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Compact Commutators of Riesz Transforms Associated to Schrödinger Operator

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Abstract: In this paper, we consider the compactness of some commutators of Riesz transforms associated to Schrödinger operator $L = -\triangle + V$ on $R^n, n \geq 3$, where V is non-zero, nonnegative and belongs to the reverse Hölder class B_q for $q > \frac{n}{2}$. We prove that if $T_1 = (-\triangle + V)^{-1}V, \quad T_2 = (-\triangle + V)^{-1/2}V^{1/2}$ and $T_3 = (-\triangle + V)^{-1/2}\nabla$, then the commutators $[b, T_j], (j = 1, 2, 3)$ are compact on $L^p(R^n)$ when p ranges in an interval and $b \in VMO(R^n)$.

Keywords: Commutator, Compactness, VMO, Schrödinger operator, Riesz transform.

Introduction

Throughout the paper, we assume that $L = -\triangle + V$ be a Schrödinger operator on $R^n, n \geq 3$ and V is a non-zero, nonnegative potential, and belongs to the reverse Hölder class B_q for $q > n/2$. Let $T_j, j = 1, 2, 3$ be the Riesz transforms associated to Schrödinger operators, namely, $T_1 = (-\triangle + V)^{-1}V, T_2 = (-\triangle + V)^{-1/2}V^{1/2}$ and $T_3 = (-\triangle + V)^{-1/2}\nabla$. The L^p boundedness of $T_j, (j = 1, 2, 3)$ was widely studied in [3]. Recently, in [1], the authors got the L^p boundedness of the commutator of $T_j, (j = 1, 2, 3)$ with the symbol $b \in BMO(R^n)$. In this paper, we will discuss the L^p compactness of the commutators $[b, T_j] = bT_j - T_jb, (j =$

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$1,2,3)$, where $b \in VMO(R^n) = \overline{C_0^\infty(R^n)}$, the closure of $C_0^\infty(R^n)$ functions in BMO norm.

A nonnegative locally L^q integrable function V on R^n is said to belong to $B_q, (1 < q < \infty)$, if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)dx\right) \tag{1}$$

holds for every ball B in R^n.

By Hölder's inequality, we can get that $B_{q_1} \subseteq B_{q_2}$, for $q_1 \geq q_2 > 1$. One remarkable feature about the B_q class is that if $V \in B_q$ for some $q > 1$ then there exists an $\varepsilon > 0$ which depends only on the dimension n and the constant C in (1), such that $V \in B_{q+\varepsilon}$. It's also well known that if $V \in B_q, q > 1$ then $V(x)dx$ is a doubling measure, namely for any $r > 0, x \in R^n$ and some constant $C_0 > 0$, one has

$$\int_{B(x,2r)} V(y)dy \leq C_0 \int_{B(x,r)} V(y)dy. \tag{2}$$

In [3], Z. Shen proved that if $V \in B_n$ then T_3 is a Calderón-Zygmund operator. According to the classical result of A.Uchiyama ([4]), for $b \in VMO(R^n)$, $[b, T_3]$ is a compact operator on $L^p, (1 < p < \infty)$ in the case. So we restrict ourselves to the case that $V \in B_q, (n/2 < q < n)$ when we consider the commutator $[b, T_3]$.

In the rest of this section, we will state some definitions and lemmas which will be used in the proofs of the main results.

Definition 0.1. For $x \in R^n$, the function $m(x, V)$ is defined by

$$\frac{1}{m(x,V)} = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leq 1\right\}. \tag{3}$$

Clearly, for every $x \in R^n$, if $r = \frac{1}{m(x,V)}$, then $\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy = 1$.

The function $m(x, V)$, as deeply studied in [3], plays an important role in estimating the kernel of $T_i, (i = 1, 2, 3)$. We list some properties of $m(x, V)$ here, and their proofs can be found in [3].

Lemma 0.2. *Assume $V \in B_q$ for $q > n/2$, there exist $C > 0, c > 0, k_0 > 0$ such that, for any $x, y \in R^n, 0 < r < R < \infty$,*

(1) *(a)* $0 < m(x, V) < \infty$;
(2) *(b)* $m(x, V) \sim m(y, V)$, *if* $|x - y| \leq \frac{c}{m(x,V)}$;
(3) *(c)* $\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq C(\frac{R}{r})^{n/q-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)dy$.

By (a) and (c) of Lemma 0.2 in [1], the authors got:

Lemma 0.3. *([1], Lemma 1) Suppose* $V \in B_q$ *for some* $q > n/2$ *and let* $K > \log_2 C_0 + 1$, *where* C_0 *is the constant in (2). Then for any* $x \in R^n$ *and* $R > 0$, *we have*

$$\frac{1}{\{1 + m(x, V)R\}^K} \int_{B(x,R)} V(y)dy \leq CR^{n-2}. \qquad (4)$$

We also list some results concerning the L^p boundedness of $T_j, (j = 1, 2, 3)$ and refer the reader to [3] for further details. We will adopt the notation $1/p' = 1-1/p$ for $p \geq 1$ throughout the paper.

Theorem 0.4. *Suppose* $V \in B_q$ *and* $q \geq n/2$, *we have:*

(1) *(i) ([3], Theorem 3.1, Page 526)* $\|(-\triangle + V)^{-1}V f\|_p \leq C_p\|f\|_p$ *for* $q' \leq p \leq \infty$.
(2) *(ii) ([3], Theorem 5.10, Page 542)* $\|(-\triangle+V)^{-1/2}V^{1/2}f\|_p \leq C_p\|f\|_p$ *for* $(2q)' \leq p \leq \infty$.
(3) *(iii) ([3], Theorem 0.5, Page 514)* $\|(-\triangle+V)^{-1/2}\nabla f\|_p \leq C_p\|f\|_p$ *for* $p_0' \leq p < \infty$,

where $1/p_0 = 1/q - 1/n \quad n/2 \leq q < n$.

In [1], using Theorem 0.4 and a pointwise estimation of the kernel of $T_i, (i = 1, 2, 3)$, the authors got the L^p boundedness of commutator $[b, T_i], (i = 1, 2, 3)$, where $b \in BMO(R^n)$.

Theorem 0.5. *([1], Theorem 1) (i) Suppose* $V \in B_q, q \geq n/2$. *If* $b \in BMO(R^n)$, *then for* $q' \leq p \leq \infty$,

$$\|[b, T_1]f\|_p \leq C_p\|b\|_{BMO}\|f\|_p.$$

(ii) Suppose $V \in B_q, q \geq n/2$. *If* $b \in BMO(R^n)$, *then for* $(2q)' \leq p < \infty$,

$$\|[b, T_2]f\|_p \leq C_p\|b\|_{BMO}\|f\|_p.$$

(iii) Suppose $V \in B_q, n/2 \leq q < n$. *If* $b \in BMO(R^n)$, *then for* $(p_0)' \leq p < \infty$ *and* $1/p_1 = 1/q - 1/n$,

$$\|[b, T_3]f\|_p \leq C_p\|b\|_{BMO}\|f\|_p.$$

Our proof of the compactness follows the well known Frechet-Kolmogorov theorem.

Theorem 0.6. *(Frechet-Kolmogorov)A subset G of $L^p(R^n), 1 \leq p < \infty$ is strongly precompact if and only if it satisfies:*

$(c1)$ $\sup_{f\in G} \|f\|_p < \infty$;

$(c2)$ *For any $\varepsilon > 0$, there exist a closed region K_ε and $\delta_\varepsilon > 0$ such that $\|f\|_{L^p(K_\varepsilon^c)} < \varepsilon$ for any $f \in G$;*

$(c3)$ *For any $f \in G$, $\lim_{|z|\to 0} \|f(\cdot + z) - f(\cdot)\|_p = 0$, uniformly.*

Therefore, in order to prove the compactness of the commutators $[b, T_i], (i = 1, 2, 3)$, we only need to test the following three conditions for the commutator $[b, T_i], (i = 1, 2, 3)$:

$(c1)'$ $\sup_{\|f\|_p \leq 1} \|[b, T_i]f\|_p \leq C$;

$(c2)'$ For any $\varepsilon > 0$, there exists a ball B such that $(\int_{B^c} |[b, T_i]f(x)|^p dx)^{1/p} < \varepsilon, \quad \|f\|_p \leq 1$;

$(c3)'$ For any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|z| < \delta$, we have

$$\|[b, T_i]f(\cdot + z) - [b, T_i]f(\cdot)\|_p < \varepsilon, \quad \|f\|_p \leq 1.$$

Remark 0.7. Because $VMO(R^n)$ is the closure of $C_0^\infty(R^n)$ in BMO norm, by density, we easily see that if $[b, T_i]$ is a compact operator on $L^p(R^n)$ for $b \in C_0^\infty(R^n)$, then for $b \in VMO(R^n)$, $[b, T_i]$ is also a compact operator on $L^p(R^n)$. So in what follows, we always assume $b \in C_0^\infty(R^n)$.

Remark 0.8. By Theorem 0.5, we know that the operators $[b, T_i], (i = 1, 2, 3)$ are bounded on $L^p(R^n)$ for some $p > 1$, so that each $[b, T_i]$ satisfies condition $(c1)'$ obviously.

