepsilon^{r_2}_3$ |
| $r_1$ even, $r_2$ odd | $\varepsilon^{r_1}_2$ | $\varepsilon^{r_1}_4 + \varepsilon^{r_1}_2 \varepsilon^{r_2}_3$ |
| $r_1$ odd, $r_2$ even | $\varepsilon^{r_2}_2$ | $\varepsilon^{r_2}_4 + \varepsilon^{r_1}_3 \varepsilon^{r_2}_2$ |

*Proof.* Let $\gamma = a_1\varpi_1 + a_2\varpi_2$ be an element of $\Gamma$ and consider the partition $\pi(\gamma) = (\pi_1, \pi_2)$ associated with $\gamma$. We consider the sequences

$$\begin{aligned}
&\xi^1 = (\pi_1 + 4, \pi_2), && \xi^2 = (\pi_1, \pi_2 + 4), && \xi^3 = (\pi_1 - 4, \pi_2 - 4),\\
&\xi^4 = (\pi_1, \pi_2 - 2), && \xi^5 = (\pi_1 - 2, \pi_2), && \xi^6 = (\pi_1 + 2, \pi_2 + 2),\\
&\eta^1 = (\pi_1 + 2, \pi_2), && \eta^2 = (\pi_1, \pi_2 + 2), && \eta^3 = (\pi_1 - 2, \pi_2 - 2)
\end{aligned}$$

associated with the partition $\pi(\gamma)$. If $a_1$ and $a_2$ are even integers, we see that a partition belonging to $\Sigma'(\gamma, 4)$ (resp. to $\Sigma'(\gamma, 2)$) is a subset of $\{\xi^1, \dots, \xi^6\}$ (resp. of $\{\eta^1, \eta^2, \eta^3\}$). We also consider the sequences

$$\begin{aligned}
&\xi^7 = (\pi_1, \pi_2 + 1), && \xi^8 = (\pi_1 + 2, \pi_2 - 1), && \xi^9 = (\pi_1 - 2, \pi_2 - 3),\\
&\xi^{10} = (\pi_1 + 1, \pi_2), && \xi^{11} = (\pi_1 - 1, \pi_2 + 2), && \xi^{12} = (\pi_1 - 3, \pi_2 - 2),\\
&\xi^{13} = (\pi_1 + 3, \pi_2 + 1), && \xi^{14} = (\pi_1 + 1, \pi_2 + 3), && \xi^{15} = (\pi_1 - 1, \pi_2 - 1),\\
&\eta^4 = (\pi_1, \pi_2 - 1), && \eta^5 = (\pi_1 - 1, \pi_2), && \eta^6 = (\pi_1 + 1, \pi_2 + 1)
\end{aligned}$$

associated with the partition $\pi(\gamma)$. If $a_1$ is even and $a_2$ is odd (resp. $a_1$ and $a_2$ are odd integers), we see that a partition belonging to $\Sigma'(\gamma,4)$ is a subset of $\{\xi^7,\xi^8,\xi^9\}$ (resp. of $\{\xi^{13},\xi^{14},\xi^{15}\}$), and that $\Sigma'(\gamma,2)$ is empty when $a_1=2$ and $a_2=1$ (resp. when $a_1=a_2=1$) and is equal to $\{\eta^4\}$ (resp. $\{\eta^6\}$) otherwise. Moreover, if $a_1$ is odd and $a_2$ is even, we see that a partition belonging to $\Sigma'(\gamma,4)$ is a subset of $\{\xi^{10},\xi^{11},\xi^{12}\}$, and that $\Sigma'(\gamma,2)=\{\eta^5\}$ when $a_2\geq 2$ and is empty otherwise. Using these remarks, we are able to compute the integers $N_2(\gamma)$ and $N_4(\gamma)$, from which we obtain the integer $M_4(\gamma)$. $\square$

## 4. The special Lagrangian Grassmannians

Let $n$ be a given integer $\geq 3$. Let $G$ be the group $SU(n)$ and let $K$ be the subgroup $SO(n)$, which is equal to the set of fixed points of the involution $s$ of $G$ sending a matrix into its complex conjugate. Then $(G,K)$ is a Riemannian symmetric pair. In the Cartan decomposition

$$\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$$

of the Lie algebra $\mathfrak{g}_0$ of $G$ corresponding to this involution, which we considered in [4, §6], we know that $\mathfrak{k}_0$ is the Lie algebra of $K$ and that the $K$-submodule $\mathfrak{p}_0$ is the space of all symmetric purely imaginary $n\times n$ matrices of trace zero. We denote by $\phi_*$ the action of an element $\phi$ of $G$ on the tangent bundle of $X$. We identify $\mathfrak{p}_0$ with the tangent space of $X$ at the coset $x_0$ of the identity element $e_0$ of $G$; in fact, if $\phi$ is an element of $K$, we have $\phi(x_0)=x_0$ and, if $\xi$ is an element of $\mathfrak{p}_0$, the vector $\phi_*\xi$ of $T_{x_0}$ is given by

$$\phi_*\xi=\operatorname{Ad}\phi\cdot\xi. \tag{4.1}$$

We consider the $G$-invariant metric $g$ and the symmetric 3-form $\sigma=\sigma_3$ on $X$ introduced in [4, §6]; they are the unique $G$-invariant symmetric forms whose restrictions to the space $T_{x_0}=\mathfrak{p}_0$ are equal to the restrictions of the symmetric forms $g'$ and $\sigma'_3$, respectively, to the space $\mathfrak{p}_0$. In fact, they are determined by

$$g(\xi_1,\xi_2)=-\operatorname{Tr}(\xi_1\cdot\xi_2),\qquad \sigma(\xi_1,\xi_2,\xi_3)=i\operatorname{Tr}(\xi_1\cdot\xi_2\cdot\xi_3),$$

for all $\xi_1,\xi_2,\xi_3\in\mathfrak{p}_0$. The Riemannian manifold $(X,g)$ is an irreducible symmetric space (which is not Hermitian), called the special Lagrangian Grassmannian. We shall consider the objects associated in §§1 and 2 with the symmetric space $X$ and the group $G$ and use the notation introduced there.

The set of matrices $\{B_{jk}, C_l\}$, with $1 \leq j < k \leq n$ and $1 \leq l \leq n-1$, form a basis of $\mathfrak{p}_0$. For $1 \leq j \leq n$, we consider the element

$$\tilde{C}_j = \frac{1}{n}\Big( \sum_{k=j}^{n-1} (n-k)C_k - \sum_{k=1}^{j-1} kC_k \Big)$$

of $\mathfrak{p}_0$. According to (2.1) and (2.2), we have

$$g(C_j, C_j) = 2, \qquad g(C_l, C_{l+1}) = -1, \qquad g(C_k, C_q) = 0, \tag{4.2}$$

for all $1 \leq j, k, q \leq n-1$ and $1 \leq l \leq n-2$, with $q \geq k+2$, and

$$g(C_l, B_{jk}) = 0, \qquad g(B_{jk}, B_{rs}) = 2\delta_{jr}\delta_{ls}, \tag{4.3}$$

for all $1 \leq l \leq n-1$ and $1 \leq j, k, r, s \leq n$ and, with $j < k$ and $r < s$.

We consider the $G$-equivariant monomorphism

$$\tilde{\sigma} : T^* \to S^2T^*$$

induced by the symmetric 3-form $\sigma$ and the differential operator $D_\sigma$ associated with $\sigma$ in §1. Let $\varphi$ be an element of $T^*_{x_0}$. In [4, §6], we saw that

$$\tilde{\sigma}(\varphi)(C_j, C_l) = 0, \tag{4.4}$$

for $1 \leq j, l \leq n-1$, with $l > j+1$, and that

$$\tilde{\sigma}(\varphi)(C_j, C_j) = \varphi(\tilde{C}_j + \tilde{C}_{j+1}), \tag{4.5}$$

for all $1 \leq j \leq n-1$; moreover, for all $1 \leq j \leq n-2$, we have

$$\tilde{\sigma}(\varphi)(C_j, C_{j+1}) = -\varphi(\tilde{C}_{j+1}). \tag{4.6}$$

From the relation (3.7) of [4], we deduce that

$$\tilde{\sigma}(\varphi)(B_{jk}, B_{jk}) = \varphi(\tilde{C}_j + \tilde{C}_k), \tag{4.7}$$

for $1 \leq j < k \leq n$.

The following lemma is a direct consequence of the expressions for the symmetric forms $g$ and $\sigma$.

**Lemma 4.1.** *Let $A, B, C$ be elements of $\mathfrak{p}_0$ and $c \in \mathbb{R}$ satisfying*

$$A \cdot B + B \cdot A = icC. \tag{4.8}$$

*Then we have*

$$\tilde{\sigma}(\varphi)(A, B) = \frac{c}{2}\varphi(C),$$

*for all $\varphi \in T^*_{x_0}$.*

Let $\{\omega_j, \varpi_{jk}\}$, with $1 \leq j < k \leq n$, be the basis of $T^*_{x_0}$ determined by

$$\omega_j(C_l) = \delta_{jl}, \quad \varpi_{jk}(C_l) = 0, \quad \omega_j(B_{rs}) = 0, \quad \varpi_{jk}(B_{rs}) = \delta_{jr}\delta_{ks},$$

for all $1 \leq l \leq n-1$ and $1 \leq j, k, r, s \leq n$, with $j < k$ and $r < s$. For $1 \leq j, k \leq n$, we consider the elements $\omega_0$, $\omega_n$, $\varpi_{jj}$ and $\varpi_{kj}$ of $T^*_{x_0}$ defined by

$$\omega_0 = \omega_n = \varpi_{jj} = 0, \qquad \varpi_{kj} = \varpi_{jk}.$$

Let $\mathcal{U}$ be the set of vectors $\{B_{jk}\}$ of $\mathfrak{p}_0$, with $1 \leq j < k \leq n$. If $A, B$ are elements of $\mathcal{U}$, with $A \neq B$, and $1 \leq l \leq n-1$ is a given integer, then we easily verify that the relation (4.8) holds, where $C$ is an element of $\mathcal{U}$ and $c = 0$ or 1, and that

$$A \cdot C_l + C_l \cdot A = icA,$$

where $c = 0$ or $\pm 1$. According to this remark, Lemma 4.1 and the relations (4.4)–(4.7), we see that

$$2\tilde{\sigma}(\varpi_{jk}) = (\omega_j - \omega_{j-1} + \omega_k - \omega_{k-1}) \cdot \varpi_{jk} + \sum_{l=1}^{n} \varpi_{jl} \cdot \varpi_{kl}, \tag{4.9}$$

for all $1 \leq j < k \leq n$; moreover, when $n = 3$, we have

$$\begin{aligned} \tilde{\sigma}(\omega_1) &= \frac{1}{6}\,(\omega_1^2 + 2\omega_1 \cdot \omega_2 - 2\omega_2^2 + \varpi_{12}^2 + \varpi_{13}^2 - 2\varpi_{23}^2), \\ \tilde{\sigma}(\omega_2) &= \frac{1}{6}\,(2\omega_1^2 - 2\omega_1 \cdot \omega_2 - \omega_2^2 + 2\varpi_{12}^2 - \varpi_{13}^2 - \varpi_{23}^2). \end{aligned} \tag{4.10}$$

Using *Maple* and the formulas (4.9) and (4.10), we see that the operator $D_\sigma$ is of finite type when $n = 3$; in fact, the morphism

$$S^3T^* \otimes (T \oplus F) \to S^2T^* \otimes S^2T^*, \tag{4.11}$$

which is equal to the second prolongation of the symbol of $D_\sigma$, is injective. Therefore by Proposition 6.2 of [6], we obtain:

**Proposition 4.2.** *When $n = 3$, the differential operator $D_\sigma$ on $X$ is elliptic.*

According to the relation (7.1) of [4], the equality of $G$-modules

$$C^\infty_{\bar{\gamma}}(S^pT^*_{\mathbb{C}}) = \overline{C^\infty_\gamma(S^pT^*_{\mathbb{C}})} \tag{4.12}$$

holds for all $\gamma \in \Gamma$ and $p \geq 0$.

We consider the line bundle $\{g\}$ generated by the section $g$ of $S^2T^*$, the subbundle $E_1$ of $S^2T^*$ introduced in [4, §6] and the isomorphisms

$$S_0^2U^* \to T^*_{x_0,\mathbb{C}}, \qquad S_0^4U^* \to E_{1,\mathbb{C},x_0}$$

of $K$-modules considered there; if $\gamma$ is an element of $\Gamma$, from the Frobenius reciprocity theorem, we obtain

$$\text{(4.13)} \qquad \operatorname{Mult} C^\infty_\gamma(T^*_{\mathbb{C}}) = M_2(\gamma), \qquad \operatorname{Mult} C^\infty_\gamma(E_{1,\mathbb{C}}) = M_4(\gamma).$$

In this section, we henceforth suppose that the integer $n$ is equal to 3. By (4.13), we see that the following proposition is an immediate consequence of Lemma 3.1.

**Proposition 4.3.** *Suppose that $n = 3$. Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma_{r_1,r_2}$ of $\Gamma$. Then the multiplicities of the $G$-modules $C^\infty_\gamma(T^*_{\mathbb{C}})$ and $C^\infty_\gamma(E_{1,\mathbb{C}})$ are given by the following table:*

| Conditions on $r_1$ and $r_2$ | $\operatorname{Mult} C^\infty_\gamma(T^*_{\mathbb{C}})$ | $\operatorname{Mult} C^\infty_\gamma(E_{1,\mathbb{C}})$ |
|---|---|---|
| $r_1, r_2$ even | $\varepsilon_2^{r_1} + \varepsilon_2^{r_2}$ | $\varepsilon_2^{r_1}\varepsilon_2^{r_2} + \varepsilon_4^{r_1} + \varepsilon_4^{r_2}$ |
| $r_1, r_2$ odd | 1 | $\varepsilon_3^{r_1} + \varepsilon_3^{r_2}$ |
| $r_1$ even, $r_2$ odd | $\varepsilon_2^{r_1}$ | $\varepsilon_4^{r_1} + \varepsilon_2^{r_1}\varepsilon_3^{r_2}$ |
| $r_1$ odd, $r_2$ even | $\varepsilon_2^{r_2}$ | $\varepsilon_4^{r_2} + \varepsilon_3^{r_1}\varepsilon_2^{r_2}$ |

If $r_1, r_2 \geq 0$ are given integers, we consider the elements

$$\begin{aligned}
\gamma^1_{r_1,r_2} &= (2r_1 + 2r_2)\lambda_1 + 2r_2\lambda_2, \\
\gamma^2_{r_1,r_2} &= (2r_1 + 2r_2 + 2)\lambda_1 + (2r_2 + 1)\lambda_2, \\
\gamma^3_{r_1,r_2} &= (2r_1 + 2r_2 + 3)\lambda_1 + (2r_2 + 1)\lambda_2, \\
\gamma^4_{r_1,r_2} &= (2r_1 + 2r_2 + 3)\lambda_1 + (2r_2 + 2)\lambda_2
\end{aligned}$$

of $\Gamma$. We easily verify that

$$\text{(4.14)} \qquad \overline{\gamma^j_{r_1,r_2}} = \gamma^j_{r_2,r_1}, \qquad \overline{\gamma^3_{r_1,r_2}} = \gamma^4_{r_2,r_1},$$

for $j = 1, 2$.

In [4, §7], we saw that the subset $\Gamma_0$ of $\Gamma$ is given by

$$\Gamma_0 = \{\, \gamma^1_{r_1,r_2} \mid r_1, r_2 \geq 0 \,\}.$$

Clearly we have $\gamma_0 = \gamma^1_{0,0}$ and we also know that $\gamma_1 = \gamma^2_{0,0}$. The $G$-invariant orthogonal decomposition (6.3) of [4] becomes

$$S^2T^* = \{g\} \oplus E_1 \oplus \tilde{\sigma}(T^*);$$

thus we obtain the equality

$$C^\infty_\gamma(S^2T^*_{\mathbb{C}}) = C^\infty_\gamma(X)\cdot g \oplus C^\infty_\gamma(E_{1,\mathbb{C}}) \oplus \tilde{\sigma} C^\infty_\gamma(T^*_{\mathbb{C}}),$$

for all $\gamma \in \Gamma$. Hence by Proposition 4.3, we have

$$C^\infty_{\gamma_0}(S^2T^*_{\mathbb{C}}) = C^\infty_{\gamma_0}(X)\cdot g = \mathbb{C}\cdot g.$$

If $\gamma$ is an element of $\Gamma$, we see that

$$\text{(4.15)} \qquad \operatorname{Mult} C^\infty_\gamma(S^2T^*_{\mathbb{C}}) = 1 + \operatorname{Mult} C^\infty_\gamma(E_{1,\mathbb{C}}) + \operatorname{Mult} C^\infty_\gamma(T^*_{\mathbb{C}})$$

whenever $\gamma$ belongs to $\Gamma_0$ and is not equal to $\gamma_0$, and that

$$\text{(4.16)} \qquad \operatorname{Mult} C^\infty_\gamma(S^2T^*_{\mathbb{C}}) = \operatorname{Mult} C^\infty_\gamma(E_{1,\mathbb{C}}) + \operatorname{Mult} C^\infty_\gamma(T^*_{\mathbb{C}})$$

whenever $\gamma$ does not belong to $\Gamma$; here the multiplicities $\operatorname{Mult} C^\infty_\gamma(E_{1,\mathbb{C}})$ and $\operatorname{Mult} C^\infty_\gamma(T^*_{\mathbb{C}})$ are given by Proposition 4.3. From this proposition, we now deduce that the $G$-modules $C^\infty_\gamma(T^*_{\mathbb{C}})$ and $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$ both vanish if $\gamma$ is not of the form $\gamma^j_{r_1,r_2}$, for some integers $r_1, r_2 \geq 0$ and $1 \leq j \leq 4$.

## 5. Symmetric forms

Let $n \geq 3$ be a given integer. For $1 \leq j, k \leq n$, we denote by $z_{jk}$ the function on the space of matrices $M_n$ which sends a matrix of $M_n$ into its $(j,k)$-th entry, and we consider the complex vector field

$$\xi_{jk} = \sum_{l=1}^n z_{jl} \frac{\partial}{\partial z_{kl}}$$

on $M_n$. For $1 \leq j \leq n$, we consider the $\mathbb{C}^n$-valued function $Z_j$ on $M_n$ which sends a matrix of $M_n$ into its $j$-th row.

If $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are elements of $\mathbb{C}^n$, we write

$$\langle z, w\rangle = \sum_{j=1}^n z_j w_j.$$

Clearly we have

$$\langle Z_j, Z_l\rangle(e_0) = \delta_{jl},$$

for $1 \leq j, l \leq n$. Let $1 \leq k \leq n-1$ be a given integer; we consider the $M_k$-valued function $A_k$ on $M_n$ defined by

$$A_k = (\langle Z_j, Z_l \rangle)_{1 \leq j,l \leq k}$$

and the complex-valued function

$$f_k = \det A_k$$

on $M_n$ which satisfies $f_k(e_0) = 1$. In particular, we have

$$f_1 = \langle Z_1, Z_1 \rangle, \qquad f_2 = \langle Z_1, Z_1 \rangle \langle Z_2, Z_2 \rangle - \langle Z_1, Z_2 \rangle^2;$$

we also write $f_1' = \langle Z_1, Z_2 \rangle$.

We consider the group $G = SU(n)$ as a real submanifold of the complex manifold $M_n$. The left action of the group $G$ on the manifold $M_n$ induces a morphism $\Phi$ from $\mathfrak{g}_0$ to the Lie algebra of vector fields on $M_n$, which are tangent to the submanifold $G$ of $M_n$. We recall that

$$\Phi(B_{jk}) = i(\bar{\xi}_{jk} + \bar{\xi}_{kj} - \xi_{jk} - \xi_{kj}), \quad \Phi(C_l) = i(\bar{\xi}_{ll} - \bar{\xi}_{l+1,l+1} + \xi_{l+1,l+1} - \xi_{ll}),$$

for $1 \leq j < k \leq n$ and $1 \leq l \leq n-1$.

According to equation (4.7) of [4], we have

$$\Phi(C_l) f_k = -2i\delta_{kl} f_k, \tag{5.1}$$

for $1 \leq k, l \leq n-1$. We now easily verify that

$$\Phi(B_{1j}) f_1 = -2i\langle Z_1, Z_j \rangle, \quad \Phi(B_{jk}) f_1 = 0, \quad \Phi(B_{12}) f_2 = \Phi(B_{rs}) f_2 = 0,$$
$$\Phi(B_{1k}) f_2 = 2i(\langle Z_1, Z_2 \rangle \langle Z_2, Z_k \rangle - \langle Z_2, Z_2 \rangle \langle Z_1, Z_k \rangle),$$
$$\Phi(B_{2k}) f_2 = 2i(\langle Z_1, Z_2 \rangle \langle Z_1, Z_k \rangle - \langle Z_1, Z_1 \rangle \langle Z_2, Z_k \rangle),$$
$$\Phi(B_{1k}) f_1' = -i\langle Z_2, Z_k \rangle, \quad \Phi(B_{2k}) f_1' = -i\langle Z_1, Z_k \rangle,$$
$$\Phi(B_{12}) f_1' = -i(\langle Z_1, Z_1 \rangle + \langle Z_2, Z_2 \rangle), \quad \Phi(C_l) f_1' = -i\delta_{2l} f_1',$$

for all $2 \leq j < k \leq n$, $3 \leq r < s \leq n$ and $1 \leq l \leq n-1$. Thus we have

$$\begin{gathered} (\Phi(B_{jk}) f_1)(e_0) = (\Phi(B_{jk}) f_2)(e_0) = 0, \\ (\Phi(C_l) f_1')(e_0) = 0, \quad (\Phi(B_{jk}) f_1')(e_0) = -2i\delta_{1j}\delta_{2k}, \end{gathered} \tag{5.2}$$

for all $1 \leq j < k \leq n$ and $1 \leq l \leq n-1$.

