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Global Classical Solutions of Initial-boundary Value Problem for the Equations of Time-like Extremal Surfaces in the Minkowski Space

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Abstract: In this paper, we consider the global existence of classical solutions of the mixed initial-boundary value problem for the equations of time-like extremal surfaces in the $(1+n)$-dimensional Minkowski space. Under some suitable assumptions, we prove the global existence and uniqueness of the C^2 solution to this kind of problem.
Keywords: initial-boundary value problem, time-like extremal surface, global classical solution.

1. Introduction

Let $(t, x, y_1, \cdots, y_n)$ be a point in the $(1 + (1 + n))$-dimensional Minkowski space. Consider a time-like surface taking the form

$$y = \phi(t, x), \tag{1}$$

where $y = (y_1, \cdots, y_n)^T$ and $\phi = (\phi_1, \cdots, \phi_n)^T$. This surface is called to be an *extremal surface* if ϕ is the critical point of the following area functional

$$I = \iint \sqrt{1 - |\phi_t|^2 + |\phi_x|^2 - |\phi_t|^2|\phi_x|^2 + \langle \phi_t, \phi_x \rangle^2} \mathrm{d}x \, \mathrm{d}t, \tag{2}$$

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where $\langle\cdot,\cdot\rangle$ stands for the inner product. The corresponding Euler-Lagrange equation is (see Kong, Sun and Zhou [8])

$$\Big(\frac{(1+|\phi_x|^2)\phi_t-\langle\phi_t,\phi_x\rangle\phi_x}{\sqrt{1-|\phi_t|^2+|\phi_x|^2-|\phi_t|^2|\phi_x|^2+\langle\phi_t,\phi_x\rangle^2}}\Big)_t-\Big(\frac{(1-|\phi_t|^2)\phi_x+\langle\phi_t,\phi_x\rangle\phi_t}{\sqrt{1-|\phi_t|^2+|\phi_x|^2-|\phi_t|^2|\phi_x|^2+\langle\phi_t,\phi_x\rangle^2}}\Big)_x=0. \tag{3}$$

If

$$1-|\phi_t|^2+|\phi_x|^2-|\phi_t|^2|\phi_x|^2+\langle\phi_t,\phi_x\rangle^2>0,$$

the system (3) is the equation for time-like extremal surfaces in the Minkowski space $\mathbb{R}^{1+(1+n)}$. It is an interesting model in Lorentzian geometry. It also arises in some physical contexts and has been investigated by several authors (e.g., [6]-[11]). Kong et al investigated the Cauchy problem for the equations of time-like extremal surfaces in the Minkowski space $\mathbb{R}^{1+n}$, which corresponds to the motion of an open string in $\mathbb{R}^{1+n}$ (see [7]-[8]). Kong and Zhang [10]-[11] study the motion of relativistic (in particular, closed) strings moving in the Minkowski space $\mathbb{R}^{1+n}$ and show an interesting and important nonlinear phenomenon: the space-periodicity implies that time-periodicity in the motion of relativistic closed string in $\mathbb{R}^{1+n}$. Recently, Huang and Kong [2] investigate the equations for the motion of relativistic torus in the Minkowski space $\mathbb{R}^{1+n}$, and obtain some interesting results.

The mixed initial-boundary value problem for the equation (3) plays an important role in electrodynamics and particle physics (see [1]). Recently, Liu and Zhou [13] have investigated the initial-boundary value problem for the equations of the time-like extremal surfaces in the Minkowski space. Based on a result in Li and Peng [12], under some small assumptions they prove the global existence and uniqueness of the C^2 solution of this kind problem. However, for the mixed initial-boundary value problem with two boundaries case, the assumptions in [13] are very strong and not easy to apply, and it seems to me that the result in Li and Peng [12] does not directly work in this case. In this paper, we shall weaken these assumptions in [13], improve the proof of the global existence of the solutions for the mixed initial-boundary value problem with two boundaries conditions and show the global existence for the problem with one boundary condition in a different way.

This paper is organized as follows. In Section 2, we state the main results. Section 3 is devoted to prove Theorems 2.1-2.2 for the problem with two boundary conditions. Theorems 2.3-2.4 for the problem with one boundary condition are proved in Section 4.

2. Statement of Main Results

Following Kong et al [8], let

$$u = \phi_x, \qquad v = \phi_t. \tag{4}$$

Then the system (3) can be reduced to

$$\begin{cases} u_t - v_x = 0, \\ v_t - \dfrac{2\langle u, v\rangle}{1+|u|^2} v_x - \dfrac{1-|v|^2}{1+|u|^2} u_x = 0. \end{cases} \tag{5}$$

The above system has two district eigenvalues with constant multiplicity n, denoted by

$$\lambda_\pm(u,v) = \frac{1}{1+|u|^2}\big(-\langle u, v\rangle \pm \sqrt{\triangle(u,v)}\big), \tag{6}$$

where

$$\triangle(u,v) = 1 - |v|^2 + |u|^2 - |u|^2|v|^2 + \langle u, v\rangle^2 > 0. \tag{7}$$

As in [8], introducing

$$R_i = v_i + \lambda_- u_i, \qquad S_i = v_i + \lambda_+ u_i \qquad (i = 1, \cdots, n), \tag{8}$$

by a direct computation, we have (see [8])

$$\begin{cases} \partial_t \lambda_+ + \lambda_- \partial_x \lambda_+ = 0, \\ \partial_t \lambda_- + \lambda_+ \partial_x \lambda_- = 0, \\ \partial_t S_i + \lambda_- \partial_x S_i = 0, \\ \partial_t R_i + \lambda_+ \partial_x R_i = 0. \end{cases} \tag{9}$$

In this paper, we consider the global existence of classical solutions of the mixed initial-boundary value problem for the system (3) in time-like case with the initial condition

$$t = 0: \qquad \phi = f(x), \qquad \phi_t = g(x), \tag{10}$$

where f is a vector-valued C^2 function and g is a vector-valued C^1 function. In terms of f and g, the initial date for $\lambda_\pm$ and $R_i,\ S_i\ (i=1,\cdots,n)$ is given by

$$t=0: \qquad \begin{aligned} &\lambda_+=\Lambda_+(x), \qquad \lambda_-=\Lambda_-(x),\\ &R_i=R_i^0(x), \qquad S_i=S_i^0(x) \quad (i=1,\cdots,n), \end{aligned} \tag{11}$$

where

$$\Lambda_\pm(x)=(1+|f'|^2)^{-1}\big(-\langle f',g\rangle\pm\sqrt{1-|g|^2+|f'|^2-|f'|^2|g|^2-\langle f',g\rangle^2}\big) \tag{12}$$

and

$$R_i^0(x)=g_i(x)+\Lambda_-(x)f_i'(x), \qquad S_i^0(x)=g_i(x)+\Lambda_+(x)f_i'(x). \tag{13}$$

In what follows, we state our main results in this paper.