1. L^p Boundedness of Maximal Operator of $T_i, (i = 1, 2, 3)$

In this section, we discuss the L^p boundedness of maximal operators of T_i,(i=1,2,3). We define the maximal operators of T_i as follows:

Definition 1.1. Suppose $V \in B_q$ for $q > n/2$. Let $T_1 = (-\triangle + V)^{-1}V, T_2 = (-\triangle + V)^{-1/2}V^{1/2}$ and $T_3 = (-\triangle + V)^{-1/2}\nabla$ be the Riesz transforms associated to Schrödinger operators. Then the maximal operator $T_{i,Max}$ of $T_i, (i = 1, 2, 3)$ is defined by:

$$T_{i,Max} = \sup_{r>0} \left| \int_{|x-y|>r} K_i(x,y) f(y) dy \right|, \quad (i = 1, 2, 3).$$

Lemma 1.2. *Suppose $V \in B_q, q > n/2$. Then the maximal operator of T_1 is bounded on $L^p(R^n)$ for $p > q'$.*

The proof of Lemma 1.2 needs the following lemma.

Lemma 1.3. *([1], Lemma 2) Suppose $V \in B_q$ for some $q > n/2$. Then there exists $\delta > 0$ such that for any integer $l > 0$, $0 < h < |x-y|/16$,*

$$|K_1(x,y)| \leq \frac{C_l}{\{1 + m(x,V)|x-y|\}^l} \frac{1}{|x-y|^{n-2}} V(y), \tag{5}$$

$$|K_1(x+h,y) - K_1(x,y)| \leq \frac{C_l}{\{1 + m(x,V)|x-y|\}^l} \frac{|h|^\delta}{|x-y|^{n-2+\delta}} V(y). \tag{6}$$

Proof of Lemma 1.2 We set $T_{1,r} f(x) = \int_{|x-y|>r} K_1(x,y) f(y) dy$, $B = B(x, r/16)$ and divide f into $f = f_1 + f_2$, where $f_1 = f\chi_{16B}$, so we get

$$\begin{aligned}
|T_{1,r} f(x)| &= \frac{1}{|B|} \int_B |T_{1,r} f(x)| dy \\
&\leq \frac{1}{|B|} \int_B |T_1 f(y)| dy + \frac{1}{|B|} \int_B |T_1 f_1(y)| dy + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1,r} f(x)| dy \\
&\leq M(T_1 f)(x) + \frac{1}{|B|^{1/q'}} \|T_1 f_1\|_{q'} + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1,r} f(x)| dy \\
&\leq M(T_1 f)(x) + \Big(\frac{1}{|B|} \int_{16B} |f(y)|^{q'} dy\Big)^{1/q'} + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1,r} f(x)| dy \\
&\leq M(T_1 f)(x) + C(M(|f|^{q'})(x))^{1/q'} + \frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1,r} f(x)| dy.
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
&\frac{1}{|B|} \int_B |T_1 f_2(y) - T_{1,r} f(x)| dy \\
&= \frac{1}{|B|} \int_B \Big| \int_{(16B)^c} K_1(y,\xi) f_2(\xi) d\xi - \int_{|x-\xi|>r} K_1(x,\xi) f(\xi) d\xi \Big| dy \\
&\leq \frac{1}{|B|} \int_B \Big[\int_{|x-\xi|>r} |K_1(y,\xi) - K_1(x,\xi)| |f(\xi)| d\xi \Big] dy.
\end{aligned}$$

Now, if we set $I_y^1 = \int_{|x-\xi|>r} |K_1(y,\xi) - K_1(x,\xi)||f(\xi)|d\xi$ and $h = |y-x|$, then by $|y-x| < r/16 < \frac{1}{16}|x-\xi|$ for $y \in B$ and (6) of Lemma 1.3, we have

$$
\begin{aligned}
I_y^1 &= \int_{|x-\xi|>r} |K_1(y,\xi) - K_1(x,\xi)||f(\xi)|d\xi \\
&\le C\sum_{k=0}^{\infty} \int_{2^k r<|x-\xi|\le 2^{k+1}r} \frac{C_l}{\{1+m(x,V)|x-\xi|\}^l} \frac{|y-x|^\delta}{|x-\xi|^{n-2+\delta}} V(\xi)|f(\xi)|d\xi \\
&\le \sum_{k=0}^{\infty} \frac{C_l}{\{1+m(x,V)2^k r\}^l} \frac{r^\delta}{(2^k r)^{n-2+\delta}} \left(\int_{|x-\xi|\le 2^{k+1}r} V^q(\xi)d\xi\right)^{1/q} \left(\int_{|x-\xi|\le 2^{k+1}r} |f(\xi)|^{q'}d\xi\right)^{1/q'} \\
&\le C\sum_{k=0}^{\infty} \frac{C_l(M(|f|^{q'})(x))^{1/q'}}{\{1+m(x,V)2^k r\}^l} \frac{r^\delta}{(2^k r)^{n-2+\delta}} (2^{k+1}r)^{n/q-n} (\int_{B(x,2^k r)} V(\xi)d\xi)(2^{k+1}r)^{n/q'} \\
&\le C(M(|f|^{q'})(x))^{1/q'} \sum_{k=0}^{\infty} \frac{r^\delta}{(2^k r)^{n-2+\delta}} (2^k r)^{n-2} \\
&\le C(M(|f|^{q'})(x))^{1/q'}.
\end{aligned}
$$

Here we have used Lemma 0.3 for $R = 2^k r$. Finally, we have

$$
\frac{1}{|B|}\int_B I_y^1 dy \le C(M(|f|^{q'})(x))^{1/q'} \quad \textit{and} \quad T_{1,Max}f(x) \le M(T_1 f)(x) + C(M(|f|^{q'})(x))^{1/q'}.
$$

By use of (i) of Theorem 0.4, we have

$$
\|T_{1,Max}(f)\|_p \le \|M(T_1 f)\|_p + C\|(M(|f|^{q'}))^{1/q'}\|_p \le C\|f\|_p, \text{for } p > q' > 1.
$$

This completes the proof of Lemma 1.2.

Similarly, for $T_{2,Max}(f)$, we have the following lemma.

Lemma 1.4. *Suppose $V \in B_q, q > n/2$. The maximal operator of T_2 is bounded on $L^p(R^n)$, for $p > (2q)'$.*

The proof of Lemma 1.4 needs the following lemma.

Lemma 1.5. *([1], Lemma 3) Suppose $V \in B_q$ for some $q > n/2$. Then there exists $\delta > 0$ such that for any integer $k > 0$, $0 < h < |x-y|/16$,*

$$
|K_2(x,y)| \le \frac{C_l}{\{1+m(x,V)|x-y|\}^l} \frac{1}{|x-y|^{n-1}} V^{1/2}(y), \tag{7}
$$

$$
|K_2(x+h,y) - K_2(x,y)| \le \frac{C_l}{\{1+m(x,V)|x-y|\}^l} \frac{|h|^\delta}{|x-y|^{n-1+\delta}} V^{1/2}(y). \tag{8}
$$

Proof. We set $T_{2,r}f(x) = \int_{|x-y|>r} K_2(x,y)f(y)dy$ and $B = B(x, \frac{r}{16})$, and divide f into $f_1 + f_2$, where $f_1 = f\chi_{16B}$. Then we have

$$\begin{aligned}
&|T_{2,r}f(x)| = \frac{1}{|B|}\int_B |T_{2,r}f(x)|dy \\
&\leq \frac{1}{|B|}\int_B |T_2 f(y)|dy + \frac{1}{|B|}\int_B |T_2 f_1(y)|dy + \frac{1}{|B|}\int_B |T_2 f_2(y) - T_{2,r}f(x)|dy \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Clearly,we have $I_1 = \frac{1}{|B|}\int_B |T_2 f(y)|dy \leq M(T_2 f)(x)$. By use of the L^p boundedness of T_2, we have

$$I_2 \leq \frac{1}{|B|^{1/(2q)'}}\|T_2 f_1\|_{(2q)'} \leq C\frac{1}{|B|^{1/(2q)'}}\|f_1\|_{(2q)'} \leq C\left(M(|f|^{(2q)'})(x)\right)^{1/(2q)'}.$$

At last we estimate I_3,

$$\begin{aligned}
I_3 &= \frac{1}{|B|}\int_B \left|\int K_2(y,\xi)f_2(\xi)d\xi - \int_{|x-\xi|>r} K_2(x,\xi)f(\xi)d\xi\right| dy \\
&\leq \frac{1}{|B|}\int_B \int_{|x-\xi|>r} |K_2(y,\xi) - K_2(x,\xi)||f(\xi)|d\xi dy.
\end{aligned}$$

Write $I_y^2 = \int_{|x-\xi|>r} |K_2(y,\xi) - K_2(x,\xi)||f(\xi)|d\xi$. Because $y \in B$ implies $h = |y-x| < 1/16r < |x-\xi|$, by (8) of Lemma 1.5 and Hölder's inequality, we have

$$\begin{aligned}
&I_y^2 \leq C\int_{|x-\xi|>r} \frac{1}{\{1+m(x,V)|x-\xi|\}^l}\frac{|y-x|^\delta}{|x-\xi|^{n-1+\delta}}V^{1/2}(\xi)|f(\xi)|d\xi \\
&\leq C\sum_{k=0}^\infty \int_{2^k r<|x-\xi|\leq 2^{k+1}r} \frac{1}{\{1+m(x,V)|x-\xi|\}^l}\frac{r^\delta}{|x-\xi|^{n-1+\delta}}V^{1/2}(\xi)|f(\xi)|d\xi \\
&\leq C\sum_{k=0}^\infty \frac{1}{\{1+m(x,V)2^k r\}^l}\frac{r^\delta}{(2^k r)^{n-1+\delta}}\left(\int_{|x-\xi|<2^{k+1}r} V^{2q}(\xi)d\xi\right)^{1/2q} \times \\
&\quad \left(\int_{|x-\xi|<2^{k+1}r} |f(\xi)|^{(2q)'}d\xi\right)^{1/(2q)'}.
\end{aligned}$$