We consider the subgroup $K = SO(n)$ of $G$, the symmetric space $X = SU(n)/SO(n)$ and the natural projection $\rho : G \to X$. A function $f$ on $G$ which is invariant under the right action of $K$ on $G$ determines a function $\tilde{f}$ on $X$ satisfying $\rho^*\tilde{f} = f$. Let $\xi$ be an element of $\mathfrak{g}_0$. The vector field $\Phi(\xi)$ on $G$ is right-invariant; thus the vector field $\Phi(\xi)$ is $\rho$-projectable and the vector field $\hat{\xi} = \rho_*\Phi(\xi)$ on $X$ is a Killing vector field. Moreover, if $f$ is a function on $G$ which is invariant under the right action of $K$, we have the relations

$$(\hat{\xi} \cdot \tilde{f})(x_0) = (\Phi(\xi)f)(e_0) = \begin{cases} -\xi \cdot \tilde{f} & \text{if } \xi \in \mathfrak{p}_0, \\ 0 & \text{if } \xi \in \mathfrak{k}_0; \end{cases} \tag{5.3}$$

here in the expression $\xi \cdot \tilde{f}$, the vector $\xi$ is viewed as an element of $T_{x_0}$. In fact, the $G$-module $\mathcal{K}$ of all Killing vector fields on $X$ is given by

$$\mathcal{K} = \{\, \hat{\xi} \mid \xi \in \mathfrak{g}_0 \,\}$$

and we know that the equality (1.5) holds. Moreover, the vector field $\eta_0$ on $X$ induced by the vector field

$$\tfrac{1}{2}(\Phi(A_{1n}) - i\Phi(B_{1n}))$$

on $G$ is a highest weight vector of the $G$-module $\mathcal{K}_{\mathbb{C}}$. By (5.3), we have

$$\eta_0(x_0) = \frac{i}{2}\, B_{1n},$$

where the vector $B_{1n}$ of $\mathfrak{p}_0$ is viewed as an element of $T_{x_0}$; thus by (4.3), we obtain

$$g^\flat(\eta_0) = i\varpi_{1n}. \tag{5.4}$$

The left action of $G$ on $M_n$ induces a structure of $G$-module on the space $C^\infty(M_n)$; we consider the $G$-submodule $C^\infty(M_n)^K$ of $C^\infty(M_n)$ consisting of all functions which are invariant under the right action of $K$. If $f$ is an element of $C^\infty(M_n)^K$, the restriction of $f$ to $G$ induces by passage to the quotient a function $\tilde{f}$ on $X$. The function $\langle Z_j, Z_l\rangle$, with $1 \leq j, l \leq n$, belongs to $C^\infty(M_n)^K$. Let $\{e_1, \dots, e_n\}$ be the standard basis of $U$ and $U_{\mathbb{C}}$; we easily verify that the mappings

$$\phi : S^2U_{\mathbb{C}}^* \to C^\infty(M_n)^K, \qquad \phi' : S^2U_{\mathbb{C}} \to C^\infty(M_n)^K,$$

which send an element $h$ of $S^2U_{\mathbb{C}}^*$ into the function

$$\sum_{j,k=1}^{n} h(e_j, e_k)\langle Z_j, Z_k\rangle$$

and the element $e_j \cdot e_k$ of $S^2U_{\mathbb{C}}$, with $1 \leq j, k \leq n$, into the function $\langle Z_j, Z_k \rangle$, are morphisms of $G$-modules. Hence the image $\mathcal{H}$ of $\phi$ is a $G$-submodule of $C^\infty(M_n)^K$; clearly, the image of $\phi'$ is equal to $\overline{\mathcal{H}}$. Since the function $f_1$ belonging to $\mathcal{H}$ is non-zero and the $G$-modules $S^2U^*_{\mathbb{C}}$ and $S^2U_{\mathbb{C}}$ are irreducible, the mappings $\phi$ and $\phi'$ are injective and $\mathcal{H}$ is an irreducible $G$-module. Since $\mathcal{H}$ is a submodule of $C^\infty(M_n)^K$, the $G$-submodule

$$\tilde{\mathcal{H}} = \{ \tilde{f} \mid f \in \mathcal{H} \}$$

of $C^\infty(X)$ is isomorphic to $\mathcal{H}$ and therefore also to $S^2U^*_{\mathbb{C}}$. Since the highest weights of the irreducible $G$-modules $S^2U^*_{\mathbb{C}}$ and $S^2U_{\mathbb{C}}$ are equal to $2\varpi_{n-1}$ and $2\varpi_1$, respectively, we obtain the equalities

$$C^\infty_{2\varpi_{n-1}}(X) = \tilde{\mathcal{H}}, \qquad C^\infty_{2\varpi_1}(X) = \overline{\tilde{\mathcal{H}}}.$$

Moreover, for $1 \leq k \leq n-1$, the complex-valued function $f_k$ on $M_n$ belongs to $C^\infty(M_n)^K$ and so it induces a function $\tilde{f}_k$ on $X$. The complex conjugate $\hat{f}_k$ of $\tilde{f}_k$ is equal to the function on $X$ induced by the function $\bar{f}_k$. We also consider the function $\tilde{f}'_1$ on $X$ induced by the element $f'_1$ of $\mathcal{H}$ and the complex conjugate $\hat{f}'_1$ of $\tilde{f}'_1$. Clearly we have $\tilde{f}_k(x_0) = 1$ and $\tilde{f}'_1(x_0) = 0$; from the relations (5.1)–(5.3), we infer that

$$(d\tilde{f}_1)(x_0) = 2i\omega_1, \qquad (d\tilde{f}_2)(x_0) = 2i\omega_2, \qquad (d\tilde{f}'_1)(x_0) = 2i\varpi_{12}; \tag{5.5}$$

thus the 1-form

$$\omega = \tilde{f}_1 d\tilde{f}'_1 - \tilde{f}'_1 d\tilde{f}_1$$

satisfies

$$\omega(x_0) = 2i\varpi_{12}. \tag{5.6}$$

The mapping

$$\chi : \bigwedge^2 C^\infty(M_n)^K \to C^\infty(T^*_{\mathbb{C}}),$$

defined by

$$\chi(f \wedge f') = \tilde{f} d\tilde{f}' - \tilde{f}' d\tilde{f},$$

for $f, f' \in W$, is a morphism of $G$-modules. The 1-form

$$\chi(\bar{f}_1 \wedge \bar{f}'_1) = \hat{f}_1 d\hat{f}'_1 - \hat{f}'_1 d\hat{f}_1$$

on $X$ is equal to the complex conjugate $\bar{\omega}$ of the 1-form $\omega$ and is therefore non-zero. We easily see that the highest weight of the $G$-module $\bigwedge^2(S^2U_{\mathbb{C}})$ is $3\lambda_1 + \lambda_2$ and that its highest weight vectors are the non-zero multiples of the vector $v = (e_1 \cdot e_1) \wedge (e_1 \cdot e_2)$. Thus $v$ is a highest weight vector of an irreducible $G$-submodule of $\bigwedge^2(S^2U_{\mathbb{C}})$. From the equalities

$$\bar{\omega} = \chi(\bar{f}_1 \wedge \bar{f}'_1) = \chi(\phi'(e_1 \cdot e_1) \wedge \phi'(e_1 \cdot e_2)),$$

we now deduce that $\bar{\omega}$ is the highest weight vector of an irreducible $G$-submodule of $C^\infty(T^*_{\mathbb{C}})$ whose highest weight is $3\lambda_1 + \lambda_2$.

In this section, we henceforth suppose that $n = 3$. By Lemma 9.3 of [4], we have

$$(5.7) \qquad \tilde{\sigma}(d\tilde{f}_1) = -\frac{i}{6}\big(4\tilde{f}_1 g + 3\,\mathrm{Hess}\,\tilde{f}_1\big), \qquad \tilde{\sigma}(d\tilde{f}_2) = \frac{i}{6}\big(4\tilde{f}_2 g + 3\,\mathrm{Hess}\,\tilde{f}_2\big).$$

Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma^1_{r_1,r_2}$ of $\Gamma_0$. The complex-valued function $\tilde{f}_{r_1,r_2} = \tilde{f}_1^{r_1} \cdot \tilde{f}_2^{r_2}$ is equal to the function on $X$ induced by the function $f_{r_1,r_2} = f_1^{r_1} \cdot f_2^{r_2}$ on $G$; the complex conjugate $\hat{f}_{r_1,r_2} = \hat{f}_1^{r_1} \cdot \hat{f}_2^{r_2}$ of $\tilde{f}_{r_1,r_2}$ is equal to the function on $X$ induced by the function $\bar{f}_{r_1,r_2}$ on $G$. If $r_1, r_2 \in \mathbb{Z}$, when at least one of the integers is $< 0$, we set

$$\tilde{f}_{r_1,r_2} = 0.$$

We consider the 1-forms

$$\varphi_1 = \tilde{f}_{r_1-1,r_2}\, d\tilde{f}_1, \qquad \varphi_2 = \tilde{f}_{r_1,r_2-1}\, d\tilde{f}_2;$$

we also consider the subspace $V_{r_1,r_2}$ of $C^\infty(S^2T^*_{\mathbb{C}})$ generated by the symmetric 2-forms

$$h_0 = \tilde{f}_{r_1,r_2} g, \quad h'_1 = \tilde{f}_{r_1-1,r_2}\tilde{\sigma}(d\tilde{f}_1), \quad h'_2 = \tilde{f}_{r_1,r_2-1}\tilde{\sigma}(d\tilde{f}_2),$$
$$h_3 = \tilde{f}_{r_1-2,r_2}\, d\tilde{f}_1 \cdot d\tilde{f}_1, \quad h_4 = \tilde{f}_{r_1,r_2-2}\, d\tilde{f}_2 \cdot d\tilde{f}_2, \quad h_5 = \tilde{f}_{r_1-1,r_2-1}\, d\tilde{f}_1 \cdot d\tilde{f}_2$$

on $X$, and the subspace $V'_{r_1,r_2}$ of $V_{r_1,r_2}$ generated by the 2-forms $h'_1 = \tilde{\sigma}(\varphi_1)$ and $h'_2 = \tilde{\sigma}(\varphi_2)$. We write

$$h_1 = \tilde{f}_{r_1-1,r_2}\mathrm{Hess}\,\tilde{f}_1, \qquad h_2 = \tilde{f}_{r_1,r_2-1}\mathrm{Hess}\,\tilde{f}_2;$$

according to the equalities (5.7), we see that $\{h_j\}_{0 \leq j \leq 5}$ is also a set of generators of $V_{r_1,r_2}$.

According to [4, §7] (cf. Proposition 4.3), we know that the following result is true:

**Lemma 5.1.** *Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma^1_{r_1,r_2}$ of $\Gamma_0$.*

(i) *The function $\hat{f}_{r_1,r_2}$ on $X$ is a highest weight vector of the irreducible $G$-module $C^\infty_\gamma(X)$.*

(ii) *The non-zero members of the family $\{\bar{\varphi}_1, \bar{\varphi}_2\}$ form a basis of the weight space of the $G$-module $C^\infty_\gamma(T^*_{\mathbb{C}})$ corresponding to its highest weight.*

If the form $h_j$ (resp. $h'_k$) is non-zero, with $0 \leq j \leq 5$ (resp. with $k = 1, 2$), from Lemma 5.1,(i) it follows that $\bar{h}_j$ (resp. $\bar{h}'_k$) is a highest weight vector of the $G$-module $C^\infty_\gamma(S^2 T^*_{\mathbb{C}})$.

**Proposition 5.2.** *Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma^1_{r_1,r_2}$ of $\Gamma$.*

(i) *The non-zero vectors of the set $\{h_j\}_{0 \leq j \leq 5}$ form a basis of the vector space $V_{r_1,r_2}$, and we have*

$$\dim V_{r_1,r_2} = 1 + \varepsilon_1^{r_1} + \varepsilon_1^{r_2} + \varepsilon_1^{r_1}\varepsilon_1^{r_2} + \varepsilon_2^{r_1} + \varepsilon_2^{r_2}.$$

(ii) *The weight space of the $G$-module $C^\infty_\gamma(S^2 T^*_{\mathbb{C}})$ corresponding to the highest weight $\gamma$ is equal to the complex conjugate $\overline{V}_{r_1,r_2}$ of the space $V_{r_1,r_2}$.*

*Proof.* From the relations (4.3), (4.7), (5.5) and (5.7), we deduce assertion (i). The second assertion now follows from Proposition 4.3, the equality (4.15) and the remark made above concerning the forms $\bar{h}_j$ as highest weight vectors. □

Since the vector field $\eta_0$ is a highest weight vector of the $G$-module $\mathcal{K}_{\mathbb{C}}$, the 1-form $g^\flat(\eta_0)$ is a highest weight vector of the irreducible $G$-module $C^\infty_{\gamma_1}(T^*_{\mathbb{C}})$, where $\gamma_1 = \gamma^2_{0,0}$. Clearly, the complex conjugate $\xi_0$ of $\eta_0$ also belongs to $\mathcal{K}_{\mathbb{C}}$. We saw above that the complex conjugate $\bar{\omega}$ of the 1-form $\omega$ is a highest weight vector of the $G$-module $C^\infty_\gamma(T^*_{\mathbb{C}})$, where $\gamma = \gamma^3_{0,0}$; according to Proposition 4.3, we know that this $G$-module is irreducible.

Let $r_1, r_2 \geq 0$ be given integers. From Lemma 5.1,(i), for $j = 2$ (resp. $j = 3$), it follows that $\hat{f}_{r_1,r_2} g^\flat(\eta_0)$ (resp. $\hat{f}_{r_1,r_2}\bar{\omega}$) is a highest weight vector of the irreducible $G$-module $C^\infty_\gamma(T^*_{\mathbb{C}})$, where $\gamma = \gamma^j_{r_1,r_2}$. We consider the symmetric 2-forms

$$k_1 = i\tilde{f}_{r_1,r_2}\tilde{\sigma}(g^\flat(\xi_0)), \quad k_2 = \tilde{f}_{r_1-1,r_2}\, d\tilde{f}_1 \cdot g^\flat(\xi_0), \quad k_3 = \tilde{f}_{r_1,r_2-1}\, d\tilde{f}_2 \cdot g^\flat(\xi_0),$$

$$k'_1 = i\tilde{f}_{r_1,r_2}\tilde{\sigma}(\omega), \quad k'_2 = \tilde{f}_{r_1-1,r_2}\, d\tilde{f}_1 \cdot \omega, \quad k'_3 = \tilde{f}_{r_1,r_2-1}\, d\tilde{f}_2 \cdot \omega$$

on $X$, and the subspaces $W_{r_1,r_2}$ and $W'_{r_1,r_2}$ of $C^\infty(S^2T^*_{\mathbb{C}})$ generated (over $\mathbb{C}$) by the forms $\{k_1, k_2, k_3\}$ and $\{k'_1, k'_2, k'_3\}$, respectively.

If the form $k_j$ (resp. $k'_j$) is non-zero, with $1 \leq j \leq 3$, by Lemma 5.1,(i) we see that $\bar{k}'_j$ is a highest weight vector of the $G$-module $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$, where $\gamma = \gamma^j_{r_1,r_2}$, with $j = 2$ (resp. $j = 3$).

**Proposition 5.3.** *Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma^2_{r_1,r_2}$ (resp. $\gamma^3_{r_1,r_2}$) of $\Gamma$.*

(i) *The non-zero generators of the vector space $W_{r_1,r_2}$ (resp. $W'_{r_1,r_2}$) form a basis of this vector space, and its dimension is equal to $1 + \varepsilon_1^{r_1} + \varepsilon_1^{r_2}$.*

(ii) *The weight space of the $G$-module $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$ corresponding to the highest weight $\gamma$ is equal to the complex conjugate $\overline{W}_{r_1,r_2}$ (resp. $\overline{W}'_{r_1,r_2}$) of the space $W_{r_1,r_2}$ (resp. $W'_{r_1,r_2}$)*

*Proof.* From the relations (4.9) and (5.4)–(5.6), we deduce assertion (i). The second assertion now follows from Proposition 4.3, the equality (4.16) and the remark made above concerning the forms $\bar{k}_j$ and $\bar{k}'_j$ as highest weight vectors. □

## 6. Symmetric forms and the Guillemin condition

Let $n \geq 3$ be a given integer and $X$ be the special Lagrangian Grassmannian $SU(n)/SO(n)$. The isometry $\tau$ of $X$ defined in [4, §10] generates a cyclic group $\Sigma$ of order $n$, and the reduced space $Y$ of $X$ (which we call the reduced Lagrangian Grassmannian) is equal to the quotient of $X$ by $\Sigma$ (see [4, §10]).

According to the relations (10.4) of [4], we have

$$\tau^* \tilde{f}_k = e^{2i\pi/n} \tilde{f}_k, \qquad \tau^* \tilde{f}'_1 = e^{2i\pi/n} \tilde{f}'_1$$

for $1 \leq k \leq n-1$. The 1-form $\omega$ on $X$ therefore satisfies

$$\tau^*\omega = e^{4i\pi/n}\omega; \tag{6.1}$$

according to the relations (10.4) of [4], if $r_1, \dots, r_{n-1} \geq 0$ are integers, we then obtain

$$\tau^* \tilde{f}_{r_1,\dots,r_{n-1}} = e^{2i\pi(r_1+2r_2+\cdots+(n-1)r_{n-1})/n} \tilde{f}_{r_1,\dots,r_{n-1}}. \tag{6.2}$$

Thus the 1-form $\tilde{f}_{r_1,\dots,r_{n-1}}\omega$ is invariant under the isometry $\tau$ if and only if the relation

$$r_1 + 2r_2 + \cdots + (n-1)r_{n-1} + 2 \equiv 0 \bmod n.$$

holds.

In this section, we henceforth suppose that $n = 3$. Let $r_1, r_2 \geq 0$ be given integers. According to (6.2), the function $\tilde{f}_{r_1,r_2}$ on $X$ is invariant under the isometry $\tau$ if and only if

$$r_1 \equiv r_2 \bmod 3. \tag{6.3}$$

We remark that the vector field $\xi_0$ is $\tau$-invariant. Therefore the 1-form $\theta_1 = \tilde{f}_{r_1,r_2} g^\flat(\xi_0)$ is invariant under the isometry $\tau$ if and only if the relation (6.3) holds; moreover, if the symmetric 2-form $h_l$ or $k_j$, associated with the integers $r_1, r_2$ in §5, with $1 \leq l \leq 6$ and $1 \leq j \leq 3$, is non-zero, by formula (10.2) of [4] it is invariant under the isometry $\tau$ if and only if the relation (6.3) holds. On the other hand, by (6.1) and (6.2) we see that the 1-form $\theta_2 = \tilde{f}_{r_1,r_2}\omega$ invariant under the isometry $\tau$ if and only if the relation

$$r_1 \equiv r_2 + 1 \bmod 3. \tag{6.4}$$

holds; moreover, if the symmetric 2-form $k'_j$, associated with the integers $r_1, r_2$ in §5, with $1 \leq j \leq 3$, is non-zero, by formula (10.2) of [4] it is invariant under the isometry $\tau$ if and only if the relation (6.4) holds.

From the preceding remarks, Lemma 5.1 and Propositions 5.2,(ii) and 5.3,(ii), we now obtain the following:

**Proposition 6.1.** *Let $r_1, r_2 \geq 0$ and $1 \leq j \leq 3$ be given integers and let $\gamma$ be the element $\gamma^j_{r_1,r_2}$ of $\Gamma$.*

(i) *If $j = 1$, the $G$-module $C^\infty_\gamma(X)^\Sigma$ is equal to $C^\infty_\gamma(X)$ if and only if $r_1$ and $r_2$ satisfy* (6.3).