2.1. **Two Boundaries Case.** We first consider the global existence of classical solutions of the mixed initial-boundary value problem for the system (3) in time-like case on the strip domain

$$D=\{(t,x)|\ t\geq 0,\ 0\leq x\leq L\}$$

with the initial data (10) and

(i) the Neumann boundary conditions

$$\begin{aligned} x=0: &\qquad \phi_x(t,0)=h_1(t),\\ x=L: &\qquad \phi_x(t,L)=h_2(t), \end{aligned} \tag{14}$$

where h_1 and h_2 are two vector-valued C^1 functions, or

(ii) the Dirichlet boundary conditions

$$\begin{aligned} x=0: &\qquad \phi(t,0)=H_1(t),\\ x=L: &\qquad \phi(t,L)=H_2(t), \end{aligned} \tag{15}$$

where H_1 and H_2 are two vector-valued C^2 functions.

For the mixed initial-boundary value problem for the system (3) with the initial condition (10) and the Neumann boundary condition (14), we suppose that the following compatibility conditions are satisfied at point $(0,0)$ and $(0,L)$,

$$\begin{cases} f'(0)=h_1(0), \qquad h_1'(0)=g'(0),\\ f'(L)=h_2(0), \qquad h_2'(0)=g'(L). \end{cases} \tag{16}$$

Thus by (4), the initial-boundary value problem (3), (10) and (14) can be rewritten as

$$\begin{cases} u_t - v_x = 0, \\ v_t - \dfrac{2\langle u, v\rangle}{1+|u|^2} v_x - \dfrac{1-|v|^2}{1+|u|^2} u_x = 0, \\ t = 0: \qquad u = f'(x), \qquad v = g(x), \\ x = 0: \qquad u = h_1(t), \\ x = L: \qquad u = h_2(t). \end{cases} \tag{17}$$

Throughout this paper, for the case of the strip domain D, we always suppose that the initial data satisfies

$$-1 \le \sup_{x\in[0,L]} \Lambda_-(x) \le -a < 0 < a \le \inf_{x\in[0,L]} \Lambda_+(x) \le 1, \tag{18}$$

where a is positive constant.

Let $F(t)$ be a positive function satisfying

$$F(t) > 0 \quad \text{is decreasing on } [0, +\infty), \quad \textit{and} \quad F(0) + \frac{\int_0^{+\infty} F(t)dt}{L} \le \frac{a}{4}. \tag{19}$$

If the Neumann boundary data (14) satisfies

$$|h_1(t)| \le F(t), \quad |h_2(t)| \le F(t), \quad \forall\, t \in [0, +\infty), \tag{20}$$

then in Section 3 we shall prove the following global existence result on the classical solutions of the initial-boundary value problem (3), (10) and (14).

Theorem 2.1. *Suppose that the initial data* (10) *and the Neumann boundary* (14) *satisfy* (18), (20) *and the* C^2 *compatibility* (16)*, then the initial-boundary value problem* (3), (10) *and* (14) *admits a unique global* C^2 *solution* $\phi = \phi(t, x)$ *on the strip domain* D.

If the Dirichlet boundary condition (15) satisfies

$$|H_1'(t)| \le 2F(t), \quad |H_2'(t)| \le 2F(t), \quad \forall\, t \in [0, +\infty), \tag{21}$$

and the following C^2 compatibility conditions,

$$\begin{cases} f(0) = H_1(0), \qquad H_1'(0) = g(0), \\ H_1''(0) - \dfrac{2f'(0)\cdot g(0)}{1+f'^2(0)}g'(0) - \dfrac{1-g^2(0)}{1+f'^2(0)}f''(0) = 0, \\ f(L) = h_2(0), \qquad H_1'(0) = g(L), \\ H_2''(0) - \dfrac{2f'(L)\cdot g(L)}{1+f'^2(L)}g'(L) - \dfrac{1-g^2(L)}{1+f'^2(L)}f''(L) = 0, \end{cases} \tag{22}$$

then in Section 3 we can prove the following global existence result on the strip domain D.

Theorem 2.2. *Suppose that the initial data* (10) *and the Dirichlet boundary* (15) *satisfy* (18), (21) *and the conditions of C^2 compatibility* (22), *then the initial-boundary value problem* (3), (10) *and* (15) *admits a unique global C^2 solution $\phi = \phi(t,x)$ on the strip domain D.*

Remark 2.1. *If the boundary conditions of the system* (3) *on $x = 0$ and $x = L$ are not of the same type, for example, one is Neumann boundary condition and the other is Dirichlet boundary condition, then under some assumptions similar to that in Theorems 2.1-2.2, we can prove the global existence of classical solutions of the system* (3) *similarly.*

Remark 2.2. *The condition* (18) *is necessary. Otherwise, the solution may blow up in finite time (see* [9]*). For the details on blowup phenomena, we refer to Kong* [3]-[5] .

2.2. **One Boundary Case.** We next consider the global existence of classical solutions of the mixed initial-boundary value problem for the system (3) in time-like case on the domain

$$\Omega = \{(t,x)|\ t \geq 0,\ x \geq 0\}$$

with the initial data (10) and

(i) the Neumann boundary condition

$$x = 0: \qquad \phi_x(t,0) = h(t), \tag{23}$$

where h is a vector-valued C^1 function, or

(ii) the Dirichlet boundary condition

$$x = 0: \qquad \phi(t, 0) = H(t), \tag{24}$$

where H is a vector-valued C^2 function.

For the initial condition (10) and the Neumann boundary condition (23), we suppose that the following compatibility conditions are satisfied at point $(0, 0)$,

$$f'(0) = h(0), \qquad h'(0) = g'(0). \tag{25}$$

Thus by (4), the initial-boundary value problem (3), (10) and (23) can be rewritten as

$$\begin{cases} u_t - v_x = 0, \\ v_t - \dfrac{2\langle u, v\rangle}{1+|u|^2} v_x - \dfrac{1-|v|^2}{1+|u|^2} u_x = 0, \\ t = 0: \qquad u = f'(x), \qquad v = g(x), \\ x = 0: \qquad u = h(t). \end{cases} \tag{26}$$

Throughout this paper, for the domain Ω we suppose that the initial data satisfies

$$\begin{cases} \sup\limits_{x\in\mathbb{R}^+} \Lambda_-(x) \le -a < 0 < b \le \inf\limits_{x\in\mathbb{R}^+} \Lambda_+(x), \\ M \triangleq \sup\limits_{x\in\mathbb{R}^+} \Big\{ |f'(x)| + |g(x)| \Big\} < \infty, \\ M' \triangleq \sup\limits_{x\in\mathbb{R}^+} \Big\{ |f''(x)| + |g'(x)| \Big\} < \infty, \end{cases} \tag{27}$$

where a and b are two positive constants. Without loss of generality, we assume $a < b$ (Otherwise, we can always replace a smaller number a').

If the Neumann boundary condition (23) satisfies

$$|h(t)| \le \frac{b-a}{2}, \tag{28}$$

then we shall prove the following global existence result in Section 4.

Theorem 2.3. *Suppose that the initial data* (10) *and the Neumann boundary condition* (23) *satisfy* (27), (28) *and the* C^2 *compatibility* (25), *then the initial-boundary value problem* (3), (10) *and* (23) *admits a unique global* C^2 *solution* $\phi = \phi(t, x)$ *on the domain* Ω.