Because of $V \in B_q$, by Lemma 0.3 and the double property of $V(x)dx$, we can get

$$
\begin{aligned}
I_y^2 &\le C\sum_{k=0}^{\infty}\frac{(2^{k+1}r)^{n/2q-n/2}}{\{1+m(x,V)2^kr\}^l}\frac{r^\delta}{(2^kr)^{n-1+\delta}}\left(\int_{|x-\xi|<2^{k+1}r}V(\xi)d\xi\right)^{1/2}\times\\
&\quad\left(\int_{|x-\xi|<2^{k+1}r}|f(\xi)|^{(2q)'}d\xi\right)^{1/(2q)'}\\
&\le C\sum_{k=0}^{\infty}\frac{r^\delta}{(2^kr)^{n-1+\delta}}(2^{k+1}r)^{n/2q-n/2}(2^kr)^{n/2-1}(2^{k+1}r)^{n/(2q)'}(M(|f|^{(2q)'})(x))^{1/(2q)'}\\
&\le C\sum_{k=0}^{\infty}\frac{r^\delta}{(2^kr)^{n-1+\delta}}(2^kr)^{n-1}(M(|f|^{(2q)'})(x))^{1/(2q)'}\\
&\le C(M(|f|^{(2q)'})(x))^{1/(2q)'}.
\end{aligned}
$$

Consequently $I_3 \le \frac{1}{|B|}\int_B I_y^2 dy \le C(M(|f|^{(2q)'})(x))^{1/(2q)'}$. Finally we have

$$T_{2,Max}(f)(x) \le M(T_2f)(x) + C(M(|f|^{(2q)'})(x))^{1/(2q)'}.$$

This completes the proof of Lemma 1.4. □

It remains to handle the L^p boundedness of maximal operator $T_{3,Max}$. For this propose, we need the following lemma.

Lemma 1.6. *([1], Lemma 4) Suppose $V \in B_q$ for some $n/2 < q < n$. Then there exists $\delta > 0$ such that for any integer $k > 0$ and $0 < h < |x-y|/16$,*

$$
\begin{aligned}
&|K_3(x,y)|\\
&\le \frac{C_l}{\{1+m(x,V)|x-y|\}^l}\frac{1}{|x-y|^{n-1}}\left(\int_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}d\xi+\frac{1}{|x-y|}\right),
\end{aligned}\tag{9}
$$

$$
\begin{aligned}
&|K_3(x+h,y)-K_3(x,y)|\\
&\le \frac{C_l}{\{1+m(x,V)|x-y|\}^l}\frac{|h|^\delta}{|x-y|^{n-1+\delta}}\left(\int_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}d\xi+\frac{1}{|x-y|}\right).
\end{aligned}\tag{10}
$$

Lemma 1.7. *Suppose $V \in B_q, n/2 < q < n$. The maximal operator $T_{3,Max}$ is bounded on L^p, for $p_1' \le p < \infty$, where $1/p_1 = 1/q - 1/n$.*

Proof. For any x, let $T_{3,r}f(x) = \int_{|x-y|>r} K_3(x,y)f(y)dy$ and $B = B(x, \frac{r}{16})$, we divide f into $f_1 + f_2$, where $f_1 = f\chi_{16B}$, similarly, we have

$$\begin{aligned}
&|T_{3,r}f(x)| = \frac{1}{|B|}\int_B |T_{3,r}f(x)|dy \\
&\le \frac{1}{|B|}\int_B |T_3 f(y)|dy + \frac{1}{|B|}\int_B |T_3 f_1(y)|dy + \frac{1}{|B|}\int_B |T_3 f_2(y) - T_{3,r}f(x)|dy \\
&\le M(T_3 f)(x) + (M(|f|^{p_0'})(x))^{1/p_0'} + \frac{1}{|B|}\int_B |T_3 f_2(y) - T_{3,r}f(x)|dy.
\end{aligned}$$

For the third term in the last inequality, we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B |T_3 f_2(y) - T_{3,r}f(x)|dy \\
&\le \frac{1}{|B|}\int_B \left|\int_{(16B)^c} K_3(y,\xi)f_2(\xi)d\xi - \int_{|x-\xi|>r} K_3(x,\xi)f(\xi)d\xi\right| dy \\
&\le \frac{1}{|B|}\int_B \int_{|x-\xi|>r} |K_3(y,\xi) - K_3(x,\xi)||f(\xi)|d\xi dy.
\end{aligned}$$

Let $I_{3,x} = \int_{|x-\xi|>r} |K_3(y,\xi) - K_3(x,\xi)||f(\xi)|d\xi$. Because $y \in B, h = |y-x| < 1/16r < |x-\xi|$, by use of (10) of Lemma 1.6 we have

$$\begin{aligned}
&I_{3,x} = \int_{|x-\xi|>r} |K_3(y,\xi) - K_3(x,\xi)||f(\xi)|d\xi \\
&\le C\int_{|x-\xi|>r} \frac{C_l}{\{1+m(x,V)|x-\xi|\}^l}\frac{r^\delta}{|x-\xi|^{n-1+\delta}}\left[\int_{B(\xi,|\xi-x|)} \frac{V(u)}{|\xi-u|^{n-1}}du\right]|f(\xi)|d\xi \\
&+ \int_{|x-\xi|>r} \frac{r^\delta}{|x-\xi|^{n+\delta}}|f(\xi)|d\xi \\
&= I^1_{3,x} + I^2_{3,x}.
\end{aligned}$$

For $I^2_{3,x}$, we have

$$\begin{aligned}
I^2_{3,x} &= \int_{|x-\xi|>r} \frac{r^\delta}{|x-\xi|^{n+\delta}}|f(\xi)|d\xi \\
&\le Cr^\delta \sum_{k=0}^{\infty} \int_{2^k r<|x-\xi|\le 2^{k+1}r} \frac{1}{(2^k r)^{n+\delta}}|f(\xi)|d\xi \\
&\le Cr^\delta \sum_{k=0}^{\infty} \frac{1}{(2^k r)^\delta}(M(|f|^{p_1'})(x))^{1/p_1'} \\
&\le C(M(|f|^{p_1'})(x))^{1/p_1'}.
\end{aligned}$$

Because $|u-\xi|<|x-\xi|$ yields $|x-u|\le|x-\xi|+|\xi-u|\le 2|x-\xi|$, by (10) of Lemma 1.6 and the fractional integral for $\frac{1}{p_1}=\frac{1}{q}-\frac{1}{n}$, we have

$$\begin{aligned}
I^1_{3,x} &\le C\sum_{k=0}^{\infty}\int_{2^k r<|x-\xi|\le 2^{k+1}r}\frac{C_l|f(\xi)|}{\{1+m(x,V)|x-\xi|\}^l}\frac{r^\delta}{|x-\xi|^{n-1+\delta}}\times\\
&\quad\left[\int_{B(\xi,2^{k+1}r)}\frac{V(u)}{|\xi-u|^{n-1}}du\right]d\xi\\
&\le C\sum_{k=0}^{\infty}\frac{C_l}{\{1+m(x,V)2^k r\}^l}\frac{r^\delta}{(2^k r)^{n-1+\delta}}\left\|\int\frac{V(u)\chi_{B(x,2^{k+1}r)}(u)}{|\xi-u|^{n-1}}du\right\|_{L^{p_1}(d\xi)}\times\\
&\quad\left(\int_{B(x,2^{k+1}r)}|f(\xi)|^{p_1'}d\xi\right)^{1/p_1'}\\
&\le C\sum_{k=0}^{\infty}\frac{(M(|f|^{p_1'})(x))^{1/p_1'}}{\{1+m(x,V)2^k r\}^l}\frac{r^\delta}{(2^k r)^{n-1+\delta}}\left(\int_{B(x,2^{k+1}r)}V^q(\xi)d\xi\right)^{1/q}(2^{k+1}r)^{n/p_1'}\\
&\le C\sum_{k=0}^{\infty}\frac{r^\delta}{(2^k r)^{n-1+\delta}}(2^{k+1}r)^{n/q-n}(2^(k+1)r)^{n/p_1'}\frac{1}{\{1+m(x,V)2^k r\}^l}\left(\int_{B(x,2^{k+1}r)}V(\xi)d\xi\right)\\
&\le C(M(|f|^{p_1'})(x))^{1/p_1'}\sum_{k=0}^{\infty}\frac{r^\delta}{(2^k r)^{n-1+\delta}}(2^k r)^{n/q-n+(n/p_1')+n-2}\\
&\le C(M(|f|^{p_1'})(x))^{1/p_1'}.
\end{aligned}$$

Here we have used the fact that $n/q-n+(n/p_1')+n-2=n-1$ and $1/p_1=1/q-1/n$. Finally, in a similar manner to proving Lemma 1.4, we can get $T_{3,Max}f(x)\le M(T_3f)(x)+C(M(|f|^{p_1'})(x))^{1/p_1'}$. This completes the proof of Lemma 1.7. □

2. The compactness of $[b,T_i],(i=1,2,3)$

First of all, we discuss the compactness of $[b,T_1]$ on L^p.

Theorem 2.1. *Suppose $V\in B_q$, $q>n/2$. If $T_1=(-\triangle+V)^{-1}V$, and $b\in VMO(R^n)$, then $[b,T_1]$ is a compact operator on L^p for $q'<p<\infty$.*

Proof. According to Remark 0.8, we only need to prove that $[b,T_1]$ satisfies the conditions $(c2)'$ and $(c3)'$.

Step I: The Proof of $(c2)'$. According to Remark 0.7, we may assume that $b\in C_0^\infty(R^n)$ with supp $b\subset B(0,R)$, the ball of radius R with center at origin.