(ii) *For $j, p = 1, 2$, the $G$-module $C^\infty_\gamma(S^pT^*_{\mathbb{C}})^\Sigma$ is equal to $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ if the relation* (6.3) *holds and it vanishes otherwise.*

(iii) *For $j = 3$ and $p = 1, 2$, the $G$-module $C^\infty_\gamma(S^pT^*_{\mathbb{C}})^\Sigma$ is equal to $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ if the relation* (6.4) *holds and it vanishes otherwise.*

Proposition 6.1,(i) tells us that the subset $\Gamma_1$ of $\Gamma_0$ is given by

$$\Gamma_1 = \{\, \gamma^1_{r_1,r_2} \mid r_1, r_2 \geq 0 \text{ and } r_1 \equiv r_2 \bmod 3 \,\}.$$

In §10, we shall prove the following three results:

**Proposition 6.2.** *Let $r_1, r_2$ be given integers satisfying the relation* (6.3) *and* $0 \leq r_1 \leq r_2$. *Then the function $\tilde{f}_{r_1,r_2}$ on $X$ does not satisfy the Guillemin condition.*

**Proposition 6.3.** *Let $r_1, r_2$ be given integers satisfying the relation* (6.3) *and* $0 \leq r_1 \leq r_2$.

(i) *Suppose that $(r_1, r_2) \neq (0,0), (1,1)$. Then we have*

$$V_{0,0} \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}, \qquad \dim (V_{1,1} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 2,$$
$$\dim (V_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1 + \varepsilon_1^{r_1} + \varepsilon_1^{r_2}.$$

(ii) *Suppose that $r_1, r_2 \geq 1$. Then we have*

$$\dim (V'_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1.$$

**Proposition 6.4.** *Let $r_1, r_2 \geq 0$ be given integers.*

(i) *Suppose that the relation* (6.3) *holds and that $0 \leq r_1 \leq r_2$. Then the symmetric 2-form $k_1 = i\tilde{f}_{r_1,r_2}\tilde{\sigma}(g^\flat(\xi_0))$ does not satisfy the Guillemin condition. Moreover, if $r_1 + r_2 > 0$, we have*

$$\dim (W_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1.$$

(ii) *Suppose that the relation* (6.4) *holds. Then the symmetric 2-form $k'_1 = i\tilde{f}_{r_1,r_2}\tilde{\sigma}(\omega)$ does not satisfy the Guillemin condition. Moreover, we have*

$$\dim (W'_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1.$$

We remark that the first assertion of Proposition 6.4,(i) implies that

$$W_{0,0} \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}.$$

Using the equalities (4.12) and (4.14), from Propositions 4.3, 5.2,(ii), 5.3,(ii), 6.3,(i) and 6.4, we obtain the following result:

**Proposition 6.5.** *Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma^j_{r_1,r_2}$ of $\Gamma$, with $1 \leq j \leq 3$.*

(i) *If $j = 1$ and $r_1 = r_2 = 0$, we have*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}}) = \{0\}.$$

(ii) *If $j = 1$ and the relation* (6.3) *holds, we have*

$$\mathrm{Mult}\,(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_\mathbb{C})) \leq 1 + \mathrm{Mult}\,C^\infty_\gamma(T^*_\mathbb{C})$$

*whenever* $(r_1, r_2) \neq (0, 0), (1, 1)$*, and*

$$\mathrm{Mult}\,(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_\mathbb{C})) \leq \mathrm{Mult}\,C^\infty_\gamma(T^*_\mathbb{C}) \tag{6.5}$$

*otherwise.*

(iii) *If $j = 2$ and $r_1 = r_2 = 0$, we have*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_\mathbb{C}) = \{0\}.$$

*Suppose that the relation* (6.3) (*resp.* (6.4)) *holds; if $j = 2$ and $r_1 + r_2 > 0$* (*resp. if $j = 3$*)*, then the inequality* (6.5) *is true.*

By means of the equalities (4.12) and (4.14), from Propositions 6.1,(iii) and 6.5,(iii) we deduce the following:

**Proposition 6.6.** *Let $r_1, r_2 \geq 0$ be given integers and let $\gamma$ be the element $\gamma^4_{r_1,r_2}$ of* $\Gamma$.

(i) *For $p = 1, 2$, the $G$-module $C^\infty_\gamma(S^pT^*_\mathbb{C})^\Sigma$ is equal to $C^\infty_\gamma(S^pT^*_\mathbb{C})$ if the relation*

$$r_1 \equiv r_2 + 2 \bmod 3. \tag{6.6}$$

*holds and vanishes otherwise.*

(ii) *If the relation* (6.6) *holds, then the inequality* (6.5) *is true.*

## 7. Main results

We consider the symmetric space $X = SU(3)/SO(3)$ and its reduced space $Y$.

From Proposition 2.29,(i) of [2], Lemma 5.1,(i) and Propositions 6.1,(i) and 6.2, we deduce the following result:

**Proposition 7.1.** *Let $Y$ be the reduced Lagrangian Grassmannian equal to the reduced space of $X = SU(3)/SO(3)$. The maximal flat Radon transform for functions on the symmetric space $Y$ is injective.*

If $\gamma_2$ is the element $\gamma^1_{1,1}$ of $\Gamma_1$, according to Proposition 9.1 of [4] we have

$$D_0 C^\infty(T_{\mathbb{C}}) \cap \tilde{\sigma} d C^\infty(X) = \tilde{\sigma} d C^\infty_{\gamma_2}(X).$$

Thus the irreducible symmetric space $X$ and the symmetric 3-form $\sigma$ satisfy hypothesis (b) of Proposition 1.1. On the other hand, Lemma 6.2 of [4] and Proposition 4.2 tell us that hypotheses (a) and (c) of this proposition also hold. According to the remark which appears at the end of §4 and Propositions 6.1, 6.5 and 6.6, we see that assertion (iv) of Proposition 1.1 is true for $X$, $\sigma$ and the group $\Sigma$. Then from Proposition 1.1, we deduce the following result:

**Theorem 7.2.** *Let $Y$ be the reduced Lagrangian Grassmannian equal to the reduced space of $X = SU(3)/SO(3)$. Then the equality*

$$\mathcal{N}_{2,Y} = D_0 C^\infty(T_Y) + \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y)$$

*holds.*

Since $\bar{\gamma}_2 = \gamma_2$, according to the relation (4.12), with $p = 0$, we know that the irreducible $G$-module $\mathcal{B} = C^\infty_{\gamma_2}(X)$ is invariant under conjugation and hence is equal to the complexification of the $G$-submodule

$$\mathcal{B}_{\mathbb{R}} = \{ f \in \mathcal{B} \mid f = \bar{f} \}$$

of $C^\infty_{\mathbb{R}}(X)$. Thus since $\gamma_2$ belongs to $\Gamma_1$, the $G$-module $\mathcal{B}_Y = C^\infty_{\gamma_2}(Y)$ is equal to the complexification of the subspace

$$\mathcal{B}_{Y,\mathbb{R}} = \{ f \in \mathcal{B}_Y \mid f = \bar{f} \}$$

of $C^\infty_{\mathbb{R}}(Y)$ and the mapping $\varpi$ induces an isomorphism $\varpi^* : \mathcal{B}_{Y,\mathbb{R}} \to \mathcal{B}_{\mathbb{R}}$.

If $P$ denotes the orthogonal projection corresponding to the decomposition (1.1) on the space $Y$, according to Lemma 1.1 of [3] and Lemma 6.2 of [4] the mapping

$$P_{\sigma_Y} = P \tilde{\sigma}_Y d : C^\infty_{\mathbb{R}}(Y) \to I(Y)$$

is well-defined. We denote by $\mathcal{F}_Y$ the orthogonal complement of the finite-dimensional space $\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{B}_Y$ in $C^\infty_{\mathbb{R}}(Y)$. From Proposition 1.2 of [3], Proposition 9.1 of [4] and Theorem 7.2, we obtain:

**Theorem 7.3.** *Let $Y$ be the reduced Lagrangian Grassmannian equal to the reduced space of $X = SU(3)/SO(3)$. Then the equality*

$$I(Y) = P \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y)$$

*holds and the mapping*

$$P\tilde{\sigma}_Y d : \mathcal{F}_Y \to I(Y)$$

*is an isomorphism.*

The preceding theorem is a complement to Theorem 10.2 of [4] with $n = 3$. According to Lemma 6.2 of [4], with $p = 3$, and Proposition 4.3 and the observations which follow it, and the remarks concerning 1-forms which precede Proposition 5.3, we see that Lemma 5.1,(ii), the equalities (4.12) and (4.14), and Propositions 1.2, 6.1, 6.3,(ii), 6.4, and 6.6,(i) give us the following result:

**Theorem 7.4.** *Let $Y$ be the reduced Lagrangian Grassmannian equal to the reduced space of $X = SU(3)/SO(3)$. A 1-form on $Y$ satisfies the Guillemin condition if and only it is exact.*

## 8. Some algebraic identities

If $p, q$ are integers, we define the binomial coefficient $\binom{p}{q}$ to be equal to zero whenever $q > p$, or whenever one of the integers $p, q$ is negative.

Let $m \geq 0$ be a given integer; for $1 \leq j \leq 16$, we define functions $\varphi_j$ on $\mathbb{N}$ by

$$\varphi_1(r) = \sum_{k\geq 0} \binom{r}{k}\binom{2m+r}{m+k}, \qquad \varphi_2(r) = \sum_{k\geq 0} \binom{r-1}{k}\binom{2m+r}{m+k},$$

$$\varphi_3(r) = \sum_{k\geq 0} k\binom{r}{k}\binom{2m+r}{m+k}, \qquad \varphi_4(r) = \sum_{k\geq 0} k\binom{r-1}{k}\binom{2m+r}{m+k},$$

$$\varphi_5(r) = \sum_{k\geq 0} \binom{r}{k}\binom{2m+r}{m+k+1}, \qquad \varphi_6(r) = \sum_{k\geq 0} k\binom{r}{k}\binom{2m+r}{m+k+1},$$

$$\varphi_7(r) = \sum_{k\geq 0} \binom{r-2}{k}\binom{2m+r}{m+k}, \qquad \varphi_8(r) = \sum_{k\geq 0} \binom{r-2}{k}\binom{2m+r}{m+k+1},$$

$$\varphi_9(r) = \sum_{k\geq 0} k\binom{r-2}{k}\binom{2m+r}{m+k}, \qquad \varphi_{10}(r) = \sum_{k\geq 0} k\binom{r-2}{k}\binom{2m+r}{m+k+1},$$

$$\varphi_{11}(r) = \sum_{k\geq 0} \binom{r}{k}\binom{2m+r-1}{m+k}, \qquad \varphi_{12}(r) = \sum_{k\geq 0} k\binom{r}{k}\binom{2m+r-1}{m+k},$$

$$\varphi_{13}(r) = \sum_{k\geq 0} k\binom{r-1}{k}\binom{2m+r}{m+k+1}, \qquad \varphi_{14}(r) = \sum_{k\geq 0} \binom{r-3}{k}\binom{2m+r}{m+k+1}$$

and

$$\varphi_{15}(r) = \sum_{k\geq 0} k\binom{r-3}{k}\binom{2m+r}{m+k+1}, \qquad \varphi_{16}(r) = \sum_{k\geq 0} \binom{r-4}{k}\binom{2m+r}{m+k+2},$$

for $r \geq 0$. By elementary computations, we verify that

$$2\varphi_3(r) = r\varphi_1(r), \tag{8.1}$$

$$\varphi_9(r) = \varphi_{10}(r), \tag{8.2}$$

for $r \geq 0$, and

$$\varphi_1(r) = 2\varphi_2(r) = 2\varphi_{11}(r), \tag{8.3}$$

for $r \geq 1$.

Standard techniques of WZ theory, as described in the book [8] and implemented by the EKHAD package for *Maple*, can be used to show that, for $j = 1, 4, 5, 6, 7, 8$, the function $\varphi_j$ satisfies a recurrence of order 1. We then easily deduce that

$$\varphi_1(r) = 2^r\binom{2m}{m}\prod_{k=1}^{r}\frac{2m+2k-1}{m+k} \tag{8.4}$$

and

$$\varphi_5(r) = \frac{m+r}{m+r+1}\,\varphi_1(r), \qquad \varphi_6(r) = \frac{r(m+r-1)}{2(m+r+1)}\,\varphi_1(r), \tag{8.5}$$

for $r \geq 0$. Moreover, if $r \geq 1$, we have

$$\varphi_4(r) = \frac{(r-1)(m+r)}{2(2m+2r-1)}\,\varphi_1(r) \tag{8.6}$$

and, if $r \geq 2$, we see that

$$\varphi_7(r) = \frac{m+r-1}{2(2m+2r-1)}\,\varphi_1(r), \quad \varphi_8(r) = \frac{m+r}{2(2m+2r-1)}\,\varphi_1(r). \tag{8.7}$$

From the formulas (8.2) and (8.6), we obtain the relations

$$\varphi_9(r) = \varphi_{10}(r) = \frac{(r-2)(m+r)}{4(2m+2r-1)}\,\varphi_1(r), \tag{8.8}$$

for $r \geq 2$. From the relations (8.1), (8.4) and (8.5), we easily deduce that

$$\varphi_{12}(r) = \frac{r(m+r-1)}{2(2m+2r-1)}\,\varphi_1(r), \tag{8.9}$$

for $r \geq 0$; from the formulas (8.4) and (8.9), we obtain the equality

$$\varphi_{13}(r) = \frac{(r-1)(m+r-1)}{2(2m+2r-1)}\,\varphi_1(r), \tag{8.10}$$

for $r \geq 1$. From the relations (8.3), (8.4) and (8.6), we easily deduce that

$$\begin{aligned} \varphi_{14}(r) &= \frac{m+r}{4(2m+2r-1)}\,\varphi_1(r), \\ \varphi_{15}(r) &= \frac{(r-3)(m+r)(m+r-1)}{4(2m+2r-1)(2m+2r-3)}\,\varphi_1(r), \end{aligned} \tag{8.11}$$

for $r \geq 3$; on the other hand, from the relations (8.4) and (8.7), we obtain

$$\varphi_{16}(r) = \frac{(m+r-1)(m+r)}{4(2m+2r-3)(2m+2r-1)}\,\varphi_1(r), \tag{8.12}$$

for $r \geq 4$.

## 9. Computing trigonometric integrals

Let $Y$ be an indeterminate over $\mathbb{C}$. If $P$ is an element of $\mathbb{C}[Y]$, we denote by $c_j(P)$ its coefficient of degree $j$ and write $c(P) = c_0(P)$; if $P$ is non-zero, we denote by $\ell(P)$ its leading coefficient.

Let $r, s$ be given integers satisfying $0 \leq r \leq s$ and the relation

$$r \equiv s \mod 3; \tag{9.1}$$

let $m \geq 0$ be the integer such that $s = r + 3m$. Let $e, e' \geq 0$ be given integers. If $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}$ are integers satisfying

$$\alpha_1 + \alpha_2 + \alpha_3 = e - e', \tag{9.2}$$

we consider the integers

$$\begin{aligned} C^{e,e'}_{\alpha_1,\alpha_2,\alpha_3} &= \binom{3m+r-e'}{m+\alpha_1} \sum_{k\geq 0} \binom{r-e}{k}\binom{2m+r-e'-\alpha_1}{m+k+\alpha_2}, \\ \tilde{C}^{e,e'}_{\alpha_1,\alpha_2,\alpha_3} &= \binom{3m+r-e'}{m+\alpha_1} \sum_{k\geq 0} k\binom{r-e}{k}\binom{2m+r-e'-\alpha_1}{m+k+\alpha_2}; \end{aligned}$$

we easily verify that

$$C^{e,e'}_{\alpha_1,\alpha_2,\alpha_3} = C^{e,e'}_{\alpha_1,\alpha_3,\alpha_2}. \tag{9.3}$$

Now let $d_1, d_2$ be given integers satisfying

$$e - e' \equiv d_1 - d_2 \pmod 3). \tag{9.4}$$

We write

$$\varepsilon_1 = \frac{2d_1 + d_2 + e - e'}{3}, \qquad \varepsilon_2 = \frac{d_2 - d_1 + e - e'}{3},$$
$$\varepsilon_3 = \frac{e - e' - d_1 - 2d_2}{3};$$

we verify that

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = e - e'.$$

The assumption (9.4) implies that $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are integers. We consider the set

$$A = \{\, (a_1, a_2, a_3) \in \mathbb{N}^3 \mid a_1 + a_2 + a_3 = r - e,\ m + a_j + \varepsilon_j \geq 0,\ \text{for } j = 1, 2, 3 \,\}.$$

We consider the functions $\psi$ and $\tilde{\psi}$ on $\mathbb{R}^3$ defined by

$$\psi(x, y, v) = ve^{ix} + (1 - v)e^{-iy} + e^{i(y-x)},$$
$$\tilde{\psi}(x, y, v) = ve^{ix} + (1 - v)e^{-iy} - e^{i(y-x)},$$

for all $(x, y, v) \in \mathbb{R}^3$; if $p, q \geq 0$ are integers, we also consider the functions

$$\psi_{p,q} = \psi^p \cdot \overline{\psi^q}, \qquad \tilde{\psi}_{p,q} = \tilde{\psi}^p \cdot \overline{\tilde{\psi}^q}$$

on $\mathbb{R}^3$. If $r_1, r_2 \geq 0$ and $a, b \in \mathbb{Z}$ are given integers, we see that the functions $\phi$ and $\tilde{\phi}$ on $\mathbb{R}$ defined by

$$\phi(v) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \psi_{r_1,r_2}(x, y, v) \cdot e^{i(ax+by)}\, dx\, dy,$$
$$\tilde{\phi}(v) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \tilde{\psi}_{r_1,r_2}(x, y, v) \cdot e^{i(ax+by)}\, dx\, dy,$$

for $v \in \mathbb{R}$, are in fact polynomials belonging to $\mathbb{Q}[Y]$.

We now suppose that $r \geq e$ and $s = 3m + r \geq e'$. We easily verify that

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \psi_{r-e,s-e'}(x, y, v) \cdot e^{i(d_1x+d_2y)}\, dx\, dy = F(v), \tag{9.5}$$

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \tilde{\psi}_{r-e,s-e'}(x, y, v) \cdot e^{i(d_1x+d_2y)}\, dx\, dy = \tilde{F}(v), \tag{9.6}$$

for all $v \in \mathbb{R}$, where $F$ and $\tilde{F}$ are the elements of $\mathbb{Q}[Y]$ given by

$$F(Y) = \sum_{(a,b,c)\in A} F_{a,b,c} \cdot Y^{2a+m+\varepsilon_1}(1-Y)^{2c+m+\varepsilon_3},$$
$$\tilde{F}(Y) = \sum_{(a,b,c)\in A} F_{a,b,c} \cdot Y^{2a+m+\varepsilon_1}(Y-1)^{2c+m+\varepsilon_3},$$

with

$$F_{a,b,c} = \frac{(r-e)!\,(3m+r-e')!}{a!\,b!\,c!\,(m+a+\varepsilon_1)!\,(m+b+\varepsilon_2)!\,(m+c+\varepsilon_3)!},$$

for $(a,b,c) \in A$. Clearly we also have the equality

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \psi_{s-e',r-e}(x,y,v)\cdot e^{-i(d_1x+d_2y)}\,dx\,dy = F(v). \tag{9.7}$$

In particular, if

$$m + \varepsilon_j \geq 0, \tag{9.8}$$

for $j = 1, 2, 3$, then we may write

$$F(Y) = Y^{m+\varepsilon_1}(1-Y)^{m+\varepsilon_3}\cdot Q(Y), \tag{9.9}$$

$$\tilde{F}(Y) = (-1)^{d_2}\cdot Y^{m+\varepsilon_1}(Y-1)^{m+\varepsilon_3}\cdot Q(Y), \tag{9.10}$$

where $Q = Q^{e,e'}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is the polynomial of $\mathbb{Q}[Y]$ of degree $2(r-e)$ equal to

$$\sum_{a+b+c=r-e} \frac{(r-e)!\,(3m+r-e')!}{a!\,b!\,c!\,(a+m+\varepsilon_1)!\,(b+m+\varepsilon_2)!\,(c+m+\varepsilon_3)!}\,Y^{2a}(Y-1)^{2c}.$$

The constant term $c(Q)$ and the leading coefficient $\ell(Q)$ of $Q$ are non-zero; in fact, we verify that

$$c(Q) = C^{e,e'}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}, \quad c_1(Q) = -2\,\tilde{C}^{e,e'}_{\varepsilon_1,\varepsilon_3,\varepsilon_2}, \quad \ell(Q) = C^{e,e'}_{\varepsilon_2,\varepsilon_1,\varepsilon_3}. \tag{9.11}$$

If $e = e' = d_1 = d_2 = 0$, the relation (9.8) holds and the polynomial $Q = Q^{0,0}_{0,0,0}$ is equal to the polynomial $P$ of degree $2r$ given by

$$P(Y) = \sum_{a+b+c=r} \frac{r!\,(3m+r)!}{a!\,b!\,c!\,(m+a)!\,(m+b)!\,(m+c)!}\,Y^{2a}(Y-1)^{2b}; \tag{9.12}$$

by formula (8.1), we see that the constant term and the leading coefficient of $P$ are positive integers and that

$$(9.13)\qquad \begin{aligned} c(P) = \ell(P) &= \binom{3m+r}{m}\varphi_1(r), \\ c_1(P) = -2\binom{3m+r}{m}\varphi_3(r) &= -r\binom{3m+r}{m}\varphi_1(r). \end{aligned}$$

## 10. Proofs of Propositions 6.2, 6.3 and 6.4

We consider the group $G = SU(3)$, the symmetric space $X = G/SO(3)$ and the natural projection $\rho : G \to X$. We consider the mapping

$$(10.1)\qquad \iota' : \mathbb{R}^2 \to G$$

of §2, which sends $\theta = (x, y) \in \mathbb{R}^2$ into the diagonal matrix

$$\iota'(\theta) = \operatorname{diag}\left(e^{ix}, e^{i(y-x)}, e^{-iy}\right)$$

of $G$; we also consider the mapping $\iota = \rho \circ \iota' : \mathbb{R}^2 \to X$. If $\{e'_1, e'_2\}$ is the standard basis of $\mathbb{R}^2$ and $\Lambda$ is the lattice of $\mathbb{R}^2$ generated by the basis $\{\pi e'_1, \pi e'_2\}$ of $\mathbb{R}^2$, the mapping $\iota$ induces by passage to the quotient an imbedding

$$\iota : \mathbb{R}^2/\Lambda \to X.$$

In [4, §6], we saw that the image $Z$ of $\iota$ is a maximal flat totally geodesic torus of $X$. Clearly we have $\iota(0) = x_0$.