If the Dirichlet boundary condition (24) satisfies

$$|H'(t)| \le b - a \tag{29}$$

and the following C^2 compatibility conditions,

$$\begin{cases} f(0) = H(0), \qquad H'(0) = g(0), \\ H''(0) - \dfrac{2f'(0)\cdot g(0)}{1+f'^2(0)}g'(0) - \dfrac{1-g^2(0)}{1+f'^2(0)}f''(0) = 0, \end{cases} \tag{30}$$

then in Section 4 we can prove the following global existence result on the domain Ω.

Theorem 2.4. *Suppose that the initial data* (10) *and the Dirichlet boundary condition* (24) *satisfy* (27), (29) *and the conditions of C^2 compatibility* (30), *then the initial-boundary value problem* (3), (10) *and* (24) *admits a unique global C^2 solution $\phi = \phi(t,x)$ on the domain Ω.*

Remark 2.3. *As shown in Remark 2.2, the first inequality in* (27) *is necessary.*

3. Proof of Theorems 2.1-2.2

To prove Theorem 2.1, we need the following Lemmas.

Lemma 3.1. *Under the assumptions* (16), (18) *and* (20), *the following Cauchy problem*

$$\begin{cases} \partial_t \lambda_+ + \lambda_- \partial_x \lambda_+ = 0, \\ \partial_t \lambda_- + \lambda_+ \partial_x \lambda_- = 0, \\ t = 0: \quad \lambda_+ = \Lambda_+(x), \quad \lambda_- = \Lambda_-(x) \end{cases} \tag{31}$$

has a unique global smooth solution $\lambda = \lambda_\pm(t,x)$ on the strip domain D. Furthermore, on D it holds that

$$-1 \le \lambda_-(t,x) \le -\frac{a}{2} < 0 < \frac{a}{2} \le \lambda_+(t,x) \le 1. \tag{32}$$

Proof. The global existence and uniqueness of the smooth solution to the Cauchy problem (31) comes from Kong, Sun and Zhou [8] (see Property 2.1 in [8]). Moreover, it holds that

$$-1 < \lambda_+(t,x) \le 1, \qquad -1 \le \lambda_-(t,x) < 1 \tag{33}$$

(see Property 2.2 in [8]).

Denoting

$$D_i \triangleq \Big\{(t,x) \mid iL \le t \le (i+1)L, \quad 0 \le x \le L\Big\} \qquad (i = 0, 1, \cdots), \tag{34}$$

we prove this Lemma on every D_i.

We claim that

$$-1 \le \lambda_-(t,x) < 0 < \lambda_+(t,x) \le 1, \quad \forall\ (t,x) \in D. \tag{35}$$

We prove (35) by contradiction.

Suppose that (35) is not true, then

$$T_0 \triangleq \inf\Big\{ t \in (0,+\infty) \ \Big|\ \text{there is a } x_0 \in [0,L] \text{ such that } \lambda_-(t,x_0) = 0 \quad or \quad \lambda_+(t,x_0) = 0 \Big\} > 0. \tag{36}$$

For any constant $\varepsilon > 0$, we have by continuity

$$-1 \le \lambda_-(t,x) < 0 < \lambda_+(t,x) \le 1, \quad \forall\ (t,x) \in D_\varepsilon,$$

where $D_\varepsilon \triangleq \Big\{(t,x) \mid 0 \le t \le T_0 - \varepsilon,\ 0 \le x \le L\Big\}$. Therefore for $t \le T_0 - \varepsilon$, noting (6), we have

$$\sqrt{\triangle(u,v)} > |\langle u,v\rangle|.$$

It follows that

$$1 - |v|^2 + |u|^2 - |u|^2|v|^2 = (1-|v|^2)(1+|u|^2) > 0,$$

that is

$$|v| < 1. \tag{37}$$

For any fixed $(t,x) \in D_0 \cap D_\varepsilon$, we draw the forward characteristic. According to (33), there are only the following two possibilities shown in Figure 1.

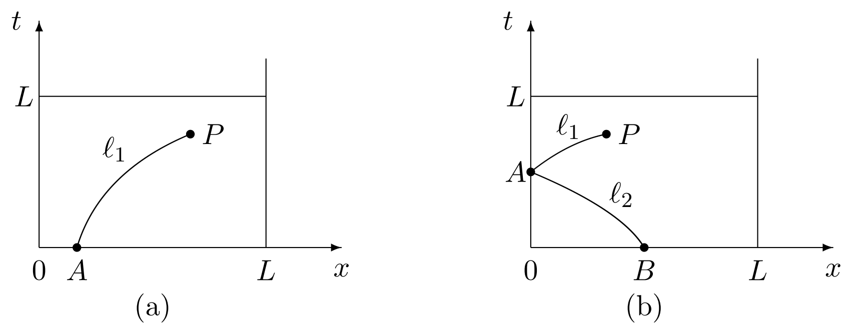

Figure 1: Forward characteristic passing through the point P in D_0.

Case 1. The forward characteristic $\ell_1 : x = x_1(t,x)$ intersects x-axis at a point $A(0,\alpha)$ (see Figure 1(a)), where ℓ_1 is defined by

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \lambda_+(t, x_1(t,\alpha)), \\ x_1(0,\alpha) = \alpha. \end{cases}$$

By the second equation in (31), $\lambda_-(t,x)$ is constant along ℓ_1, i.e.,

$$\lambda_-(P) = \lambda_-(A).$$

It follows from (18) that

$$-1 \le \lambda_-(P) \le -a. \tag{38}$$

Case 2. The forward characteristic $\ell_1 : x = x_1(t,x)$ intersects t-axis at a point $A(\gamma,0)$ and the backward characteristic $\ell_2 : x = x_2(t,x)$ passing through the point A intersects x-axis at a point $B(0,\beta)$ (see Figure 1(b)), where ℓ_2 is defined by

$$\begin{cases} \dfrac{dx_2(t)}{dt} = \lambda_-(t, x_2(t,\beta)), \\ x_2(0,\beta) = \beta. \end{cases}$$

Then we have

$$\begin{cases} \lambda_-(P) = \lambda_-(A), \\ \lambda_+(A) = \lambda_+(B). \end{cases}$$

Noting (6) and (37), we have

$$\begin{aligned} \lambda_-(P) + \lambda_+(B) &= \lambda_-(A) + \lambda_+(A) \\ &= -\frac{2\langle h_1, v\rangle}{1+|h_1|^2}(0,\gamma) \le \frac{2|h_1|}{1+|h_1|^2}(\gamma). \end{aligned}$$

It follows that

$$\lambda_-(P) \le -\lambda_+(B) + \frac{2|h_1|}{1+|h_1|^2}(\gamma) \le -\left(a - \frac{2|h_1|}{1+|h_1|^2}(\gamma)\right). \tag{39}$$

