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On a Family of Hypersurfaces in $\mathbb{C}^2$ with Stability Groups Determined by High Jet-Order

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Abstract: In this paper, we study the collection of all real-analytic hypersurfaces in $\mathbb{C}^2$ of the form $M = \{(z,w) : \operatorname{Im} w = \operatorname{Re} w\, \theta(|z|^2)\}$. We compute all local automorphisms for such hypersurfaces, providing new examples of hypersurfaces with stability groups determined by arbitrary jet-orders. Moreover, we show that for such hypersurfaces, if the stability group is not determined by 1-jets, then the hypersurface is "generically" spherical. That is, such hypersurfaces are locally spherical at every point except those along a specific complex curve.

1. Introduction

Two germs of real hypersurfaces (M, p) and (M', p') in $\mathbb{C}^2$ are biholomorphically equivalent if there exists a *local biholomorphism* between them, i.e. a holomorphic mapping $H : \mathbb{C}^2 \to \mathbb{C}^2$, defined and invertible in a neighborhood of p, that sends p to p' and satisfies $H(M) \subseteq M'$. A germ (M, p) is called *spherical* if it is biholomorphically equivalent to a germ of the 3-dimensional sphere in $\mathbb{C}^2$. For example, the Lewy hypersurface

$$L = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = |z|^2\} \tag{1}$$

is spherical at 0, as seen by the local biholomorphism

$$(z,w) \mapsto \left(\frac{2z}{w+i}, \frac{w-i}{w+i}\right).$$

A basic invariant related to biholomorphic equivalence is the *stability group* of (M, p), defined to be the set $\operatorname{Aut}(M, p)$ of all local biholomorphisms sending the

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germ (M,p) to itself. We say that $\mathrm{Aut}(M,p)$ is *determined by k-jets* if for every pair $H_1, H_2 \in \mathrm{Aut}(M,p)$, we have $H_1 \equiv H_2$ as power series at p whenever

$$\frac{\partial^{|\lambda|} H_1}{\partial Z^\lambda}(p) = \frac{\partial^{|\lambda|} H_2}{\partial Z^\lambda}(p) \quad \forall\, \lambda \in \mathbb{N}^2,\, 0 \le |\lambda| \le k.$$

There exists a large body of work concerning the jet-determinacy of stability groups of hypersurfaces in $\mathbb{C}^2$. For example, Poincaré [10] proved that the stability group of any spherical hypersurface is determined by 2-jets, but not by 1-jets. Chern and Moser [3] proved that the stability group of any Levi-nondegenerate hypersurface is determined by at most 2-jets and is at most 5-dimensional, while Beloshapka [2] provided a converse of sorts, proving that unless such a hypersurface was spherical, then 1-jets suffice. These results have been extended more recently to ever more general hypersurfaces. For example, Ebenfelt, Lamel, and Zaitsev [5] proved the stability group of any Levi nonflat hypersurface (M,p) in $\mathbb{C}^2$ is determined by k-jets for some finite number k; moreover, if M is of *finite type* at p (i.e. contains no complex hypersurface passing through p), then 2-jets will always suffice. We also cite the work of Kolǎr [6], which shows that the stability group of any finite type hypersurface has dimension at most 5.

If the hypersurface is not of finite type, however, then 2-jets may be insufficient. In [7], the author gave the first known example of a Levi nonflat hypersurface, necessarily of infinite type, for which 2-jets were insufficient to determine the stability group; in fact, 3-jets were required. In [8], the author extended this example to the family of hypersurfaces

$$M^n := \left\{ (z,w) \in \mathbb{C}^2 : \mathrm{Im}\, w = \frac{(\mathrm{Re}\, w)\, \mathrm{Im}\left(S\big(|z|^2\big)^{2/n}\right)}{1 + \mathrm{Re}\left(S\big(|z|^2\big)^{2/n}\right)} \right\} \tag{2}$$

where $S(t) := i\,t + (1-t^2)^{1/2}$, $n \ge 2$ is an integer, and the principle branch of each complex root function is used. It is shown in that paper that $\mathrm{Aut}(M^n, 0)$ is determined by $(n+1)$-jets, but *not* by n-jets.

The example of [7] was independently generalized by Zaitsev in [11] to a second family of hypersurfaces with stability groups determined by an arbitrary jet. In that survey, Zaitsev noted that both his hypersurface examples and those provided in [8] shared a common property, which he referred to as being generically spherical, which we state precisely as follows.

Definition 1.1. A germ of a hypersurface (M, p) to be *generically spherical* if it is spherical at every point outside a proper subvariety $V \subset M$ containing p.

Zaitsev then poses the following question:

Question 1.2. Does there exist a germ of a Levi nonflat, real-analytic hypersurface (M, p) in $\mathbb{C}^2$ that is not generically spherical at p, but whose stability group $\operatorname{Aut}(M, p)$ is not determined by 2-jets?

In this paper, we investigate these ideas for a family $\mathcal{G}$ of real-analytic hypersurfaces of infinite type. This family will not include each of the hypersurfaces M^n defined by (2), but provides further evidence that the answer to Question 1.2 is "*No.*"

Specifically, we shall consider the set $\mathcal{G}$ of all germs of hypersurfaces $(M, 0)$, where M can be expressed in some holomorphic coordinates (z, w) as

$$M = \left\{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = (\operatorname{Re} w)\theta\left(|z|^2\right)\right\}$$

for a nonzero, real-analytic function $\theta : \mathbb{R} \to \mathbb{R}$ satisfying $\theta(0) = 0$.

To state the main result of the paper, let us indicate some notation we shall use consistently throughout it. We shall let $\mathbb{R}^* \subset \mathbb{R}$ denote the set of nonzero real numbers, and $\mathbb{U} \subset \mathbb{C}$ denote the set of unimodular complex numbers. Moreover, we shall always assume any complex power function $\zeta \mapsto \zeta^p$ denotes the *principal* branch of that mapping whenever p is not an integer, unless another branch is made explicit.

Our major result is the following.

Theorem 1.3. *For any germ of a hypersurface $(M, 0) \in \mathcal{G}$, there exists a 4-tuple $(a, b, c, d) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{N}$ and a subset $\mathcal{P} \subseteq \mathbb{U} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{C}$, such that*

$$\operatorname{Aut}(M, 0) = \left\{H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu} : (\varepsilon, \rho, \sigma, \nu) \in \mathcal{P}\right\},$$

where $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is the mapping

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{\varepsilon\left(z + \overline{\nu}\, w^a\right)}{\left(1 - i\, 2c\, \nu\, z\, w^a - (\sigma + i\, c\, |\nu|^2)\, w^{2a}\right)^{\frac{1}{2d}(1 - i\, b)}} \\[2ex] \dfrac{\rho\, w}{\left(1 - i\, 2c\, \nu\, z\, w^a - (\sigma + i\, c\, |\nu|^2)\, w^{2a}\right)^{\frac{1}{2a}}} \end{pmatrix}.$$

Moreover, if $\operatorname{Aut}(M,0)$ *is not determined by* 1*-jets, then* $(M,0)$ *is generically spherical.*

A few words are in order about the statement of this theorem. The reader will note that if a is not an integer, then the mapping $w \mapsto w^a$ is not holomorphic at $w = 0$. Thus, if the parameter a is not a natural number, then in order for the mapping $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ to be holomorphic at $(0,0)$ it must follow that $\mathcal{P}$ is a subset of $\mathbb{U} \times \mathbb{R}^* \times \mathbb{R} \times 0$, so that the parameter ν is forced to be 0; in particular, the automorphisms of M will take the form

$$H^{a,b,c,d}_{\varepsilon,\rho,\sigma,0}\big(z,w\big) = \left(\frac{\varepsilon\, z}{\big(1-\sigma\, w^{2a}\big)^{\frac{1}{2d}(1-i\,b)}}, \frac{\rho\, w}{\big(1-\sigma\, w^{2a}\big)^{\frac{1}{2a}}} \right).$$

Further, should $2a$ not be a natural number, then we must similarly conclude that $\mathcal{P}$ is a subset of $\mathbb{U} \times \mathbb{R}^* \times 0 \times 0$, so that the parameter σ will also always be 0, and the automorphisms of M will take the trivial form of

$$H^{a,b,c,d}_{\varepsilon,\rho,0,0}\big(z,w\big) = \big(\varepsilon\, z, \rho\, w\big).$$

We shall revisit these observations later with Theorem 4.1, which gives a more precise articulation of Theorem 1.3.

We conclude this section with some remarks concerning the consequences of Theorem 1.3. In addition to supporting a negative answer to Question 1.2, the final statement of the theorem may be viewed as the analog of Beloshapka's result applicable to the (infinite type) collection $\mathcal{G}$: unless a hypersurface is generically spherical, its stability group is determined by 1-jets. For the sake of completeness, we note that the complementary question of those infinite type hypersurfaces with *very few* automorphisms is addressed in the recent paper [4] by Ebenfelt, Lamel, and Zaitsev.

