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Mappings of Bounded Distortion Between Complex Manifolds

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Abstract: We obtain Liouville type theorems for holomorphic mappings with bounded s-distortion between $\mathbb{C}^n$ and positively curved Kähler manifolds.
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1 Introduction

The classic Liouville theorem states that- every bounded holomorphic function on the entire complex plane is constant. H.Grötzsch observed that the classic Liouville theorem can be extended to quasi-conformal mappings. A smooth mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called quasi-conformal, if it is orientation preserving, and locally it is a diffeomorphism such that

$$|df|^n \leq KJ(f), \quad a.e. \tag{1.1}$$

for some positive constant $K \geq 1$, where $|df|$ is the operator norm of the Jacobian matrix df and $J(f)$ is the determinant of df. It is well-known that every holomorphic mapping $f : \mathbb{C} \longrightarrow \mathbb{C}$ is quasi-conformal with $K = 1$. More

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precisely, $|df|^2 = J(f)$. However, in higher dimensional case, holomorphic mappings are not necessarily quasi-conformal. For example, the holomorphic mapping $f : D = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_2| > \frac{1}{2}\} \longrightarrow \mathbb{C}^2$ with $f(z_1, z_2) = (z_1, z_2^2)$ is not quasi-conformal, but it satisfies $|df|^2 = J(f)$. Here $2 < \dim_{\mathbb{R}} \mathbb{C}^2 = 4$. For more details, one can see Example 2.2 and Example 2.14. Hence, we can consider smooth mappings with bounded s-distortion,

$$|df|^s \leq KJ(f) \tag{1.2}$$

for some $s \in (0, \infty)$.

In this paper, we consider holomorphic mappings with bounded s-distortion between complex manifolds and obtain Liouville type theorem for such mappings. Let's recall several classic Liouville type theorem for holomorphic mappings and quasi-conformal mappings between manifolds.

Theorem 1.1 (Yau's Schwarz Lemma)**.** *Let M be a complete Kähler manifold with Ricci curvature bounded from below by K_1. Let N be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant K_2. Then if there is a non-constant holomorphic mapping f from M into N, we have $K_1 \leq 0$ and*

$$f^*\left(dS_N^2\right) \leq \frac{K_1}{K_2} dS_M^2$$

In particular, if $K_1 \geq 0$, every holomorphic mapping from M into N is constant.

For more details about Schwarz Lemma and related Liouville type theorem, we refer the reader to Ahlfors([1]), Yau([20]), Kobayashi([14]), Chen-Yang([4]), Tossati([19]) and reference therein.

The study of the Schwarz Lemma and Liouville type theorem for non-holomorphic quasi-conformal(quasi-regular) mapping was started from Kiernan([12]) in our knowledge. From then, there are many mathematicians study the harmonic mappings with various bounded distortion, for example, Chern, Goldberg, Har'El, Ishihara, Petridis, Shen([3],[6], [7], [8],[9],[17]), etc. We summarize their works wildly in the following:

Theorem 1.2 (Generalized Schwarz Lemma)**.** *Let M, N be complete Riemannian manifolds. Suppose the Ricci curvature of M is bounded below by $-K_1$ and the* *sectional curvature of N is bounded from above by $-K_2$ where $K_1, K_2 > 0$. If $f : M \longrightarrow N$ is a harmonic K-quasi-regular mapping, then*

$$f^*(ds_N^2) \leq C\frac{K_1}{K_2}ds_M^2$$

where C is a positive constant depending on K and the dimension of the manifolds. In particular, if $f : \mathbb{R}^m \longrightarrow N$ is a harmonic K-quasi-regular mapping, then f is a constant.

The common conditions in the Schwarz Lemma and Liouville type theorem are

(1) The target should be negatively curved;

(2) The mapping should satisfy certain bounded distortion condition.