We consider the standard coordinate system $(x, y)$ on $\mathbb{R}^2$ and endow this space with the flat Riemannian metric

$$\tilde{g} = dx \cdot dx + dy \cdot dy - dx \cdot dy.$$

According to the relation (6.4) of [4], we know that

$$(10.2)\qquad \iota^* g = \tilde{g};$$

hence if $f$ is a function on $X$, we easily see that

$$(10.3)\qquad \int_Z f\, dZ = \sqrt{3} \int_0^\pi \int_0^\pi f(\iota(x, y))\, dx\, dy.$$

As in [4, §6], we also consider the parallel vector fields $\zeta_1$ and $\zeta_2$ on $Z$ which are determined by

$$(10.4)\qquad \iota_*(\partial/\partial x)(x, y) = \zeta_1(\iota(x, y)), \qquad \iota_*(\partial/\partial y)(x, y) = \zeta_2(\iota(x, y)),$$

for $(x, y) \in \mathbb{R}^2$. If $\varphi$ is a 1-form on $X$, according to the formulas (6.11) and (6.12) of [4], we have

$$\iota^*\tilde{\sigma}(\varphi)(\partial/\partial x, \partial/\partial x) = \frac{1}{3}\big(\iota^*\varphi(\partial/\partial x) + 2\iota^*\varphi(\partial/\partial y)\big). \tag{10.5}$$

For $\alpha \in \mathbb{R}$, we consider the element

$$\phi_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\alpha & 1 & \sin\alpha \\ -\sqrt{2}\sin\alpha & 0 & \sqrt{2}\cos\alpha \\ \cos\alpha & -1 & \sin\alpha \end{pmatrix}$$

of $SO(3)$ and the maximal flat totally geodesic torus $Z_\alpha = \phi_\alpha(Z)$ of $X$. If $f$ is a function on $X$, we have

$$\int_{Z_a} f\, dZ_\alpha = \int_Z \phi_\alpha^* f\, dZ,$$

for all $\alpha \in \mathbb{R}$. For $\alpha \in \mathbb{R}$, we verify that

$$\begin{aligned} \operatorname{Ad}\phi_\alpha \cdot C_1 &= \frac{1}{2}\big((1+\cos^2\alpha)B_{13} - \sin^2\alpha\,(C_1 - C_2) \\ &\qquad\qquad - \sqrt{2}\cos\alpha\cdot\sin\alpha\,(B_{12}+B_{23})\big), \\ \operatorname{Ad}\phi_\alpha \cdot C_2 &= \frac{1}{2}\big(\cos^2\alpha\,(C_1 - C_2) - (1+\sin^2\alpha)B_{13} \\ &\qquad\qquad - \sqrt{2}\cos\alpha\cdot\sin\alpha\,(B_{12}+B_{23})\big). \end{aligned} \tag{10.6}$$

Since $\xi_0$ is a Killing vector field on $X$ and the mapping

$$\phi_\alpha \circ \iota : (\mathbb{R}^2, \tilde{g}) \to (X, g),$$

sending $\theta \in \mathbb{R}^2$ into $\phi_\alpha(\iota(\theta))$ is totally geodesic, for any element $\alpha \in \mathbb{R}$, there is a parallel vector field $\xi'_\alpha$ on $\mathbb{R}^2$ such that

$$\iota^*\phi_\alpha^*\, g^\flat(\xi_0) = \tilde{g}^\flat(\xi'_\alpha);$$

in fact, for all $\theta \in \mathbb{R}^2$, the tangent vector $(\phi_{\alpha *}\xi'_\alpha)(\iota(\theta))$ is equal to the orthogonal projection of the vector $\xi_0(\phi_\alpha(\iota(\theta)))$ onto the tangent space of the totally geodesic torus $Z_\alpha$ at the point $\phi_\alpha(\iota(\theta))$. Therefore, for $\alpha \in \mathbb{R}$, the functions $\iota^*\phi_\alpha^*\, g^\flat(\xi_0)(\partial/\partial x)$ and $\iota^*\phi_\alpha^*\, g^\flat(\xi_0)(\partial/\partial y)$ on $\mathbb{R}^2$ are constant. According to the equality (6.7) of [4] and the relations (4.1), (5.4), (10.4) and (10.6), we see that the equalities

$$\begin{aligned}\iota^*\phi^*_\alpha\, g^\flat(\xi_0)(\partial/\partial x)(\theta) &= \langle \zeta_1, \phi^*_\alpha\, g^\flat(\xi_0)\rangle(\iota(\theta)) = -\frac{i}{2}\,(1+\cos^2\alpha),\\ \iota^*\phi^*_\alpha\, g^\flat(\xi_0)(\partial/\partial y)(\theta) &= \langle \zeta_2, \phi^*_\alpha\, g^\flat(\xi_0)\rangle(\iota(\theta)) = \frac{i}{2}\,(1+\sin^2\alpha).\end{aligned} \tag{10.7}$$

are true when $\theta$ is the origin of $\mathbb{R}^2$; in fact, the previous remark implies that they hold for all $\theta \in \mathbb{R}^2$.

If $f$ is a function on $\mathbb{R}^3$ and $v \in \mathbb{R}$, we consider the function $f_v$ on $\mathbb{R}^2$ defined by

$$f_v(x,y) = f(x,y,v),$$

for all $(x,y) \in \mathbb{R}^2$. We consider the function $\psi$ on $\mathbb{R}^3$ defined in §9; we define a function $\psi'$ and a 1-form $\check{\omega}_v$ on $\mathbb{R}^2$, with $v \in \mathbb{R}$, by

$$\psi'(x,y) = e^{-iy} - e^{ix}, \qquad \check{\omega}_v = \psi_v\, d\psi' - \psi' d\psi_v,$$

for $(x,y) \in \mathbb{R}^2$. We also consider the dilation $\tau$ of $\mathbb{R}^2$ defined by $\tau(x) = 2x$, for all $x \in \mathbb{R}^2$.

We consider the functions $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}'_1$ on $X$. For $\alpha \in \mathbb{R}$, if $v = \cos^2\alpha$, we verify that the relations

$$\begin{aligned}\iota^*\phi^*_\alpha \tilde{f}_1 &= \frac{1}{2}\,\tau^*\psi_v, &\qquad \iota^*\phi^*_\alpha \tilde{f}'_1 &= \frac{1}{\sqrt{2}}\cos\alpha\cdot\sin\alpha\cdot\tau^*\psi',\\ \iota^*\phi^*_\alpha \tilde{f}_2 &= \frac{1}{2}\,\tau^*\bar{\psi}_v, &\qquad \iota^*\phi^*_\alpha\,\omega &= \frac{1}{2\sqrt{2}}\cos\alpha\cdot\sin\alpha\cdot\tau^*\check{\omega}_v,\end{aligned} \tag{10.8}$$

hold; hence we have

$$\iota^*\phi^*_\alpha \tilde{f}_{r,s} = \frac{1}{2^{r+s}}\,\tau^*\psi_{r,s,v}, \tag{10.9}$$

and, by (10.5), we obtain

$$(\iota^*\phi^*_\alpha\,\tilde{\sigma}(\omega))(\partial/\partial x, \partial/\partial x) = -\frac{i\cos\alpha\cdot\sin\alpha}{\sqrt{2}}\cdot\big(e^{-2ix} + e^{2i(x-y)}\big). \tag{10.10}$$

Let $r_1, r_2 \geq 0$ be given integers. We consider the symmetric 2-forms $h_l$, $k_j$ and $k'_j$, with $0 \leq l \leq 5$ and $j = 1,2,3$, associated in §5 with the integers $r_1$ and $r_2$, and the corresponding subspaces $V_{r_1,r_2}$, $W_{r_1,r_2}$ and $W'_{r_1,r_2}$ of $C^\infty(S^2T^*_{\mathbb{C}})$ generated by these forms. By means of the relations (10.2)–(10.5) and (10.7)–(10.10), we easily see that there exist polynomials $J$, $J_l$, $L_j$ and $M_j$ belonging to $\mathbb{Q}[X]$ such that the equalities

$$
\begin{aligned}
\frac{1}{\pi^2\sqrt{3}}\int_Z \phi_\alpha^*(\tilde{f}_{r_1,r_2})\,dZ &= J(\cos^2\alpha),\\
\frac{1}{\pi^2\sqrt{3}}\int_Z (\phi_\alpha^* h_l)(\zeta_1,\zeta_2)\,dZ &= J_l(\cos^2\alpha),\\
\frac{1}{\pi^2\sqrt{3}}\int_Z (\phi_\alpha^* k_j)(\zeta_1,\zeta_1)\,dZ &= L_j(\cos^2\alpha),\\
\frac{1}{\pi^2\sqrt{3}}\int_Z (\phi_\alpha^* k'_j)(\zeta_1,\zeta_1)\,dZ &= \frac{\cos\alpha\cdot\sin\alpha}{\sqrt{2}}\,M_j(\cos^2\alpha),
\end{aligned}
\tag{10.11}
$$

for $0 \leq l \leq 5$ and $1 \leq j \leq 3$ and all $\alpha \in \mathbb{R}$. The linear mappings

$$
\Phi_{r_1,r_2} : V_{r_1,r_2} \to \mathbb{C}[Y], \qquad \Psi_{r_1,r_2} : W_{r_1,r_2} \to \mathbb{C}[Y],
$$

$$
\Psi'_{r_1,r_2} : W'_{r_1,r_2} \to \mathbb{C}[Y],
$$

sending the elements $h \in V_{r_1,r_2}$, $k \in W_{r_1,r_2}$ and $k' \in W'_{r_1,r_2}$ into the polynomials $\Phi_{r_1,r_2}(h)$, $\Psi_{r_1,r_2}(k)$ and $\Phi'_{r_1,r_2}(k')$ of $\mathbb{C}[Y]$, respectively, determined by

$$
\begin{aligned}
\Phi_{r_1,r_2}(h)(\cos^2\alpha) &= \frac{1}{\pi^2\sqrt{3}}\int_Z (\phi_\alpha^* h)(\zeta_1,\zeta_2)\,dZ,\\
\Psi_{r_1,r_2}(k)(\cos^2\alpha) &= \frac{1}{\pi^2\sqrt{3}}\int_Z (\phi_\alpha^* k)(\zeta_1,\zeta_2)\,dZ,\\
\frac{\cos\alpha\cdot\sin\alpha}{\sqrt{2}}\,\Psi'_{r_1,r_2}(k')(\cos^2\alpha) &= \frac{1}{\pi^2\sqrt{3}}\int_Z (\phi_\alpha^* k')(\zeta_1,\zeta_1)\,dZ,
\end{aligned}
$$

for all $\alpha \in \mathbb{R}$, are well-defined. Clearly, by (10.11) we have

$$
\Phi_{r_1,r_2}(h_l) = J_l, \qquad \Psi_{r_1,r_2}(k_j) = L_j, \qquad \Psi'_{r_1,r_2}(k'_j) = M_j,
$$

for $0 \leq l \leq 5$ and for $j = 1,2,3$; hence the rank of $\Phi_{r_1,r_2}$ (resp. $\Psi_{r_1,r_2}$, $\Psi'_{r_1,r_2}$) is equal to the dimension of the subspace of $\mathbb{C}[Y]$ generated by the polynomials $J_l$, with $0 \leq l \leq 5$ (resp. $L_j$, $M_j$, with $j = 1,2,3$). An element of $V_{r_1,r_2}$ (resp. $W_{r_1,r_2}$, $W'_{r_1,r_2}$) satisfying the Guillemin condition belongs to the kernel of $\Phi_{r_1,r_2}$ (resp. $\Psi_{r_1,r_2}$, $\Psi'_{r_1,r_2}$); thus we have the inclusions

$$
\begin{gathered}
V_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}} \subset \operatorname{Ker}\Phi_{r_1,r_2}, \qquad W_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}} \subset \operatorname{Ker}\Psi_{r_1,r_2},\\
W'_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}} \subset \operatorname{Ker}\Psi'_{r_1,r_2}.
\end{gathered}
\tag{10.12}
$$

**Proposition 10.1.** *Let $r_1, r_2$ be given integers satisfying the relations* (6.3) *and $0 \leq r_1 \leq r_2$. Let $m \geq 0$ be the integer such that $r_2 = r_1 + 3m$.*

(i) *The relation*

$$J(Y) = \frac{1}{2^{r_1+r_2}}\, Y^m (1-Y)^m \cdot P(Y)$$

*holds.*

(ii) *For $l = 0, 1, 3$, there exists a polynomial $P_l \in \mathbb{Q}[Y]$ such that*

$$J_l(Y) = \frac{1}{2^{r_1+r_2}}\, Y^m (1-Y)^m \cdot P_l(Y). \tag{10.13}$$

*The polynomial $P_0$ is equal to $-P$ and is of degree $2r_1$. If $r_1 \geq 1$, the polynomial $P_1$ is non-zero and its degree is $\leq 2r_1 - 2$. If $r_1 \geq 2$, the degree of the polynomial $P_3$ is equal to $2r$ and the determinant of the matrix*

$$\begin{pmatrix} c(P_0)\, c_1(P_0)\, c_{2r}(P_0) \\ c(P_1)\, c_1(P_1)\, c_{2r}(P_1) \\ c(P_3)\, c_1(P_3)\, c_{2r}(P_3) \end{pmatrix}$$

*is non-zero; moreover, the polynomials $P_0$, $P_1$ and $P_3$ are linearly independent.*

(iii) *For $j = 1, 2$, there exists a polynomial $Q_j \in \mathbb{Q}[Y]$ such that*

$$L_j(Y) = \frac{1}{2^{r_1+r_2}}\, Y^m (1-Y)^m \cdot Q_j(Y). \tag{10.14}$$

*The degree of the polynomial $Q_1$ is equal to $2r_1 + 1$. If $r_1 \geq 1$, the determinant of the matrix*

$$\begin{pmatrix} c(Q_1)\, c_1(Q_1) \\ c(Q_2)\, c_1(Q_2) \end{pmatrix}$$

*is non-zero; moreover, the polynomials $Q_1$ and $Q_2$ are linearly independent.*

*Proof.* We write $r = r_1$ and $s = r_2$. By means of formula (9.5), we obtain explicit expressions for the polynomials $J$, $J_l$ and $L_j$, with $l = 0, 1, 3$ and $j = 1, 2$. Assertion (i) is an immediate consequence of formulas (9.5) and (9.9), with $d_1 = d_2 = 0$. Using formulas (9.5) and (9.9), we demonstrate the existence of a polynomial $P_l$ satisfying the relation (10.13) and a polynomial $Q_j$ satisfying the relation (10.14). In fact, $P_0$ is equal to $-P$. If $r \geq 1$, we see that

$$J_1(v) = \frac{1}{2^{r+s-2}} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \psi_{r-1,s,v}(x,y) e^{i(y-x)}\, dx\, dy,$$

for all $v \in \mathbb{R}$; hence $P_1$ is equal to the polynomial $4Q^{1,0}_{0,1,0}$ of degree $2r-2$. If $r \geq 2$, we have

$$J_3(v) = \frac{1}{2^{r+s-3}} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\psi_{r-2,s,v} \cdot \chi)(x, y, v)\, dx\, dy,$$

for all $v \in \mathbb{R}$, where $\chi$ is the function on $\mathbb{R}^3$ defined by

$$\chi(x, y, v) = (1 - v)(ve^{i(x-y)} - e^{-ix}) + e^{2i(y-x)} - ve^{iy},$$

for $(x, y, v) \in \mathbb{R}^3$, and so $P_3$ is the polynomial defined by

$$P_3(Y) = 8\big(Y^2(1 - Y)^2 Q^{2,0}_{1,0,1} - (1 - Y)^2 Q^{2,0}_{0,1,1} + Q^{2,0}_{0,2,0} - Y^2 Q^{2,0}_{1,1,0}\big).$$

Thus by (9.11), we obtain

$$c(P_1) = 4C^{1,0}_{0,1,0} = 4\binom{3m + r}{m}\varphi_2(r),$$

$$c_1(P_1) = -2\tilde{C}^{1,0}_{0,0,1} = -8\binom{3m + r}{m}\varphi_4(r)$$

when $r \geq 1$, and

$$c(P_3) = 8\big(-C^{2,0}_{0,1,1} + C^{2,0}_{0,2,0}\big) = 8\binom{3m + r}{m}(\varphi_7 - \varphi_8)(r),$$

$$c_1(P_3) = 16\big(\tilde{C}^{2,0}_{0,1,1} + C^{2,0}_{0,1,1} - \tilde{C}^{2,0}_{0,0,2}\big) = 16\binom{3m + r}{m}(\varphi_{10} - \varphi_9 + \varphi_8)(r),$$

$$c_{2r}(P_3) = 8C^{2,0}_{0,1,1} = 8\binom{3m + r}{m}\varphi_8(r)$$

when $r \geq 2$. By means of the formulas (8.2), (8.3), (8.6) and (8.7), we express these coefficients as multiples of $\varphi_1(r)$, and then easily see that the determinant of the matrix of assertion (ii) is negative when $r \geq 2$. From these results, we deduce all the properties of the polynomials $P_l$ given in (ii). Similarly, we also find that

$$2Q_1(Y) = -(1 - Y)P(Y),$$

and for $r \geq 1$ that

$$Q_2 = 4(1 + Y)\big(Y^2 Q^{1,0}_{1,0,0}(Y) - Q^{1,0}_{0,1,0}\big)(Y);$$

then using formulas (9.11), (9.13), (8.3) and (8.6), we obtain the properties of the polynomials $Q_j$ described in (iii). □

The proof of the following result is entirely similar to that of assertions (ii) and (iii) of the preceding proposition and will be omitted; the formulas (8.1), (8.3), (8.4) and (8.6) and the first relation of (8.5) are the only results of §8 required here.

**Proposition 10.2.** *Let $r_1, r_2$ be given integers satisfying the relations* (6.4) *and* $0 \leq r_1 \leq r_2$. *Let $m \geq 1$ be the integer such that $r_2 = r_1 + 3m - 1$. For $j = 1, 2$, there exists a polynomial $R_j$ belonging to $\mathbb{Q}[Y]$ such that*

$$M_j(Y) = \frac{1}{2^{r_1+r_2}} Y^{m-1}(1-Y)^{m-1} \cdot R_j(Y). \tag{10.15}$$

*The polynomial $R_1$ is non-zero. If $r_1 \geq 1$, the determinant of the matrix*

$$\begin{pmatrix} c(R_1) \, c_1(R_1) \\ c(R_2) \, c_1(R_2) \end{pmatrix}$$

*is non-zero; moreover, the polynomials $R_1$ and $R_2$ are linearly independent.*

**Proposition 10.3.** *Let $r_1, r_2$ be given integers satisfying the relations* (6.4) *and* $0 \leq r_2 \leq r_1$. *Let $m \geq 0$ be the integer such that $r_1 = r_2 + 3m + 1$; we set $m' = \sup(m-1, 0)$. For $j = 1, 2$, there exists a polynomial $R'_j$ belonging to $\mathbb{Q}[Y]$ such that*

$$M_j(Y) = \frac{1}{2^{r_1+r_2}} Y^{m'}(1-Y)^{m'} \cdot R'_j(Y). \tag{10.16}$$

*The polynomial $R'_1$ is non-zero. If $r_1 \geq 1$, the determinant of the matrix*

$$\begin{pmatrix} c_1(R'_1) \, c_l(R'_1) \\ c_1(R'_2) \, c_l(R'_2) \end{pmatrix}$$

*is non-zero, where $l = 2$ when $m \geq 1$, and $l = 0$ when $m = 0$; moreover, the polynomials $R'_1$ and $R'_2$ are linearly independent.*

*Proof.* By means of formula (9.7), we obtain explicit expressions for the polynomials $M_j$, with $j = 1, 2$. Using formulas (9.7) and (9.9), we demonstrate the existence of a polynomial $R'_j$ satisfying the relation (10.16). Then using the formulas (9.11), (8.1), (8.3), (8.5) and (8.10), we obtain all the remaining properties of these polynomials described in this proposition. □

Let $r_1, r_2 \geq 0$ be given integers. First, suppose that the relations (6.3) and $0 \leq r_1 \leq r_2$ are true. According to Propositions 10.1, the polynomials $J$ and $L_1$ are non-zero; therefore there exists $\alpha_0 \in \mathbb{R}$ such that $J(\cos^2 \alpha_0) \neq 0$. Hence by (10.11), the function $\tilde{f}_{r_1,r_2}$ and the 2-form $\tilde{f}_{r_1,r_2}\tilde{\sigma}(g^\flat(\xi_0))$ do not satisfy the Guillemin condition; thus Proposition 6.2 and the first assertion of Proposition 6.4,(i) hold. From Proposition 10.1,(ii), we also deduce that

$$\operatorname{rank} \Phi_{r_1,r_2} \geq \min(r_1 + 1, 3).$$

According to the first inclusion of (10.12) and Proposition 5.2,(i), we obtain the results of Proposition 6.3,(i). The inequality

$$\operatorname{rank} \Psi_{r_1,r_2} \geq \min(r_1 + 1, 2)$$

is a direct consequence of Proposition 10.1,(iii); from the second inclusion of (10.12) and Proposition 5.3,(i), we then obtain the second assertion of Proposition 6.4,(i). Now, assume moreover that $r_1 \geq 1$ and consider the forms $h'_1$ and $h'_2$ associated in §5 with the integers $r_1, r_2$ and the space $V'_{r_1,r_2}$ generated by these forms. According to (5.7) and Proposition 10.1,(ii), for all $\alpha \in \mathbb{R}$, we see that

$$\Phi_{r_1,r_2}(ih'_1)(v) = \frac{1}{6}\, v^m(1-v)^m \cdot (4P_0 + 3P_1)(v),$$

where $v = \cos^2\alpha$; moreover, by (9.13) the coefficient of degree $2r_1$ of the polynomial $4P_0 + 3P_1$ is equal to $4\ell(P_0) = 4\varphi_1(r_1)$. By the first inclusion of (10.12), it follows that the inequality

$$\dim(V'_{r_1,r_2} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1$$

holds. This completes the proof of Proposition 6.3.