For $v > 0$, set $B^c = \{x \in R^n : |x| > vR\}$. Then have

$$\left(\int_{|x|>vR} |[b, T_1]f(x)|^p dx\right)^{1/p} \le \left(\int_{|x|>vR} \left(\int_{|y|<R} |K_1(x, y)||b(y)||f(y)|dy\right)^p dx\right)^{1/p}.$$

Lemma 2.2. *For any $x \in B^c$, we have uniformly*

$$I_x = \int_{|y|<R} |K_1(x, y)||b(y)||f(y)|dy \le C|x|^{n/q-n}\|f\|_p R^{n/q'-n/p}.$$

Proof of Lemma 2.2. Because $|x| > vR$ and $|y| < R$ imply $|x - y| > (1 - \frac{1}{v})|x|$ for $v > 2$, by use of (5) of Lemma 1.3, we have

$$\begin{aligned} I_x &\le C_l \int_{|y|<R} \frac{1}{\{1 + m(x, V)|x - y|\}^l} \frac{1}{|x - y|^{n-2}} V(y)|f(y)|dy \\ &\le \frac{C_l}{\{1 + m(x, V)(1 - 1/v)|x|\}^l} \frac{1}{(1 - \frac{1}{v})^{n-2}|x|^{n-2}} \left(\int_{|y|<R} V^q(y)dy\right)^{1/q} \times \\ &\quad \left(\int_{|y|<R} |f(y)|^{q'}|b(y)|^{q'} dy\right)^{1/q'} \\ &\le \frac{C_l}{\{1 + m(x, V)(1 - 1/v)|x|\}^l} \frac{1}{(1 - \frac{1}{v})^{n-2}|x|^{n-2}} \left(\int_{|y|<R} V^q(y)dy\right)^{1/q} \times \\ &\quad \left(\int_{|y|<R} |f(y)|^p dy\right)^p R^{n(\frac{1}{q'} - \frac{1}{p})}. \end{aligned}$$

In the last inequality, we have used $p > q'$, $\|b\|_\infty \le C$ and Hölder's inequality. Notice that for $|x| > vR$ and $|y| < R$, we have $|y| < \frac{1}{v}|x|$. So if $|y| < R$, then $|x - y| < (1 + \frac{1}{v})|x| < 2|x|$. As a result we get

$$I_x \le \frac{C_K}{\{1 + m(x, V)(1 - 1/v)|x|\}^l} \frac{1}{(1 - \frac{1}{v})^{n-2}|x|^{n-2}} \left(\int_{B(x,2|x|)} V^q(y)dy\right)^{1/q} \|f\|_p R^{n(\frac{1}{q'} - \frac{1}{p})}.$$

For every $y \in B(x, 2|x|)$, $2|x| = \frac{2}{1-\frac{1}{v}}(1 - \frac{1}{v})|x| = (2 + \frac{2}{v-1})(1 - \frac{1}{v})|x| \le 3(1 - \frac{1}{v})|x|$ and $(\frac{1}{1-1/v})^{n-2} = (1 + \frac{1}{v-1})^{n-2} \le C$ when $v \ge 3$. By $V \in B_q$, Lemma 0.3 and the doubling property of $V(x)dx$, we have

$$\begin{aligned}
I_x \leq & \frac{C_l}{\{1+m(x,V)(1-1/v)|x|\}^l} \frac{\|f\|_p R^{n(\frac{1}{q'}-\frac{1}{p})}}{(1-\frac{1}{v})^{n-2}|x|^{n-2}} |x|^{n/q-2} \left(\int_{B(x,2|x|)} V(y)dy \right) \\
\leq & \frac{C_l}{\{1+m(x,V)(1-1/v)|x|\}^l} \frac{\|f\|_p R^{n(\frac{1}{q'}-\frac{1}{p})}}{(1-\frac{1}{v})^{n-2}} |x|^{2+n/q-2n} \left(\int_{B(x,2|x|)} V(y)dy \right) \\
\leq & C\|f\|_p R^{n(\frac{1}{q'}-\frac{1}{p})} |x|^{2+n/q-2n} \frac{C_l}{\{1+m(x,V)(1-1/v)|x|\}^l} \left(\int_{B(x,3(1-1/v)|x|)} V(y)dy \right) \\
\leq & C\|f\|_p R^{n(\frac{1}{q'}-\frac{1}{p})} |x|^{2+n/q-2n} |x|^{n-2} \\
\leq & C|x|^{n/q-n} \|f\|_p R^{n(\frac{1}{q'}-\frac{1}{p})}.
\end{aligned}$$

This completes the proof of Lemma 2.2.

Now, by use of Lemma 2.2, we can complete the proof of condition $(c2)'$. In fact, for $p > q'$, we have $np - np/q - n + 1 = np/q' - n + 1$,

$$\begin{aligned}
\left(\int_{|x|>vR} |[b,T_1]f(x)|^p dx \right)^{1/p} \leq & C\|f\|_p R^{n/q'-n/p} \left(\int_{|x|>vR} |x|^{np/q-np} dx \right)^{1/p} \\
\leq & C\|f\|_p R^{n/q'-n/p} (vR)^{n/p-n/q'} \\
\leq & \frac{C}{v^{n/q'-n/p}}.
\end{aligned}$$

Since $p > q'$, for every $\varepsilon > 0$, we can choose v large enough such that $\frac{1}{v^{n/q'-n/p}} < \varepsilon$.

Step II: The proof of $(c3)'$. We will prove: for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $\|[b,T_1]f(\cdot+z) - [b,T_1]f(\cdot)\|_p < \varepsilon$ if $|z| < \delta_\varepsilon$.

For every x, we divide $[b,T_1]f(x+z)-[b,T_1]f(x)$ into four parts as follows:

$$\begin{aligned}
&[b,T_1]f(x+z)-[b,T_1]f(x)\\
&=\int K_1(x+z,y)[b(x+z)-b(y)]f(y)dy-\int K_1(x,y)[b(x)-b(y)]f(y)dy\\
&=\int_{|x-y|>a|z|} K_1(x,y)[b(x)-b(x+z)]f(y)dy\\
&+\int_{|x-y|>a|z|}[K_1(x,y)-K_1(x+z,y)][b(x+z)-b(y)]f(y)dy\\
&+\int_{|x-y|<a|z|} K_1(x,y)[b(x)-b(y)]f(y)dy\\
&-\int_{|x-y|<a|z|} K_1(x+z,y)[b(x+z)-b(y)]f(y)dy\\
&=I_{1,x}+I_{2,x}+I_{3,x}+I_{4,x}.
\end{aligned}$$

This derives $\|[b,T_1]f(\cdot+z)-[b,T_1]f(\cdot)\|_p\le\sum_{i=1}^4\|I_{i,x}\|_p$.

Clearly, by Definition 1.1 and $b\in C_0^\infty$, we have $|I_{1,x}|\le|z|T_{1,Max}f(x)$. So for $p>q'$, by Lemma 1.2, we have $\|I_{1,x}\|_p\le|z|\|T_{1,Max}f\|_p\le C|z|\|f\|_p$.

For $I_{2,x}$, we write $a > 16$. By (6) of Lemma 1.3, Lemma 0.3 and $\|b\|_\infty \le C$, we have

$$
\begin{aligned}
|I_{2,x}| &\le \int_{|x-y|>a|z|} |k_1(x+z,y) - K_1(x,y)||b(x+z)-b(y)||f(y)|dy \\
&\le C\sum_{k=0}^{\infty}\int_{2^k a|z|<|x-y|\le 2^{k+1}a|z|} \frac{C_l}{\{1+m(x,V)|x-y|\}^l}\frac{|z|^\delta}{|x-y|^{n-2+\delta}}V(y)|f(y)|dy \\
&\le C\sum_{k=0}^{\infty}\frac{|z|^\delta}{\{1+m(x,V)2^k a|z|\}^l}\frac{1}{(2^k a|z|)^{n-2+\delta}}\int_{|x-y|\le 2^{k+1}a|z|} V(y)|f(y)|dy \\
&\le C\sum_{k=0}^{\infty}\frac{|z|^\delta(2^k a|z|)^{-(n-2+\delta)}}{\{1+m(x,V)2^k a|z|\}^l}\Big(\int_{|x-y|\le 2^{k+1}a|z|} V^q(y)dy\Big)^{1/q}\Big(\int_{|x-y|\le 2^{k+1}a|z|} |f(y)|^{q'}dy\Big)^{1/q'} \\
&\le C\sum_{k=0}^{\infty}\frac{|z|^\delta(2^{k+1}a|z|)^{n/q-n+n/q'}}{\{1+m(x,V)2^k a|z|\}^l}\frac{(M(|f|^{q'})(x))^{1/q'}}{(2^k a|z|)^{n-2+\delta}}\Big(\int_{B(x,2^{k+1}a|z|)} V(y)dy\Big) \\
&\le C\sum_{k=0}^{\infty}\frac{|z|^\delta}{(2^k a|z|)^{n-2+\delta}}\frac{(M(|f|^{q'})(x))^{1/q'}}{\{1+m(x,V)2^k a|z|\}^l}\int_{B(x,2^k a|z|)} V(y)dy \\
&\le C\sum_{k=0}^{\infty}|z|^\delta\frac{(2^k a|z|)^{n-2}}{(2^k a|z|)^{n-2+\delta}}(M(|f|^{q'})(x))^{1/q'} \\
&\le C\frac{1}{a^\delta}(M(|f|^{q'})(x))^{1/q'}.
\end{aligned}
$$

So we have $\|I_{2,x}\|_p \le C\frac{1}{a^\delta}\|(M(|f|^{q'}))^{1/q'}\|_p \le C\frac{1}{a^\delta}\|f\|_p$, for $p > q'$.