Combining Case 1 and Case 2 gives

$$\begin{aligned} &-1 \le \lambda_-(t,x) \le -\left(a - \frac{2|h_1|}{1+|h_1|^2}(t_0)\right), \\ &\qquad \forall\ (t,x) \in D_0 \cap D_\varepsilon, \quad \exists\ t_0 \in \big[\ 0,\ \min\{L,\ T_0 - \varepsilon\}\big]. \end{aligned} \tag{40}$$

In the similar way, we can get

$$a - \frac{2|h_2|}{1+|h_2|^2}(t_0) \le \lambda_+(t,x) \le 1,$$
$$\forall\ (t,x) \in D_0 \cap D_\varepsilon, \quad \exists\ t_0 \in \big[\ 0,\ \ \min\{L,\ T_0-\varepsilon\}\big]. \qquad (41)$$

Taking $\lambda_\pm(L,x)$ as the new initial data on $t = L$ if $L < T_0 - \varepsilon$ and repeating the previous procedure, then in $D_1 \cap D_\varepsilon$ we have

$$-1 \le \lambda_-(t,x) \le -\left(a - \sum_{i=0}^{1} \frac{2|h|}{1+|h|^2}(t_i)\right) < 0 < a - \sum_{i=0}^{1} \frac{2|h|}{1+|h|^2}(t_i) \le \lambda_+(t,x) \le 1,$$
$$\forall\ (t,x) \in D_1 \cap D_\varepsilon, \quad \exists\ t_i \in \big[iL,\ \ \min\{(i+1)L,\ T_0-\varepsilon\}\big]\ (i=0,1), \qquad (42)$$

where $h(t) = \max\big\{h_1(t), h_2(t)\big\}$.

Repeating this procedure at most $N = \left[\dfrac{T_0-\varepsilon}{L}\right] + 1$ times, we get

$$\begin{aligned}
&-1 \le \lambda_-(t,x) \le -\left(a - \sum_{i=0}^{n} \frac{2|h|}{1+|h|^2}(t_i)\right) \le -\left(a - \sum_{i=0}^{n} 2|h|(t_i)\right),\\
&1 \ge \lambda_+(t,x) \ge a - \sum_{i=0}^{n} \frac{2|h|}{1+|h|^2}(t_i) \ge a - \sum_{i=0}^{n} 2|h|(t_i),\\
&\forall\ (t,x) \in D_n \cap D_\varepsilon\ (0 \le n \le N-1),\\
&\exists\ t_i \in \big[iL,\ \ \min\{(i+1)L,\ T_0-\varepsilon\}\big]\ (i=0,1,\cdots,n).
\end{aligned} \qquad (43)$$

Noting (20), we have (see Figure 2)

$$\begin{aligned}
\sum_{i=0}^{\infty} |h|(t_i) &\le \sum_{i=0}^{\infty} \sup_{t\in[iL,\ (i+1)L]} |h|(t) \le \sum_{i=0}^{\infty} \sup_{t\in[iL,\ (i+1)L]} F(t)\\
&= \sum_{i=0}^{\infty} F(iL) = \frac{\mathcal{A}}{L}\\
&\le \frac{F(0)\cdot L + \int_0^{+\infty} F(t)dt}{L}\\
&= F(0) + \frac{\int_0^{+\infty} F(t)dt}{L},
\end{aligned}$$

where $\mathcal{A}$ denotes the area of the shaded part in Figure 2.

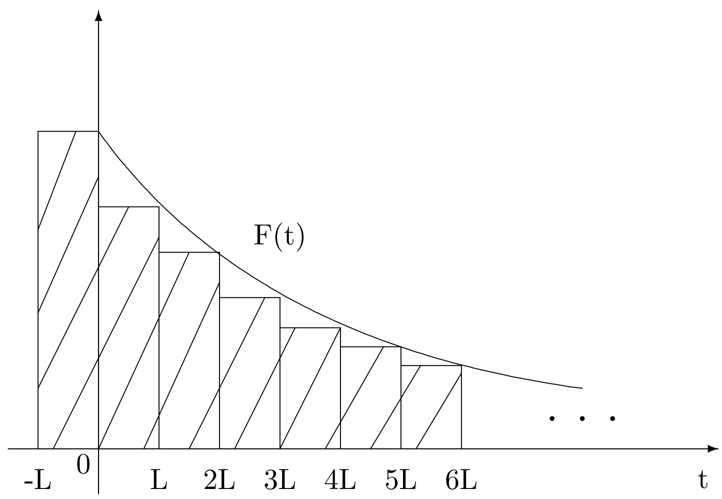

Figure 2: the figure of curve $F(t)$ and series $F(iL)$ $(i = 0, 1, \cdots)$.

Therefore by (19), it follows from (43) that

$$\begin{aligned} &-1 \le \lambda_-(t,x) \le -(a - \frac{a}{2}) = -\frac{a}{2} < 0, \\ &1 \ge \lambda_+(t,x) \ge a - \frac{a}{2} = \frac{a}{2} > 0, \quad \forall\ (t,x) \in [0,\ T_0 - \varepsilon] \times [0,\ L]. \end{aligned} \tag{44}$$

By the randomicity of ε and the continuity of $\lambda_\pm(t,x)$, let $\varepsilon \to 0$, we can get

$$-1 \le \lambda_-(T_0,x) < 0 < \lambda_+(T_0,x) \le 1, \quad \forall\ x \in [0,\ L]. \tag{45}$$

This is a contradiction with (36), so (35) holds.

(32) can be proved similar to (44). Thus the proof of Lemma 3.1 is completed. ■

For the following Lemmas 3.2-3.3, we denote

$$\begin{cases} M_0 \triangleq \sup\limits_{x\in[0,\ L]} \Big\{|f'(x)| + |g(x)|\Big\}, \\ M_0' \triangleq \sup\limits_{x\in[0,\ L]} \Big\{|f''(x)| + |g'(x)|\Big\}. \end{cases} \tag{46}$$

Noting (20) and $f', g \in C^1[0,\ L]$, we have

$$M_0 < \infty, \qquad M_0' < \infty.$$

Lemma 3.2. *Assume that R_i and $S_i (i = 1, \cdots, n)$ satisfy* (9) *and* (11)*, then*

$$\max_{i=1,\cdots,n} \Big\{ |R_i(t,x)|, \quad |S_i(t,x)| \Big\} \le C_0, \quad \forall (t,x) \in D, \tag{47}$$

where C_0 is a positive constant only depending on a and M_0.

Proof. By (13) and (33), we can use the initial data and boundary condition to estimate R_i and S_i $(i = 1, \cdots, n)$.

For any fixed point $P : (t,x) \in D$, we draw the forward characteristic. According to (33), there are only the following two possibilities shown in Figure 3.

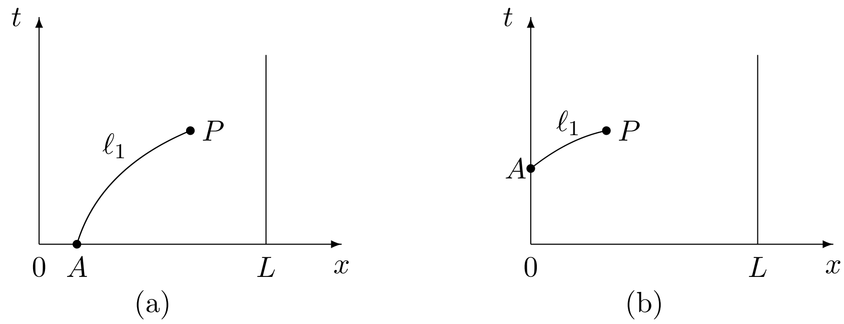

Figure 3: Forward characteristic passing through the point P in D.