One consequence of the proof of Theorem 1.3 will show that the hypersurfaces M^n given by equation (2) are elements of $\mathcal{G}$ and are, in some sense, the *only* hypersurfaces in $\mathcal{G}$ to have a maximal 5-dimensional stability group.

More generally, the proof of Theorem 1.3 will show that for any choice of a 4-tuple $(\varepsilon,\rho,\sigma,\nu) \in \mathbb{U} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{C}$, there exists a natural number a and a hypersurface $M \in \mathcal{G}$ such that $H^{a,0,1,1}_{\varepsilon,\rho,\sigma,\nu} \in \operatorname{Aut}(M,0)$, so this collection $\mathcal{G}$ provides further explicit examples of hypersurfaces of infinite type with stability groups determined by jets of arbitrary order.

We conclude this introduction with an outline of the rest of the paper. In Section 2, we introduce some basic notation and preliminary results concerning the hypersurfaces in the family $\mathcal{G}$ and their stability groups. In Section 3, we specify an important subset $\mathcal{G}_S \subseteq \mathcal{G}$ consisting of generically spherical hypersurfaces. In Section 4 we prove that Theorem 1.3 is true for the elements of this subset $\mathcal{G}_S$. Finally, in Section 5, we extend this proof to the whole of $\mathcal{G}$.

2. Preliminaries and notation

In this section, we investigate the family $\mathcal{G}$ and the structure of its elements and their stability groups. We begin with the following basic result.

Proposition 2.1. *For the germ of a hypersurface $(M, 0)$ in $\mathbb{C}^2$, the following are equivalent.*

- *There exists a nonzero, real-analytic function θ defined on a neighborhood of $0 \in \mathbb{R}$ such that $\theta(0) = 0$ and*

$$M = \left\{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = (\operatorname{Re} w)\, \theta\big(|z|^2\big)\right\}. \tag{3}$$

- *There exists a nonconstant, holomorphic function S defined on a neighborhood of $0 \in \mathbb{C}$ such that $S(0) = 1$, $S(t)$ is unimodular for real values of t, and*

$$M = \left\{(z, w) \in \mathbb{C}^2 : w = \overline{w}\, S\big(|z|^2\big)\right\}. \tag{4}$$

Moreover, the functions θ and S are related the pair of equations

$$S(t) = \frac{1 + i\,\theta(t)}{1 - i\,\theta(t)}, \qquad \theta(t) = i\,\frac{1 - S(t)}{1 + S(t)}, \tag{5}$$

for all real t sufficiently close to 0.

We shall prove this proposition in a moment, but for the moment let us use it to give a precise definition of the family $\mathcal{G}$ of hypersurfaces under consideration in this paper.

Definition 2.2. A germ of a hypersurface $(M, 0)$ in $\mathbb{C}^2$ is an element of $\mathcal{G}$ if there exist holomorphic coordinates (z, w) under which M can be written in either of the forms indicated by Proposition 2.1. As is traditional, we shall abuse notation and simply write $M \in \mathcal{G}$, rather than $(M, 0) \in \mathcal{G}$.

For a hypersurface $M \in \mathcal{G}$, we shall use the phrase "M expressed with θ" to mean M is written in form (3), and similarly, we shall use the phrase "M is expressed with S" to mean that M is written in the form (4).

Since each hypersurface $M \in \mathcal{G}$ contains the complex line $\{w = 0\}$, it follows that $\mathcal{G}$ consists of (germs of) hypersurfaces of infinite type. (More specifically, each element of $\mathcal{G}$ is of 1-infinite type, in the language of Meylan [9].) We now prove the proposition above.

Proof. Fix the germ of a hypersurface $(M, 0)$ in $\mathbb{C}^2$.

First assume that M is expressed with θ. Replacing $\operatorname{Re} w$ and $\operatorname{Im} w$ with $(w + \overline{w})/2$ and $(w - \overline{w})/(2i)$ respectively, we can solve equation (3) for w to rewrite it as

$$M = \left\{ (z, w) \in \mathbb{C}^2 : w = \overline{w} \left(\frac{1 + i\,\theta(|z|^2)}{1 - i\,\theta(|z|^2)} \right) \right\}.$$

Define the complex curve $q : \mathbb{R} \to \mathbb{C}$ by

$$q(t) := \frac{1 + i\,t}{1 - i\,t}.$$

It is a simple calculation to show that $q(0) = 1$ and $|q(t)| \equiv 1$, and so M is expressed with S using $S = q \circ \theta$. Note that this also proves the first equation of (5).

Conversely, suppose that M is expressed with S. Replacing w and $\overline{w}$ with $(\operatorname{Re} w + i \operatorname{Im} w)$ and $(\operatorname{Re} w - i \operatorname{Im} w)$ respectively, we can solve equation (4) for $\operatorname{Im} w$ to obtain

$$M = \left\{ (z, w) \in \mathbb{C}^2 : \operatorname{Im} w = \operatorname{Re} w \left(\frac{i(1 - S(|z|^2)}{1 + S(|z|^2)} \right) \right\}.$$

If it can be shown that the expression in parentheses above is real valued, this will complete the proof. Multiplying the numerator and denominator of the ratio above by the conjugated expression $1 + \overline{S(|z|^2)}$ and using the unimodularity assumption, it follows that this equation can be rewritten as

$$\begin{aligned} \operatorname{Im} w &= (\operatorname{Re} w)\, \frac{i(1 - S(|z|^2))(1 + \overline{S(|z|^2)})}{(1 + S(|z|^2))(1 + \overline{S(|z|^2)})} \\ &= (\operatorname{Re} w) \frac{i\big(- 2i \operatorname{Im} S(|z|^2)\big)}{2 + 2 \operatorname{Re} S(|z|^2)}. \end{aligned}$$

Hence, we find that M is expressed with θ, with

$$\theta(t) = \frac{i\big(1 - S(t)\big)}{1 + S(t)} = \frac{\operatorname{Im} S(t)}{1 + \operatorname{Re} S(t)}.$$

Since $\theta(0) = 0$, the proof is complete. □

As the main result of this paper deals with the structure of the local automorphisms of elements of $\mathcal{G}$, let us establish some basic notation and results for such automorphisms. Our investigation begins with the following preliminary result.

Proposition 2.3. *Let $M \in \mathcal{G}$, and suppose $H \in \operatorname{Aut}(M, 0)$, i.e. H is a local biholomorphism of $\mathbb{C}^2$ sending $(M, 0)$ to $(M, 0)$. Then H takes the form*

$$H(z, w) = \big(f(z, w),\, w\, g(z, w)\big), \tag{6}$$

where $f, g : \mathbb{C}^2 \to \mathbb{C}$ are holomorphic functions satisfying

$$\frac{\partial f}{\partial z}(0, 0) \neq 0, \qquad g(0, 0) \neq 0. \tag{7}$$

Conversely, if M is expressed with S, then any mapping H of the form (6) *satisfying* (7) *is a local automorphism of $(M, 0)$ if and only if it satisfies the power series identity*

$$S(z\chi)\, g\big(z, \tau\, S(z\chi)\big) \equiv \overline{g}(\chi, \tau)\, S\big(f\big(z, \tau\, S(z\chi)\big)\, \overline{f}(\chi, \tau)\big), \tag{8}$$

where (z, χ, τ) are indeterminates, and $\overline{f}$ and $\overline{g}$ denote the conjugated power series to f and g respectively.

Proof. The first half of the proposition is proved as Lemma 9.4.4 in [1], Chapter IX. As for establishing the converse statement, note the conditions (7) ensure that any H of the form (6) is a locally invertible mapping of $(\mathbb{C}^2, 0)$ into itself. Such an H maps the hypersurface M into itself if and only if the complex functions f and g satisfy the equation

$$\begin{aligned} w\, g(z, w) &= \overline{w}\, \overline{g(z, w)}\, S\big(f(z, w)\, \overline{f(z, w)}\big) \\ &= \overline{w}\, \overline{g}(\overline{z}, \overline{w})\, S\big(f(z, w)\, \overline{f}(\overline{z}, \overline{w})\big) \end{aligned}$$

whenever $w = \overline{w}\, S(z\overline{z})$. If we make the substitutions $\overline{z} = \chi$, $\overline{w} = \tau$, and $w = \tau\, S(z\chi)$ in the equation above, we find that this is equivalent to identity (8). □

As a consequence of Proposition 2.3, we shall always assume any automorphism of a hypersurface $M \in \mathcal{G}$ is written in the form (6). In our later computation of the stability groups of the hypersurfaces in $\mathcal{G}$, we shall find it convenient to further expand the functions f and g in the variable w as

$$f(z,w) = \sum_{n=0}^{\infty} f_n(z)\,\frac{w^n}{n!}, \qquad g(z,w) = \sum_{n=0}^{\infty} g_n(z)\,\frac{w^n}{n!}.$$

We shall also find it convenient to formally expand the conjugated functions $\overline{f}$ and $\overline{g}$ as power series of the form

$$\overline{f}(\chi,\tau) = \sum_{m,n=0}^{\infty} \lambda_n^m\,\frac{\chi^m}{m!}\,\frac{\tau^n}{n!}, \qquad \overline{g}(\chi,\tau) = \sum_{m,n=0}^{\infty} \mu_n^m\,\frac{\chi^m}{m!}\,\frac{\tau^n}{n!}$$

Note that the coefficients of expansions of f and g are related to the coefficients of the expansions of $\overline{f}$ and $\overline{g}$ (respectively) by the equations

$$f_n^{(m)}(0) = \overline{\lambda_n^m}, \qquad g_n^{(m)}(0) = \overline{\mu_n^m}, \qquad m,n = 0,1,2,\ldots.$$

As a result, it is possible to express the functions f_n and g_n as power series involving the coefficients $\overline{\lambda_n^m}$ and $\overline{\mu_n^m}$.