In this paper, we consider the Liouville type theorem for positively curved targets instead of negatively curved ones. By a geometric interpretation of inequality (1.2), we obtain:

Main Theorem Let (N, h) be a complete Kähler manifold of complex dimension n, and $f : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \longrightarrow (N, h)$ be holomorphic. If f is a mapping with bounded $2s$-distortion and N has the curvature property (Q_s), then f is constant.

In fact, the curvature condition (Q_s) has a geometric explanation. It is equivalent to the Griffiths positivity of the (formal) vector bundle $G = T^*N \otimes K_N^{*1/s}$. For any *compact* Kähler manifold with $c_1(M) > 0$, the anti-canonical line bundle K_N^* is a positive line bundle, so there exists some small $s \in (0, \infty)$ such that the vector bundle G is Griffiths positive, i.e., the curvature condition Q_s can be satisfied automatically(Theorem 2.10).

Corollary 1.3. *Let M be a compact Kähler manifold with $c_1(M) > 0$. Then there exist a Kähler metric ω and some $s_0 \in (0, \infty)$ such that any holomorphic mapping $f : \mathbb{C}^n \longrightarrow (M, \omega)$ with bounded s-distortion, $s \in (0, s_0)$, is constant.*

In particular, for $\mathbb{P}^n$, we obtain

Corollary 1.4. *If $f : \mathbb{C}^n \longrightarrow \mathbb{P}^n$ is a holomorphic mapping with bounded s-distortion, $0 < s < n+1$, with respect to the canonical metrics, then f is a constant.*

There do exist holomorphic mappings of bounded s-distortion between $\mathbb{C}^n$ and $\mathbb{P}^n$ for some s. For example, the canonical map $f : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \longrightarrow (\mathbb{P}^n, \omega_{FS})$

$$f(z_1, \cdots, z_n) = [1, z_1, \cdots, z_n]$$

is a holomorphic mapping with bounded $(2n+2)$-distortion and it fails to be a mapping of bounded s-distortion for any $s \in (0, 2n+2)$. In particular, it is not quasiconformal.

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2 Mappings of bounded s-distortion between manifolds

In the paper [18], the authors consider a generalized version of mappings with bounded distortion.

Definition 2.1. A smooth mapping $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has bounded s-distortion $(0 < s < \infty)$ if it is a constant or a local diffeomorphism and

$$|df|^s \leq KJ(f), \quad \text{for} \quad x \in \mathbb{R}^n \tag{2.1}$$

for some positive constant K.

It is obvious that, mappings of bounded n-distortion are quasiconformal. But in general, a holomorphic mapping can be bounded s-distortion for some s but not quasiconformal.

Example 2.2. If $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, f(z_1, z_2) = (z_1, z_2^2)$ and $z_1 = x + \sqrt{-1}y, z_2 = s + \sqrt{-1}t$, then $(df)^t \cdot df$ is a 4×4 diagonal matrix with diagonal entries $1, 1, 4(s^2 + t^2), 4(s^2 + t^2)$. Therefore

$$J(f) = 4(s^2 + t^2), \quad |df| = \max\{1, 2\sqrt{s^2 + t^2}\}$$

There is no positive constant K such that

$$|df|^4 \leq KJ(f)$$

holds on $\mathbb{C}^2$. In fact, if $s^2+t^2>1$,

$$\frac{|df|^4}{J(f)} = 4(s^2+t^2)$$

which is undounded on $\mathbb{C}^2$. However, on $D = \{(z_1,z_2) \in \mathbb{C}^n \mid |z_2| > \frac{1}{2}\}$,

$$\frac{|df|^2}{J(f)} = 1$$

Hence, $f : D \longrightarrow \mathbb{C}^2, f(z_1,z_2) = (z_1, z_2^2)$ is a holomorphic mapping of bounded 2-distortion, but it is not a mapping of bounded 4-distortion.