For $I_{3,x}$, by use of (5) of Lemma 1.3 and $b \in C_0^\infty$ we have

$$
\begin{aligned}
|I_{3,x}| &\le \int_{|x-y|<a|z|} |K_1(x,y)||x-y||f(y)|dy \\
&\le \int_{|x-y|<a|z|} \frac{C_l}{\{1+m(x,V)|x-y|\}^l}\frac{1}{|x-y|^{n-3}}V(y)|f(y)|dy \\
&\le \sum_{j=-\infty}^{0}\int_{2^{j-1}a|z|<|x-y|\le 2^j a|z|} \frac{C_K}{\{1+m(x,V)|x-y|\}^l}\frac{1}{|x-y|^{n-3}}V(y)|f(y)|dy \\
&\le \sum_{j=-\infty}^{0}\frac{1}{(2^{j-1}a|z|)^{n-3}}\frac{C_l}{\{1+m(x,V)2^{j-1}a|z|\}^l}\int_{2^{j-1}a|z|<|x-y|\le 2^j a|z|} V(y)|f(y)|dy.
\end{aligned}
$$

Notice that $V \in B_q$. So using Hölder's inequality and Lemma 0.3 we can get

$$
\begin{aligned}
|I_{3,x}| &\le \sum_{j=-\infty}^{0} \frac{C_l (2^{j-1}a|z|)^{3-n}}{\{1+m(x,V)2^{j-1}a|z|\}^l} \left(\int_{|x-y|<2^j a|z|} V^q(y)dy \right)^{1/q} \times \\
&\quad \left(\int_{|x-y|<2^j a|z|} |f(y)|^{q'} dy \right)^{1/q'} \\
&\le \sum_{j=-\infty}^{0} \frac{1}{(2^{j-1}a|z|)^{n-3}} \frac{C_l (M(|f|^{q'})(x))^{1/q'}}{\{1+m(x,V)2^{j-1}a|z|\}^l} (2^j a|z|)^{n/q-n+n/q'} \left(\int_{B(x,2^j a|z|)} V(y)dy \right) \\
&\le \sum_{j=-\infty}^{0} \frac{1}{(2^{j-1}a|z|)^{n-3}} (2^j a|z|)^{n-2} (M(|f|^{q'})(x))^{1/q'} \\
&\le Ca|z|(M(|f|^{q'})(x))^{1/q'} \sum_{j=-\infty}^{0} 2^{j(n-2)} \\
&\le Ca|z|(M(|f|^{q'})(x))^{1/q'}.
\end{aligned}
$$

Thus, we have $\|I_{3,x}\|_p \le Ca|z|\|(M(|f|^{q'}))^{1/q'}\|_p \le Ca|z|\|f\|_p$ for $p > q'$.

Similarly we can estimate $I_{4,x}$. Because $|x-y| < a|z|$, we have $|x+z-y| < (a+1)|z|$. Notice that $V \in B_q$. So, by use of (5) of Lemma 1.3 and Hölder's inequality we have

$$
\begin{aligned}
|I_{4,x}| &\le \int_{|x+z-y|<(a+1)|z|} |K_1(x+z,y)||b(x+z)-b(y)||f(y)|dy \\
&\le \int_{|x+z-y|<(a+1)|z|} \frac{C_l}{\{1+m(x+z,V)|x+z-y|\}^l} \frac{1}{|x+z-y|^{n-3}} V(y)|f(y)|dy \\
&\le \sum_{j=-\infty}^{0} \int_{2^{j-1}(a+1)|z| \le |x+z-y| < 2^j(a+1)|z|} \frac{C_l |x+z-y|^{3-n}}{\{1+m(x+z,V)|x+z-y|\}^l} V(y)|f(y)|dy \\
&\le \sum_{j=-\infty}^{0} \frac{C_l (2^{j-1}(a+1)|z|)^{3-n}}{\{1+m(x+z,V)2^{j-1}(a+1)|z|\}^l} \int_{2^{j-1}(a+1)|z| \le |x+z-y| < 2^j(a+1)|z|} V(y)|f(y)|dy \\
&\le \sum_{j=-\infty}^{0} \frac{(M(|f|^{q'})(x))^{1/q'}}{(2^{j-1}(a+1)|z|)^{n-3}} \frac{C_l (2^j(a+1)|z|)^{n/q'}}{\{1+m(x+z,V)2^{j-1}(a+1)|z|\}^l} \left(\int_{B(x+z,2^j(a+1)|z|)} V^q(y)dy \right)^{1/q}.
\end{aligned}
$$

Then, by Hölder's inequality and Lemma 0.3 we get

$$
\begin{aligned}
|I_{4,x}| &\leq \sum_{j=-\infty}^{0} \frac{(M(|f|^{q'})(x))^{1/q'}}{(2^{j-1}(a+1)|z|)^{n-3}} \frac{C_l(2^j(a+1)|z|)^{n/q-n+n/q'}}{\{1+m(x+z,V)2^{j-1}(a+1)|z|\}^l} \times \\
&\qquad \left(\int_{B(x+z,2^j(a+1)|z|)} V(y)dy\right) \\
&\leq \sum_{j=-\infty}^{0} \frac{1}{(2^{j-1}(a+1)|z|)^{n-3}} (2^j(a+1)|z|)^{n-2} (M(|f|^{q'})(x))^{1/q'} \\
&\leq C(a+1)|z|(M(|f|^{q'})(x))^{1/q'}.
\end{aligned}
$$

We have $\|I_{4,x}\|_p \leq C(a+1)|z|\|(M(|f|^{q'}))^{1/q'}\|_p \leq C(a+1)|z|\|f\|_p$ for $p > q'$.

Finally, we get

$$
\begin{aligned}
&\|[b,T_1]f(\cdot+z) - [b,T_1]f(\cdot)\|_p \\
&\leq \sum_{i=1}^{4} \|I_{i,x}\|_p \\
&\leq C|z|\|f\|_p + C\frac{1}{a^\delta}\|f\|_p + Ca|z|\|f\|_p + C(a+1)|z|\|f\|_p.
\end{aligned}
$$

Consequently, for every $\varepsilon > 0$ we can choose a large enough such that $\max\{\frac{1}{a^2}, \frac{1}{(a+1)^2}, \frac{1}{a^\delta}, \} < \varepsilon$, and set $|z|$ be small enough, say $|z| < \min\{\frac{1}{a^2}, \frac{1}{(a+1)^2}\}$. From this we can see that the δ_ε in $(c3)'$ is $\max\{\frac{1}{a^2}, \frac{1}{(a+1)^2}, \frac{1}{a^\delta}\}$. This completes the proof of Theorem 2.1. □

By duality, we have the following corollary.

Corollary 2.3. *Suppose $V \in B_q$, $q > n/2$. Let $T_1^* = V(-\triangle + V)^{-1}$ be the dual operator of T_1. If $b \in VMO(R^n)$, then have $[b, T_1^*]$ is a compact operator on $L^p(R^n)$, $1 < p < q$.*

In order to prove the compactness of the commutator $[b, T_2]$, we only need to prove the following lemma.

Lemma 2.4. *Suppose $b \in C_0^\infty(R^n)$ with supp b=$B(0,R)$. Then for any $x, |x| > vR$ and $p > (2q)'$ we have*

$$
A_x = \int_{|y|<R} |K_2(x,y)||b(y)||f(y)|dy \leq C|x|^{n/2q-n}\|f\|_p R^{n/(2q)'-n/p}.
$$

Proof. By (7) of Lemma 1.5, the implication: $|x| > vR, |y| < R \Longrightarrow |x-y| > (1-\frac{1}{v})|x|$ for $v > 2$ and Hölder's inequality, we have

$$\begin{aligned} A_x &= \int_{|y|<R} |K_2(x,y)||b(y)||f(y)|dy \\ &\leq \int_{|y|<R} \frac{C_l}{\{1+m(x,V)|x-y|\}^l} \frac{1}{|x-y|^{n-1}} V^{1/2}(y)|f(y)||b(y)|dy \\ &\leq \frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l} \frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} \int_{|y|<R} V^{1/2}(y)|f(y)||b(y)|dy \\ &\leq \frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l} \frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} \left(\int_{|y|<R} V^q(y)dy\right)^{1/2q} \times \\ &\quad \left(\int_{|y|<R} (|f(y)||b(y)|)^{(2q)'}dy\right)^{1/(2q)'}. \end{aligned}$$

Because $p > (2q)'$, using $b \in L^\infty$ and Hölder's inequality again, we have

$$A_x \leq \frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l} \frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} (\int_{|y|<R} V^q(y)dy)^{1/2q} \|f\|_p R^{n(\frac{1}{(2q)'}-\frac{1}{p})}.$$

For every x, we have $2|x| = \frac{2}{1-\frac{1}{v}}(1-\frac{1}{v})|x| = (2+\frac{2}{v-1})(1-\frac{1}{v})|x| \leq 3(1-\frac{1}{v})|x|$ and $(\frac{1}{1-1/v})^{n/2} = (1+\frac{1}{v-1})^{n/2} \leq C$ when $v \geq 3$. So, by $V \in B_q$, Lemma 0.3 and the double property of $V(x)dx$ we have

$$\begin{aligned} A_x &\leq \frac{C_l|x|^{n/2q-n/2}}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l} \frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} (\int_{B(x,3|x|)} V(y)dy)^{1/2} \|f\|_p R^{n(\frac{1}{(2q)'}-\frac{1}{p})} \\ &\leq C\|f\|_p R^{n(\frac{1}{(2q)'}-\frac{1}{p})} \frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} (1-\frac{1}{v})^{\frac{n}{2}-1}|x|^{\frac{n}{2}-1}|x|^{n/2q-n/2} \\ &\leq C\frac{1}{(1-\frac{1}{v})^{n/2}} \|f\|_p R^{n(\frac{1}{(2q)'}-\frac{1}{p})}|x|^{n/2q-n} \\ &\leq C\|f\|_p R^{n(\frac{1}{(2q)'}-\frac{1}{p})}|x|^{n/2q-n}. \end{aligned}$$