What is perhaps more surprising is that we can express the functions f_n and g_n as power series involving the the *non-conjugated* coefficients λ_k^ℓ and μ_k^ℓ. For example:

Proposition 2.4. *If $M \in \mathcal{G}$ and $H \in \operatorname{Aut}(M,0)$, then λ_0^1 a unimodular complex number and μ_0^0 is real and nonzero. Moreover*

$$f_0(z) = \frac{1}{\lambda_0^1}\,z, \qquad g_0(z) = \mu_0^0,$$

Proof. Since $H \in \operatorname{Aut}(M,0)$, it satisfies identity (8). Setting $\tau = 0$ yields

$$S(z\chi)g_0(z) \equiv \overline{g}_0(\chi)\,S\big(f_0(z)\overline{f_0}(\chi)\big). \tag{9}$$

Setting $\chi = 0$ in this identity and using the fact that $S(0) = 1$, we have $g_0(z) = \mu_0^0$. However, we also know $g_0(0) = \overline{\mu_0^0}$ by the definition of μ_k^ℓ, and $g_0(0) \neq 0$ by Proposition 2.3, whence μ_0^0 is a nonzero real number, proving that g_0 has the desired form.

Now, if we replace $g_0(z) = \overline{g_0}(\chi) = \mu_0^0$, then identity (9) simplifies to

$$S(z\chi) \equiv S\big(f_0(z)\overline{f_0}(\chi)\big).$$

Since S is nonconstant, some derivative of S must not vanish at 0. Let $k \geq 1$ be the smallest value such that $S^{(k)}(0) \neq 0$. Differentiating the identity above k

times in χ and setting $\chi = 0$ yields

$$S^{(k)}(0)z^k \equiv S^{(k)}(0)\big(\lambda_0^1\big)^k f_0(z)^k.$$

Taking k-th roots of this equation and solving for $f_0(z)$ implies

$$f_0(z) = \xi\,\frac{z}{\lambda_0^1}$$

for some k-th root of unity ξ. Differentiating this once and evaluating at $z = 0$ yields

$$\overline{\lambda_0^1} = f_0{}'(0) = \frac{\xi}{\lambda_0^1},$$

or $\xi = |\lambda_0^1|^2$. In particular, this implies that ξ is real and positive, whence $\xi = 1$. This in turn implies that λ_0^1 is unimodular, which gives the desired form for f_0. □

The method above can be generalized to obtain similar (albeit more complicated) parameterizations of the functions f_n and g_n in terms of the coefficients λ_k^ℓ and μ_k^ℓ, an explicit structure we shall find useful in Section 4. Specifically, we have the following.

Proposition 2.5. *Let $M \in \mathcal{G}$, and assume M is expressed with θ. Let $d \geq 1$ denote the smallest integer such that $\theta^{(d)}(0) \neq 0$, and let $\Delta \in \{0,1\}$ be defined to be 1 if $d = 1$ and 0 otherwise.*

Then for each $n \in \mathbb{N}$, there exist holomorphic functions $\mathcal{R}_n, \mathcal{T}_n : \mathbb{C}^5 \times \mathbb{C}^{4(n-1)} \to \mathbb{C}$ satisfying $\mathcal{R}_n(\zeta, \mathbf{0}) \equiv \mathcal{T}_n(\zeta, \mathbf{0}) = 0$ such that

$$\begin{aligned}
f_n(z) &= \Delta\left[\frac{i\,2n\,\theta'(0)^2 - \theta''(0)}{\theta'(0)(\lambda_0^1)^2}\,z^2 - \frac{i\,\mu_n^1}{2\,\theta'(0)\,\mu_0^0}\right] + \left(\frac{n\,\mu_n^0}{d\,\lambda_0^1\,\mu_0^0} - \frac{\lambda_n^1}{(\lambda_0^1)^2}\right)z \\
&\quad + \mathcal{R}_n\left(\left(z, \frac{1}{\lambda_0^1}, \frac{1}{\mu_0^0}, \lambda_0^1, \mu_0^0\right), \left(\lambda_k^0, \lambda_k^1, \mu_k^0, \mu_k^1\right)_{k=1}^{n-1}\right) \\
g_n(z) &= \left(\frac{i\,2\,\lambda_n^0\,\mu_0^0\theta'(0)}{\lambda_0^1}\right)z + \mu_n^0 \\
&\quad + \mathcal{T}_n\left(\left(z, \frac{1}{\lambda_0^1}, \frac{1}{\mu_0^0}, \lambda_0^1, \mu_0^0\right), \left(\lambda_k^0, \lambda_k^1, \mu_k^0, \mu_k^1\right)_{k=1}^{n-1}\right).
\end{aligned}$$

for any $H \in \mathrm{Aut}(M, 0)$.

Note that if $d > 1$, then $\Delta = 0$, which simply means that we ignore the portion in square brackets in the formula for f_n. Proposition 2.5 is actually proved in greater generality as Proposition 5.2 in [8], and the interested reader is directed there for further details.

We conclude this section with another result from [8] that provides a useful criterion for uniquely identifying an automorphism of a hypersurface $M \in \mathcal{G}$ based on a finite number of the coefficients λ_k^ℓ and μ_k^ℓ. To state this result precisely, we need the following, rather technical definition.

Definition 2.6. Suppose $M \in \mathcal{G}$ is expressed with θ. For each integer $n \geq 0$, define the holomorphic mappings $\Upsilon^n : \mathbb{C}^2 \to \mathbb{C}^4$ by

$$\Upsilon^n(z,\chi) = \Big(\upsilon_1^n(z\chi),\, \upsilon_2^n(z\chi),\, z\,\upsilon_3^n(z\chi),\, \chi\,\upsilon_4^n(z\chi)\Big),$$

where each $\upsilon_j^n : \mathbb{C} \to \mathbb{C}$ is defined as

$$\begin{aligned}
\upsilon_1^n(t) &:= t\,\theta'(t)\left[\left(\frac{1+i\,\theta(t)}{1-i\,\theta(t)}\right)^n - 1\right] \\
\upsilon_2^n(t) &:= \big(1+\theta(t)^2\big)\left[\left(\frac{1+i\,\theta(t)}{1-i\,\theta(t)}\right)^n - 1\right] - \frac{i\,2n}{d}\,t\,\theta'(t) \\
\upsilon_n^3(t) &:= \Delta\Bigg\{\theta'(t) - \left(\frac{1+i\,\theta(t)}{1-i\,\theta(t)}\right)^n \\
&\qquad \times \left[\theta'(0)\big(1+\theta(t)^2\big) + \frac{\theta''(0) - i\,2n\,\theta'(0)^2}{\theta'(0)}\,t\,\theta'(t)\right]\Bigg\} \\
\upsilon_4^n(t) &:= \Delta\Bigg\{1+\theta(t)^2 + \frac{\theta'(t)}{\theta'(0)}\left[\frac{\theta''(0) - i\,2n\,\theta'(0)^2}{\theta'(0)}\,t - \left(\frac{1+i\,\theta(t)}{1-i\,\theta(t)}\right)^n\right]\Bigg\},
\end{aligned}$$

and d and Δ are defined in Proposition 2.5. Note that if $d > 1$, then this simply means $\upsilon_3^n \equiv \upsilon_4^n \equiv 0$.

For each n, consider the complex linear subspace $\mathcal{S}^n$ in $\mathbb{C}^4$ spanned by the set of vectors

$$\left\{\frac{\partial^{k+\ell}\Upsilon^n}{\partial z^k \partial \chi^\ell}(0,0) : k,\ell = 0,1,2,\cdots\right\}.$$

Note that the maximum dimension of the subspace $\mathcal{S}^n$ is 4, unless $d > 1$, in which case $\mathcal{S}^n$ is at most 2-dimensional.

Finally, define $\mathcal{U}$ to be the set of integers $n \geq 0$ such that $\mathcal{S}^n$ is not of maximal dimension; that is, set

$$\mathcal{U} := \left\{n \geq 0 : \dim_{\mathbb{C}}\left(\mathcal{S}^n\right) < 2 + 2\Delta\right\}.$$

Note that 0 is always an element of $\mathcal{U}$, since υ_1^0 vanishes identically.