Now we go to define mappings of bounded distortion between manifolds. Let $f : (M,g) \longrightarrow (N,h)$ be a smooth mapping between oriented n-dimensional Riemannian manifolds. In the local coordinates (x^α) and (y^i) on M and N respectively, we set

$$J(x,f) = \frac{f^* dv_h}{dv_g} = \sqrt{\frac{\det(h_{ij})(f(x))}{\det(g_{\alpha\beta})(x)}} \det\left(\frac{\partial f^i}{\partial x^\alpha}\right) \tag{2.2}$$

where $f^i = y^i \circ f$. The pointwise operator norm of df with respect to the metrics g and h is given by

$$|df(x)|^2 = \max_{X\neq 0} \frac{|f_*X|_h^2}{|X|_g^2} = \max_{X\neq 0} \frac{\sum\limits_{i,j,\alpha,\beta} h_{ij} f_\alpha^i f_\beta^j X^\alpha X^\beta}{\sum\limits_{\alpha,\beta} g_{\alpha\beta} X^\alpha X^\beta} \tag{2.3}$$

where $f_\alpha^i = \frac{\partial f^i}{\partial x^\alpha}$ and $X = X^\alpha \frac{\partial}{\partial x^\alpha}$. Here and henceforth we sometimes adopt the Einstein convention for summation. It is obvious that $|df(x)|^2$ is the maximal eigenvalue of the positive definite matrix $A = (A_{\alpha\beta})$ with respect to the metric g where $A_{\alpha\beta} = \sum\limits_{i,j} h_{ij} f_\alpha^i f_\beta^j$. So $J(x,f)$ and $|df(x)|$ are well defined and do not depend on the local coordinates.

Definition 2.3. A smooth mapping $f : (M, g) \longrightarrow (N, h)$ between oriented n-dimensional manifolds is said to have bounded s-distortion ($0 < s < \infty$) with respect to the metrics g and h if it is a constant or a local diffeomorphism with

$$|df(x)|^s \leq KJ(x, f) \tag{2.4}$$

for some positive constant K.

Now we recall some notations on Kähler manifolds. Let $\{z^\alpha\}_{\alpha=1}^n$ be the local holomorphic coordinates on the Kähler manifold (M, g), then the metric g is locally represented by a Hermitian positive matrix $(g_{\alpha\overline{\beta}})$, that is

$$g_{\alpha\overline{\beta}} = g\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right)$$

If ∇ is the complexified Levi-Civita connection on M, the curvature of ∇ is locally given by

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = -\frac{\partial^2 g_{\alpha\overline{\beta}}}{\partial z^\gamma \partial \overline{z}^\delta} + g^{\lambda\overline{\mu}} \frac{\partial g_{\alpha\overline{\mu}}}{\partial z^\gamma} \frac{\partial g_{\lambda\overline{\beta}}}{\partial \overline{z}^\delta}$$

The Ricci curvature of ∇ is

$$Ric(g) = \frac{\sqrt{-1}}{2} R_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$$

where

$$R_{\alpha\overline{\beta}} = g^{\gamma\overline{\delta}} R_{\alpha\overline{\beta}\gamma\overline{\delta}} = -\frac{\partial^2 \log\det(g_{\lambda\overline{\mu}})}{\partial z^\alpha \partial \overline{z}^\beta}$$

For more basic notations of complex geometry, we refer the reader to the book [5].

Let $f : (M, g) \longrightarrow (N, h)$ be a holomorphic mapping between Kähler manifolds with complex dimension n. In the local holomorphic coordinates w^α and z^i on M and N respectively,

$$df = f^i_\alpha dw^\alpha \otimes \frac{\partial}{\partial z^i} \in \Gamma\left(M, T^{1,0}M \otimes f^*(T^{1,0}N)\right) \tag{2.5}$$

where $f^i = z^i \circ f$ and $f^i_\alpha = \frac{\partial f^i}{\partial w^\alpha}$ since f is holomorphic. The operator norm $|df|^2$ is