This completes the proof of Lemma 2.4. □

Theorem 2.5. *Suppose $V \in B_q$, $q > n/2$ and let $T_2 = (-\triangle + V)^{-1/2}V^{1/2}$. If $b \in VMO(R^n)$, then commutator $[b, T_2]$ is a compact operator on L^p for $(2q)' < p < \infty$.*

Proof. The proof is similar to that of Theorem 2.1, we omit the details. □

Corollary 2.6. *Suppose $V \in B_q$, $q > n/2$ and let $T_2^* = V^{/1/2}(-\triangle + V)^{-1/2}$ the dual operator of T_2. If $b \in VMO(R^n)$, then the commutator $[b, T_2^*]$ is a compact operator on L^p for $1 < p < 2q$.*

Theorem 2.7. *Suppose $V \in B_q$, $q > n/2$ and let $T_3 = (-\triangle + V)^{-1/2}\nabla$. If $b \in VMO(R^n)$, the commutator $[b, T_3]$ is a compact operator on L^p for $(p_1)' < p < \infty$ and $1/p_1 = 1/q - 1/n$.*

Proof. By Remark 0.7, we only need to prove that $[b, T_3]$ satisfies the conditions $(c2)'$ and $(c3)'$. We divide the proof into two steps.

Step I: The proof of $(c2)'$. Suppose the support set of b is $B(0, R)$. For $v > 0$ we have

$$I \leq \left(\int_{|x|>vR} \left(\int_{|y|<R} |K_3(x,y)||b(y)||f(y)|dy \right)^p dx \right)^{1/p}.$$

By (9) of Lemma 1.6, we have

$$|K_3(x,y)| \leq \frac{C_l}{\{1 + m(x,V)|x-y|\}^l} \frac{1}{|x-y|^{n-1}} \left(\int_{B(y,|x-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi \right) + \frac{1}{|x-y|^n}.$$

Then we divide I into I_1 and I_2, where

$$I_1 \leq \left(\int_{|x|>vR} \left(\int_{|y|<R} \frac{C_l |b(y)||f(y)|}{\{1 + m(x,V)|x-y|\}^l} \frac{1}{|x-y|^{n-1}} (\int_{B(y,|x-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi) dy \right)^p dx \right)^{1/p},$$

$$I_2 = \left(\int_{|x|>vR} \left(\int_{|y|<R} \frac{1}{|x-y|^n} |b(y)||f(y)|dy \right)^p dx \right)^{1/p}.$$

For I_2, because $|x| > vR$ and $|y| < R$, one has $|y| < \frac{1}{v}|x|$ and $|x - y| \geq (1 - \frac{1}{v})|x|$ for $v > 2$. Notice that for $b \in C_0^\infty(R^n)$ we have $\|b\|_\infty \leq C$. Therefore, by Hölder's inequality we have

$$
\begin{aligned}
I_2 &= \left(\int_{|x|>vR} (\int_{|y|<R} \frac{1}{|x-y|^n} |b(y)||f(y)|dy)^p dx \right)^{1/p} \\
&\leq \left(\int_{|x|>vR} \frac{1}{(1-\frac{1}{v})^{np}|x|^{np}} (\int_{|y|<R} |b(y)||f(y)|dy)^p dx \right)^{1/p} \\
&\leq \left(\int_{|x|>vR} \frac{1}{(1-\frac{1}{v})^{np}|x|^{np}} \|f\|_p^p (\int_{|y|<R} |b(y)|^{p'} dx)^{p/p'} dx \right)^{1/p} \\
&\leq \frac{1}{(1-\frac{1}{v})^n} \|f\|_p R^{n/p'} \left(\int_{|x|>vR} \frac{|x|^{n-1}}{|x|^{np}} d|x| \right)^{1/p} \\
&\leq C(1+\frac{1}{v-1})^n \|f\|_p R^{n/p'} \frac{1}{(vR)^{n-n/p}} \\
&\leq \frac{C}{v^{n/p'}} \|f\|_p,
\end{aligned}
$$

where in the last inequality, we have used the fact that for $v > 2$, $(1+\frac{1}{v-1})^n < 2^n$.

It remains to estimate I_1. For every $|x| > vR$, we write

$$
I_{1,x} = \int_{|y|<R} \frac{C_K}{\{1+m(x,V)|x-y|\}^K} \frac{1}{|x-y|^{n-1}} \left(\int_{B(y,|x-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi \right) |b(y)||f(y)|dy.
$$

Lemma 2.8. *For $|x| > vR$, Let $I_{1,x}$ be the same as before. Then*

$$
|I_{1,x}| \leq CR^{n(1/p_1'-1/p)}|x|^{-n/p_1'}\|f\|_p.
$$

Proof of Lemma 2.8. Because $|x| > vR$ and $|y| < R$ yield $|x-y| > (1-\frac{1}{v})|x|$, setting $v > 2$ we use the fractional integral for $\frac{1}{p_1} = \frac{1}{q} - \frac{1}{n}$ to obtain

$$\begin{aligned}
I_{1,x} &\le \frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l}\frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} \times \\
&\qquad \int_{|y|<R}\left(\int_{B(y,(1+\frac{1}{v})|x|)}\frac{V(\xi)}{|y-\xi|^{n-1}}d\xi\right)|b(y)||f(y)|dy \\
&\le \frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l}\frac{1}{(1-\frac{1}{v})^{n-1}|x|^{n-1}} \times \\
&\qquad \int_{|y|<R}\left(\int_{R^n}\frac{V(\xi)\chi_{B(x,2(1+1/v)|x|)}}{|y-\xi|^{n-1}}d\xi\right)|b(y)||f(y)|dy \\
&\le \frac{C_l|x|^{1-n}}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l}\left\|\int_{R^n}\frac{V(\xi)\chi_{B(x,2(1+1/v)|x|)}}{|y-\xi|^{n-1}}d\xi\right\|_{L^{p_1}(dy)} \times \\
&\qquad \left(\int_{|y|<R}|b(y)|^{p_1'}|f(y)|^{p_1'}dy\right)^{1/p_1'} \\
&\le C\frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l}\frac{1}{|x|^{n-1}}\left(\int_{B(x,2(1+1/v)|x|)}V^q(\xi)d\xi\right)^{1/q}\|f\|_p R^{n(1-\frac{p_1'}{p})\frac{1}{p_1'}} \\
&\le C\frac{C_l}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l}\frac{1}{|x|^{n-1}}\left(\int_{B(x,2|x|)}V^q(\xi)d\xi\right)^{1/q}\|f\|_p R^{n(1-\frac{p_1'}{p})\frac{1}{p_1'}}.
\end{aligned}$$

In the last inequality, we have used the double property of $V(x)dx$ and $1+\frac{1}{v} < 2$ for $v > 2$.

As before, $2|x| = \frac{2}{1-\frac{1}{v}}(1-\frac{1}{v})|x| = (2+\frac{2}{v-1})(1-\frac{1}{v})|x| \le 3(1-\frac{1}{v})|x|$ for $v \ge 3$ and $(\frac{1}{1-1/v})^{n-2} = (1+\frac{1}{v-1})^{n-2} \le C$ for $v \ge 3$. By use of $V \in B_q$, Lemma 0.3 and the double property of $V(x)dx$, we have

$$\begin{aligned}
I_{1,x} &\le \frac{C_l\|f\|_p R^{n(1/p_1'-1/p)}}{\{1+m(x,V)(1-\frac{1}{v})|x|\}^l|x|^{n-1}}[(1-\frac{1}{v})|x|]^{n/q-n}\left(\int_{B(x,3(1-\frac{1}{v})|x|)}V(\xi)d\xi\right) \\
&\le C[(1-\frac{1}{v})|x|]^{n/q-n}[(1-\frac{1}{v})|x|]^{n-2}\|f\|_p R^{n(1/p_1'-1/p)} \\
&\le C(1+\frac{1}{v-1})^{2-n/q}|x|^{n/q-2}\|f\|_p R^{n(1/p_1'-1/p)} \\
&\le C|x|^{n/q-2}\|f\|_p R^{n(1/p_1'-1/p)}.
\end{aligned}$$

In the last inequality, we have used the fact $n/2 < q < n$ and $2 - n/q > 0$ imply $(1 + \frac{1}{v-1})^{2-n/q} \leq C$ when v large enough. This completes the proof of Lemma 2.8.

Now we return to the proof of Step I of Theorem 2.7. By Lemma 2.8, we have

$$I_1 \leq (\int_{|x|>vR} |I_{1,x}|^p dx)^{1/p} \leq CR^{n/p_1'-n/p}\|f\|_p(\int_{|x|>vR} \frac{1}{|x|^{np/p_1'-n+1}} d|x|)^{1/p}$$
$$\leq CR^{n/p_1'-n/p}\|f\|_p \frac{1}{(vR)^{n/p_1'-n/p}} \leq \frac{C}{v^{n/p_1'-n/p}}\|f\|_p.$$

Because $p > p_1'$, for every $\varepsilon > 0$ we can choose v large enough so that $\frac{C}{v^{n/p_1'-n/p}} < \varepsilon$ and $B = B(0, vR)$. This completes the proof of $(c2)'$.