Case 1. The forward characteristic $\ell_1 : x = x_1(t,x)$ intersects x-axis at a point $A(0,\alpha)$ (see Figure 3(a)),where ℓ_1 satisfies

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \lambda_+(t, x_1(t,\alpha)), \\ x_1(0,\alpha) = \alpha. \end{cases}$$

By the last equation in the system (9), $R_i(t,x)$ is constant along ℓ_1, i.e.,

$$R_i(P) = R_i^0(A) \quad (i = 1, \cdots, n).$$

By (13) and (33), we have

$$|R_i^0(A)| \le |g_i(A)| + |f_i'(A)| \le M_0 \quad (i = 1, \cdots, n).$$

It yields

$$|R_i(P)| \le M_0 \quad (i = 1, \cdots, n). \tag{48}$$

Case 2. The forward characteristic $\ell_1 : x = x_1(t,x)$ intersects t-axis at a point $A(\gamma, 0)$ (see Figure 3(b)).

Then by (8), (33) and (37), we get

$$\begin{aligned} |R_i(P)| = |R_i(A)| &\le |v_i(A)| + |\lambda_-||u_i(A)| \\ &\le 1 + |h_1(A)| \\ &\le 1 + \frac{a}{4} \quad (i = 1, \cdots, n). \end{aligned} \tag{49}$$

Combining (48) and (49) gives

$$|R_i(t,x)| \le K_1 \qquad (i = 1, \cdots, n), \tag{50}$$

where K_1 is a positive constant only depending on a and M_0.

By the same way, we can obtain

$$|S_i(t,x)| \le K_2 \qquad (i = 1, \cdots, n), \tag{51}$$

where K_2 is a positive constant only depending on a and M_0. Thus the proof of Lemma 3.2 is completed. ■

Denote

$$N_0(T) \triangleq \sup_{t\in[0,T]} \Big\{ |h_1'(t)| + |h_2'(t)| \Big\}. \tag{52}$$

Since $h_1(t),\ h_2(t) \in C^1(\mathbb{R}^+)$, for any given T we have

$$N_0(T) < \infty.$$

Next, we estimate the C^1 norm of $\lambda_\pm$, R_i and S_i $(i = 1, \cdots, n)$.

Lemma 3.3. *Assume that $\lambda_\pm$, R_i and S_i $(i = 1, \cdots, n)$ satisfy* (9) *and* (11), *then for any given T_0,*

$$\max_{(t,x)\in\ \widetilde{D}(T_0)} \Big\{ |\partial_x \lambda_\pm(t,x)|, \quad \max_{i=1,\cdots,\ n} \Big\{ |\partial_x R_i(t,x)|,\ |\partial_x S_i(t,x)| \Big\} \Big\} \le C_1, \tag{53}$$

where $\widetilde{D}(T_0) \triangleq \{(t,x)|\ 0 \le t \le T_0,\ 0 \le x \le L\}$ and C_1 is a positive constant only depending on a, M_0, M_0', $N_0(T_0)$ and T_0.

Proof. Noting (33), by divide the strip domain D into $L \times L$ areas (see the notation (34)), we can find that the characteristic in every area intersect the boundary only one time. Then we can prove this Lemma by establishing a connection between

$|\partial_x\lambda_\pm(t,x)|$, $|\partial_x R_i(t,x)|$, $|\partial_x S_i(t,x)|$ and $|\partial_x\Lambda_\pm(x)|$, $|\partial_x R_i^0(x)|$, $|\partial_x S_i^0(x)|$ $(i = 1,\cdots,n)$ in every area.

By direct computations, from (9) we can get

$$\begin{cases}(\partial_t+\lambda_-\partial_x)\big((\lambda_+-\lambda_-)\partial_x\lambda_+\big)=0,\\(\partial_t+\lambda_+\partial_x)\big((\lambda_+-\lambda_-)\partial_x\lambda_-\big)=0,\\(\partial_t+\lambda_-\partial_x)\big((\lambda_+-\lambda_-)\partial_x S_i\big)=0,\\(\partial_t+\lambda_+\partial_x)\big((\lambda_+-\lambda_-)\partial_x R_i\big)=0.\end{cases} \tag{54}$$

For any fixed point $P(t,x)\in D_0$, where D_0 is defined in (34), we draw the forward characteristic through it. According to (33), there are only the following two possibilities shown in Figure 1.

Case 1. The forward characteristic $\ell_1 : x = x_1(t,x)$ intersects x-axis at a point $A(0,\alpha)$ (see Figure 1(a)),where ℓ_1 satisfies

$$\begin{cases}\dfrac{dx_1(t)}{dt}=\lambda_+(t,x_1(t,\alpha)),\\x_1(0,\alpha)=\alpha.\end{cases}$$

By (54), we get

$$(\lambda_+(t,x)-\lambda_-(t,x))\partial_x\lambda_-(t,x)=(\Lambda_+(\alpha)-\Lambda_-(\alpha))\partial_x\Lambda_-(\alpha). \tag{55}$$

Noting (32), we have

$$\begin{aligned}|\partial_x\lambda_-(t,x)|&=\left|\frac{\Lambda_+(\alpha)-\Lambda_-(\alpha)}{\lambda_+(t,x)-\lambda_-(t,x)}\right||\partial_x\Lambda_-(\alpha)|\\&\le\frac{2}{a}|\partial_x\Lambda_-(\alpha)|.\end{aligned} \tag{56}$$

Case 2. The forward characteristic $\ell_1 : x = x_1(t,x)$ intersects t-axis at a point $A(\gamma,0)$ and the backward characteristic $\ell_2 : x = x_2(t,x)$ passing through the point A intersects x-axis at a point $B(0,\beta)$ (see Figure 1(b)), where ℓ_2 satisfies

$$\begin{cases}\dfrac{dx_2(t)}{dt}=\lambda_-(t,x_2(t,\beta)),\\x_2(0,\beta)=\beta.\end{cases}$$

By (8), on the t-axis we have

$$\partial_t v_i(\gamma,0)=\partial_t S_i(\gamma,0)-\partial_t\lambda_+(\gamma,0)h_{1i}(\gamma)-\lambda_+(\gamma,0)h'_{1i}(\gamma)\qquad(i=1,\cdots,n).$$

Along ℓ_2, by (9), (32) and (20), we have

$$\begin{aligned} |\partial_t v_i(\gamma,0)| &\le |\partial_t S_i(\gamma,0)| + |\partial_t \lambda_+(\gamma,0)|\cdot|h_{1i}(\gamma)| + |\lambda_+(\gamma,0)|\cdot|h'_{1i}(\gamma)| \\ &\le |\lambda_-\partial_x S_i(\gamma,0)| + |\lambda_-\partial_x\lambda_+(\gamma,0)|\cdot|h_{1i}(\gamma)| + |\lambda_+(\gamma,0)|\cdot|h'_{1i}(\gamma)| \\ &\le |\partial_x S_i(\gamma,0)| + \frac{a}{4}|\partial_x\lambda_+(\gamma,0)| + N_0(T_0), \end{aligned} \tag{57}$$

where $i = 1,\cdots,n$.