A useful alternative characterization of $\mathcal{U}$ is the following: If $d = 1$, then $n \notin \mathcal{U}$ means that the four complex functions

$$\big\{ \upsilon_1^n(z\chi),\ \upsilon_2^n(z\chi),\ z\,\upsilon_3^n(z\chi),\ \chi\,\upsilon_4^n(z\chi) \big\}$$

form a linearly independent set; if $d > 1$, then $n \notin \mathcal{U}$ means that the two complex functions

$$\big\{ \upsilon_1^n(z\chi),\ \upsilon_2^n(z\chi) \big\}$$

are linearly independent.

The importance of the set $\mathcal{U}$ is detailed in the following result, with which we conclude this section. As noted above, it is proved in [8] in greater generality as Theorem 4.1.

Theorem 2.7. *Let $M \in \mathcal{G}$, and let $\mathcal{U}$ be as in Definition 2.6. Then any automorphism $H \in \mathrm{Aut}(M,0)$ is uniquely determined by those coefficients $\big(\lambda_n^0, \lambda_n^1, \mu_n^0, \mu_n^1\big)$ with $n \in \mathcal{U}$. That is, if H and $\widetilde{H}$ are two automorphisms of $(M,0)$, and*

$$\big(\lambda_n^0, \lambda_n^1, \mu_n^0, \mu_n^1\big) = \Big(\widetilde{\lambda}_n^0, \widetilde{\lambda}_n^1, \widetilde{\mu}_n^0, \widetilde{\mu}_n^1\Big) \quad \text{for } n \in \mathcal{U},$$

then $H \equiv \widetilde{H}$ as germs of biholomorphisms.

3. The generically spherical hypersurfaces of $\mathcal{G}$

In proving Theorem 1.3, it will be important to identify those elements of $\mathcal{G}$ that are generically spherical. To do so, we begin with the following lemma.

Lemma 3.1. *For the ordered pair $(a,b) \in \mathbb{R}^2$, consider the equation*

$$S^{2a} = 1 + 2i\,z\,S^{a(1+i\,b)} \tag{10}$$

in the complex variables z and S. If $a \neq 0$, then this equation admits a unique holomorphic solution $S = S_{a,b}(z)$ satisfying $S_{a,b}(0) = 1$.

Moreover, this solution is given by

$$S_{a,b}(z) := \exp\left(i\,\frac{\phi_b(z)}{a} \right), \tag{11}$$

where ϕ_b denotes the unique (local) holomorphic inverse to the mapping

$$z \mapsto e^{bz}\sin(z)$$

satisfying $\phi_b(0) = 0$.

As we noted in the Introduction, we shall always assume that any mapping $S \mapsto S^p$ denotes the principal branch unless explicitly indicated otherwise.

Proof. Note first that ϕ_b is well-defined by the Inverse Function Theorem, since

$$\frac{d}{dz}\big(e^{bz}\sin(z)\big)\Big|_{z=0} = 1 \neq 0;$$

in fact, using standard power series techniques we may expand ϕ_b as

$$\phi_b(z) = z - b\,z^2 + \frac{1}{6}(1+9b^2)\,z^3 + O(z^4).$$

As a result, the function $S_{a,b}$ is well-defined and holomorphic, and using (11) we can expand $S_{a,b}$ as

$$\begin{aligned} S_{a,b}(z) = 1 + \frac{i}{a}\,z - \frac{1+i\,2ab}{2a^2}\,z^2 \\ - \frac{i(1+a+i\,3ab)(1-a+i\,3ab)}{6a^3}\,z^3 + O(z^4), \end{aligned} \tag{12}$$

which we shall find useful in later calculations.

To complete the proof, we need only show that $S_{a,b}$ solves equation (10), as the uniqueness of such a solution follows immediately from the Implicit Function Theorem. To begin, let us simplify the left- and right-hand sides of (10) using the explicit formula for $S_{a,b}$ given in equation (11). Applying (the complex) Euler's formula $\exp(i\,z) = \cos(z) + i\,\sin(z)$, we find the left-hand side of (10) simplifies to

$$\begin{aligned} \big(S_{a,b}(z)\big)^{2a} &= \exp\left(i\,\frac{\phi_b(z)}{a}\right)^{2a} \\ &= \exp(i\,2\phi_b(z)) \\ &= \cos\big(2\phi_b(z)\big) + i\,\sin\big(2\phi_b(z)\big), \end{aligned}$$

while the right-hand side becomes

$$\begin{aligned} 1 + 2i\,z\,\big(S_{a,b}(z)\big)^{a(1+i\,b)} &= 1 + 2i\,z\,\exp\left(i\,\frac{\phi_b(z)}{a}\right)^{a(1+i\,b)} \\ &= 1 + 2i\,z\,\exp\big((i-b)\phi_b(z)\big) \\ &= \big(1 - 2z\,e^{-b\phi_b(z)}\sin\big(\phi_b(z)\big)\big) \\ &\qquad + i\,\big(2\,z\,e^{-b\phi_b(z)}\cos\big(\phi_b(z)\big)\big). \end{aligned}$$

Thus, to show that $S_{a,b}$ satisfies equation (10), it suffices to prove that ϕ_b satisfies the following pair of equations:

$$\cos\big(2\phi_b(z)\big) = 1 - 2z\, e^{-b\phi_b(z)} \sin\big(\phi_b(z)\big), \tag{13}$$

$$\sin\big(2\phi_b(z)\big) = 2\, e^{-b\phi_b(z)} \cos\big(\phi_b(z)\big). \tag{14}$$

(Note that while equations (13) and (14) appear to be equating the real and imaginary parts of equation (10), since ϕ_b is complex-valued neither of these equations involve real numbers.)

Using the double angle formula for cosine, we can rewrite equation (13) as

$$1 - 2\sin^2\big(\phi_b(z)\big) = 1 - 2z\, e^{-b\phi_b(z)} \sin\big(\phi_b(z)\big),$$

which holds identically if and only if

$$e^{b\phi_b(z)} \sin\big(\phi_b(z)\big) = z, \tag{15}$$

and this is true by the definition of ϕ_b. Similarly, applying the double angle formula for sine to (14), we obtain

$$2\sin\big(\phi_b(z)\big)\cos\big(\phi_b(z)\big) = 2z\, e^{-b\phi_b(z)} \cos\big(\phi_b(z)\big),$$

which is equivalent to equation (15) as well. □

Note that $S_{a,b}$ is a holomorphic function that satisfies $S_{a,b}(0) = 1$. Moreover, since $e^{bt}\sin(t)$ is real-valued for any real t, the definition of ϕ_b given in Lemma 3.1 implies that it too is real whenever t is. As a consequence, $S_{a,b}(t) = \exp(i\,\phi_b(t)/a)$ is unimodular for any *real* value t. Thus, according to Proposition 2.1, the set

$$M^{a,b,c,d} := \left\{(z,w) \in \mathbb{C}^2 : w = \overline{w}\, S_{a,b}\big(c|z|^{2d}\big)\right\}, \tag{16}$$

defines a hypersurface in $\mathcal{G}$ for any choice of $(a,b,c,d) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{N}$.

Definition 3.2. Define the subset $\mathcal{G}_S \subseteq \mathcal{G}$ as follows: the germ of a hypersurface $(M,0)$ is an element of $\mathcal{G}_S$ if there exist holomorphic coordinates (z,w) under which $M = M^{a,b,c,d}$ for some 4-tuple $(a,b,c,d) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{N}$. As is customary, we shall also write $M \in \mathcal{G}_S$ to mean $(M,0) \in \mathcal{G}_S$.

Proposition 3.3. *Every element of $\mathcal{G}_S$ is the germ of a generically spherical hypersurface in $\mathbb{C}^2$.*

Proof. Let $M^{a,b,c,d} \in \mathcal{G}_S$. It useful to note first that

$$M^{a,b,c,d} = M^{-a,-b,-c,d} \quad \text{for any } (a,b,c,d). \tag{17}$$

This follows immediately if it can be shown that $S_{a,b}(t) = S_{-a,-b}(-t)$ for real values of t, but this is a straightforward calculation:

$$S_{a,b}(t) = \big(S_{-a,b}(t)\big)^{-1} = \frac{1}{S_{-a,b}(t)} = \overline{S_{-a,b}(t)} = S_{-a,-b}(-t),$$

where the bar denotes complex conjugation.

We now prove that $M^{a,b,c,d}$ is spherical at any point $(z,w) \in M^{a,b,c,d}$ off of the plane $\{w=0\}$; this will suffice to prove that the germ $(M^{a,b,c,d},0)$ is generically spherical. To this end, fix a point $(z_0,w_0) \in M^{a,b,c,d}$ with $w_0 \neq 0$. Using equation (17), we may without loss of generality assume $c>0$, so define the mapping

$$(z,w) \mapsto (Z,W) = \left(\sqrt{c}\, z^d w^{a(1+i\,b)}, w^{2a}\right),$$

where in this case we take a branch of $w \mapsto w^\lambda$ that is analytic *on a neighborhood of* w_0. (Note that this will coincide with the principal branch whenever w_0 is not a negative real number.) Since the Jacobian of this mapping is well-defined and nonsingular at (z_0,w_0), it defines a local biholomorphism of $\mathbb{C}^2$ near (z_0,w_0). Moreover, we claim that if $(z,w) \in M^{a,b,c,d}$ is sufficiently close to (z_0,w_0), then $(Z,W) \in L$, where L is the Lewy hypersurface defined by equation (1) in the Introduction. Since L is itself spherical, this will complete the proof.