Step II: The proof of $(c3)'$. For every x, we divide $[b, T_3]f(x+z) - [b, T_3]f(x)$ into four parts. In fact, we have

$$[b, T_3]f(x + z) - [b, T_3]f(x)$$
$$= \int K_3(x + z, y)[b(x + z) - b(y)]f(y)dy - \int K_3(x, y)[b(x) - b(y)]f(y)dy$$
$$= B_{1,x} + B_{2,x} + B_{3,x} + B_{4,x},$$

where

$$B_{1,x} = \int_{|x-y|>a|z|} K_3(x, y)[b(x) - b(x + z)]f(y)dy$$
$$B_{2,x} = \int_{|x-y|>a|z|} [K_3(x, y) - K_1(x + z, y)][b(x + z) - b(y)]f(y)dy$$
$$B_{3,x} = \int_{|x-y|<a|z|} K_3(x, y)[b(x) - b(y)]f(y)dy$$
$$B_{4,x} = \int_{|x-y|<a|z|} K_3(x + z, y)[b(x + z) - b(y)]f(y)dy.$$

Obviously $\|[b, T_3]f(\cdot + z) - [b, T_3]f(\cdot)\|_p \leq \sum_{i=1}^{4} \|B_{i,x}\|_p$. In the following we estimate $B_{i,x}, (i = 1, 2, 3, 4)$ separately.

For $B_{1,x}$, because $b \in C_0^\infty$, we have $|b(x + z) - b(x)| \leq C|z|$ and then

$$|B_{1,x}| = |\int_{|x-y|>a|z|} K_3(x, y)[b(x) - b(x + z)]f(y)dy| \leq C|z|T_{3,Max}(f)(x).$$

So by Lemma 1.7, for $p > p_1'$ and $1/p_1 = 1/q-1/n$ we have $\|B_{1,x}\|_p \leq C|z|\|T_{3,Max}f\|_p \leq C|z|\|f\|_p$.

For $B_{2,x}$, by (10) of Lemma 1.6 and letting $a > 16$, we have

$$|B_{2,x}| \leq \int_{|x-y|>a|z|} |K_3(x+z,y) - K_3(x,y)||b(x+z) - b(y)||f(y)|dy$$
$$\leq \int_{|x-y|>a|z|} \frac{C_l|b(x+z) - b(y)||f(y)|}{\{1+m(x,V)|x-y|\}^l} \frac{|z|^\delta}{|x-y|^{n-1+\delta}}[(\int_{B(y,|x-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}}d\xi) + \frac{1}{|x-y|}]dy$$
$$\leq B^1_{2,x} + B^2_{2,x}.$$

For $B^2_{2,x}$, because $\|b\|_\infty \leq C$, we have

$$B^2_{2,x} = \int_{|x-y|\geq a|z|} \frac{|z|^\delta}{|x-y|^{n+\delta}}|f(y)|dy$$
$$\leq |z|^\delta \sum_{j=0}^{\infty} \int_{2^ja|z|\leq|x-y|<2^{j+1}a|z|} \frac{1}{|x-y|^{n+\delta}}|f(y)|dy$$
$$\leq C\sum_{j=0}^{\infty} \frac{|z|^\delta}{(2^ja|z|)^{n+\delta}} \int_{B(x,2^{j+1}a|z|)} |f(y)|dy$$
$$\leq C\sum_{j=0}^{\infty} \frac{|z|^\delta}{(2^ja|z|)^\delta} M(f)(x) \leq \frac{C}{a^\delta}M(f)(x).$$

For $B^1_{2,x}$, by Hölder's inequality and $V \in B_q$, we have

$$|B^1_{2,x}| \leq \int_{2^ja|z|\leq|x-y|<2^{j+1}a|z|} \frac{C_l|f(y)|}{\{1+m(x,V)2^ja|z|\}^l} \frac{|z|^\delta}{(2^ja|z|)^{n-1+\delta}} \times$$
$$(\int_{B(x,2^{j+3}a|z|)} \frac{V(\xi)}{|y-\xi|^{n-1}}d\xi)dy$$
$$\leq \sum_{j=0}^{\infty} \frac{C_l}{\{1+m(x,V)2^ja|z|\}^l} \frac{|z|^\delta}{(2^ja|z|)^{n-1+\delta}} \|\int_{R^n} \frac{V(\xi)\chi_{B(x,2^{j+3}a|z|)}}{|y-\xi|^{n-1}}d\xi\|_{p_1} \times$$
$$(\int_{|x-y|<2^{j+1}a|z|} |f(y)|^{p'_1}dy)^{1/p'_1}$$
$$\leq C\sum_{j=0}^{\infty} \frac{C_l(2^{j+1}a|z|)^{n/p'_1}(M(|f|^{p'_1})(x))^{1/p'_1}}{\{1+m(x,V)2^ja|z|\}^l} \frac{|z|^\delta}{(2^ja|z|)^{n-1+\delta}}(\int_{B(x,2^{j+3}a|z|)} V^q(\xi)d\xi)^{1/q}$$
$$\leq \sum_{j=0}^{\infty} \frac{C_l(M(|f|^{p'_1})(x))^{1/p'_1}}{\{1+m(x,V)2^ja|z|\}^l} \frac{|z|^\delta}{(2^ja|z|)^{n-1+\delta}}(2^{j+3}a|z|)^{n/q-n+n/p'_1}(\int_{B(x,2^{j+3}a|z|)} V(\xi)d\xi)$$
$$\leq C\sum_{j=0}^{\infty} \frac{|z|^\delta}{(2^ja|z|)^{n-1+\delta}}(2^ja|z|)^{n/q-n+n/p'_1} \frac{C_l(M(|f|^{p'_1})(x))^{1/p'_1}}{\{1+m(x,V)2^ja|z|\}^l} \int_{B(x,2^ja|z|)} V(\xi)d\xi.$$

Then by Lemma 0.3, we have

$$
\begin{aligned}
|B^1_{2,x}| &\leq C\sum_{j=0}^{\infty}\frac{|z|^\delta}{(2^ja|z|)^{n-1+\delta}}(2^ja|z|)^{n/q-n+n/p_1'}(2^ja|z|)^{n-2}(M(|f|^{p_1'})(x))^{1/p_1'}\\
&\leq C\sum_{j=0}^{\infty}|z|^\delta(2^ja|z|)^{n/q-n+n/p_1'-n+1-\delta+n-2}(M(|f|^{p_1'})(x))^{1/p_1'}\\
&\leq \frac{C}{a^\delta}(M(|f|^{p_1'})(x))^{1/p_1'}\sum_{j=0}^{\infty}\frac{1}{2^{j\delta}}\\
&\leq \frac{C}{a^\delta}(M(|f|^{p_1'})(x))^{1/p_1'}.
\end{aligned}
$$

In the above, we have used the fact: for $p > p_1'$, $n/q-n+n/p_1'-n+1-\delta+n-2 = -\delta$ because $1/p_1 = 1/q - 1/n$. Then we have

$$
\|B_{2,x}\|_p \leq \frac{C}{a^\delta}\|M(f)\|_p + \frac{C}{a^\delta}\|(M(|f|^{p_1'}))^{1/p_1'}\|_p \leq \frac{C}{a^\delta}\|f\|_p.
$$

For $B_{3,x}$, by (9) of Lemma 1.6 we have

$$
\begin{aligned}
|B_{3,x}| &\leq \int_{|x-y|<a|z|}|K_3(x,y)||b(x)-b(y)||f(y)|dy\\
&\leq \int_{|x-y|<a|z|}\frac{C_l}{\{1+m(x,V)|x-y|\}^l}\frac{|x-y|}{|x-y|^{n-1}}\Big(\int_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}d\xi+\frac{1}{|x-y|}\Big)|f(y)|dy\\
&\leq \int_{|x-y|<a|z|}\frac{C_l}{\{1+m(x,V)|x-y|\}^l}\frac{1}{|x-y|^{n-2}}\Big(\int_{B(y,|x-y|)}\frac{V(\xi)}{|y-\xi|^{n-1}}d\xi\Big)|f(y)|dy\\
&\quad+\int_{|x-y|<a|z|}\frac{1}{|x-y|^{n-1}}|f(y)|dy\\
&= B^1_{3,x}+B^2_{3,x}.
\end{aligned}
$$

Next, we estimate $B^1_{3,x}$ and $B^2_{3,x}$ separately. For $B^2_{3,x}$, we have

$$
B^2_{3,x} \leq C\sum_{j=-\infty}^{0}\frac{1}{(2^{j-1}a|z|)^{n-1}}\int_{B(x,2^ja|z|)\setminus B(x,2^{j-1}a|z|)}|f(y)|dy \leq Ca|z|M(f)(x).
$$

For $B^1_{3,x}$, because $\xi \in B(y,|x-y|)$, one has $|x-\xi| \le |x-y| + |y-\xi| \le 2|x-y|$, so by Hölder's inequality, we have

$$
\begin{aligned}
|B^1_{3,x}| &\le C \sum_{j=-\infty}^{0} \int_{B(x,2^j a|z|)\setminus B(x,2^{j-1}a|z|)} \frac{C_l|x-y|^{2-n}}{\{1+m(x,V)|x-y|\}^l} |f(y)|dy \times \\
&\quad (\int_{B(x,2|x-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi) \\
&\le C \sum_{j=-\infty}^{0} \frac{C_l(2^{j-1}a|z|)^{2-n}}{\{1+m(x,V)2^{j-1}a|z|\}^l} \int_{B(x,2^j a|z|)\setminus B(x,2^{j-1}a|z|)} |f(y)| \times \\
&\quad (\int_{B(x,2^{j+1}a|z|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi) dy \\
&\le C \sum_{j=-\infty}^{0} \frac{C_l(2^{j-1}a|z|)^{2-n}}{\{1+m(x,V)2^{j-1}a|z|\}^l} \| \int \frac{V(\xi)\chi_{B(x,2^{j+1}a|z|)}(\xi)}{|y-\xi|^{n-1}} d\xi \|_{L^{p_1}(dy)} \times \\
&\quad (\int_{B(x,2^j a|z|)\setminus B(x,2^{j-1}a|z|)} |f(y)|^{p_1'} dy)^{1/p_1'}.
\end{aligned}
$$