It follows from (54) that

$$\begin{cases} (\lambda_+(\gamma,0)-\lambda_-(\gamma,0))\partial_x\lambda_+(\gamma,0) = (\Lambda_+(\beta)-\Lambda_-(\beta))\partial_x\Lambda_+(\beta), \\ (\lambda_+(\gamma,0)-\lambda_-(\gamma,0))\partial_x S_i(\gamma,0) = (\Lambda_+(\beta)-\Lambda_-(\beta))\partial_x S_i^0(\beta), \end{cases} \tag{58}$$

where $i = 1,\cdots,n$. Hence, by (32) we have

$$\begin{cases} |\partial_x\lambda_+(\gamma,0)| \le \dfrac{2}{a}|\partial_x\Lambda_+(\beta)|, \\ |\partial_x S_i(\gamma,0)| \le \dfrac{2}{a}|\partial_x S_i^0(\beta)|. \end{cases} \tag{59}$$

where $i = 1,\cdots,n$. Then (57) becomes

$$|\partial_t v_i(\gamma,0)| \le K_1(|\partial_x\Lambda_+(\beta)| + |\partial_x S_i^0(\beta)| + 1), \tag{60}$$

where $i = 1,\cdots,n$ and K_1 is a positive constant only dependent of a and $N_0(T_0)$. By (54) we have, along ℓ_1,

$$(\lambda_+(t,x)-\lambda_-(t,x))\partial_x\lambda_-(t,x) = (\lambda_+(\gamma,0)-\lambda_-(\gamma,0))\partial_x\lambda_-(\gamma,0). \tag{61}$$

On the other hand, by (6) and (37),

$$\begin{aligned} &|\nabla_u\lambda_-(h_1,v)| \\ =&\left|\frac{1}{(1+h_1^2)^2}\left[(1+h_1^2)\left(-v-\frac{2(1-v^2)h_1+2\langle h,v\rangle v}{2\sqrt{\triangle}}\right)+2(\langle h,v\rangle+\sqrt{\triangle})h\right]\right| \\ \le& K_2, \end{aligned} \tag{62}$$

where K_2 is a positive constant only dependent of a. Here, we have made use of

$$\frac{1}{\sqrt{\triangle(u,v)}} \le \frac{2}{a}, \tag{63}$$

which is derived from (6) and (32) by

$$\frac{2\sqrt{\triangle(u,v)}}{1+|u|^2} = \lambda_+ - \lambda_- \ge a. \tag{64}$$

Similarly, we have

$$|\nabla_v \lambda_-(h_1, v)| \le K_3, \tag{65}$$

where K_3 is a positive constant only dependent of a.

Then by (32), (9), (60), (62) and (65), we have

$$\begin{aligned}
|\partial_x \lambda_-(t,x)| &\le \frac{2}{a}|\partial_x \lambda_-(\gamma,0)| \\
&= \frac{2}{a}\left| -\frac{1}{\lambda_+(\gamma,0)}\partial_t \lambda_-(\gamma,0)\right| \\
&\le \frac{4}{a^2}|\partial_t \lambda_-(\gamma,0)| \\
&\le \frac{4}{a^2}\left[|\nabla_u \lambda_-(h_1,v)|\cdot|h_1'(\gamma)| + |\nabla_v \lambda_-(h_1,v)|\cdot|\partial_t v(\gamma)|\right] \\
&\le K_4\left(|\partial_x \Lambda_+(\beta)| + |\partial_x S_i^0(\beta)| + 1\right),
\end{aligned} \tag{66}$$

where $i = 1, \cdots, n$ and K_4 is a positive constant only dependent of a and $N_0(T_0)$.

Combining (56) and (66) leads to

$$|\partial_x \lambda_-(t,x)| \le K_5\left(|\partial_x \Lambda_+(\beta)| + |\partial_x S_i^0(\beta)| + 1\right), \quad \forall\ (t,x) \in D_0, \tag{67}$$

where $i = 1, \cdots, n$ and K_5 is a positive constant only dependent of a and $N_0(T_0)$.

Similar estimates can be obtained in D_0 for $|\partial_x \lambda_+(t,x)|$, $\max\limits_{i=1,\cdots,n}\left\{|\partial_x R_i(t,x)|\right\}$ and $\max\limits_{i=1,\cdots,n}\left\{|\partial_x S_i(t,x)|\right\}$. Thus we can get

$$\begin{aligned}
&|\partial_x \lambda_+(t,x)| + |\partial_x \lambda_-(t,x)| + \max_{i=1,\cdots,n}\left\{|\partial_x R_i(t,x)|\right\} + \max_{i=1,\cdots,n}\left\{|\partial_x S_i(t,x)|\right\} + 1 \\
\le& K_6\left(\sup|\partial_x \Lambda_+| + \sup|\partial_x \Lambda_-| + \max_{i=1,\cdots,n}\left\{\sup|\partial_x R_i^0|\right\} + \max_{i=1,\cdots,n}\left\{\sup|\partial_x S_i^0|\right\} + 1\right), \\
&\forall\ (t,x) \in D_0,
\end{aligned} \tag{68}$$

where K_6 is a positive constant only dependent of a and $N_0(T_0)$. Here we may assume that $K_6 \ge 1$.

Taking $\partial_x \lambda_\pm(L,x)$, $\partial_x R_i(L,x)$ and $\partial_x S_i(L,x)$ $(i = 1, \cdots, n)$ as the new initial data on $t = L$ and repeating the previous procedure, then for any (t,x) in D_1 we have

$$
\begin{aligned}
&|\partial_x\lambda_+(t,x)|+|\partial_x\lambda_-(t,x)|+\max_{i=1,\cdots,n}\Big\{|\partial_x R_i(t,x)|\Big\}+\max_{i=1,\cdots,n}\Big\{|\partial_x S_i(t,x)|\Big\}+1\\
\le& K_6\Big(\sup|\partial_x\lambda_+(L,\cdot)|+\sup|\partial_x\lambda_-(L,\cdot)|+\max_{i=1,\cdots,n}\Big\{\sup|\partial_x R_i(L,\cdot)|\Big\}\\
&\qquad+\max_{i=1,\cdots,n}\Big\{\sup|\partial_x S_i(L,\cdot)|\Big\}+1\Big)\\
\le& K_6K_7\Big(\sup|\partial_x\Lambda_+|+\sup|\partial_x\Lambda_-|+\max_{i=1,\cdots,n}\Big\{\sup|\partial_x R_i^0|\Big\}\\
&\qquad+\max_{i=1,\cdots,n}\Big\{\sup|\partial_x S_i^0|\Big\}+1\Big).
\end{aligned}\tag{69}
$$

where K_7 is a positive constant only dependent of a and $N_0(T_0)$.