Finally, suppose that only the relation (6.4) holds. From Propositions 10.2 and 10.3, we deduce that the polynomial $M_1$ is non-zero and that the inequality

$$\operatorname{rank} \Psi'_{r_1,r_2} \geq \min(r_1 + 1, 2)$$

holds. Therefore there exists $\alpha_1 \in \mathbb{R}$ such that $M_1(\cos^2\alpha_1) \neq 0$ and $\cos\alpha_1 \cdot \sin\alpha_1 \neq 0$. By (10.11), we infer that the 2-form $\tilde{f}_{r_1,r_2}\tilde{\sigma}(\omega)$ does not satisfy the Guillemin condition. From the last inclusion of (10.12) and Proposition 5.3,(i), we then obtain the second assertion of Proposition 6.4,(ii). This completes the proof of Proposition 5.4.

# CHAPTER II: The unitary groups

## 11. The special and the reduced unitary groups

Let $G$ be a simple Lie group; we suppose that $X = G$. If $B$ denotes the Killing form of the Lie algebra $\mathfrak{g}_0$ of $G$, we endow $X$ with the bi-invariant Riemannian metric $g_0$ induced by $-B$. As usual, we identify the $G$-module $\mathfrak{g}_0$ with the tangent space of $X$ at the identity element $e_0$ of $G$. We consider the involutive automorphism $s$ of the group $\tilde{G} = G \times G$ which sends $(g_1, g_2)$ into $(g_2, g_1)$. The fixed point set of $s$ is the diagonal subgroup $G^*$ of $G \times G$; thus the pair $(\tilde{G}, G^*)$ is a Riemannian symmetric pair. Since the homogeneous space $\tilde{G}/G^*$ is diffeomorphic to the group $G$ under the mapping $\tilde{G}/G^* \to G$, sending $(g_1, g_2)G^*$ into $g_1 g_2^{-1}$, where $g_1, g_2 \in G$, we may identify $X$ with the homogeneous space $\tilde{G}/G^*$. Then the action of the group $\tilde{G}$ on the space $X$ is given by

$$(g_1, g_2) \cdot a = g_1 a g_2^{-1}$$

for all $g_1, g_2, a \in G$; it induces $\tilde{G}$-module structures on the spaces $C^\infty(G)$ and $C^\infty(S^p T^*_{\mathbb{C}})$. A symmetric form on $X$ is $\tilde{G}$-invariant if and only if it is bi-invariant under the action of $G$. Thus the metric $g_0$ on $X$ is $\tilde{G}$-invariant and the manifold $X$ endowed with this metric is an irreducible symmetric space.

We denote by $\Gamma$ the dual of the Lie group $G$; we may identify the dual of the group $\tilde{G}$ with the product $\Gamma \times \Gamma$. We consider the $\tilde{G}$-module structures on $C^\infty(G)$, $C^\infty(T_{\mathbb{C}})$ and $C^\infty(S^p T^*_{\mathbb{C}})$ induced by the action of $\tilde{G}$ on $X$. If $(\gamma, \gamma')$ is an element of $\Gamma \times \Gamma$, as in §1 we consider the isotypic components $C^\infty_{(\gamma,\gamma')}(G)$, $C^\infty_{(\gamma,\gamma')}(T_{\mathbb{C}})$ and $C^\infty_{(\gamma,\gamma')}(S^p T^*_{\mathbb{C}})$ of the $\tilde{G}$-modules $C^\infty(G)$, $C^\infty(T_{\mathbb{C}})$ and $C^\infty(S^p T^*_{\mathbb{C}})$, respectively, corresponding to $(\gamma, \gamma')$. The two $\tilde{G}$-modules $C^\infty_{(\gamma,\gamma')}(T_{\mathbb{C}})$ and $C^\infty_{(\gamma,\gamma')}(T^*_{\mathbb{C}})$ are clearly isomorphic. If $E_\gamma$ and $E_{\gamma'}$ are irreducible $G$-modules corresponding to $\gamma$ and $\gamma'$, respectively, a $\tilde{G}$-submodule $W$ of the isotypic component $C^\infty_{(\gamma,\gamma')}(S^p T^*_{\mathbb{C}}))$ is isomorphic to $k$ copies of the irreducible $\tilde{G}$-module $E_\gamma \otimes E_{\gamma'}$; this integer $k$ is called the multiplicity of the $\tilde{G}$-module $W$ and shall be denoted by $\operatorname{Mult} W$. The spaces $C^\infty(G)$ and $C^\infty(S^p T^*_{\mathbb{C}})$ inherit structures of $G$-modules arising from the left (resp. right) action of $G$ on $X$. The corresponding representation $\pi$ (resp. $\pi'$) of $G$ on $C^\infty(G)$ is the left (resp. right) regular representation; we shall also consider the corresponding representation $(\pi, C^\infty(S^p T^*_{\mathbb{C}}))$ (resp. $(\pi', C^\infty(S^p T^*_{\mathbb{C}})))$ of $G$ on $C^\infty(S^p T^*_{\mathbb{C}})$. We shall denote by $C^\infty_\gamma(G)$ and $C^\infty_\gamma(S^p T^*_{\mathbb{C}})$ the isotypic components of the $G$-modules $(\pi, C^\infty(G))$ and $(\pi, C^\infty(S^pT^*_{\mathbb{C}}))$, respectively, corresponding to an element $\gamma$ of $\Gamma$. If $(\gamma, \gamma')$ is an element of $\Gamma \times \Gamma$, the isotypic component $C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})$ is a $\tilde{G}$-submodule of $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$.

The complexification $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}_0$ is an irreducible $G$-module, and, by means of its Killing form, we are able to identify this $G$-module with its dual. We view the complexification $\mathcal{A}_{\mathbb{C}}$ of the space $\mathcal{A}$ of left-invariant 1-forms on $G$ as a $\tilde{G}$-submodule of $C^\infty(T^*_{\mathbb{C}})$; more generally, if $p$ is an integer $\geq 1$, we view the $p$-th symmetric power $S^p\mathcal{A}$ of $\mathcal{A}$ and its complexification $S^p\mathcal{A}_{\mathbb{C}}$ as $\tilde{G}$-submodules of $C^\infty(S^pT^*_{\mathbb{C}})$. Clearly, the space $\mathcal{A}_{\mathbb{C}}$ is a trivial $G$-submodule of $(\pi, C^\infty(T^*_{\mathbb{C}}))$ and a $G$-submodule of $(\pi', C^\infty(S^pT^*_{\mathbb{C}}))$ isomorphic to the irreducible $G$-module $\mathfrak{g}$. Thus the space $S^p\mathcal{A}_{\mathbb{C}}$ is a trivial $G$-submodule of $(\pi, C^\infty(S^pT^*_{\mathbb{C}}))$ and is also a $G$-submodule of $(\pi', C^\infty(S^pT^*_{\mathbb{C}}))$. If $V_p$ is the $G$-module equal to the $p$-th symmetric power $S^p\mathfrak{g}$ of $\mathfrak{g}$ endowed with the trivial action of $G$ and $\gamma$ is an element of $\Gamma$, since the cotangent bundle $T^*$ of $G$ is trivial, the isotypic component $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ is isomorphic to $C^\infty_\gamma(G) \otimes V_p$.

The left (resp. right) action of the group $G$ on itself induces a morphism $\Phi$ (resp. $\Phi'$) from $\mathfrak{g}_0$ to the Lie algebra of vector fields on $G$, whose image is the space of left-invariant (resp. right-invariant) vector fields on $G$. The mappings $\Phi$ and $\Phi'$ extend to $\mathbb{C}$-linear morphisms from $\mathfrak{g}$ to the space of all complex vector fields on $G$. For $\xi \in \mathfrak{g}_0$, the restriction of $-\Phi(\xi)$ (resp. $\Phi'(\xi)$) to $G$ is the right-invariant (resp. left-invariant) vector field on $G$ whose value at $e_0$ is the vector $\xi$ of $\mathfrak{g}_0$ viewed as a tangent vector at $e_0$.

Let $\gamma_0$ be the element of $\Gamma$ corresponding to the trivial irreducible $G$-module $\mathbb{C}$. The Lie algebra $\mathfrak{g}$ is an irreducible $G$-module corresponding to an element $\gamma_1$ of $\Gamma$. The space $\mathcal{K}$ of all Killing vector fields on $X$, i.e., the space of all solutions $\xi \in C^\infty(T)$ of the equation $D_0\xi = 0$, is equal to the direct sum $\Phi(\mathfrak{g}_0) \oplus \Phi'(\mathfrak{g}_0)$. Thus we may view its complexification $\mathcal{K}_{\mathbb{C}}$ as the $\tilde{G}$-submodule $\Phi(\mathfrak{g}) \oplus \Phi'(\mathfrak{g})$ of $C^\infty(T_{\mathbb{C}})$, where

$$\begin{gathered} \mathcal{K}_{\mathbb{C}} \subset C^\infty_{(\gamma_0,\gamma_1)}(T_{\mathbb{C}}) \oplus C^\infty_{(\gamma_1,\gamma_0)}(T_{\mathbb{C}}), \\ \Phi(\mathfrak{g}) \subset C^\infty_{(\gamma_0,\gamma_1)}(T_{\mathbb{C}}), \qquad \Phi'(\mathfrak{g}) \subset C^\infty_{(\gamma_1,\gamma_0)}(T_{\mathbb{C}}). \end{gathered} \tag{11.1}$$

Throughout this chapter, we shall suppose henceforth that $X = G$ is the group $SU(n)$, with $n \geq 3$, and always consider the symmetric space $X$ endowed with the Riemannian metric $g = \sigma'_2$. We consider the abelian subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, the subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ and the linear forms $\lambda_j$ on $\mathfrak{h}$ introduced in §2. Also we shall always identify the dual $\Gamma$ of the group $G = SU(n)$ with the set of linear forms on $\mathfrak{h}$ described in §2. If $\gamma$ is an element of $\Gamma$, let $E_\gamma$ be an irreducible $G$-module corresponding to $\gamma$.

In the following, by $C_j$, $A_{kl}$ and $B_{kl}$ we shall mean the left-invariant vector fields on $G$ determined by the corresponding elements of $\mathfrak{g}_0$. Throughout this chapter, we consider the left-invariant 1-forms $\{\omega_0, \omega_j, \omega_{jk}, \varpi_{jk}\}$ on $G$, with $1 \leq j, k \leq n$, determined by

$$\omega_0 = \omega_n = \varpi_{jj} = 0, \qquad \omega_{jk} = -\omega_{kj}, \qquad \varpi_{jk} = \varpi_{kj},$$

for $1 \leq j, k \leq n$, and

$$\begin{aligned} &\omega_j(C_l) = \delta_{jl}, && \omega_{jk}(C_l) = 0, && \varpi_{jk}(C_l) = 0, \\ &\omega_j(A_{rs}) = 0, && \omega_{jk}(A_{rs}) = \delta_{jr}\delta_{ks}, && \varpi_{jk}(A_{rs}) = 0, \\ &\omega_j(B_{rs}) = 0, && \omega_{jk}(B_{rs}) = 0, && \varpi_{jk}(B_{rs}) = \delta_{jr}\delta_{ks}, \end{aligned}$$

for all $1 \leq l \leq n-1$ and $1 \leq j, k, r, s \leq n$, with $j < k$ and $r < s$. For $1 \leq j < k \leq n$, we set

$$\theta_{jk} = \omega_{jk} - i\varpi_{jk}, \qquad \bar{\theta}_{jk} = \omega_{jk} + i\varpi_{jk}$$

of $\mathcal{A}_{\mathbb{C}}$; then the 1-forms $\{\omega_l, \theta_{jk}, \bar{\theta}_{jk}\}$, with $1 \leq l \leq n-1$ and $1 \leq j < k \leq n$, form a basis for the $G$-module $\mathcal{A}_{\mathbb{C}}$.

If $n = 3$, according to Lemma 2.1 of [5] and the remark following this lemma, and the relations (2.4)–(2.6) of [5], we easily verify that

$$\begin{aligned} 6\tilde{\sigma}(\omega_1) &= \omega_1^2 + 2\omega_1 \cdot \omega_2 - 2\omega_2^2 + \theta_{12} \cdot \bar{\theta}_{12} + \theta_{13} \cdot \bar{\theta}_{13} - 2\theta_{23} \cdot \bar{\theta}_{23}, \\ 6\tilde{\sigma}(\omega_2) &= 2\omega_1^2 - 2\omega_1 \cdot \omega_2 - \omega_2^2 + 2\theta_{12} \cdot \bar{\theta}_{12} - \theta_{13} \cdot \bar{\theta}_{13} - \theta_{23} \cdot \bar{\theta}_{23}; \end{aligned} \tag{11.2}$$

from the relations (2.10) of [5], we deduce that

$$\begin{aligned} 2\tilde{\sigma}(\theta_{12}) = \omega_2 \cdot \theta_{12} - i\theta_{13} \cdot \bar{\theta}_{23}, \qquad 2\tilde{\sigma}(\theta_{23}) = -\omega_1 \cdot \theta_{23} - i\theta_{13} \cdot \bar{\theta}_{12}, \\ 2\tilde{\sigma}(\theta_{13}) = (\omega_1 - \omega_2) \cdot \theta_{13} + i\theta_{12} \cdot \theta_{23}. \end{aligned} \tag{11.3}$$

Throughout this chapter, we shall consider the symmetric 3-form $\sigma = \sigma_3'$ on $X$, the $\tilde{G}$-equivariant monomorphism $\tilde{\sigma} : T^* \to S^2T^*$ and the first-order differential operator

$$D_\sigma : \mathcal{T} \oplus \mathcal{F} \to S^2\mathcal{T}^*$$

associated with $\sigma$ in §1, where $F$ is the trivial real line bundle over $G$. Using *Maple* and the formulas (11.2) and (11.3), we see that the operator $D_\sigma$ is of finite type when $n = 3$; in fact, the morphism (4.11), which is equal to the second prolongation of the symbol of $D_\sigma$, is injective. Therefore by Proposition 6.2 of [6], we obtain:

**Proposition 11.1.** *The differential operator $D_\sigma$ on $X = SU(3)$ is elliptic.*

Let $\gamma, \gamma'$ be elements of $\Gamma$. The contragredient $G$-module $E^*_\gamma$ of $E_\gamma$ is isomorphic to the $G$-module $E_{\bar\gamma}$. Hence according to the Peter-Weyl theorem, the isotypic component $C^\infty_{(\gamma,\gamma')}(G)$ vanishes unless $\gamma' = \bar\gamma$; moreover, the isotypic component $C^\infty_{(\gamma,\bar\gamma)}(G)$ is an irreducible $\tilde G$-submodule of $C^\infty(G)$ equal to the submodule $C^\infty_\gamma(G)$ of the $G$-module $(\pi, C^\infty(G))$, and the weight subspace $\mathcal{C}_\gamma$ of the $G$-submodule $C^\infty_\gamma(G)$ corresponding to the highest weight $\gamma$ is an irreducible $G$-submodule of $(\pi', C^\infty(G))$ isomorphic to $E_{\bar\gamma}$. Since the isotypic component $C^\infty_\gamma(S^pT^*_{\mathbb{C}})$ is isomorphic to $C^\infty_\gamma(G) \otimes V_p$, its weight subspace $\mathcal{C}_\gamma(S^pT^*_{\mathbb{C}})$ corresponding to the highest weight $\gamma$ is the $G$-submodule $\mathcal{C}_\gamma \cdot S^p\mathcal{A}_{\mathbb{C}}$ of $(\pi', C^\infty(S^pT^*_{\mathbb{C}}))$, and is therefore isomorphic to $E_{\bar\gamma} \otimes S^p\mathfrak{g}$ as a $G$-module. The weight space $\mathcal{C}_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})$ of the $\tilde G$-module $C^\infty(S^pT^*_{\mathbb{C}})$ corresponding to the highest weight $(\gamma, \gamma')$ is contained in the weight space of the $G$-submodule $\mathcal{C}_\gamma \cdot S^p\mathcal{A}_{\mathbb{C}}$ of $(\pi', C^\infty_\gamma(S^pT^*_{\mathbb{C}}))$ corresponding to the weight $\gamma'$; in fact, we have

$$(11.4)\qquad \mathcal{C}_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}}) = \left\{ u \in \mathcal{C}_\gamma \cdot S^p\mathcal{A}_{\mathbb{C}} \;\middle|\; \begin{array}{c} \Phi'(\eta)u = 0,\ \Phi'(\xi)u = \gamma'(\xi)u, \\ \text{for all } \eta \in \mathfrak{n}^+,\ \xi \in \mathfrak{h}_0 \end{array} \right\}$$

(see [5, §4]). From these observations, we infer that the multiplicity of the $\tilde G$-module $C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})$, which is equal to the dimension of the weight subspace $\mathcal{C}_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})$, is given by

$$(11.5)\qquad \operatorname{Mult} C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}}) = \dim \operatorname{Hom}_G\,(E_{\gamma'}, E_{\bar\gamma} \otimes S^p\mathfrak{g}).$$

A linear form $\lambda$ on $\mathfrak{h}$ is a weight of the $G$-module $C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})$ with respect to the representation $\pi$ (resp. $\pi'$) if and only if $-\lambda$ is a weight of its complex conjugate $\overline{C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})}$. Therefore, if $\delta$ is the element $\overline{\gamma'}$ of $\Gamma$, we have the equality

$$(11.6)\qquad C^\infty_{(\bar\gamma,\delta)}(S^pT^*_{\mathbb{C}}) = \overline{C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})}$$

of $\tilde G$-modules.

The highest weight of the irreducible $G$-module $\mathfrak{g}$ is equal to $2\lambda_1+\lambda_2$. The Cartan product $F$ of the irreducible $G$-module $\mathfrak{g}$ with itself is the unique $G$-submodule of $S^2\mathfrak{g}$ whose highest weight is equal to $4\lambda_1+2\lambda_2$. As above, we identify $S^p\mathfrak{g}$ with the symmetric $p$-th power $S^p\mathfrak{g}^*$ of $\mathfrak{g}^*$ by means of the Killing form $B$; thus $B$ generates a trivial $G$-submodule $\{B\}$ of $S^2\mathfrak{g}$. As we identify the $G$-modules $\mathfrak{g}_0$ and $T_{e_0}$, the complexification of the morphism $\tilde{\sigma} : T_{e_0} \to S^2T^*_{e_0}$ determines a monomorphism $\tilde{\sigma} : \mathfrak{g} \to S^2\mathfrak{g}$ of $G$-modules. Then $\tilde{\sigma}(\mathfrak{g})$ is an irreducible $G$-submodule of $S^2\mathfrak{g}$ whose highest weight is equal to $2\lambda_1+\lambda_2$. It follows that the sum $\{B\} \oplus F \oplus \tilde{\sigma}(\mathfrak{g})$ is direct and is a $G$-submodule of $S^2\mathfrak{g}$. When $n = 3$, it is easily verified the equality

$$S^2\mathfrak{g} = \{B\} \oplus F \oplus \tilde{\sigma}(\mathfrak{g})$$

holds; in this case, for $\gamma, \gamma' \in \Gamma$, by Schur's lemma we therefore see that

$$\begin{aligned} \dim \mathrm{Hom}_G\,(E_{\gamma'}, E_\gamma \otimes S^2\mathfrak{g}) = \delta_{\gamma,\gamma'} &+ \dim \mathrm{Hom}_G\,(E_{\gamma'}, E_\gamma \otimes F) \\ &+ \dim \mathrm{Hom}_G\,(E_{\gamma'}, E_\gamma \otimes \mathfrak{g}), \end{aligned} \tag{11.7}$$

for all $\gamma \in \Gamma$, where $\delta_{\gamma,\gamma'}$ is equal to 1 if $\gamma' = \gamma$ and 0 otherwise.