To prove the claim, consider a spherical neighborhood of (z_0,w_0) that does not intersect the plane $\{w=0\}$. If $(z,w) \in M^{a,b,c,d}$ is in this neighborhood, then we know $w = \overline{w}\, S_{a,b}(c|z|^{2d})$ and $w \neq 0$. To show that this point is mapped to the Lewy hypersurface, we must show that $\operatorname{Im} W = (\operatorname{Re} W)|Z|^2$, or, equivalently, that $W = \overline{W} + i\,2Z\overline{Z}$. Using the defining property of $S_{a,b}$, we compute that

$$\begin{aligned}
\overline{W} + i\,2Z\overline{Z} &= \overline{w^{2a}} + i\,2\big(\sqrt{c}\, z^d w^{a(1+i\,b)}\big)\overline{\big(\sqrt{c}\, z^d w^{a(1+i\,b)}\big)} \\
&= \overline{w}^{2a} + i\,2\left(\sqrt{c}\, z^d \left[\overline{w}\, S_{a,b}(c|z|^{2d})\right]^{a(1+i\,b)}\right)\big(\sqrt{c}\, \overline{z}^d \overline{w}^{a(1-i\,b)}\big) \\
&= \overline{w}^{2a} + i\,2c\, z^d \overline{z}^d\, S_{a,b}(c|z|^{2d})^{a(1+i\,b)} \overline{w}^{a(1+i\,b)} \overline{w}^{a(1-bi)} \\
&= \overline{w}^{2a}\left[1 + i\,2c\,|z|^{2d}\, S_{a,b}(c|z|^{2d})^{a(1+i\,b)}\right] \\
&= \overline{w}^{2a}\, S_{a,b}(c|z|^{2d})^{2a} = \big(\overline{w}\, S_{a,b}(c|z|^{2d})\big)^{2a} = w^{2a} = W,
\end{aligned}$$

as desired. $\square$

We conclude this section with the following observation. Although we shall not need this explicit description for the present paper, it is instructive to note that

$$S_{a,0}(z) = \big(i\,z + (1-z^2)^{1/2}\big)^{1/a}.$$

In particular, this implies that the hypersurface M^n given in equation (2) in the Introduction is the same as the hypersurface $M^{\frac{n}{2},0,1,1}$ in the set $\mathcal{G}_S$.

4. The stability groups in $\mathcal{G}_S$

Before proving Theorem 1.3 in general for the family $\mathcal{G}$, in this section we first prove that it applies to the subset $\mathcal{G}_S$ of generically spherical elements of $\mathcal{G}$.

To prepare for the proof, let us define for any 4-tuple $(a,b,c,d) \in \mathbb{R}^4$ with $ad \neq 0$ and any 4-tuple $(\varepsilon,\rho,\sigma,\nu) \in \mathbb{C}^4$ the mapping $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu} : \mathbb{C}^2 \to \mathbb{C}^2$ by the formula

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \left(\begin{array}{c} \dfrac{\varepsilon(z + \overline{\nu}\,w^a)}{\big(1 - i\,2c\,\nu\,z\,w^a - (\sigma + i\,c\,|\nu|^2)\,w^{2a}\big)^{\frac{1}{2d}(1-i\,b)}} \\[2ex] \dfrac{\rho\,w}{\big(1 - i\,2c\,\nu\,z\,w^a - (\sigma + i\,c\,|\nu|^2)\,w^{2a}\big)^{\frac{1}{2a}}} \end{array} \right),$$

where in each case we assume the principle branch of a complex power function is being used. Observe that if a is a natural number, then $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ defines a local biholomorphism of $\mathbb{C}^2$ in a neighborhood of $(z,w) = (0,0)$.

However, as we noted in the Introduction, the mapping $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ fails to be holomorphic (or even continuous!) along the plane $\{w = 0\}$ whenever a is not a positive integer. Nevertheless, the mapping

$$H^{a,b,c,d}_{\varepsilon,\rho,0,0}(z,w) = \big(\varepsilon\,z,\ \rho\,w\big)$$

is a global biholomorphism for any value of a, whereas the mapping

$$H^{a,b,c,d}_{\varepsilon,\rho,\sigma,0}(z,w) = \left(\frac{\varepsilon\,z}{(1-\sigma\,w^{2a})^{\frac{1}{2d}(1-i\,b)}},\ \frac{\rho\,w}{(1-\sigma\,w^{2a})^{\frac{1}{2a}}} \right)$$

is a local biholomorphism of $(\mathbb{C}^2, 0)$ whenever $2a$ is a positive integer.

We are now ready to prove the following result.

Theorem 4.1. *Let* $M^{a,b,c,d} \in \mathcal{G}_S$*, and assume that* $a > 0$*. Define*

$$\mathcal{P} := \begin{cases} \mathbb{U} \times \mathbb{R}^* \times \mathbb{R} \times \mathbb{C}, & \text{if } a \in \mathbb{N},\ b = 0,\ \text{and } d = 1, \\ \mathbb{U} \times \mathbb{R}^* \times 0 \times 0, & \text{if } 2a \notin \mathbb{N}, \\ \mathbb{U} \times \mathbb{R}^* \times \mathbb{R} \times 0, & \text{otherwise.} \end{cases}$$

Then $\operatorname{Aut}\big(M^{a,b,c,d}, 0\big) = \Big\{ H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu} : (\varepsilon,\rho,\sigma,\nu) \in \mathcal{P} \Big\}$.

Recall that Equation (17) implies that any hypersurface $M^{a,b,c,d}$ in $\mathcal{G}_S$ can be written with $a > 0$, so that Theorem 4.1 completely classifies the automorphism group of any hypersurface in $\mathcal{G}_S$. Note too that the discussion preceding Theorem 4.1 implies that each mapping $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ is (at least) a local biholomorphism of $(\mathbb{C}^2, 0)$ under the allowable set of parameters in $\mathcal{P}$.

For the remainder of this section, let us fix a hypersurface $M^{a,b,c,d} \in \mathcal{G}_S$, and assume $a > 0$. Define $\mathcal{P}$ as in the statement of Theorem 4.1. We shall prove the result of the theorem by demonstrating that each set involved is a subset of the other. We do this in a pair of lemmas.

Lemma 4.2. $\Big\{ H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu} : (\varepsilon,\rho,\sigma,\nu) \in \mathcal{P} \Big\} \subseteq \operatorname{Aut}\big(M^{a,b,c,d}, 0\big)$.

Proof. Fix a 4-tuple $(\varepsilon,\rho,\sigma,\nu) \in \mathcal{P}$; we must show the mapping $H := H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ is a local automorphism of $(M^{a,b,c,d}, 0)$. We have already argued that $(\varepsilon,\rho,\sigma,\nu) \in \mathcal{P}$ ensures that the mapping H is a local biholomorphism of $(\mathbb{C}^2, 0)$, so we need only show that it maps $M^{a,b,c,d}$ into itself.

Write H in the form $H = \big(f, w\,g\big)$. According to Proposition 2.3, we need only prove that the identity

$$\frac{S_{a,b}(c\,z^d\chi^d)\,g\big(z, \tau\,S_{a,b}(cz^d\chi^d)\big)}{\overline{g}(\chi,\tau)} \equiv S_{a,b}\Big(c\,f\big(z, \tau\,S_{a,b}(c\,z^d\chi^d)\big)^d\,\overline{f}(\chi,\tau)^d\Big)$$

is satisfied.

Observe that the right-hand side of this identity, more or less by the definition given in Lemma 3.1, is the unique holomorphic solution $\mathcal{S}(z,\chi,\tau)$ to the complex equation

$$\mathcal{S}^{2a} = 1 + i\,2c\,f\big(z, \tau\,S_{a,b}(c\,z^d\chi^d)\big)^d\,\overline{f}(\chi,\tau)^d \cdot \mathcal{S}^{a(1+i\,b)} \tag{18}$$

satisfying $\mathcal{S}(0,0,0) = 1$. Hence, if it can be shown that the left-hand side of this identity is also a solution, then the lemma is proved.