Repeating this procedure at most $N=\left[\dfrac{T_0}{L}\right]+1$ times, we get

$$
\begin{aligned}
&|\partial_x\lambda_+(t,x)|+|\partial_x\lambda_-(t,x)|+\max_{i=1,\cdots,n}\Big\{|\partial_x R_i(t,x)|\Big\}+\max_{i=1,\cdots,n}\Big\{|\partial_x S_i(t,x)|\Big\}+1\\
\le& K_8(N)\Big(\sup|\partial_x\Lambda_+|+\sup|\partial_x\Lambda_-|+\max_{i=1,\cdots,n}\Big\{\sup|\partial_x R_i^0|\Big\}\\
&\qquad+\max_{i=1,\cdots,n}\Big\{\sup|\partial_x S_i^0|\Big\}+1\Big),\qquad \forall\ t\in[0,T_0],
\end{aligned}\tag{70}
$$

where $K_8(N)$ is a positive constant only dependent of a, $N_0(T_0)$ and T_0.

Noting (12), (13) and (33), we have

$$
|\partial_x\Lambda_+|+|\partial_x\Lambda_-|+\max_{i=1,\cdots,n}\Big\{|\partial_x R_i^0|\Big\}+\max_{i=1,\cdots,n}\Big\{|\partial_x S_i^0|\Big\}\le K_9,\tag{71}
$$

where K_9 is a positive constant only dependent of M_0 and M_0'.

So for any constant $(t,x)\in[0,\ T_0]\times[0,\ L]$, we finally have

$$
|\partial_x\lambda_+(t,x)|+|\partial_x\lambda_-(t,x)|+\max_{i=1,\cdots,n}\Big\{|\partial_x R_i(t,x)|\Big\}+\max_{i=1,\cdots,n}\Big\{|\partial_x S_i(t,x)|\Big\}+1\le K,\tag{72}
$$

where K is a positive constant only depending on a, M_0, M_0', $N_0(T_0)$ and T_0. Thus the proof of Lemma 3.3 is completed. ■

Proof of Theorem 2.1. By (4), if the classical solution of system (17) exist globally, then we can obtain the conclusion of Theorem 2.1.

Noting (8), we have

$$
u_i(t,x)=\frac{S_i(t,x)-R_i(t,x)}{\lambda_+(t,x)-\lambda_-(t,x)},\quad v_i(t,x)=\frac{\lambda_+R_i(t,x)-\lambda_-S_i(t,x)}{\lambda_+(t,x)-\lambda_-(t,x)},\tag{73}
$$

where $i = 1, \cdots, n$.

For the initial-boundary value problem (3), (10) and (14), noting (73), under the assumptions of Theorem 2.1, by Lemmas 3.1-3.3 we can get a priori uniform estimate of C^1 norm of u and v, i.e. system (17) admits a unique global C^1 solution. Thus the proof of Theorem 2.1 is completed. ∎

Similar to Lemmas 3.1-3.3, we have the following Lemmas for Dirichlet boundary conditions.

Lemma 3.4. *Under the assumptions* (22), (18) *and* (21), *the Cauchy problem* (31) *has a unique global smooth solution* $\lambda = \lambda_{\pm}(t,x)$ *on the strip domain D. Furthermore, on D it holds that*

$$-1 \le \lambda_-(t,x) \le -\frac{a}{2} < 0 < \frac{a}{2} \le \lambda_+(t,x) \le 1. \tag{74}$$

The proof of Lemma 3.4 is similar to that of Lemma 3.1, so we omit the details here. The only difference is that similar to (43) we can get

$$\begin{aligned}
&-1 \le \lambda_-(t,x) \le -\left(a - \sum_{i=0}^{n} \frac{2\langle u, H'\rangle}{1+|u|^2}(t_i)\right) \le -\left(a - \sum_{i=0}^{n} |H'|(t_i)\right),\\
&1 \ge \lambda_+(t,x) \ge a - \sum_{i=0}^{n} \frac{2\langle u, H'\rangle}{1+|u|^2}(t_i) \ge a - \sum_{i=0}^{n} |H'|(t_i),\\
&\forall\ (t,x) \in D_n \cap D_\varepsilon\ (0 \le n \le N-1),\\
&\exists\ t_i \in \big[iL,\ \min\{(i+1)L,\ T_0-\varepsilon\}\big]\ (i = 0, 1, \cdots, n),
\end{aligned} \tag{75}$$

where $H(t) = \max\Big\{H_1(t),\ H_2(t)\Big\}$ and D_n, D_ε, N are defined as before. Then by (21) we can get the similar conclusion to (44).

Lemma 3.5. *Assume that* R_i *and* S_i $(i = 1, \cdots, n)$ *satisfy* (9) *and* (11), *then*

$$\max_{i=1,\cdots,n} \Big\{|R_i(t,x)|, \quad |S_i(t,x)|\Big\} \le C_0, \quad \forall\ (t,x) \in D, \tag{76}$$

where C_0 *is a positive constant only dependent of* a *and* M_0.

Proof. The proof of Lemma 3.5 is similar to that of Lemma 3.2, the only thing should be detailed here is the estimate of $|u(t,x)|$ on the boundaries. In what follows, we estimate $|u|$.

Let $\theta \in [0, \pi]$ be the angle between the vectors u and H' (H' denotes H_1' or H_2'). Then it follows from (6) that

$$(1+|u|^2)\lambda_\pm + |u||H'|\cos\theta = \sqrt{1-|H'|^2+|u|^2-|u|^2|H'|^2+|u|^2|H'|^2\cos^2\theta}$$

on the boundaries. It yields that

$$(1+|u|^2)\lambda_\pm^2 + |H'|^2 + 2\lambda_\pm|u||H'|\cos\theta = 1. \tag{77}$$

By $|\cos\theta| \le 1$ we have

$$(1+|u|^2)\lambda_\pm^2 + |H'|^2 - 2\lambda_\pm|u||H'| \le 1.$$

That is,

$$(|u|\lambda_\pm - |H'|)^2 \le 1-\lambda_\pm^2.$$

It follows from (74) that

$$|u| \le \frac{|H'|+\sqrt{1-\lambda_\pm^2}}{|\lambda_\pm|} \le \frac{a+2}{a}. \tag{78}$$

Thus the proof of Lemma 3.5 is completed. ■

Lemma 3.6. *Assume that $\lambda_\pm$, R_i and S_i $(i=1,\cdots,n)$ satisfy* (9) *and* (11), *then for any given T_0,*

$$\max\Big\{|\partial_x\lambda_\pm(t,x)|, \quad \max_{i=1,\cdots,n}\Big\{|\partial_x R_i(t,x)|, \quad |\partial_x S_i(t,x)|\Big\}\Big\} \le C_1 \tag{79}$$

for any $(t,x) \in \widetilde{D}(T_0) \triangleq \{(t,x)|\ 0\le t\le T_0,\ 0\le x\le L\}$, where C_1 is a positive constant only dependent of a, M_0, M_0', $N_0(T_0)$ and T_0.