The center of $G = SU(n)$ is the cyclic subgroup $S$ of order $n$ generated by the element

$$a_0 = e^{2i\pi/n} I_n$$

of $G$, where $I_n$ is the $n \times n$ identity matrix. The group $Y = \check{G} = G/S$ is a symmetric space of compact type, which is the reduced space of the symmetric space $G$ and which we call the reduced unitary group; it is isomorphic to the adjoint group of $\mathfrak{su}(n)$ (see [5, §7]).

Let $\check{\Gamma}$ be the subset of $\Gamma$ consisting of all elements $\gamma_{r_1,\dots,r_{n-1}}$ of $\Gamma$, where $r_1, \dots, r_{n-1}$ are non-negative integers satisfying the relation

$$r_1 + 2r_2 + \cdots + (n-1)r_{n-1} \equiv 0 \bmod n. \tag{11.8}$$

If $E$ is a $G$-module, we denote by $E^S$ the $G$-submodule of $E$ consisting of all $S$-invariant elements of $E$.

We consider the natural projection $\pi : G \to \check{G}$. If $\gamma$ is an element of $\Gamma$, the isomorphism $\pi^* : C^\infty(Y) \to C^\infty(X)^S$ induces an isomorphism of $G$-modules

$$\pi^* : C^\infty_\gamma(Y) \to C^\infty_\gamma(X)^S$$

of $G$-modules; according to Lemma 5.1,(ii) of [4], we know that

$$C^\infty_\gamma(X)^S = C^\infty_\gamma(X) \tag{11.9}$$

if and only if $\gamma$ belongs to $\check\Gamma$.

The element $\gamma_1$ of $\Gamma$ is equal to $\varpi_1 + \varpi_{n-1}$ and so belongs to $\check\Gamma$. Since $\bar\gamma_1 = \gamma_1$, according to the relation (4.6) of [5], the irreducible $G$-module $\mathcal{B} = C^\infty_{\gamma_1}(X)$ is invariant under conjugation and hence it is equal to the complexification of the $G$-submodule

$$\mathcal{B}_{\mathbb{R}} = \{\, f \in \mathcal{B} \mid f = \bar f \,\}$$

of $C^\infty_{\mathbb{R}}(X)$. Thus since $\gamma_1$ belongs to $\check\Gamma$, the $G$-module $\mathcal{B}_Y = C^\infty_{\gamma_1}(Y)$ is equal to the complexification of the subspace

$$\mathcal{B}_{Y,\mathbb{R}} = \{\, f \in \mathcal{B}_Y \mid f = \bar f \,\}$$

of $C^\infty_{\mathbb{R}}(Y)$ and the mapping $\pi$ induces an isomorphism $\pi^* : \mathcal{B}_{Y,\mathbb{R}} \to \mathcal{B}_{\mathbb{R}}$.

The symmetric form $\sigma$ induces a symmetric 3-form $\sigma_Y$ on $Y$ such that

$$\sigma = \pi^*\sigma_Y$$

and we consider the morphism of vector bundles

$$\tilde\sigma_Y : T^*_Y \to S^2T^*_Y$$

induced by the symmetric 3-form $\sigma_Y$. If $\varphi$ is a 1-form on $Y$, we have

$$\pi^*\tilde\sigma_Y(\varphi) = \tilde\sigma(\pi^*\varphi). \tag{11.10}$$

According to Lemma 2.3 of [5] or Lemma 3.1 of [4], a 1-form $\varphi$ on $X$ satisfies the Guillemin condition if and only if the symmetric 2-form $\tilde\sigma(\varphi)$ on $X$ satisfies the Guillemin condition. Thus for all $\gamma \in \Gamma$, since the differential operators $D_0$ and $\tilde\sigma d$ are homogeneous, we have the inclusions

$$\begin{aligned} D_0C^\infty(T)^S + \tilde\sigma dC^\infty_{\mathbb{R}}(X)^S &\subset \mathcal{N}_2 \cap C^\infty(S^2T^*)^S, \\ D_0C^\infty_\gamma(T_{\mathbb{C}})^S + \tilde\sigma dC^\infty_\gamma(X)^S &\subset \mathcal{N}_{2,\mathbb{C}} \cap C^\infty_\gamma(S^2T^*_{\mathbb{C}})^S, \end{aligned} \tag{11.11}$$

for all $\gamma \in \Gamma$.

According to Proposition 6.2 of [5] and its proof, we know that

$$\begin{aligned} D_0C^\infty(T) \cap \tilde\sigma dC^\infty_{\mathbb{R}}(X) &= \tilde\sigma d\mathcal{B}_{\mathbb{R}}, \\ D_0C^\infty(T_{\mathbb{C}}) \cap \tilde\sigma dC^\infty(X) &= \tilde\sigma d\mathcal{B}, \end{aligned} \tag{11.12}$$

where $\mathcal{B} = C^\infty_{\gamma_1}(X)^S$.

**Proposition 11.2.** *Let $X$ be the symmetric space equal to the simple Lie group $SU(n)$, with $n \geq 3$, and $Y$ be the symmetric space $SU(n)/S$. Assume that the operator $D_\sigma$ is elliptic. Then the following assertions are equivalent:*

(i) *The equality*

$$\mathcal{N}_{2,Y} = D_0 C^\infty(T_Y) + \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y)$$

*holds.*

(ii) *The equality*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty(S^2 T^*_{\mathbb{C}})^S = D_0 C^\infty(T_{\mathbb{C}})^S + \tilde{\sigma} d C^\infty(X)^S$$

*holds.*

(iii) *We have*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{(\gamma_0,\gamma_0)}(S^2 T^*_{\mathbb{C}})^S = \{0\}, \tag{11.13}$$

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{(\gamma_1,\gamma_1)}(S^2 T^*_{\mathbb{C}})^S = D_0 C^\infty_{(\gamma_1,\gamma_1)}(T_{\mathbb{C}})^S, \tag{11.14}$$

*and the equality*

$$\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{(\gamma,\gamma')}(S^2 T^*_{\mathbb{C}})^S = D_0 C^\infty_{(\gamma,\gamma')}(T_{\mathbb{C}})^S + \tilde{\sigma} d C^\infty_{(\gamma,\gamma')}(X)^S \tag{11.15}$$

*holds for all $\gamma, \gamma' \in \Gamma$, whenever $(\gamma, \gamma')$ is not equal to $(\gamma_0, \gamma_0)$ or $(\gamma_1, \gamma_1)$.*

(iv) *The equality* (11.13) *holds and*

$$\operatorname{Mult}\big(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{(\gamma,\gamma')}(S^2 T^*_{\mathbb{C}})^S\big) \leq \operatorname{Mult} C^\infty_{(\gamma,\gamma')}(T^*_{\mathbb{C}})^S - 1$$

*whenever the element $(\gamma, \gamma')$ of $\Gamma \times \Gamma$ is equal to $(\gamma_0, \gamma_1)$ or $(\gamma_1, \gamma_0)$; moreover, if $\gamma$ is an element of $\check{\Gamma}$ which is not equal to $\gamma_0$ or $\gamma_1$, the inequality*

$$\operatorname{Mult}\big(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{(\gamma,\bar{\gamma})}(S^2 T^*_{\mathbb{C}})^S\big) \leq \operatorname{Mult} C^\infty_{(\gamma,\bar{\gamma})}(T^*_{\mathbb{C}})^S + 1$$

*holds, and the inequality*

$$\operatorname{Mult}\big(\mathcal{N}_{2,\mathbb{C}} \cap C^\infty_{(\gamma,\gamma')}(S^2 T^*_{\mathbb{C}})^S\big) \leq \operatorname{Mult} C^\infty_{(\gamma,\gamma')}(T^*_{\mathbb{C}})^S$$

*holds for all elements $(\gamma, \gamma')$ of $\Gamma \times \Gamma$ satisfying one of the following conditions:*

(a) *$\gamma' \neq \bar{\gamma}$ and $(\gamma, \gamma') \neq (\gamma_0, \gamma_1), (\gamma_1, \gamma_0)$;*

(b) *$\gamma' = \bar{\gamma}$ and $\gamma$ does not belong to $\check{\Gamma}$;*

(c) *$\gamma = \gamma' = \gamma_1$.*

*Proof.* Lemma 2.17 of [2], together with the relations (2.6) of [2] and (11.10), gives us the equivalence of (i) and (ii). Since $D_\sigma$ is an elliptic homogeneous differential operator, from Proposition 2.2,(iii) and the inclusions (2.12) of [2], by (11.11) we infer that assertion (ii) is equivalent to the fact that the equality (11.15) holds for all $\gamma \in \Gamma$. According to the relations (11.12), the equality (11.15), with $\gamma = \gamma' = \gamma_1$, is equivalent to (11.14). When $\gamma = \gamma' = \gamma_0$, we know that $\bar\gamma = \gamma$ and that the spaces $dC^\infty_{(\gamma,\bar\gamma)}(X)$ and $C^\infty_{(\gamma,\bar\gamma)}(T_{\mathbb{C}})$ vanish. Thus the assertions (ii) and (iii) are equivalent. Since $\mathrm{Mult}\, C^\infty_{(\gamma,\bar\gamma)}(X)^S$ is equal to 1 when the element $\gamma$ of $\Gamma$ belongs to $\check\Gamma$ and vanishes otherwise, and since the $\tilde G$-modules $C^\infty_{(\gamma,\gamma')}(T_{\mathbb{C}})$ and $C^\infty_{(\gamma,\gamma')}(T^*_{\mathbb{C}})$, with $\gamma,\gamma' \in \Gamma$, are isomorphic, the equivalence of (iii) and (iv) follows from the relations (11.1), (11.11) and (11.12). □

## 12. Highest weight vectors and multiplicities

Henceforth in this paper, we shall suppose that $n = 3$ and that $X$ is the symmetric space $G = SU(3)$. We consider $G$ as a real submanifold of the complex manifold $M_3$ and denote by $z_{jk}$ the restriction to $G$ of the function $z_{jk}$ on $M_3$ defined in §5. Here we consider the subalgebra $\mathfrak{n}^+$ generated by the elements $E_{jk}$ of $\mathfrak{g}$, with $j < k$.

From the relations (3.10) of [5], we deduce that

$$
\begin{aligned}
&\Phi'(\eta)z_{31} = 0, && \Phi'(\eta)\bar z_{13} = 0,\\
&\Phi'(E_{12})z_{32} = z_{31}, && \Phi'(E_{12})z_{33} = 0,\\
&\Phi'(E_{23})z_{32} = 0, && \Phi'(E_{23})z_{33} = z_{32},\\
&\Phi'(E_{13})z_{32} = 0, && \Phi'(E_{13})z_{33} = z_{31},\\
&\Phi'(E_{12})\bar z_{11} = -\bar z_{12}, && \Phi'(E_{12})\bar z_{12} = 0,\\
&\Phi'(E_{23})\bar z_{11} = 0, && \Phi'(E_{23})\bar z_{12} = -\bar z_{13},\\
&\Phi'(E_{13})\bar z_{11} = -\bar z_{13}, && \Phi'(E_{13})\bar z_{12} = 0,
\end{aligned}
\tag{12.1}
$$

for all $\eta \in \mathfrak{n}^+$. We also verify that

$$
\begin{aligned}
&\Phi'(\eta)\theta_{13} = 0, && \Phi'(E_{23})\bar\theta_{13} = \bar\theta_{12},\\
&\Phi'(E_{12})\bar\theta_{13} = -\bar\theta_{23}, && \Phi'(E_{13})\bar\theta_{13} = i(\omega_1 + \omega_2)
\end{aligned}
\tag{12.2}
$$

and

$$
\begin{aligned}
&\Phi'(E_{12})\theta_{12} = 0, && \Phi'(E_{12})\bar\theta_{12} = i(2\omega_1 - \omega_2),\\
&\Phi'(E_{23})\theta_{12} = -\theta_{13}, && \Phi'(E_{23})\bar\theta_{12} = 0,\\
&\Phi'(E_{13})\theta_{12} = 0, && \Phi'(E_{13})\bar\theta_{12} = \theta_{23},\\
&\Phi'(E_{12})\theta_{23} = \theta_{13}, && \Phi'(E_{12})\bar\theta_{23} = 0,\\
&\Phi'(E_{23})\theta_{23} = 0, && \Phi'(E_{23})\bar\theta_{23} = i(2\omega_2 - \omega_1),\\
&\Phi'(E_{13})\theta_{23} = 0, && \Phi'(E_{13})\bar\theta_{23} = -\theta_{12},\\
&\Phi'(E_{12})\omega_1 = -i\theta_{12}, && \Phi'(E_{12})\omega_2 = 0,\\
&\Phi'(E_{23})\omega_1 = 0, && \Phi'(E_{23})\omega_2 = -i\theta_{23},\\
&\Phi'(E_{13})\omega_1 = -i\theta_{13}, && \Phi'(E_{13})\omega_2 = -i\theta_{13},
\end{aligned}
\tag{12.3}
$$

for all $\eta \in \mathfrak{n}^+$. Clearly, the 1-forms

$$
\begin{aligned}
&\vartheta_1 = \theta_{13},\\
&\vartheta_2 = z_{32}\theta_{13} - z_{31}\theta_{23}, \qquad \vartheta_3 = z_{32}\theta_{12} + z_{33}\theta_{13} - iz_{31}\omega_1,\\
&\vartheta_4 = \bar z_{12}\theta_{13} - \bar z_{13}\theta_{12}, \qquad \vartheta_5 = \bar z_{11}\theta_{13} + \bar z_{12}\theta_{23} + i\bar z_{13}\omega_2,\\
&\vartheta_6 = \sum_{1\le j<k\le 3} (z_{3j}\bar z_{1k}\bar\theta_{jk} - z_{3k}\bar z_{1j}\theta_{jk})\\
&\qquad + i(z_{31}\bar z_{11} - z_{32}\bar z_{12})\omega_1 + i(z_{32}\bar z_{12} - z_{33}\bar z_{13})\omega_2,\\
&\vartheta_7 = z_{32}z_{33}\theta_{13} - z_{31}z_{33}\theta_{23} + z_{32}^2\theta_{12} + z_{31}^2\bar\theta_{12} - iz_{31}z_{32}(2\omega_1 - \omega_2),\\
&\vartheta_8 = \bar z_{11}\bar z_{12}\theta_{13} + \bar z_{12}^2\theta_{23} - \bar z_{11}\bar z_{13}\theta_{12} + \bar z_{13}^2\bar\theta_{23} + i\bar z_{12}\bar z_{13}(2\omega_2 - \omega_1)
\end{aligned}
$$

on $X$ are all non-zero. Using the relations (12.1), (12.2) and (12.3), we easily verify that

$$
\Phi'(\eta)\vartheta_j = 0, \tag{12.4}
$$

for all $\eta \in \mathfrak{n}^+$ and $1 \le j \le 8$.

If $r, s \ge 0$ are integers, we consider the function

$$
f_{r,s} = z_{31}^r \bar z_{13}^s
$$

on $G$. If $r, s \in \mathbb{Z}$, with $r < 0$ or $s < 0$, we set $f_{r,s} = 0$. If $\gamma$ is the element $s\varpi_1 + r\varpi_2$ of $\Gamma$, with $r, s \ge 0$, in [5, §4] we saw that the function $f_{r,s}$ is a highest weight vector of the irreducible $G$-submodule $\mathcal{C}_\gamma$ of $(\pi', C^\infty(G))$ and that its weight is equal to $\bar{\gamma} = r\varpi_1 + s\varpi_2$; in other words, we have

$$\Phi'(\eta) f_{r,s} = 0, \qquad \Phi'(\xi) f_{r,s} = \bar{\gamma}(\xi) f_{r,s}, \tag{12.5}$$

for all $\eta \in \mathfrak{n}^+$ and $\xi \in \mathfrak{h}_0$.

If $\{v_\alpha\}_{\alpha\in A}$ is a family of elements belonging to a vector space $V$, we shall also denote by $\{v_\alpha\}$ the subspace of $V$ generated by this family. We denote by $\mathcal{P}$ the subset

$$(4,2),\ (3,3),\ (3,0),\ (2,4),\ (2,1),\ (2,-2),\ (1,2),\ (1,-1),$$
$$(0,3),\ (0,0),\ (0,-3),\ (-1,1),\ (-1,-2),\ (-2,2),$$
$$(-2,-1),\ (-2,-4),\ (-3,0),\ (-3,-3),\ (-4,-2)$$

of $\mathbb{Z} \times \mathbb{Z}$. We consider the subsets

$$\mathcal{P}_0 = \{(4,2),\ (2,1),\ (0,0),\ (-2,-1),\ (-4,-2)\},$$
$$\mathcal{P}_1 = \{(3,0),\ (2,-2),\ (1,-1),\ (0,-3),\ (-1,-2),\ (-2,-4),\ (-3,-3)\},$$
$$\mathcal{P}' = \{(2,1),\ (1,2),\ (1,-1),\ (0,0),\ (-1,1),\ (-1,-2),\ (-2,-1)\}$$

of $\mathcal{P}$. We also consider the involution $\Psi$ of $\mathcal{P}$ which is determined by the relations $\Psi(q) = q$, for all $q \in \mathcal{P}_0$, and

$$\Psi(3,3) = (3,0), \quad \Psi(2,4) = (2,-2), \quad \Psi(1,2) = (1,-1), \quad \Psi(0,3) = (0,-3),$$
$$\Psi(-1,1) = (-1,-2), \quad \Psi(-2,2) = (-2,-4), \quad \Psi(-3,0) = (-3,-3).$$

We note that $\mathcal{P}$ is the disjoint union of the subsets $\mathcal{P}_0$, $\mathcal{P}_1$ and $\Psi(\mathcal{P}_1)$.

Now let $r, s \geq 0$ be given integers. For $(a,b) \in \mathcal{P}$, we now define subspaces $V'_{a,b}$ of $C^\infty(T^*_{\mathbb{C}})$ by

$$V'_{0,0} = \{f_{r-1,s}\vartheta_3, f_{r,s-1}\vartheta_5\},$$
$$V'_{2,1} = \{f_{r,s}\vartheta_1\}, \qquad V'_{-2,-1} = \{f_{r-1,s-1}\vartheta_6\},$$
$$V'_{1,2} = \{f_{r-1,s}\vartheta_2\}, \qquad V'_{-1,-2} = \{f_{r,s-2}\vartheta_8\},$$
$$V'_{1,-1} = \{f_{r,s-1}\vartheta_4\}, \qquad V'_{-1,1} = \{f_{r-2,s}\vartheta_7\};$$

if $(a,b) \in \mathcal{P}$ does not belong to $\mathcal{P}'$, we set $V'_{a,b} = \{0\}$.

We consider the sections $h_j$ of $C^\infty(S^2T^*_{\mathbb{C}})$, with $0 \leq l \leq 35$, defined by

$$
\begin{aligned}
&h_0 = f_{r,s}g, && h_1 = if_{r,s-1}\tilde{\sigma}(\vartheta_5), && h_2 = f_{r-2,s}\vartheta_3\cdot\vartheta_3,\\
&h_3 = f_{r-1,s-1}\vartheta_3\cdot\vartheta_5, && h_4 = if_{r-1,s}\tilde{\sigma}(\vartheta_3), && h_5 = f_{r,s-2}\vartheta_5\cdot\vartheta_5,\\
&h_6 = if_{r,s}\tilde{\sigma}(\vartheta_1), && h_7 = f_{r-1,s}\vartheta_1\cdot\vartheta_3, && h_8 = f_{r,s-1}\vartheta_1\cdot\vartheta_5,\\
&h_9 = if_{r-1,s-1}\tilde{\sigma}(\vartheta_6), && h_{10} = f_{r-2,s-1}\vartheta_3\cdot\vartheta_6, && h_{11} = f_{r-1,s-2}\vartheta_5\cdot\vartheta_6,\\
&h_{12} = if_{r-1,s}\tilde{\sigma}(\vartheta_2), && h_{13} = f_{r-2,s}\vartheta_2\cdot\vartheta_3, && h_{14} = f_{r-1,s-1}\vartheta_2\cdot\vartheta_5,\\
&h_{15} = if_{r,s-1}\tilde{\sigma}(\vartheta_4), && h_{16} = f_{r-1,s-1}\vartheta_3\cdot\vartheta_4, && h_{17} = f_{r,s-2}\vartheta_4\cdot\vartheta_5,\\
&h_{18} = if_{r-2,s}\tilde{\sigma}(\vartheta_7), && h_{19} = f_{r-3,s}\vartheta_3\cdot\vartheta_7, && h_{20} = f_{r-2,s-1}\vartheta_5\cdot\vartheta_7,\\
&h_{21} = if_{r,s-2}\tilde{\sigma}(\vartheta_8), && h_{22} = f_{r-1,s-2}\vartheta_3\cdot\vartheta_8, && h_{23} = f_{r,s-3}\vartheta_5\cdot\vartheta_8,\\
&h_{24} = f_{r,s}\vartheta_1\cdot\vartheta_1, && h_{25} = f_{r-1,s}\vartheta_1\cdot\vartheta_2, && h_{26} = f_{r-2,s}\vartheta_2\cdot\vartheta_2,\\
&h_{27} = f_{r-3,s}\vartheta_2\cdot\vartheta_7, && h_{28} = f_{r-4,s}\vartheta_7\cdot\vartheta_7, && h_{29} = f_{r-3,s-1}\vartheta_6\cdot\vartheta_7,\\
&h_{30} = f_{r-2,s-2}\vartheta_6\cdot\vartheta_6, && h_{31} = f_{r,s-1}\vartheta_1\cdot\vartheta_4, && h_{32} = f_{r,s-2}\vartheta_4\cdot\vartheta_4,\\
&h_{33} = f_{r,s-3}\vartheta_4\cdot\vartheta_8, && h_{34} = f_{r,s-4}\vartheta_8\cdot\vartheta_8, && h_{35} = f_{r-1,s-3}\vartheta_6\cdot\vartheta_8.
\end{aligned}
$$

For $1 \leq l \leq 35$, note that the expression for the section $h_l$ given here is of the form $f_{r-e_l,s-e'_l}\vartheta_{j_l}\cdot\vartheta_{k_l}$ or $if_{r-e_l,s-e'_l}\tilde{\sigma}(\vartheta_{j_l})$, where $e_l, e'_l \geq 0$ and $1 \leq j_l, k_l \leq 8$ are integers independent of $r$ and $s$. We set $e_0 = e'_0 = 0$. Clearly, for all $0 \leq l \leq 35$, when $r \geq e_l$ and $s \geq e'_l$, the form $h_l$ is non-zero.