To that end, define the holomorphic function

$$\mathcal{S}(z,\chi,\tau) := \frac{S_{a,b}(c\,z^d\chi^d)\,g\big(z,\tau\,S_{a,b}(cz^d\chi^d)\big)}{\overline{g}(\chi,\tau)}.$$

For convenience, write

$$\Phi^c_{\sigma,\nu}(Z,W) := 1 - i\,2c\,\nu\,Z\,W - (\sigma + i\,c\,|\nu|^2)\,W^2,$$

so that H may be more compactly expressed as

$$H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}(z,w) := \left(\frac{\varepsilon(z+\overline{\nu}\,w^a)}{\Phi^c_{\sigma,\nu}(z^d,w^a)^{\frac{1}{2d}(1-i\,b)}}, \frac{\rho\,w}{\Phi^c_{\sigma,\nu}(z^d,w^a)^{\frac{1}{2a}}}\right).$$

In particular, we can write $\mathcal{S}$ as

$$\mathcal{S}(z,\chi,\tau) = \frac{S_{a,b}(c\,z^d\chi^d)\,\Phi^{-c}_{\sigma,\nu}(\chi^d,\tau^a)^{\frac{1}{2a}}}{\Phi^c_{\sigma,\nu}\big(z^d,\tau^a S_{a,b}(c\,z^d\chi^d)^a\big)^{\frac{1}{2a}}}.$$

Observe that

$$\mathcal{S}(0,0,0) = \frac{S_{a,b}(0)\,\Phi^{-c}_{\sigma,\nu}(0,0)^{\frac{1}{2a}}}{\Phi^c_{\sigma,\nu}(0,0)^{\frac{1}{2a}}} = 1,$$

so we need only prove it satisfies equation (18).

Note that the right-hand side of that identity evaluates to

$$1 + \frac{i\,2c\,\big(z+\overline{\nu}\,\tau^a S_{a,b}(c\,z^d\chi^d)^a\big)^d(\chi+\nu\,\tau^a)^d\,S_{a,b}(c\,z^d\chi^d)^{a(1+i\,b)}}{\Phi^c_{\sigma,\nu}\big(z^d,\tau^a S_{a,b}(c\,z^d\chi^d)\big)}.$$

We show this expression equals $\mathcal{S}(z,\chi,\tau)^{2a}$ by examining two cases, depending on the value of ν.

If $\nu = 0$, this simplifies to

$$1 + \frac{2i\,c\,z^d\chi^d\,S_{a,b}(c\,z^d\chi^d)^{a(1+i\,b)}}{1-\sigma\,\tau^{2a}S_{a,b}(c\,z^d\chi^d)^{2a}}.$$

Putting this on a common denominator and simplifying yields

$$\begin{aligned}&\frac{S_{a,b}(c\,z^d\chi^d)^{2a} - \sigma\,\tau^{2a}S_{a,b}(c\,z^d\chi^d)^{2a}}{1-\sigma\,\tau^{2a}S_{a,b}(c\,z^d\chi^d)^{2a}}\\ &\quad= \frac{S_{a,b}(c\,z^d\chi^d)^{2a}\,\Phi^{-c}_{\sigma,0}(\chi^d,\tau^a)}{\Phi^c_{\sigma,0}\big(z^d,\tau^a S_{a,b}(c\,z^d\chi^d)^{2a}\big)} = \mathcal{S}(z,\chi,\tau)^{2a},\end{aligned}$$

which proves (18) holds.

On the other hand, if $\nu \neq 0$, then the definition of $\mathcal{P}$ forces $d = 1$ and $b = 0$, so that the right-hand side evaluates to

$$\begin{aligned}
1 + &\frac{i\,2c\,(z+\overline{\nu}\,\tau^a S_{a,0}(c\,z\chi)^a)(\chi+\nu\,\tau^a)S_{a,0}(c\,z\chi)^a}{1 - i\,2c\nu\,z\,\tau^a S_{a,0}(c\,z\chi)^a - (\sigma + i\,c|\nu|^2)\tau^{2a}S_{a,0}(c\,z\chi)^{2a}} \\
&= \frac{\left(1+i\,2c\,z\chi S_{a,0}(c\,z\chi)^a\right) + \left(i\,2c\overline{\nu}\,\chi\,\tau^a - (\sigma - i\,c|\nu|^2)\tau^{2a}\right)S_{a,0}(c\,z\chi)^{2a}}{1 - i\,2c\nu\,z\,\tau^a S_{a,0}(c\,z\chi)^a - (\sigma + i\,c|\nu|^2)\tau^{2a}S_{a,0}(c\,z\chi)^{2a}} \\
&= \frac{S_{a,0}(c\,z\chi)^{2a}\,\Phi^{-c}_{\sigma,\overline{\nu}}(\chi,\tau^a)}{\Phi^{c}_{\sigma,\nu}\left(z,\tau^a S_{a,0}(c\,z\chi)^a\right)} = \mathcal{S}(z,\chi,\tau)^{2a},
\end{aligned}$$

which proves (18) holds once again. □

Lemma 4.3. $\operatorname{Aut}\left(M^{a,b,c,d},0\right) \subseteq \left\{H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu} : (\varepsilon,\rho,\sigma,\nu) \in \mathcal{P}\right\}$.

Proof. Fix an automorphism $H \in \operatorname{Aut}(M,0)$; we must prove that there exists a 4-tuple $(\varepsilon,\rho,\sigma,\nu) \in \mathcal{P}$ such that $H = H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$.

To do this, we shall appeal to Theorem 2.7. Recall the definition of the numbers λ^ℓ_k and μ^ℓ_k for the automorphism H given in Section 2. Similarly, for a mapping $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ with $(\varepsilon,\rho,\sigma,\nu) \in \mathcal{P}$, we can define the corresponding values $\widetilde{\lambda}^\ell_k$ and $\widetilde{\mu}^\ell_k$; note that these values are themselves parameterized by $(\varepsilon,\rho,\sigma,\nu)$. If we can choose a specific 4-tuple $(\varepsilon,\rho,\sigma,\nu) \in \mathcal{P}$ such that

$$\left(\lambda^0_n, \lambda^1_n, \mu^0_n, \mu^1_n\right) = \left(\widetilde{\lambda}^0_n, \widetilde{\lambda}^1_n, \widetilde{\mu}^0_n, \widetilde{\mu}^1_n\right) \quad \text{for } n \in \mathcal{U}, \tag{19}$$

then Theorem 2.7 implies that $H \equiv H^{a,b,c,d}_{\varepsilon,\rho,\sigma,\nu}$ as mappings.

Recall from Proposition 2.1 that we may express $M^{a,b,c,d}$ with θ using

$$\begin{aligned}
\theta(t) &= i\,\frac{1 - S_{a,b}(c\,t^d)}{1 + S_{a,b}(c\,t^d)} \qquad (20)\\
&= \frac{c}{2a}t^d - \frac{bc^2}{2a}\,t^{2d} + \frac{(1+2a^2+18a^2b^2)}{24a^3}\,t^{3d} + O(t^{4d}).
\end{aligned}$$

We complete the proof of Lemma 4.3 by examining two cases, depending on the value of d.

4.1. **Case 1:** $d = 1$**.** To apply Proposition 2.7, we must first compute the set $\mathcal{U}$ given in Definition 2.6. Given the expansion of θ above, we compute that the power series for υ_k^n given in that definition as

$$\upsilon_1^n(t) = \frac{i\,c^2\,n}{4a^3}\big[2a\,t^2 + i\,c(n+6abi)\,t^3\big] + O(t^4)$$

$$\upsilon_2^n(z,\chi) = -\frac{c^2 n}{6a^3}\big[3a(n - i\,2ab)\,t^2 + i\,c(n^2+2a^2+18a^2b^2+i\,6abn)\,t^3\big] + O(t^4)$$

$$\upsilon_3^n(z,\chi) = -\frac{c^3}{12a^4}(n+a-i\,ab)(n-a-i\,ab) \times \big[3a\,t^2 + i\,2c(n+i\,5ab)\,t^3\big] + O(t^4)$$

$$\upsilon_n^4(z,\chi) = \frac{c^2}{6a^3}(n+a-i\,ab)(n-a-i\,ab) \times \big[3a\,t^2 + i\,c(n+i\,10ab)\,t^3\big] + O(t^4)$$

As a result, we find that

$$\det\left[\frac{\partial^4\Upsilon^n}{\partial z^2\partial\chi^2} \quad \frac{\partial^6\Upsilon^n}{\partial z^3\partial\chi^3} \quad \frac{\partial^5\Upsilon^n}{\partial z^3\partial\chi^2} \quad \frac{\partial^5\Upsilon^n}{\partial z^2\partial\chi^3}\right]\Bigg|_{(z,\chi)=(0,0)} = -\frac{108c^{10}}{a^{10}}\,n^2\big(n^2-(2a)^2\big)\big(n^2-(a+i\,ab)^2\big)\big(n^2-(a-i\,ab)^2\big),$$

where Υ^n is the holomorphic mapping given in Definition 2.6. This implies that the four vectors

$$\left\{\frac{\partial^4\Upsilon^n}{\partial z^2\partial\chi^2}(0,0), \quad \frac{\partial^6\Upsilon^n}{\partial z^3\partial\chi^3}(0,0), \quad \frac{\partial^5\Upsilon^n}{\partial z^3\partial\chi^2}(0,0), \quad \frac{\partial^5\Upsilon^n}{\partial z^2\partial\chi^3}(0,0)\right\}$$

are linearly independent in $\mathbb{C}^4$ except at those values $n \in \mathbb{Z}$ for which this determinant is 0. It follows from this that $\mathcal{U} \subseteq \{0, a \pm abi, 2a\} \cap \mathbb{Z}$. Note that exactly one of the following is true:

- $\{0, a \pm i\,ab, 2a\} \cap \mathbb{Z} = \{0\}$.
- $\{0, a \pm i\,ab, 2a\} \cap \mathbb{Z} = \{0, 2a\}$.
- $\{0, a \pm i\,ab, 2a\} \cap \mathbb{Z} = \{0, a, 2a\}$.