$$|df|^2 = \max_{X \neq 0} \frac{|f_* X|_h^2}{|X|_g^2} = \max_{X \neq 0} \frac{\sum\limits_{i,j,\alpha,\beta} h_{i\overline{j}} f^i_\alpha \overline{f}^j_\beta X^\alpha \overline{X}^\beta}{\sum\limits_{\alpha,\beta} g_{\alpha\overline{\beta}} X^\alpha \overline{X}^\beta} \tag{2.6}$$

for any $X = X^\alpha \frac{\partial}{\partial w^\alpha} \in \Gamma(M, T^{1,0}M)$. On the other hand, the Riemannian volume and the Kähler metric is related by

$$dV = \frac{\omega^n}{n!}$$

therefore by formula (2.2),

$$J(f) = \frac{f^*\omega_h^n}{\omega_g^n} = \frac{\det(h_{i\bar{j}})}{\det(g_{\alpha\bar{\beta}})} \left|\det\left(f_\alpha^i\right)\right|^2 \tag{2.7}$$

The real dimension of the manifolds M and N is $2n$. That is, a bounded $2n$-distortion mapping is quasiconformal.

The curvatures of manifolds and $J(f)$ are related by the following formula(see [2]).

Lemma 2.4. *Let (M, g) and (N, h) be two complete Kähler manifolds of complex dimension n. Let $f : (M, g) \longrightarrow (N, h)$ be a holomorphic mapping, then at a point $J(f) \neq 0$, the following formula holds*

$$Ric(\omega_g) - f^*Ric(\omega_h) = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log J(f) \tag{2.8}$$

Proof. If φ is a holomorphic function without zero point,

$$\partial\bar{\partial}\log|\varphi|^2 = \frac{|\varphi|^2\partial\varphi\wedge\overline{\partial\varphi} - \overline{\varphi}\partial\varphi\wedge\varphi\overline{\partial\varphi}}{|\varphi|^4} = 0$$

Since $\det(f_\alpha^i)$ is holomorphic, by formula (2.7), we obtain

$$\begin{aligned}\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log J(f) &= \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\frac{\det(h_{i\bar{j}})}{\det(g_{\alpha\bar{\beta}})} \\ &= Ric(\omega_g) - f^*(Ric(\omega_h))\end{aligned}$$

□

Lemma 2.5. *Let (N, h, ω_h) be a complete Kähler manifold. If X is a holomorphic vector field of N, then on a small neighborhood of point P such that $X_P \neq 0$,*

$$T = \sqrt{-1}\left(\frac{\partial^2\log|X|_h^2}{\partial z^k\partial\bar{z}^l} + \sum_{i,j} R_{i\bar{j}k\bar{l}}\frac{X^i\overline{X}^j}{|X|_h^2}\right)dz^k\wedge d\bar{z}^\ell \tag{2.9}$$

is a semi-positive $(1,1)$ *form where* $R_{i\bar{j}k\bar{l}}$ *is the curvature of* (N,h) *given by*

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial h_{i\bar{j}}}{\partial z^k \partial \overline{z}^l} + h^{s\bar{t}} \frac{\partial h_{i\bar{t}}}{\partial z^k} \frac{\partial h_{s\bar{j}}}{\partial \overline{z}^l}$$

Proof. We can choose the normal coordinates $(z^1, \cdots, z^n)$ centered at the fixed point $P \in N$, i.e., $h_{i\bar{j}}(P) = \delta_{i\bar{j}}$ and $\frac{\partial h_{i\bar{j}}}{\partial z^k}(P) = \frac{\partial h_{i\bar{j}}}{\partial \overline{z}^l}(P) = 0$. More precisely, on the small neighborhood of P,

$$h_{i\bar{j}}(z) = \delta_{i\bar{j}} - R_{i\bar{j}k\bar{l}}(P) z^k \overline{z}^l + O(|z|^3)$$