Using $V \in B_q$ and Lemma 0.3, we get

$$
\begin{aligned}
|B^1_{3,x}| &\le \sum_{j=-\infty}^{0} \frac{C_l(M(|f|^{p_1'})(x))^{1/p_1'}}{\{1+m(x,V)2^{j-1}a|z|\}^l} \frac{(2^j a|z|)^{n/p_1'}}{(2^{j-1}a|z|)^{n-2}} \left(\int_{B(x,2^{j+1}a|z|)} V^q(\xi) d\xi \right)^{1/q} \\
&\le \sum_{j=-\infty}^{0} \frac{C_l(M(|f|^{p_1'})(x))^{1/p_1'}}{\{1+m(x,V)2^{j-1}a|z|\}^l} \frac{(2^j a|z|)^{n/p_1'}}{(2^{j-1}a|z|)^{n-2}} (2^j a|z|)^{n/q-n} (\int_{B(x,2^{j+1}a|z|)} V(\xi) d\xi) \\
&\le \sum_{j=-\infty}^{0} \frac{1}{(2^{j-1}a|z|)^{n-2}} (2^j a|z|)^{n/q-n+n/p_1'} (M(|f|^{p_1'})(x))^{1/p_1'} (2^{j-1}a|z|)^{n-2} \\
&\le C(M(|f|^{p_1'})(x))^{1/p_1'} \sum_{j=-\infty}^{0} (2^j a|z|) \le Ca|z|(M(|f|^{p_1'})(x))^{1/p_1'}.
\end{aligned}
$$

Finally we get $\|B_{3,x}\|_p \le Ca|z|\|M(f)\|_p + Ca|z|\|(M(|f|^{p_1'}))^{1/p_1'}\|_p \le Ca|z|\|f\|_p$.

At last, we estimate $B_{4,x}$. Because $|x-y|<a|z|$, we have $|x+z-y|<(a+1)|z|$. Similarly, we get

$$\begin{aligned}
|B_{4,x}| &\le \int_{|x-y|<a|z|} |K_3(x+z,y)||b(x+z)-b(y)||f(y)|dy \\
&\le \int_{|x+z-y|<(a+1)|z|} \frac{C_l}{\{1+m(x+z,V)|x+z-y|\}^l} \frac{1}{|x+z-y|^{n-1}} \times \\
&\quad (\int_{B(y,|x+z-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi + \frac{1}{|x+z-y|})|x+z-y||f(y)|dy \\
&=: B^1_{4,x} + B^2_{4,x}.
\end{aligned}$$

For $B^2_{4,x}$, we have

$$\begin{aligned}
B^2_{4,x} &= \int_{|x+z-y|<(a+1)|z|} \frac{1}{|x+z-y|^{n-1}} |f(y)|dy \\
&\le C \sum_{j=-\infty}^{0} \frac{1}{(2^{j-1}(a+1)|z|)^{n-1}} \int_{B(x+z,2^j(a+1)|z|)\setminus B(x+z,2^{j-1}(a+1)|z|)} |f(y)|dy \\
&\le C(a+1)|z|M(f)(x+z).
\end{aligned}$$

For $B^1_{4,x}$, because $\xi \in B(y,|x+z-y|)$, one has $|x+z-\xi| \le |x+z-y|+|y-\xi| \le 2|x+z-y|$. As in the proof of Theorem 1, using $V \in B_q$ and Lemma 0.3 we obtain

$$\begin{aligned}
|B^1_{4,x}| &= \int_{|x+z-y|<(a+1)|z|} \frac{C_l|x+z-y|^{2-n}|f(y)|}{\{1+m(x+z,V)|x+z-y|\}^l} \left(\int_{B(y,|x+z-y|)} \frac{V(\xi)}{|y-\xi|^{n-1}} d\xi \right) dy \\
&\le C(M(|f|^{p'_1})(x+z))^{1/p'_1} \sum_{j=-\infty}^{0} (2^j(a+1)|z|) \\
&\le C(a+1)|z|(M(|f|^{p'_1})(x+z))^{1/p'_1}.
\end{aligned}$$

So we get $\|B_{4,x}\|_p \le C(a+1)|z|\|M(f)\|_p + C(a+1)|z|\|(M(|f|^{p'_1}))^{1/p'_1}\|_p \le C(a+1)|z|\|f\|_p$. From the estimates of $B_{i,x}, (i=1,2,3,4)$ we get that for $\|f\|_p \le 1$,

$$\|[b,T_3]f(\cdot+z) - [b,T_3]f(\cdot)\|_p \le C|z| + \frac{C}{a^\delta} + Ca|z| + C(a+1)|z|.$$

Now, for every $\varepsilon > 0$ we find a $\delta_\varepsilon > 0$ such that $|z| < \delta_\varepsilon$ implies $\|[b,T_3]f(\cdot+z) - [b,T_3]f(\cdot)\|_p < \varepsilon$. This completes the proof of Theorem 2.7. □

Corollary 2.9. *Suppose $V \in B_q$, $q > n/2$ and let $T_3^* = -\nabla(-\triangle+V)^{-1/2}$ be the dual operator of T_3. If $b \in VMO(R^n)$, then the commutator $[b, T_3^*]$ is a compact operator on $L^p(R^n)$, $1 < p < p_0$.*

3. The Reverse Result

In Section 2-3, we have discussed the compactness of the commutator of $T_i, (i = 1, 2, 3)$ on $L^p(R^n)$. A natural problem is whether the reverse problem holds. Namely, if $[b, T_i], (i = 1, 2, 3)$ is a compact operator on $L^p(R^n)$, do we have $b \in VMO(R^n)$? In this section, we will study this problem.

Take $T_3^* = \nabla(-\triangle + V)^{-1/2}$ for example. If we set $V \equiv 0$, the operator reduces to the classical Riesz transform. In 1978, in [4], A.Uchiyama proved that, for a singular integral operator T, if $[b, T]$ is a compact operator on $L^p(R^n)$, then $b \in VMO(R^n)$. However, for a general nonnegative $V \in B_q$, the converse fails. In [1], the authors constructed an example to indicate that merely the L^2 boundedness of $[b, T_3^*]$ cannot guarantee $b \in BMO(R^n)$. So by the counterexample in [1], if $[b, T_3^*]$ is a compact operator on L^2, then $[b, T_3^*]$ is also a bounded operator on L^2, but b may not be in $BMO(R^n)$, and hence it may not belong to $VMO(R^n)$.

The counterexample in [1] implies that the assumption $V \in B_q$ is too weak and it cannot guarantee the function $b \in VMO(R^n)$. However if we assume V satisfies some additional conditions, then we can get the reverse result.

Theorem 3.1. *Let $T_4 = (-\triangle)^{1/2}(-\triangle + V)^{-1/2}$. If $[b, T_3^*]$ and $[b, T_4]$ are compact on L^2 and $V \in L^{n/2} \cap B_q$ for $q > n/2$, then $b \in VMO(R^n)$.*

Proof. Firstly we prove that $V^{1/2}(-\triangle)^{-1/2}$ is bounded on $L^2(R^n)$. By use of Hölder inequality and the fractional integration, we can get

$$\|V^{1/2}(-\triangle)^{-1/2}f\|_2 \le \|V^{1/2}\|_n \|(-\triangle)^{-1/2}f\|_{n'} \le \|V\|_{n/2}^{1/2}\|f\|_2.$$

Then we can get that T_4 has an inverse which is bounded on $L^2(R^n)$. In fact we have

$$\begin{aligned} T_4^{-1}f &= (-\triangle + V)^{1/2}(-\triangle)^{-1/2}f \\ &= (-\triangle + V)^{-1/2}(-\triangle + V)(-\triangle)^{-1/2}f \\ &= (-\triangle + V)^{-1/2}(-\triangle)^{1/2}f + (-\triangle + V)^{-1/2}V^{1/2}V^{1/2}(-\triangle)^{-1/2}f. \end{aligned}$$

So by the L^2 boundedness of $(-\triangle + V)^{-1/2}(-\triangle)^{1/2}$ and $(-\triangle + V)^{-1/2}V^{1/2}$, we get that T_4^{-1} is bounded on $L^2(R^n)$.

Because $\nabla(-\triangle)^{-1/2}$ is bounded on $L^2(R^n)$ and $[b, T_4]$ is compact on $L^2(R^n)$, we get that $\nabla(-\triangle)^{-1/2}[b, T_4]$ is also a compact operator on L^2. Therefore we have $[b, \nabla(-\triangle)^{-1/2}]T_4 = [b, T_3^*] - \nabla(-\triangle)^{-1/2}[b, T_4]$ is a compact operator on L^2. Moreover because we have proved that T_4^{-1} is bounded on $L^2(R^n)$, we can get that $[b, \nabla(-\triangle)^{-1/2}] = [b, \nabla(-\triangle)^{-1/2}]T_4 T_4^{-1}$ is a compact operator on $L^2(R^n)$. Finally, by use of the classical result of A.Uchiyama, we have $b \in VMO(R^n)$. $\square$

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