We complete the proof of this case by examining each of these possibilities.

Subcase (a). Suppose $\{0, a \pm i\,ab, 2a\} \cap \mathbb{Z} = \{0\}$. This implies $\mathcal{U} = \{0\}$, so we need only find a set of parameters in $\mathcal{P}$ that verify equation (19) for $n = 0$. Moreover, this also implies that $2a \notin \mathbb{N}$, and $a \notin \mathbb{N}$ or $b \neq 0$, whence we are forced by the definition of $\mathcal{P}$ to set $\sigma = \nu = 0$. Computing the values of $\widetilde{\lambda}_k^\ell$ and $\widetilde{\mu}_k^\ell$ from the explicit formula for $H^{a,b,c,d}_{\varepsilon,\rho,0,0}$, we determine that (19) equates to finding a value of $\varepsilon \in \mathbb{U}$ and $\rho \in \mathbb{R}^*$ such that the following four conditions are met:

$$\lambda^0_0 = 0, \qquad \lambda^1_0 = \overline{\varepsilon}, \qquad \mu^0_0 = \rho, \qquad \mu^1_0 = 0. \tag{21}$$

According to Proposition 2.4, the automorphism H must satisfy

$$\lambda^0_0 = 0, \qquad \lambda^1_0 \in \mathbb{U}, \qquad \mu^0_0 \in \mathbb{R}^*, \qquad \mu^1_0 = 0,$$

whence (21) is easily solved by setting

$$\varepsilon := \overline{\lambda^1_0}, \qquad \rho = \mu^0_0.$$

Subcase (b). Suppose that $\{0, a \pm i\,ab, 2a\} \cap \mathbb{Z} = \{0, 2a\}$. This implies that $\mathcal{U} \subseteq \{0, 2a\}$, so it suffices to a set of parameters in $\mathcal{P}$ that verify equation (19) for the pair of values $n = 0, 2a$. This also implies that $2a \in \mathbb{N}$, but $a \notin \mathbb{N}$ or $b \neq 0$, whence we are forced by the definition of $\mathcal{P}$ to set $\nu = 0$. Computing the values of $\widetilde{\lambda}^\ell_k$ and $\widetilde{\mu}^\ell_k$ from the explicit formula for $H^{a,b,c,d}_{\varepsilon,\rho,\sigma,0}$, we determine that (19) equates to finding a value of $\varepsilon \in \mathbb{U}$, $\rho \in \mathbb{R}^*$, and $\sigma \in \mathbb{R}$ such that the following eight conditions are met:

$$\begin{aligned} \lambda^0_0 &= 0, & \lambda^1_0 &= \overline{\varepsilon}, \\ \mu^0_0 &= \rho, & \mu^1_0 &= 0, \\ \lambda^0_{2a} &= 0, & \lambda^1_{2a} &= \frac{(2a)!(1 + i\,b)\,\overline{\varepsilon}\,\sigma}{2}, \\ \mu^1_{2a} &= 0, & \mu^0_{2a} &= (2a-1)!\,\rho\,\sigma. \end{aligned} \tag{22}$$

As above, using Proposition 2.4 implies that the first four conditions are satisfied by setting

$$\varepsilon := \overline{\lambda^1_0}, \qquad \rho = \mu^0_0.$$

To prove the final four conditions can be met, let us compute the explicit forms of the functions f_{2a} and g_{2a} for the automorphism H. Recall from Proposition 2.3 that f_n and g_n satisfy the power series identity (8). If we differentiate this identity n times in the variable τ and then set $\tau = 0$, we obtain the identity

$$S(z\chi)^{n+1} g_n(z) \equiv S(z\chi)\overline{g_n}(\chi) + \overline{g_0}(\chi)\, S'(z\chi)\Big(S(z\chi)^n \overline{f_0}(\chi)\, f_n(z) + f_0(z)\,\overline{f_n}(\chi)\Big) + \mathcal{Q}_n\Big(z, \chi, \big(f_k(z), g_k(z), \overline{f_k}(z), \overline{g_k}(z)\big)_{k=1}^{n-1}\Big), \tag{23}$$

where $\mathcal{Q}_n : \mathbb{C}^2 \times \mathbb{C}^{4(n-1)} \to \mathbb{C}$ is a holomorphic function satisfying $\mathcal{Q}_n(z, \chi, \mathbf{0}) \equiv 0$.

We claim first that $f_n = g_n = 0$ for $0 < n < 2a$, which we prove by induction on n. Suppose this holds for all f_k and g_k with $k < n$. This implies that the $\mathcal{Q}_n$ term in identity (23) vanishes, which means that the functions f_n and g_n must satisfy the reduced identity

$$S(z\chi)^{n+1} g_n(z) \equiv S(z\chi)\overline{g_n}(\chi) + \mu_0^0\, S'(z\chi)\Big(S(z\chi)^n\, \lambda_0^1\, \chi\, f_n(z) + \frac{1}{\lambda_0^1}\, z\, \overline{f_n}(\chi)\Big).$$

However, Proposition 2.5 asserts that f_n and g_n are uniquely determined from this identity by the values $(\lambda_k^\ell, \mu_k^\ell) = (0,0)$ for $\ell \in \{0,1\}$ and $k < n$. Since $f_n(z) \equiv g_n(z) \equiv 0$ is *a* solution this identity, it must therefore be the *only* such solution, completing the induction.

Given that f_n and g_n vanish for $1 < n < 2a$, we can now compute f_{2a} and g_{2a} using the expansion for θ given in (20) and the formulas given in Proposition 2.5. We find

$$f_{2a}(z) = \frac{i\, 2c(1 - i\, b)\lambda_{2a}^0}{(\lambda_0^1)^2}\, z^2 + \left(\frac{2a\,\mu_{2a}^0}{\lambda_0^1\,\mu_0^0} - \frac{\lambda_{2a}^1}{(\lambda_0^1)^2}\right) z + \frac{i\, a\, \mu_{2a}^1}{c\,\lambda_0^1\,\mu_0^0},$$

$$g_{2a}(z) = \frac{i\, c\, \mu_0^0\, \lambda_{2a}^0}{a\, \lambda_0^1}\, z + \mu_{2a}^0.$$

Armed with these explicit formulas for f_{2a} and g_{2a}, we may compute the coefficients λ_{2a}^ℓ and μ_{2a}^ℓ.

Note that $g_{2a}(0) = \mu_{2a}^0$ by the formula for g_{2a}, but $g_{2a}(0) = \overline{\mu_{2a}^0}$ by the definition of μ_k^ℓ, whence μ_{2a}^0 is real. If we set

$$\sigma := \frac{\mu_{2a}^0}{(2a-1)!\,\mu_0^0} \in \mathbb{R},$$

then it follows that $\mu_{2a}^0 = (2a-1)!\,\mu_0^0\,\sigma = (2a-1)!\,\rho\,\sigma$, which verifies the eighth condition of (22).

To verify the remaining three conditions, first note that the functions f_{2a} and g_{2a} are just polynomials, whence their conjugates have the finite series expansions

$$\overline{f_{2a}}(\chi) = \lambda_{2a}^2\,\frac{\chi^2}{2!} + \lambda_{2a}^1\,\chi + \lambda_{2a}^0, \qquad \overline{g_{2a}}(\chi) = \mu_{2a}^1\,\chi + \mu_{2a}^0.$$

Let us substitute these formulas for f_{2a}, g_{2a}, $\overline{f_{2a}}$, and $\overline{g_{2a}}$ into (23) with $n = 2a$. (Note that the $\mathcal{Q}_{2a}$ term vanishes completely!) This results in a power series equation in the indeterminates z and χ. If we equate the $z^2\chi^3$ coefficients on both sides of the identity, we find

$$\lambda_{2a}^0 = 0,$$

which satisfies the fifth condition of (22). Similarly, examining the $z^2\chi^3$ coefficients yields

$$\mu_{2a}^1 = 0,$$

leaving only the λ_{2a}^1 condition of (22) unverified. If we equate the $z^2\chi^2$ coefficients, we find

$$\lambda_{2a}^1 = \frac{a(1+i\,b)\lambda_0^1\,\mu_{2a}^0}{\mu_0^0};$$

but given our choices for ε, ρ, and σ, that means

$$\lambda_{2a}^1 = \frac{a(1+i\,b)\overline{\varepsilon}}{\rho}\cdot(2a-1)!\rho\,\sigma = \frac{(2a)!(1+i\,b)\overline{\varepsilon}\,\sigma}{2},$$

verifying all eight conditions of (22).