Now we assume $X = X^i \frac{\partial}{\partial z^i}$, $|X|_h^2 = \sum_{i,j} h_{i\bar{j}} X^i \overline{X}^j$. At point P,

$$\partial\overline{\partial}|X|_h^2 = \sum_{i,j,k,l} \left(\frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \overline{z}^l} X^i \overline{X}^j + h_{i\bar{j}} \frac{\partial X^i}{\partial z^k} \overline{\frac{\partial X^j}{\partial z^l}} \right) dz^k \wedge d\overline{z}^l$$

that is,

$$\frac{\partial^2 |X|_h^2}{\partial z^k \partial \overline{z}^l} = -\sum_{i,j} R_{i\bar{j}k\bar{l}} X^i \overline{X}^j + \sum_i \frac{\partial X^i}{\partial z^k} \overline{\frac{\partial X^i}{\partial z^l}}$$

At the fixed point P,

$$\frac{\partial^2 \log |X|_h^2}{\partial z^k \partial \overline{z}^l} = \frac{-\sum\limits_{i,j} R_{i\bar{j}k\bar{l}} X^i \overline{X}^j}{|X|_h^2} + \frac{|X|_h^2 \left(\sum\limits_i \frac{\partial X^i}{\partial z^k} \overline{\frac{\partial X^i}{\partial z^l}} \right) - \left(\sum\limits_j X^j \frac{\partial \overline{X}^j}{\partial \overline{z}^l} \right) \left(\sum\limits_i \overline{X}^i \frac{\partial X^i}{\partial z^k} \right)}{|X|_h^4}$$

If we set

$$T_{k\bar{l}} = \frac{\partial^2 \log |X|_h^2}{\partial z^k \partial \overline{z}^l} + \sum_{i,j} R_{i\bar{j}k\bar{l}} \frac{X^i \overline{X}^j}{|X|_h^2}$$

then by Schwarz inequality,

$$\sum_{k,l} T_{k\bar{l}} v_k \overline{v}_l = \frac{|X|_h^2 \left(\sum\limits_i \frac{\partial X^i}{\partial z^k} v_k \overline{\frac{\partial X^i}{\partial z^l} v_l} \right) - \left(\sum\limits_j X^j \frac{\partial \overline{X}^j}{\partial \overline{z}^l} \overline{v}_l \right) \left(\sum\limits_i \overline{X}^i \frac{\partial X^i}{\partial z^k} v_k \right)}{|X|_h^4} \geq 0$$

for any $v \in \mathbb{C}^n$. □

Definition 2.6. Let (N, h) be a complete Kähler manifold. We say that (N, h) has the curvature property (Q_s) if for any $u, v \in \mathbb{C}^n - \{0\}$,

$$\frac{1}{s} \sum_{i,j,k,\ell} R_{i\bar{j}} h_{k\bar{\ell}} u^i \overline{u}^j v^k \overline{v}^\ell - \sum_{i,j,k,\ell} R_{i\bar{j}k\bar{l}} u^i \overline{u}^j v^k \overline{v}^\ell > 0 \tag{2.10}$$

for some constant $s \in (0, \infty)$.

Remark 2.7. We have a geometric explanation of the curvature formula (2.10). If E and F are two holomorphic vector bundles with connections ∇^E and ∇^F, then the curvature of the induced connection on $E \otimes F$ is

$$R = R^E \otimes Id_F + Id_E \otimes R^F$$

where R^E and R^F are the curvatures of E and F respectively. Apply this to the (formal) vector bundle

$$G = T^*N \otimes {K_N^*}^{1/s}$$

we get the curvature of it, which is locally given by

$$\widehat{R}_{i\bar{j}k\bar{l}} = \frac{1}{s} R_{i\bar{j}} h_{k\bar{l}} - R_{i\bar{j}k\bar{l}}$$

where K_N is the canonical line bundle of the manifold N. The positivity condition in the definition is nothing but the Griffiths positivity of the vector bundle G, i.e.