The proof of Lemma 3.6 is similar to that of Lemma 3.3, so we omit it here.

Proof of Theorem 2.2. Under the assumptions of Theorem 2.2, by Lemma 3.4-3.6 and noting (73), we can get a priori uniform estimates of C^1 norm of u and v. Then the initial-boundary value problem (3), (10) and (15) has a unique global C^2 solution. Thus the proof of Theorem 2.2 is completed. ■

Remark 3.1. *Noting that we consider the boundary conditions respectively in the proof of Theorems 2.1-2.2, so Remark 2.1 is correct similarly.*

Remark 3.2. *Comparing Liu and Zhou* [13]*, we observe that Theorems 2.1-2.2 have two main different points as follows.*

(i) *the boundary conditions:*
In [13]*, the Neumann boundary datum (respectively the Dirichlet boundary datum) are small and decaying, i.e.*

$$\begin{aligned}\Big\{|h_1(t)|+|h_2(t)|\Big\} &\le \frac{\varepsilon}{(1+t)^{1+\mu}} \\ \bigg(respectively\ \Big\{|H_1'(t)|+|H_2'(t)|\Big\} &\le \frac{\varepsilon}{(1+t)^{1+\mu}}\bigg),\end{aligned} \tag{80}$$

where μ is an arbitrary positive constant and ε is a positive constant only depending on a and μ. Apparently they are special case of our conditions.

(ii) *the estimates of $||\lambda_\pm(t,x)||_{C^1}$, $||R_i(t,x)||_{C^1}$ and $||S_i(t,x)||_{C^1}$ $(i=1,\cdots,n)$:*
In [13]*, Liu and Zhou prove that $||\lambda_\pm(t,x)||_{C^1}$, $||R_i(t,x)||_{C^1}$ and $||S_i(t,x)||_{C^1}$ $(i=1,\cdots,n)$ are bounded by using Theorem 2.1 in* [12] *directly for system* (9)*. But it seems that the derived boundary conditions of system* (9) *do not satisfy the condition of Theorem 2.1 in* [12]*. We give a rigorous proof (Lemma 3.3) in a different way.*

4. Proof of Theorems 2.3-2.4

In [13], Liu and Zhou prove Theorems 2.3-2.4. We can prove them in a different way by using some results in [13] and the following Lemma.

Lemma 4.1. *Assume that for any given $T_0>0$, the following mixed initial-boundary value problem*

$$\begin{cases}\partial_t r+\lambda(s)\partial_x r=0,\\ \partial_t s+\mu(r)\partial_x s=0,\\ t=0: \quad r=r_0(x), \quad s=s_0(x),\\ x=0: \quad s=w(t,r),\end{cases} \tag{81}$$

where r_0, s_0 and w are all C^1 functions and

$$\begin{cases}\lambda(s)<0<\mu(r),\\ m\triangleq \sup_{x\in\mathbb{R}^+}\Big\{|r_0'(x)|+|s_0'(x)|\Big\}<\infty,\end{cases}$$

admits a unique C^1 solution $(r,s)=\big(r(t,x),s(t,x)\big)$ on the domain $\widetilde{D}(T)\triangleq\{(t,x)|\ 0\le t\le T,\ x\ge 0\}$ with $0<T\le T_0$, then the following a priori uniform

estimate on the C^0 *norm of solution holds*

$$||r(t,\cdot)||_{C^0} + ||s(t,\cdot)||_{C^0} \le C(T_0), \quad \forall\ t \in [0,T],$$

where $C(T_0)$ *is a positive constant depending on* T_0. *Then the mixed initial-boundary value problem* (81) *admits a unique global* C^1 *solution* $(r,s) = \big(r(t,x), s(t,x)\big)$ *on the domain* $\Omega = \{(t,x)|\ t \ge 0,\ x \ge 0\}$.

From the proof of Theorem 2.1 in [12], we can get Lemma 4.1 directly. So we omit it here.

Proof of Theorem 2.3. For the initial-boundary value problem (3), (10) and (23), under the assumptions of Theorem 2.3, Liu and Zhou have proved that $||\lambda_\pm(t,x)||_{C^0}$, $||R_i(t,x)||_{C^0}$, $||S_i(t,x)||_{C^0}$ $(i = 1,\cdots,n)$ are bounded and $\lambda_+ - \lambda_-$ has a positive lower bound on the domain Ω by using the similar methods to (40) and (47)(see Lemma 2.1 and Lemma 2.2 in [13]). Then according to Lemma 4.1, we find that $||\lambda_\pm(t,x)||_{C^1}$, $||R_i(t,x)||_{C^1}$ and $||S_i(t,x)||_{C^1}$ $(i = 1,\cdots,n)$ are bounded. Therefore by (73) we get the global existence and uniqueness of the C^1 solution of system (26). Thus the proof of Theorem 2.3 is completed. ■

Remark 4.1. *To prove Theorem 2.3, by using Lemma 4.1, we need to check that if the boundary condition of system* (9) *satisfies the conditions of Lemma 4.1. Noting* (6) *and* (8)*, we can get the boundary condition of system* (9) *as follow*

$$\lambda_-(t,0) = \lambda_+(t,0) - \frac{2\sqrt{\triangle(h(t), v(t,0))}}{1+|h(t)|^2} \qquad \forall\ t \ge 0 \tag{82}$$

for the first two equations of system (9) *and*

$$R_i(t,0) = S_i(t,0) - (\lambda_+(t,0) - \lambda_-(t,0))h_i(t) \quad (i = 1,\cdots,n) \quad \forall\ t \ge 0 \tag{83}$$

for the last two equations of system (9). *By the similar way how to get* (60)*, we can get the estimate of* $|\partial_t v|$ *on* t*-axis. That means the right term of* (82) *is* C^1 *with respect to* t*, i.e. it satisfies the boundary condition of* (81). *Then by Lemma 4.1 we have* $||\lambda_\pm(t,x)||_{C^1}$ *are bounded. So* $|\partial_t\lambda_\pm(t,x)|$ *are* C^1 *respect to* t *on* t*-axis. Then by Lemma 4.1, it follows from* (83) *that* $||R_i(t,x)||_{C^1}$ *and* $||S_i(t,x)||_{C^1}$ $(i = 1,\cdots,n)$ *are bounded.*

Proof of Theorem 2.4. For the initial-boundary value problem (3), (10) and (24), under the assumptions of Theorem 2.4, by the similar methods to the proof of Theorem 2.3, we can get the global existence and uniqueness of the C^2 solution of system (3). Thus the proof of Theorem 2.4 is completed. ■

Remark 4.2. *In the proof of Theorems 2.3-2.4, the main difference between this paper and Liu et al* [13] *is that we use the conclusion of* (60) *and Lemma 4.1 (Lemma 4.1 can be regarded as a direct extent of the result of* [13]*) to get the estimates of* $||\lambda_\pm(t,x)||_{C^1}$*,* $||R_i(t,x)||_{C^1}$ *and* $||S_i(t,x)||_{C^1}$ $(i = 1, \cdots, n)$ *directly; however, Liu and Zhou get the estimates by the characteristic method in detail.*

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