For $(a,b) \in \mathcal{P}$, we define subspaces $V_{a,b}$ of $C^\infty(S^2T^*_{\mathbb{C}})$ by

$$
\begin{aligned}
&V_{2,1} = \{h_6, h_7, h_8\}, && V_{-2,-1} = \{h_9, h_{10}, h_{11}\},\\
&V_{1,2} = \{h_{12}, h_{13}, h_{14}\}, && V_{1,-1} = \{h_{15}, h_{16}, h_{17}\},\\
&V_{-1,1} = \{h_{18}, h_{19}, h_{20}\}, && V_{-1,-2} = \{h_{21}, h_{22}, h_{23}\},
\end{aligned}
$$

$$
\begin{aligned}
&V_{4,2} = \{h_{24}\}, && V_{3,3} = \{h_{25}\}, && V_{2,4} = \{h_{26}\},\\
&V_{0,3} = \{h_{27}\}, && V_{-2,2} = \{h_{28}\}, && V_{-3,0} = \{h_{29}\},\\
&V_{-4,-2} = \{h_{30}\}, && V_{3,0} = \{h_{31}\}, && V_{2,-2} = \{h_{32}\},\\
&V_{0,-3} = \{h_{33}\}, && V_{-2,-4} = \{h_{34}\}, && V_{-3,-3} = \{h_{35}\};
\end{aligned}
$$

finally, $V_{0,0}$ is the subspace of $C^\infty(S^2T^*_{\mathbb{C}})$ generated by the functions $\{h_j\}$, with $0 \leq j \leq 5$. We remark that

$$
\tilde{\sigma}(V'_{a,b}) \subset V_{a,b}, \tag{12.6}
$$

for all $(a,b) \in \mathcal{P}$.

Since a form belonging to $\mathcal{A}_{\mathbb{C}}$ is left-invariant, it is also invariant under the right action of the center $S$. On the other hand, we easily see that

$$\pi'(a_0)z_{3j} = e^{2i\pi/3}z_{3j}, \qquad \pi'(a_0)\bar{z}_{1j} = e^{4i\pi/3}\bar{z}_{1j},$$

for $j = 1, 2, 3$. It follows that

$$\pi'(a_0)\theta = e^{2i(r+2s)\pi/3}\theta, \qquad \pi'(a_0)h = e^{2i(r+2s)\pi/3}h, \tag{12.7}$$

for all $\theta \in V'_{a,b}$ and $h \in V_{a,b}$, with $(a,b) \in \mathcal{P}$.

If $p, q \geq 0$ are given integers, let $\varepsilon^p_q$ be the integer equal to 1 if $p \geq q$ and 0 otherwise. For $(a,b) \in \mathcal{P}$, we consider the integers $N_1(a,b)$ and $N_2(a,b)$ defined by the relations

$$N_1(0,0) = \varepsilon^r_1 + \varepsilon^s_1, \qquad N_2(0,0) = \varepsilon^r_2 + \varepsilon^s_2 + \varepsilon^r_1\varepsilon^s_1$$

and the following table:

| $(a,b)$ | $N_1(a,b)$ | $N_2(a,b)$ | $(a,b)$ | $N_1(a,b)$ | $N_2(a,b)$ |
|---|---|---|---|---|---|
| $(4,2)$ | $0$ | $1$ | $(-4,-2)$ | $0$ | $\varepsilon^r_2\varepsilon^s_2$ |
| $(3,3)$ | $0$ | $\varepsilon^r_1$ | $(3,0)$ | $0$ | $\varepsilon^s_1$ |
| $(2,4)$ | $0$ | $\varepsilon^r_2$ | $(2,-2)$ | $0$ | $\varepsilon^s_2$ |
| $(2,1)$ | $1$ | $\varepsilon^r_1 + \varepsilon^s_1$ | $(-2,-1)$ | $\varepsilon^r_1\varepsilon^s_1$ | $\varepsilon^r_2\varepsilon^s_1 + \varepsilon^r_1\varepsilon^s_2$ |
| $(1,2)$ | $\varepsilon^r_1$ | $\varepsilon^r_2 + \varepsilon^r_1\varepsilon^s_1$ | $(1,-1)$ | $\varepsilon^s_1$ | $\varepsilon^s_2 + \varepsilon^r_1\varepsilon^s_1$ |
| $(0,3)$ | $0$ | $\varepsilon^r_3$ | $(0,-3)$ | $0$ | $\varepsilon^s_3$ |
| $(-1,1)$ | $\varepsilon^r_2$ | $\varepsilon^r_3 + \varepsilon^r_2\varepsilon^s_1$ | $(-1,-2)$ | $\varepsilon^s_2$ | $\varepsilon^s_3 + \varepsilon^r_1\varepsilon^s_2$ |
| $(-2,2)$ | $0$ | $\varepsilon^r_4$ | $(-2,-4)$ | $0$ | $\varepsilon^s_4$ |
| $(-3,0)$ | $0$ | $\varepsilon^r_3\varepsilon^s_1$ | $(-3,-3)$ | $0$ | $\varepsilon^r_1\varepsilon^s_3$ |

Using the expressions for the sections $\vartheta_j$ and the formulas (11.2) and (11.3), we easily verify the following:

**Lemma 12.1.** *Let $r, s \geq 0$ be given integers. If $(a,b)$ is an element of $\mathcal{P}$, the non-zero generators of the space $V_{a,b}$ (resp. $V'_{a,b}$) form a basis of this space, and we have*

$$\dim V'_{a,b} = N_1(a,b), \qquad \dim V_{a,b} = N_1(a,b) + N_2(a,b) + \delta_{0a'}\delta_{0b'},$$

*where $a' = |a|$ and $b' = |b|$.*

Let $a, b \in \mathbb{Z}$ be given integers. If $\delta$ is an element of $\Gamma$, we consider the element

$$\delta_{a,b} = \delta + a\lambda_1 + b\lambda_2$$

of $\mathfrak{h}^*$; throughout this section, this notation always supersedes the one introduced in §11. We now consider the element $\gamma = s\varpi_1 + r\varpi_2$ of $\Gamma$. Then we know that $\bar{\gamma} = r\varpi_1 + s\varpi_2$ and we see that $\bar{\gamma}_{a,b}$ belongs to $\Gamma$ if and only if

$$r + s + a \geq s + b \geq 0. \tag{12.8}$$

When the inequalities (12.8) hold, we consider the irreducible $G$-module $E_{\bar{\gamma}_{a,b}}$ corresponding to the element $\bar{\gamma}_{a,b}$ of $\Gamma$. We note that $\bar{\gamma}_{a,b}$ is equal to $\bar{\gamma}$ if and only if $a = b = 0$. Moreover, if $r = s = 0$, we have $\bar{\gamma}_{0,0} = \gamma_0$ and $\bar{\gamma}_{2,1} = \gamma_1$; if $r = s = 1$, we have $\bar{\gamma}_{0,0} = \gamma_1$ and $\bar{\gamma}_{-2,-1} = \gamma_0$.

We also consider the subset $\tilde{\mathcal{P}}$ of $\mathcal{P}$ consisting of pairs $(a, b) \in \mathcal{P}$ satisfying the relations (12.8). By means of Lemma 4.2 of [5], for $(a, b) \in \mathcal{P}$, we easily see that

$$V'_{a,b} \subset \mathcal{C}_\gamma \cdot \mathcal{A}_{\mathbb{C}}, \qquad V_{a,b} \subset \mathcal{C}_\gamma \cdot S^2\mathcal{A}_{\mathbb{C}}; \tag{12.9}$$

in fact, we have

$$V'_{a,b} = \{0\}, \qquad V_{a,b} = \{0\},$$

if $(a, b)$ does not belong to $\tilde{\mathcal{P}}$.

In [5, §5], we noted that, for $1 \leq j < k \leq 3$, the element $\omega_j$ is a vector of $\mathcal{A}_{\mathbb{C}}$ of weight 0, and that the elements $\theta_{jk}$ and $\bar{\theta}_{jk}$ are vectors of $\mathcal{A}_{\mathbb{C}}$ of weight $\lambda_j - \lambda_k$ and $\lambda_k - \lambda_j$, respectively, when we view $\mathcal{A}_{\mathbb{C}}$ as a $G$-submodule of $(\pi', C^\infty(T^*_{\mathbb{C}}))$. According to formulas (3.10) of [5], we know that

$$\Phi'(C_l) z_{jk} = i z_{jk}(\delta_{kl} - \delta_{k-1,l}),$$

for all $1 \leq j, k \leq 3$ and $l = 1, 2$. By means of the relations (11.4), (12.4), (12.5), (12.9), and the preceding remarks, we easily verify that

$$V'_{a,b} \subset \mathcal{C}_{(\gamma,\bar{\gamma}_{a,b})}(T^*_{\mathbb{C}}), \qquad V_{a,b} \subset \mathcal{C}_{(\gamma,\bar{\gamma}_{a,b})}(S^2T^*_{\mathbb{C}}), \tag{12.10}$$

for all $(a, b) \in \tilde{\mathcal{P}}$.

Let $(a, b)$ be an element of $\tilde{\mathcal{P}}$. If $(c, d)$ is the element $\Psi(a, b)$ of $\mathcal{P}$, we easily verify that $\gamma_{c,d}$ belongs to $\Gamma$ and that

$$\bar{\gamma}_{a,b} = \overline{\gamma_{c,d}};$$

from the equality (11.6), we therefore obtain the relations

$$C^\infty_{(\gamma,\bar\gamma_{a,b})}(S^pT^*_{\mathbb{C}}) = \overline{C^\infty_{(\bar\gamma,\gamma_{c,d})}(S^pT^*_{\mathbb{C}})}. \tag{12.11}$$

The Littlewood-Richardson rule (see [1, pp. 455–456]) gives us the decompositions of the $G$-modules $E_{\bar\gamma} \otimes \mathfrak{g}$ and $E_{\bar\gamma} \otimes F$ into irreducible submodules; in particular, it tells us that the spaces $\mathrm{Hom}_G(E_{\gamma'}, E_{\bar\gamma} \otimes \mathfrak{g})$ and $\mathrm{Hom}_G(E_{\gamma'}, E_{\bar\gamma} \otimes F)$ vanish unless $\gamma' = \bar\gamma_{a,b}$, with $(a,b) \in \tilde{\mathcal{P}}$, and that

$$\dim \mathrm{Hom}_G(E_{\bar\gamma_{a,b}}, E_{\bar\gamma} \otimes \mathfrak{g}) = N_1(a,b), \quad \dim \mathrm{Hom}_G(E_{\bar\gamma_{a,b}}, E_{\bar\gamma} \otimes F) = N_2(a,b),$$

for $(a,b) \in \tilde{\mathcal{P}}$.

By (11.5) and (11.7), from the above discussion we obtain the following result:

**Lemma 12.2.** *Let $r, s \geq 0$ be given integers; let $\gamma$ be the element $s\varpi_1 + r\varpi_2$ of $\Gamma$ and let $\gamma'$ be an arbitrary element of $\Gamma$. The $\tilde G$-modules $C^\infty_{(\gamma,\gamma')}(T^*_{\mathbb{C}})$ and $C^\infty_{(\gamma,\gamma')}(S^2T^*_{\mathbb{C}})$ vanish unless $\gamma' = \bar\gamma_{a,b}$, where $(a,b)$ is an element of $\tilde{\mathcal{P}}$. If $(a,b)$ is an element of $\tilde{\mathcal{P}}$ satisfying $\bar\gamma_{a,b} = \gamma'$, then the multiplicities of the $\tilde G$-modules $C^\infty_{(\gamma,\gamma')}(T^*_{\mathbb{C}})$ and $C^\infty_{(\gamma,\gamma')}(S^2T^*_{\mathbb{C}})$ are given by the relations*

$$\begin{aligned} \mathrm{Mult}\, C^\infty_{(\gamma,\bar\gamma_{a,b})}(T^*_{\mathbb{C}}) &= N_1(a,b), \\ \mathrm{Mult}\, C^\infty_{(\gamma,\bar\gamma_{a,b})}(S^2T^*_{\mathbb{C}}) &= N_1(a,b) + N_2(a,b) + \delta_{0a'}\delta_{0b'}, \end{aligned} \tag{12.12}$$

*where $a' = |a|$ and $b' = |b|$.*

From the the inclusions (12.10) and Lemmas 12.1 and 4.1, we obtain the following result:

**Lemma 12.3.** *Let $r, s \geq 0$ be given integers.*

(i) *If $(a,b)$ is an element of $\mathcal{P}'$ satisfying* (12.8)*, we have the equality*

$$\mathcal{C}_{(\gamma,\bar\gamma_{a,b})}(T^*_{\mathbb{C}}) = V'_{a,b}.$$

(ii) *If $(a,b)$ is an element of $\mathcal{P}$ satisfying* (12.8)*, we have the equality*

$$\mathcal{C}_{(\gamma,\bar\gamma_{a,b})}(S^2T^*_{\mathbb{C}}) = V_{a,b}.$$

The preceding lemma and the relations (12.7) give us the following:

**Proposition 12.4.** *Let $r, s \geq 0$ be given integers and let $\gamma$ be the element $s\varpi_1 + r\varpi_2$ of $\Gamma$. For $\gamma' \in \Gamma$ and $p = 1, 2$, the $G$-module $C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})^S$ is equal to $C^\infty_{(\gamma,\gamma')}(S^pT^*_{\mathbb{C}})$ if the relation*

$$2r + s \equiv 0 \pmod 3 \tag{12.13}$$

*holds and $\gamma'$ is equal to $\bar{\gamma}_{a,b}$, with $(a, b) \in \mathcal{P}$, and vanishes otherwise.*

In §14, we shall prove the following three results:

**Proposition 12.5.** *Let $r, s$ be given integers satisfying $0 \leq r \leq s$ and the relation (12.13). Then the function $f_{r,s}$ on $X$ does not satisfy the Guillemin condition.*

**Proposition 12.6.** *Let $r, s$ be given integers satisfying $0 \leq r \leq s$ and the relation (12.13).*

(i) *Let $(a, b)$ be a given element of $\mathcal{P} - \{(0, 0)\}$; assume that $(r, s, a, b)$ is not equal to $(0, 0, 2, 1)$ or $(1, 1, -2, -1)$. If $(a, b)$ belongs to $\mathcal{P}_1$, assume also that $s > r$. Then we have the inequality*

$$\dim (V_{a,b} \cap \mathcal{N}_{2,\mathbb{C}}) \leq N_1(a, b).$$

(ii) *If $(r, s) \neq (0, 0), (1, 1)$, we have the inequality*

$$\dim (V_{0,0} \cap \mathcal{N}_{2,\mathbb{C}}) \leq N_1(0, 0) + 1.$$

(iii) *If $(r, s) = (1, 1)$, we have the relations*

$$\dim (V_{0,0} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 2, \qquad V_{-2,-1} \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}.$$

(iv) *If $(r, s) = (0, 0)$, we have the equalities*

$$V_{0,0} \cap \mathcal{N}_{2,\mathbb{C}} = V_{2,1} \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}.$$

**Proposition 12.7.** *Let $r, s \geq 0$ be given integers satisfying $0 \leq r \leq s$ and the relation (12.13). Let $(a, b)$ be a given element of $\mathcal{P}' - \{(0, 0)\}$. If $(a, b)$ belongs to $\mathcal{P}_1$, assume also that $s > r$. Then we have the relations*

$$\tilde{\sigma}(V'_{a,b}) \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}, \qquad \dim (\tilde{\sigma}(V'_{0,0}) \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1.$$

According to Lemma 2.3 of [5] and the relation (12.6), under the hypotheses of Proposition 12.7, we see that this proposition implies that

$$V'_{a,b} \cap \mathcal{N}_{1,\mathbb{C}} = \{0\}, \qquad \dim (V'_{0,0} \cap \mathcal{N}_{1,\mathbb{C}}) \leq 1.$$

## 13. Main results

We consider the reduced space $Y = SU(3)/S$ of the symmetric space $X$. Let $r, s \geq 0$ be given integers and let $\gamma$ be the element $s\varpi_1 + r\varpi_2$ of $\Gamma$. According to Lemma 4.2 of [5] and Proposition 12.4, we know that $f_{r,s}$ is a highest weight vector of the irreducible $\tilde{G}$-module $C^\infty_{(\gamma,\bar{\gamma})}(X)$ and that the equality (11.9) holds if and only if the relation (12.13) is true. Hence from the equalities (12.11), with $p = 1$ and $(a, b) = (0, 0)$, Proposition 2.29 of [2] and Proposition 12.5, we deduce the following result:

**Proposition 13.1.** *Let $Y$ be the reduced group $SU(3)/S$. The maximal flat Radon transform for functions on the symmetric space $Y$ is injective.*

According to Lemmas 12.2 and 12.3,(ii), Propositions 12.4 and 12.6, the relations (12.11) and the remarks appearing after Lemma 12.1, we see that assertion (iv) of Proposition 11.2 is true when $n = 3$; from Propositions 11.1 and 11.2, we then deduce the following result:

**Theorem 13.2.** *Let $Y$ be the reduced group $SU(3)/S$. Then the equality*

$$\mathcal{N}_{2,Y} = D_0 C^\infty(T_Y) + \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y)$$

*holds.*

If $P$ denotes the orthogonal projection corresponding to the decomposition (1.1) on the space $Y$, according to Lemma 1.1 of [3] and Lemma 2.3 of [5] (see also Lemma 3.1 of [4]) the mapping

$$P_{\sigma_Y} = P\tilde{\sigma}_Y d : C^\infty_{\mathbb{R}}(Y) \to I(Y)$$

is well-defined. We denote by $\mathcal{F}_Y$ the orthogonal complement of the finite-dimensional space $\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{B}_Y$ in $C^\infty_{\mathbb{R}}(Y)$. From Proposition 1.2 of [3], the relations (11.12), and Theorem 13.2, we obtain:

**Theorem 13.3.** *Let $Y$ be the reduced group $SU(3)/S$. Then the equality*

$$I(Y) = P\tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y)$$

*holds and the mapping*

$$P\tilde{\sigma}_Y d : \mathcal{F}_Y \to I(Y)$$

*is an isomorphism.*

The preceding theorem is a complement to Theorem 7.4 of [5] with $n = 3$. From Lemmas 12.2 and 12.3,(i), and Propositions 1.2, 12.4 and 12.7, and the remark which follows the latter proposition, and the relations (12.11), we deduce the following result:

**Theorem 13.4.** *Let $Y$ be the reduced group $SU(3)/S$. A 1-form on $Y$ satisfies the Guillemin condition if and only it is exact.*

## 14. Proofs of Propositions 12.5, 12.6 and 12.7

In this section, we consider the symmetric space $X = G = SU(3)$ and denote by

$$\iota : \mathbb{R}^2 \to G$$

the mapping (10.1). This mapping induces by passage to the quotient an imbedding

$$\mathbb{R}^2/\Lambda' \to G,$$

whose image is the maximal torus $H$ of $G$. This torus $H$ is also a maximal flat totally geodesic torus of $X = G$ viewed as a symmetric space.