$$\widehat{R}_{i\bar{j}k\bar{l}} u^i \overline{u}^j v^k \overline{v}^\ell > 0$$

Example 2.8. Let $(\mathbb{P}^n, \omega_{FS})$ be the complex projective space with the Fubini-Study metric. Locally, it can be written as

$$\omega_{FS} = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\overline{z}^j$$

It is well-known that

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}, \qquad R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = (n+1) g_{i\bar{j}}$$

It is obvious that, for nonzero X,

$$\frac{1}{s} R_{i\bar{j}} |X|_g^2 - R_{i\bar{j}k\bar{l}} X^k \overline{X}^l = \frac{n+1-s}{s} |X|_g^2 g_{i\bar{j}} - g_{i\bar{l}} g_{k\bar{j}} X^k \overline{X}^l$$

Then basic linear algebra shows that the above matrix is Hermitian positive if and only if

$$\frac{n+1-s}{s} - 1 > 0 \Longleftrightarrow 0 < s < \frac{n+1}{2}$$

Corollary 2.9. $(\mathbb{P}^n, \omega)$ *has the property* (Q_s) *with* $0 < s < \frac{n+1}{2}$.

Formally, we have

$$T^*\mathbb{P}^n \otimes (K^*_{\mathbb{P}^n})^{\frac{1}{s}} = T^*\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}\left(\frac{n+1}{s}\right)$$

which is positive if and only if $\frac{n+1}{s} > 2$.

More generally, we have

Theorem 2.10. *If* M *is a compact Kähler manifold with* $c_1(M) > 0$, *then there exist a Kähler metric* ω *and some* $s_0 \in (0, \infty)$ *such that* (M, ω) *satisfies the curvature condition* (Q_s) *for any* $s \in (0, s_0)$.

Proof. By Yau's solution of Calabi conjecture([21]), if $c_1(M) > 0$, there exists a Kähler metric such that

$$Ric(\omega) > 0$$

Since the manifold is compact, there exists $\varepsilon > 0$ such that

$$Ric(\omega) \geq \varepsilon\omega$$

It is obvious that the holomorphic bisectional curvature $R_{i\bar{j}k\bar{l}}$ is also bounded, that is, there exists $\varepsilon_1 > 0$ such that

$$R_{i\bar{j}k\bar{l}} X^k \overline{X}^l \xi^i \overline{\xi}^j \leq \varepsilon_1 |X|^2_\omega |\xi|^2_\omega$$

Hence, there exists $s_0 \in (0, \infty)$ such that

$$\frac{|X|^2_\omega}{s_0} R_{i\bar{j}} \xi^i \overline{\xi}^j - R_{i\bar{j}k\bar{l}} X^k \overline{X}^l \xi^i \overline{\xi}^j \geq \frac{\varepsilon - s_0\varepsilon_1}{s_0} |X|^2_\omega |\xi|^2_\omega > 0$$

for any nonzero $X = \sum X^i \frac{\partial}{\partial z^i}$ and $\xi = \sum \xi^j \frac{\partial}{\partial z^j}$. The constant s_0 depends on the manifold M and ω. □

Theorem 2.11. *Let* (N, h) *be a complete Kähler manifold of complex dimension* n, *and* $f : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \longrightarrow (N, h)$ *be holomorphic. If* f *is a mapping with bounded* $2s$*-distortion and* N *has the property* (Q_s), *then* f *is constant.*

Proof. Let ω_h be the corresponding Kähler form of h. On the local holomorphic coordinates $(z^1, \cdots, z^n)$ of N, we can write

$$\omega_h = \frac{\sqrt{-1}}{2}\sum_{i,j} h_{i\bar{j}} dz^i \wedge d\overline{z}^j$$

If f is not constant, then for any nontrivial constant holomorphic vector field $Y = Y^\alpha \frac{\partial}{\partial w^\alpha}$ on $\mathbb{C}^n$, $|f_*Y|_h^2$ is nonzero everywhere by the local diffeomorphism property of mappings of bounded distortion. We claim that

$$L = \frac{\sqrt{-1}}{2} L_{\gamma\bar{\delta}} dw^\gamma \wedge d\overline{w}^\delta := \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log|f_*Y|^2 + \frac{1}{s} f^*(Ric(\omega_h)) \tag{2.11}$$

is a semi-positive $(1,1)$-form on $\mathbb{C}^n$ if N has curvature property (Q_s).

In fact, we have

$$f_*Y = \sum_{i,\alpha} \frac{\partial f^i}{\partial w^\alpha} Y^\alpha \frac{\partial}{\partial z^i}, \qquad |f_*Y|_h^2 = \sum_{i,j,\alpha,\beta} h_{i\bar{j}}(f) \frac{\partial f^i}{\partial w^\alpha}\overline{\frac{\partial f^j}{\partial w^\beta}} Y^\alpha \overline{Y}^\beta$$

By a similar computation as Lemma 2.5,

$$\frac{\partial^2 \log|f_*Y|^2}{\partial w^\alpha \partial \overline{w}^\beta} = -R_{i\bar{j}k\bar{\ell}} \cdot \frac{\partial f^k}{\partial w^\gamma}\overline{\frac{\partial f^\ell}{\partial w^\delta}} \cdot \frac{\frac{\partial f^i}{\partial w^\alpha}\overline{\frac{\partial f^j}{\partial w^\beta}} Y^\alpha \overline{Y}^\beta}{|f_*Y|_h^2} + W_{\gamma\bar{\delta}} \tag{2.12}$$

where $\left(W_{\gamma\bar{\delta}}\right)$ is a semi-positive Hermitian matrix by Schwarz inequality. In the sense of Hermitian positivity, we obtain

$$L_{\gamma\bar{\delta}} \geq \frac{1}{s} R_{i\bar{j}} \frac{\partial f^i}{\partial w^\gamma}\overline{\frac{\partial f^j}{\partial w^\delta}} - R_{i\bar{j}k\bar{\ell}} \frac{\partial f^k}{\partial w^\gamma}\overline{\frac{\partial f^\ell}{\partial w^\delta}} \frac{\frac{\partial f^i}{\partial w^\alpha}\overline{\frac{\partial f^j}{\partial w^\beta}} Y^\alpha \overline{Y}^\beta}{|f_*Y|_h^2} \geq 0 \tag{2.13}$$

where the last step follows by the the curvature condition (Q_s).

On the other hand, by Lemma 2.4,

$$\frac{1}{s} f^*(Ric(\omega_h)) = \frac{1}{s} Ric(\omega_{\mathbb{C}^n}) - \frac{\sqrt{-1}}{2s}\partial\overline{\partial}\log J(f) = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log J(f)^{\frac{1}{s}} \tag{2.14}$$

By formula (2.11), we obtain

$$L = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\left(\frac{|f_*Y|_h^2}{J(f)^{\frac{1}{s}}}\right) \geq 0 \tag{2.15}$$

The weight function

$$\Phi = \frac{|f_*Y|_h^2}{J(f)^{\frac{1}{s}}} \tag{2.16}$$

is plurisubharmonic on $\mathbb{C}^n$. By the definition (2.6) of $|df|$,

$$|f_*Y|_h^2 \leq |df|^2|Y|_g^2 \tag{2.17}$$

Then we get

$$\Phi \leq \frac{|df|^2|Y|_g^2}{J^{\frac{1}{s}}(f)} \leq K^{\frac{1}{s}}|Y|_g^2 = K^{\frac{1}{s}}\sum_\alpha |Y^\alpha|^2$$

for some positive constant K by the definition of bounded $2s$-distortion mapping. Since Y is a constant vector field on $\mathbb{C}^n$, we obtain that Φ is a plurisubharmonic function bounded from above and so Φ is constant on $\mathbb{C}^n$. That is $L = 0$. By formula (2.13), we know

$$\frac{\partial f^i}{\partial w^\gamma} = 0$$

for any i and γ. Finally, we obtain f is anti-holomorphic, and so it is constant. □