We consider the standard coordinate system $(x, y)$ on $\mathbb{R}^2$ and endow this space with the flat Riemannian metric

$$\tilde{g} = dx \cdot dx + dy \cdot dy - dx \cdot dy.$$

According to the relation (3.10) of [4], we know that

$$\iota^* g = \tilde{g}; \tag{14.1}$$

hence if $f$ is a function on $X$, we easily see that

$$\int_H f\, dH = \sqrt{3} \int_0^{2\pi} \int_0^{2\pi} f(\iota(x, y))\, dx\, dy. \tag{14.2}$$

In [4, §3], we saw that the parallel vector fields $\zeta_1$ and $\zeta_2$ on $H$ determined by

$$\iota_*(\partial/\partial x)(x, y) = \zeta_1(\iota(x, y)), \qquad \iota_*(\partial/\partial y)(x, y) = \zeta_2(\iota(x, y)), \tag{14.3}$$

for $(x, y) \in \mathbb{R}^2$, are equal to the restrictions to $H$ of the vector fields $C_1$ and $C_2$, respectively (see also [5, §2]). Thus if $\varphi$ is a 1-form on $X$, according to the formulas (3.5) and (3.6) of [4] (or formulas (2.4) and (2.5) of [5]), we have

$$\begin{aligned} \iota^*\tilde\sigma(\varphi)(\partial/\partial x,\partial/\partial x) &= \frac{1}{3}\big(\iota^*\varphi(\partial/\partial x)+2\iota^*\varphi(\partial/\partial y)\big),\\ \iota^*\tilde\sigma(\varphi)(\partial/\partial x,\partial/\partial y) &= \frac{1}{3}\big(\iota^*\varphi(\partial/\partial x)-\iota^*\varphi(\partial/\partial y)\big),\\ \iota^*\tilde\sigma(\varphi)(\partial/\partial y,\partial/\partial y) &= -\frac{1}{3}\big(2\iota^*\varphi(\partial/\partial x)+\iota^*\varphi(\partial/\partial y)\big). \end{aligned} \tag{14.4}$$

If $\phi$ is an element of $SO(3)$, we consider the maximal torus $H' = \mathrm{Ad}\,\phi\cdot H$ of $X$; if $f$ is a function on $X$, we have

$$\int_{H'} f\,dH' = \int_H (\mathrm{Ad}\,\phi)^* f\,dH,$$

and we easily see that

$$\iota^*(\mathrm{Ad}\,\phi)^* z_{jk} = \iota^*(\mathrm{Ad}\,\phi)^* z_{kj} \tag{14.5}$$

for $1 \le j,k \le 3$.

We consider the functions $\psi$ and $\tilde\psi$ on $\mathbb{R}^3$ introduced in §8. If $r,s \ge 0$ are integers, we also consider the function

$$\tilde\psi_{r,s} = \tilde\psi^r\cdot\overline{\tilde\psi^s}$$

on $\mathbb{R}^3$.

If $f$ is a function on $\mathbb{R}^3$ and $v\in\mathbb{R}$, we consider the function $f_v$ on $\mathbb{R}^2$ defined by

$$f_v(x,y) = f(x,y,v),$$

for all $(x,y)\in\mathbb{R}^2$.

For $\alpha\in\mathbb{R}$, we consider the element $\phi_\alpha$ of $SO(3)$ introduced in §10 and the maximal flat totally geodesic torus $H_\alpha = \mathrm{Ad}\,\phi_\alpha\cdot H$ of $X$. For $\alpha\in\mathbb{R}$, if we write $v = \cos^2\alpha$, we verify that

$$\begin{gathered} \iota^*(\mathrm{Ad}\,\phi_\alpha)^* z_{13} = \frac{1}{2}\,\tilde\psi_v, \qquad \iota^*(\mathrm{Ad}\,\phi_\alpha)^* z_{11} = \iota^*(\mathrm{Ad}\,\phi_\alpha)^* z_{33} = \frac{1}{2}\,\psi_v,\\ \big(\iota^*(\mathrm{Ad}\,\phi_\alpha)^* z_{j2}\big)(x,y) = \frac{1}{\sqrt2}\,\cos\alpha\cdot\sin\alpha\,(e^{-iy}-e^{ix}), \end{gathered} \tag{14.6}$$

for $j = 1, 3$ and all $(x, y) \in \mathbb{R}^2$, and

$$\begin{aligned} \iota^*(\operatorname{Ad}\phi_\alpha)^*\omega_2 &= \frac{1}{2}\big(\sin^2\alpha\cdot\iota^*\omega_1 - \cos^2\alpha\cdot\iota^*\omega_2\big) = -\iota^*(\operatorname{Ad}\phi_\alpha)^*\omega_1, \\ \iota^*(\operatorname{Ad}\phi_\alpha)^*\theta_{13} &= \frac{i}{2}\big((1+\sin^2\alpha)\cdot\iota^*\omega_2 - (1+\cos^2\alpha)\cdot\iota^*\omega_1\big), \\ \iota^*(\operatorname{Ad}\phi_\alpha)^*\theta_{jk} &= \frac{i}{\sqrt{2}}\cos\alpha\cdot\sin\alpha\cdot\iota^*(\omega_1+\omega_2), \end{aligned} \tag{14.7}$$

if $(j, k)$ is equal to $(1, 2)$ or $(2, 3)$; if $r, s \geq 0$ are integers, from (14.6) it follows that

$$\iota^*(\operatorname{Ad}\phi_\alpha)^* f_{r,s} = \frac{1}{2^{r+s}}\,\tilde{\psi}_{r,s,v}. \tag{14.8}$$

For $0 \leq l \leq 35$, we define integers $p_l$ by

$$p_l = \begin{cases} 1 & \text{for } 12 \leq l \leq 23, \text{ and } l = 25, 29, 31, 35, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y$ be an indeterminate over $\mathbb{C}$. If $P$ is an element of $\mathbb{C}[Y]$, we denote by $c_j(P)$ its coefficient of degree $j$ and write $c(P) = c_0(P)$. For $j = 1, 2$, we define projections

$$\pi_j : \mathbb{C}[Y] \oplus \mathbb{C}[Y] \oplus \mathbb{C}[Y] \to \mathbb{C}[Y] \oplus \mathbb{C}[Y]$$

by

$$\pi_1(P_1, P_2, P_3) = (P_1, P_2), \qquad \pi_2(P_1, P_2, P_3) = (P_1, P_3),$$

for $P_1, P_2, P_3 \in \mathbb{C}[Y]$.

Let $r, s \geq 0$ be given integers; we now consider the symmetric 2-forms $h_l$, with $0 \leq l \leq 35$, and the spaces $V_{a,b}$, with $(a, b) \in \mathcal{P}$, associated with the integers $r, s$.

Let $0 \leq l \leq 35$ be a given integer. By means of the relations (14.1)–(14.8), we easily see that there exist polynomials $J$ and $I_{l,j,k}$, with $j, k = 1, 2$, belonging to $\mathbb{Q}[Y]$ such that the equalities

$$\begin{aligned} \frac{1}{4\pi^2\sqrt{3}}\int_H \phi_\alpha^* f_{r,s}\,dH &= J(\cos^2\alpha), \\ \frac{1}{4\pi^2\sqrt{3}}\int_H (\phi_\alpha^* h_l)(\zeta_j, \zeta_k)\,dH &= \Big(\frac{\cos\alpha\cdot\sin\alpha}{\sqrt{2}}\Big)^{p_l}\cdot I_{l,j,k}(\cos^2\alpha), \end{aligned} \tag{14.9}$$

hold for all $\alpha \in \mathbb{R}$. We write $J_l = I_{l,1,1}$, $K_l = I_{l,2,2}$ and $M_l = I_{l,1,2}$. If $J$ (resp. $J_l$) does not vanish, there exists $\alpha_0 \in \mathbb{R}$ such that $J(\cos^2\alpha_0)$ (resp. $J_l(\cos^2\alpha_0)$)

and $\cos\alpha_0 \cdot \sin\alpha_0$ are non-zero; therefore by (14.9), the function $f_{r,s}$ (resp. the 2-form $h_l$) does not satisfy the Guillemin condition.

Let $(a,b)$ be a given element of $\mathcal{P}$; suppose that the space $V_{a,b}$ is non-zero. Let $\{h_l, h_{l+1}, \dots, h_{l+q}\}$ be the generators of the space $V_{a,b}$ considered in §12. We know that either $q = 5$ and $l = 0$, or $q = 2$ and $l$ belongs to the set of integers

$$L_1 = \{6, 9, 12, 15, 18, 21\},$$

or $q = 0$ and $24 \leq l \leq 35$; also we note that the integers $p_{l+j}$ are independent of $0 \leq j \leq q$. According to (14.9), for $j, k = 1, 2$, the linear mapping

$$\Phi^{j,k}_{a,b} : V_{a,b} \to \mathbb{C}[Y],$$

sending an element $h$ of $V_{a,b}$ into the polynomial $\Phi^{j,k}_{a,b}(h)$ determined by

$$\Big(\frac{\cos\alpha \cdot \sin\alpha}{\sqrt{2}}\Big)^{p_l} \cdot \Phi^{j,k}_{a,b}(h)(\cos^2\alpha) = \frac{1}{4\pi^2\sqrt{3}} \int_H (\phi^*_\alpha h)(\zeta_j, \zeta_k)\, dH,$$

for all $\alpha \in \mathbb{R}$, is well-defined. We consider the mappings

$$\Phi_{a,b} = \big(\Phi^{1,1}_{a,b}, \Phi^{2,2}_{a,b}, \Phi^{1,2}_{a,b}\big) : V_{a,b} \to \mathbb{C}[Y] \oplus \mathbb{C}[Y] \oplus \mathbb{C}[Y],$$

$$\Phi_{a,b,j} = \pi_j \circ \Phi_{a,b} : V_{a,b} \to \mathbb{C}[Y] \oplus \mathbb{C}[Y],$$

with $j = 1, 2$, and write $\Psi_{a,b} = \Phi^{1,1}_{a,b}$. Clearly, by (14.9) we have

$$\Phi_{a,b}(h_{l+j}) = (J_{l+j}, K_{l+j}, M_{l+j}),$$

for $0 \leq j \leq q$; hence the rank of $\Psi_{a,b}$ is equal to the dimension of the subspace of $\mathbb{C}[Y]$ generated by its elements $\{J_l, J_{l+1}, \dots, J_{l+q}\}$. An element of $V_{a,b}$ satisfying the Guillemin condition belongs to the kernel of $\Phi_{a,b}$; so we have the inclusions

$$V_{a,b} \cap \mathcal{N}_{2,\mathbb{C}} \subset \operatorname{Ker} \Phi_{a,b} \subset \operatorname{Ker} \Phi_{a,b,j} \subset \operatorname{Ker} \Psi_{a,b},$$

with $j = 1, 2$. Hence if one of the inequalities

$$\operatorname{rank} \Phi_{a,b,j} \geq N_2(a,b), \tag{14.10}$$

with $j = 1$ or 2, or the inequality

$$\operatorname{rank} \Psi_{a,b} \geq N_2(a,b) \tag{14.11}$$

holds, we know that

$$\operatorname{rank} \Phi_{a,b} \geq N_2(a,b)$$

and, from Lemma 3.1, we deduce that

$$\dim (V_{a,b} \cap \mathcal{N}_{2,\mathbb{C}}) \leq N_1(a,b) + \delta_{0a'}\delta_{0b'}, \tag{14.12}$$

where $a' = |a|$ and $b' = |b|$.

We consider the sets of integers

$$L_2 = \{15, 16, 21, 22, 31, 32, 33, 34, 35\}, \qquad L_3 = \{6, 7, 12, 13, 21, 22\}.$$

For $0 \leq l \leq 35$, we define integers $m_l \in \mathbb{Z}$ by

$$m_l = \begin{cases} m+1 & \text{for } l = 26, 27, 28, \\ m-1 & \text{for } l = 15, 16, 21, 22, 31, 35, \\ \sup(m-1, 1) & \text{for } l = 32, 33, 34, \\ m & \text{otherwise;} \end{cases}$$

when $l$ belongs to $L_2$ and $m \geq 1$, we note that the integer $m_l$ is non-negative. We also note that $e'_l = e'_{l+1}$, for all $l \in L_1$. We consider the polynomial $P \in \mathbb{Q}[Y]$ of degree $2r$ given by (9.12).

**Proposition 14.1.** *Let $r, s$ be given integers satisfying the relation* (12.13) *and $0 \leq r \leq s$. Let $m \geq 0$ be the integer such that $s = r + 3m$. Let $0 \leq l \leq 35$ be a given integer.*

(i) *The relation*

$$J(Y) = \frac{1}{2^{r+s}} Y^m (Y-1)^m \cdot P(Y)$$

*holds.*

(ii) *Assume either that $l$ belongs to $L_1$, or that $l-1$ belongs to $L_1$, or that $l$ satisfies $0 \leq l \leq 2$ or $24 \leq l \leq 35$. Suppose that $r \geq e_l$ and $s \geq e'_l$. If $l$ belongs to the set $L_2$, assume also that $m \geq 1$. Then there exists a non-zero polynomial $P_l \in \mathbb{Q}[Y]$ such that*

$$J_l(Y) = \frac{1}{2^{r+s}} Y^{m_l} (Y-1)^{m_l} \cdot P_l(Y). \tag{14.13}$$

(iii) *Suppose that $l$ is equal to* 9, 15 *or* 18*; if $l$ is equal to* 15, *assume that $m \geq 1$. If $r \geq e_{l+1}$, then we have $r \geq e_l$ and $s \geq e'_l = e'_{l+1}$, and the determinant of the* *matrix*

$$\begin{pmatrix} c(P_l) & c_1(P_l) \\ c(P_{l+1}) & c_1(P_{l+1}) \end{pmatrix}$$

*is non-zero, and the polynomials* $J_l$ *and* $J_{l+1}$ *are linearly independent.*

(iv) *Suppose that* $l$ *belongs to the set* $L_3$. *If* $l$ *is equal to* 21 *or* 22, *assume that* $m \geq 1$. *If* $r \geq e_l$, *there exists a polynomial* $Q_l \in \mathbb{Q}[Y]$, *whose constant term is non-zero, such that*

$$K_l(Y) = \frac{1}{2^{r+s}} Y^{m_l} (Y-1)^{m_l} \cdot Q_l(Y). \tag{14.14}$$

*If* $l$ *is equal to* 6, 12 *or* 21 *and if* $r \geq e_{l+1}$, *then we have* $r \geq e_l$ *and* $s \geq e'_l = e'_{l+1}$, *and the determinant of the matrix*

$$\begin{pmatrix} c(P_l) & c(Q_l) \\ c(P_{l+1}) & c(Q_{l+1}) \end{pmatrix}$$

*is non-zero.*

(v) *Suppose that* $0 \leq l \leq 2$. *If* $l = 1$, *suppose that* $s \geq 1$; *if* $l = 2$, *suppose that* $r \geq 2$. *There exists a polynomial* $R_l \in \mathbb{Q}[Y]$, *whose constant term is non-zero, such that*

$$M_l(Y) = \frac{1}{2^{r+s}} Y^{m} (Y-1)^{m} \cdot R_l(Y). \tag{14.15}$$

*If* $r \geq 1$, *the determinant of the matrix*

$$\begin{pmatrix} c(P_0) & c(R_0) \\ c(P_1) & c(R_1) \end{pmatrix}$$

*is non-zero. If* $r \geq 2$, *the determinant of the matrix*

$$\begin{pmatrix} c(P_0) & c(R_0) & c_1(R_0) \\ c(P_1) & c(R_1) & c_1(R_1) \\ c(P_2) & c(R_2) & c_1(R_2) \end{pmatrix}$$

*is non-zero.*

*Proof.* By means of formula (9.6), we obtain explicit expressions for the polynomials $J$, $J_q$, $K_q$ and $M_q$, with $0 \leq q \leq 35$. Assertion (i) is an immediate consequence of formulas (9.6) and (9.10), with $d_1 = d_2 = 0$. Next, suppose the hypotheses of (ii) hold; we saw that the integer $m_l$ is non-negative. Using formulas (9.6) and (9.10), we demonstrate the existence of a polynomial $P_l$ satisfying the relation (14.13), and a polynomial $Q_l$ satisfying the relation (14.14) when $l$ belongs to the set $L_3$, and a polynomial $M_l$ satisfying the relation (14.15) when $0 \leq l \leq 2$. By means of formulas (9.3) and (9.11), we compute the following coefficient of the polynomial $P_l$:

(i) if $l \neq 35$, its constant term $c(P_l)$;

(ii) if $l = 35$ and $m = 1$ or 2, its constant term $c(P_l)$;

(iii) if $l = 35$ and $m \geq 3$, its leading coefficient $c_{2r+3}(P_l)$.

Using the equalities (8.3)–(8.5) and (8.7), the first relation of (8.11) and the equality (8.12), we obtain an explicit expression for this coefficient, which shows that it is a non-zero multiple of $\varphi_1(r)$. We use the same methods and the equalities (7.1) and (8.3)–(8.7), the equalities (8.8)–(8.10) and the second relation of (8.11) to compute explicitly the coefficient $c_1(P_l)$ when $l$ belongs to the set $\{15, 16, 18, 19\}$, and the coefficient $c(Q_l)$ when $l$ belongs to $L_3$, and the coefficients $c(R_l)$ and $c_1(R_l)$ when $0 \leq l \leq 2$. Finally, the expressions of these coefficients allow us to show that the determinants of assertions (iii)–(v) are non-zero multiples of $\varphi_1(r)$ under the appropriate hypotheses. Under the hypotheses of (iii), the non-vanishing of the determinant of (iii) implies that the polynomials $P_l$ and $P_{l+1}$ are linearly independent; this gives us the last assertion of (iii). □

From assertions (iii), (iv) and (v) of Proposition 14.1, we deduce the assertions (i), (ii) and (iii), respectively, of the following:

**Proposition 14.2.** *Let $r, s$ be given integers satisfying the relation* (12.13) *and $1 \leq r \leq s$. Let $m \geq 0$ be the integer such that $s = r + 3m$. Let $(a, b)$ be an element of $\mathcal{P}$.*

(i) *If $r \geq 2$ and $(a, b) = (-2, -1)$, or if $m \geq 1$ and $(a, b) = (1, -1)$, or if $r \geq 3$ and $(a, b) = (-1, 1)$, the rank of the mapping $\Psi_{a,b}$ is $\geq 2$.*

(ii) *If $(a, b) = (2, 1)$, or if $r \geq 2$ and $(a, b)$ is equal to $(1, 2)$, or if $m \geq 1$ and $(a, b) = (-1, -2)$, the rank of the mapping $\Phi_{a,b,1}$ is $\geq 2$.*

(iii) *The rank of the mapping $\Phi_{0,0,2}$ is $\geq \min(r + 1, 3)$.*

Let $r, s$ be given integers satisfying the relation (12.13) and $0 \leq r \leq s$. Let $m \geq 0$ be the integer such that $s = r + 3m$. From Proposition 14.1,(i) and a remark made above concerning the polynomial $J$, we infer that the function $f_{r,s}$ on $X$ does not satisfy the Guillemin condition; thus Proposition 12.5 is true.

Let $(a, b)$ be an element of $\mathcal{P}$; if $(a, b)$ belongs to $P_1$, assume also that $m \geq 1$. Suppose that the space $V_{a,b}$ is non-zero. Let $\{h_l, h_{l+1}, \dots, h_{l+q}\}$ be the generators of the space $V_{a,b}$ considered in §12. We know that either $q = 5$ and $l = 0$, or $q = 2$ and $l \in L_1$, or $q = 0$ and $24 \leq l \leq 35$. When $(a, b) \in \mathcal{P}_1$, we verify that $l$ belongs to the set $L_2$. We easily see that the non-vanishing of $V_{a,b}$ implies that $r \geq e_l$ and $s \geq e'_l$, and so the 2-form $h_l$ is non-zero. Therefore by Proposition 14.1,(ii), the polynomials $P_l$ and $J_l$ are non-zero; it follows that

$$\operatorname{rank} \Psi_{a,b} \geq 1. \tag{14.16}$$

Thus according to a remark made above, $h_l$ does not satisfy the Guillemin condition. Hence if $q = 0$, we have proved that

$$V_{a,b} \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}. \tag{14.17}$$

Now suppose that $q = 2$. If $(a, b)$ is equal to $(-2, -1)$, $(1, -1)$, or $(-1, 1)$, from Proposition 14.2,(i) and (14.16), we obtain the relation (14.11). On the other hand, if $(a, b)$ belongs to the set $\{(2, 1), (1, 2), (-1, -2)\}$, from Proposition 14.2,(ii) and (14.16), we obtain the relation (14.10), with $j = 1$. We saw above that either one of the inequalities (14.10) and (14.11) implies the inequality (14.12); thus the relations

$$\dim (V_{a,b} \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1 = N_1(a, b)$$

always hold when $q = 2$. Since $h_l$ does not belong to $\mathcal{N}_{2,\mathbb{C}}$, we see that

$$\tilde{\sigma}(V'_{a,b}) \cap \mathcal{N}_{2,\mathbb{C}} = \{0\}$$

and that the equality (14.17) is true when $(r, s, a, b)$ is equal to $(0, 0, 2, 1)$ or $(1, 1, -2, -1)$. Finally, assume that $q = 5$ and $(a, b) = (0, 0)$. When $r \geq 1$, according to Proposition 14.2,(iii), the rank of the mapping $\Phi_{0,0,2}$ is $\geq 2$ and the inequality (14.10), with $j = 2$, is true; therefore we obtain the inequality (14.12), and the first relation of Proposition 12.6,(iii) holds when $(r, s) = (1, 1)$. When $r = 0$, we know that the inequality (14.16) holds; thus (14.12) is also true in this case, and the equality (14.17) holds when $s = 0$. Thus we have verified all the relations of Proposition 12.6 involving $V_{0,0}$. According to Proposition 6.1,(ii), when $r \geq 1$, the polynomial $P_1$ is non-zero and so $h_1$ does not satisfy the Guillemin condition;

hence the inequality

$$\dim(\tilde{\sigma}(V'_{0,0}) \cap \mathcal{N}_{2,\mathbb{C}}) \leq 1$$

holds. Thus we have completed the proof of Propositions 12.6 and 12.7.

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