Corollary 2.12. *Let M be a compact Kähler manifold with $c_1(M) > 0$. Then there exist a Kähler metric ω and some $s_0 \in (0, \infty)$ such that any holomorphic mapping $f : \mathbb{C}^n \longrightarrow (M, \omega)$ with bounded s-distortion, $s \in (0, s_0)$, is constant.*

In particular, for $\mathbb{P}^n$ we obtain

Corollary 2.13. *If $f : \mathbb{C}^n \longrightarrow \mathbb{P}^n$ is a holomorphic mapping with bounded s-distortion, $0 < s < n + 1$, with respect to the canonical metrics, then f is a constant.*

Proof. It follows easily from Corollary 2.9. □

Example 2.14. Let $f : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \longrightarrow (\mathbb{P}^n, \omega_{FS})$ be the canonical map

$$f(z_1, \cdots, z_n) = [1, z_1, \cdots, z_n]$$

Then f is a holomorphic mapping with bounded $(2n + 2)$-distortion and it fails to be a mapping of bounded s-distortion for any $s \in (0, 2n + 2)$. In particular, it is not quasiconformal.

In fact, on the chart $U_0 = \{[z_0, \cdots, z_n] \mid z_0 = 1\}$,

$$\omega_{FS} = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\left(1+\sum_{i=1}^{n}|z_i|^2\right) = \frac{\sqrt{-1}}{2}\cdot h_{i\bar{j}}\cdot dz_i\wedge d\overline{z}_j \tag{2.18}$$

where

$$h_{i\bar{j}} = \frac{(1+\sum|z_i|^2)\delta_{ij} - \overline{z}_i z_j}{(1+\sum|z_i|^2)^2}$$

It is obvious that $(h_{i\bar{j}})$ has two different eigenvalues, $\lambda_{max} = (1+\sum|z_i|^2)^{-1}$ with multiplicity $(n-1)$ and $\lambda_{min} = (1+\sum|z_i|^2)^{-2}$ with multiplicity 1. Therefore,

$$\det(h_{i\bar{j}}) = \left(1+\sum_{i=1}^{n}|z_i|^2\right)^{-(n+1)} \tag{2.19}$$

By formula (2.7),

$$J(f) = \frac{f^*(\omega_{FS}^n)}{\omega_{\mathbb{C}^n}^n} = \left(1+\sum_{i=1}^{n}|z_i|^2\right)^{-(n+1)} \tag{2.20}$$

On the other hand, on the chart U_0, $f(z_1,\cdots,z_n) = (z_1,\cdots,z_n)$ and $df = I$. By formula (2.6), $|df|^2$ is the maximal eigenvalue $\lambda_{max} = (1+\sum_{i=1}^{n}|z_i|)^{-1}$ of the Hermitian positive matrix $(h_{i\bar{j}})$. Therefore

$$|df| = \left(1+\sum_{i=1}^{n}|z_i^2|\right)^{-\frac{1}{2}} \tag{2.21}$$

By the expression of $J(f)$ and $|df|$, we obtain

$$|df|^{2(n+1)} = J(f) \tag{2.22}$$

For any $0 < s < 2(n+1)$, the ratio

$$\frac{|df|^s}{J(f)} = \left(1+\sum_{i=1}^{n}|z_i|\right)^{n+1-\frac{s}{2}} \tag{2.23}$$

is unbounded on $\mathbb{C}^n$. In summary, $f : \mathbb{C}^n \longrightarrow \mathbb{P}^n$ is a holomorphic mapping of bounded $(2n+2)$-distortion and it fails to be a mapping of bounded s-distortion for any $s \in (0, 2n+2)$.

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