Subcase (c). Suppose that $\{0, a \pm i\,ab, 2a\} \cap \mathbb{Z} = \{0, a, 2a\}$. This implies $\mathcal{U} \subseteq \{0, a, 2a\}$, so it suffices to find a set of parameters in $\mathcal{P}$ that verify equation (19) for the three values $n = 0, a, 2a$. This also implies that $a \in \mathbb{N}$ and $b = 0$. Computing the values of $\widetilde{\lambda}_k^\ell$ and $\widetilde{\mu}_k^\ell$ from the explicit formula for $H_{\varepsilon,\rho,\sigma,\nu}^{a,b,c,d}$, we determine that (19) equates to finding a value of $\varepsilon \in \mathbb{U}$, $\rho \in \mathbb{R}^*$, $\sigma \in \mathbb{R}$, and

$\nu \in \mathbb{C}$ such that the following twelve conditions are satisfied.

$$\begin{aligned} \lambda_0^0 &= 0, & \lambda_0^1 &= \overline{\varepsilon}, \\ \mu_0^0 &= \rho, & \mu_0^1 &= 0, \\ \lambda_a^0 &= a!\,\overline{\varepsilon}\,\nu, & \lambda_a^1 &= 0, \\ \mu_a^0 &= 0, & \mu_a^1 &= -i\,(a-1)!\,c\,\rho\,\overline{\nu}, \\ \lambda_{2a}^0 &= 0, & \lambda_{2a}^1 &= \frac{(2a)!\,\overline{\varepsilon}\left(\sigma - i\,3c\,|\nu|^2\right)}{2}, \\ \mu_{2a}^1 &= 0, & \mu_{2a}^0 &= (2a-1)!\,\rho\left(\sigma - i\,c\,|\nu|^2\right). \end{aligned} \tag{24}$$

As above, using Proposition 2.4 implies that the first four conditions are satisfied by setting

$$\varepsilon := \overline{\lambda_0^1}, \qquad \rho := \mu_0^0.$$

To prove the next four conditions can be met, we compute the explicit forms of the functions f_a and g_a of the automorphism H. Arguing as in Subcase (b), we find $f_n = g_n = 0$ for $0 < n < a$, and

$$\begin{aligned} f_a(z) &= \frac{i\,c\,\lambda_a^0}{(\lambda_0^1)^2}\,z^2 + \left(\frac{a\,\mu_a^0}{\lambda_0^1\,\mu_0^0} - \frac{\lambda_a^1}{(\lambda_0^1)^2}\right) z + \frac{i\,a\,\mu_a^1}{c\,\lambda_0^1\,\mu_0^0} \\ g_a(z) &= \frac{i\,c\,\mu_0^0\,\lambda_a^0}{a\,\lambda_0^1}\,z + \mu_a^0 \end{aligned}$$

From this, we may compute the coefficients λ_a^ℓ and μ_a^ℓ.

Note that based on our choice of ε, the fifth condition of (24) is met by setting

$$\nu := \frac{\lambda_a^0}{a!\,\lambda_0^1}.$$

Moreover, if we note equate $f_a(0) = \overline{\lambda_a^0}$ (from the definition of λ_k^ℓ) with the value of $f_a(0)$ from the formula above, we find

$$\mu_a^1 = -\frac{i\,c\,\lambda_0^1\,\mu_0^0\,\overline{\lambda_a^0}}{a}.$$

However, given our choices of ε, ρ, and ν (and the fact that ε is unimodular), this implies

$$\mu_a^1 = -\frac{i\,c\,\overline{\varepsilon}\,\rho\,a!\,\varepsilon\,\overline{\nu}}{a} = -i\,(a-1)!\,c\,|\varepsilon|^2\,\rho\,\overline{\nu},$$

which verifies the eighth condition of (24).

Just as in Subcase (b), to verify the remaining pair of conditions, we substitute the formulas for f_a and g_a, as well as those corresponding to the conjugate polynomials $\overline{f_a}$ and $\overline{g_a}$, into identity (23) with $n = a$. As in the previous subcase, the $\mathcal{Q}_a$ term vanishes. If we equate the $z^2\chi^2$ coefficients on either side of the equation, we obtain

$$\lambda_a^1 = 0;$$

if we equate the $z^3\chi^3$ coefficients, we find

$$\mu_a^0 = 0.$$

To show the final four conditions of (24) hold, let us compute f_{2a} and g_{2a}. We begin by noting that our work above allows us to rewrite f_a and g_a in the simpler form of

$$f_a(z) = \frac{ci\,\lambda_a^0}{(\lambda_0^1)^2}\,z^2 + \overline{\lambda_a^0}, \qquad g_a(z) = \frac{ci\,\mu_0^0\,\lambda_a^0}{a\,\lambda_0^1}\,z.$$

We next claim that $f_n = g_n = 0$ for all $a < n < 2a$. As proof, a careful inspection of Chain rule derivation of $\mathcal{Q}_n$ in (23) shows that if $n < 2a$, then each term of the power series $\mathcal{Q}_n$ contains a factor from the set $\left\{f_j, g_j, \overline{f_j}, \overline{g_j} : 1 < j < a\right\}$. The $\mathcal{Q}_n$ term must therefore vanish for such n. A similar uniqueness argument as that given in Subcase (b) completes the argument.

Finally, when $n = 2a$, a careful derivation using the chain rule shows that

$$\begin{aligned}
&\mathcal{Q}_{2a}\left(z, \chi; \left(f_k(z), g_k(z), \overline{f_k}(z), \overline{g_k}(z)\right)_{k=1}^{2a-1}\right) \\
&\quad = \frac{(2a)!}{(a!)^2}\Bigg\{\mu_0^0\,S'(z\chi)\,S(z\chi)^a\,f_a(z)\,\overline{f}_a(\chi) \\
&\qquad + \frac{1}{2}\mu_0^0\,S''(z\chi)\left(\lambda_0^1\,\chi\,S(z\chi)^n f_a(z) + \frac{z}{\lambda_0^1}\,\overline{f}_a(z)\right)^2 \\
&\qquad + \overline{g}_a(\chi)\,S'(z\chi)\left(\lambda_0^1\,\chi\,S(z\chi)^n f_a(z) + \frac{z}{\lambda_0^1}\,\overline{f}_a(z)\right)
\end{aligned}$$

Substituting this back into into (23) with $n = 2a$ and using the explicit formulas given in Proposition 2.5 yield the following explicit formulas for f_{2a} and g_{2a}:

$$
\begin{aligned}
f_{2a}(z) &= -\frac{3(2a)!\,c^2\,(\lambda_a^0)^2}{2(a!)^2(\lambda_0^1)^3}z^3 + \frac{i\,2c\,\lambda_{2a}^0}{(\lambda_0^1)^2}\,z^2 \\
&\quad + \left(\frac{2a\,\mu_{2a}^0}{\lambda_0^1\,\mu_0^0} - \frac{\lambda_{2a}^1}{(\lambda_0^1)^2} + \frac{i\,(2a)!\,c\,|\lambda_a^0|^2}{(a!)^2\,\lambda_0^1}\right)z + \frac{i\,a\,\mu_{2a}^1}{c\,\lambda_0^1\,\mu_0^0} \\
g_{2a}(z) &= -\frac{(2a)!(1+2a)\,c^2\,\mu_0^0\,(\lambda_0^0)^2}{2a^2(a!)^2(\lambda_0^1)^2}\,z^2 + \frac{i\,c\,\mu_0^0\,\lambda_{2a}^0}{a\,\lambda_0^1}\,z \\
&\quad + \left(\mu_{2a}^0 + \frac{i\,(2a)!\,c\,\mu_0^0|\lambda_a^0|^2}{a(a!)^2}\right)
\end{aligned}
$$

If we equate $\overline{g_{2a}(0)}$ with μ_{2a}^0, we obtain

$$\operatorname{Im}\mu_{2a}^0 = -\frac{(2a-1)!\,c\,\mu_0^0\,|\lambda_a^0|^2}{(a!)^2}.$$

Note that if we set

$$\sigma := \frac{\operatorname{Re}\mu_{2a}^0}{(2a-1)!\,\mu_0^0},$$

then it follows that

$$
\begin{aligned}
\mu_{2a}^0 &= \operatorname{Re}\mu_{2a}^0 + i\,\operatorname{Im}\mu_{2a}^0 = (2a-1)!\,\rho\,\sigma - i\,(2a-1)!\,c\,\rho\,|\nu|^2 \\
&= (2a-1)!\,\rho\big(\sigma - i\,c\,|\nu|^2\big),
\end{aligned}
$$

verifying yet another condition of (24).

To tackle the last three, we once again substitute the formulas for f_a, g_a, f_{2a}, g_{2a}, and their conjugates back into (23) with $n = 2a$ and equate various coefficients. If we equate the $z^3\chi^2$ or $z^2\chi^3$ coefficients, we find

$$\lambda_{2a}^0 = \mu_{2a}^1 = 0.$$

Finally, if we equate the $z^2\chi^2$ coefficients, we obtain

$$\lambda_{2a}^1 = \lambda_0^1\left(\frac{a\,\mu_{2a}^0}{\mu_0^0} - \frac{i\,(2a)!c\,|\lambda_a^0|^2}{(a!)^2}\right),$$

which given our choices for ε, ρ, σ, and ν imply

$$\begin{aligned}\lambda^1_{2a} &= \overline{\varepsilon}\bigg(\frac{a}{\rho}\,(2a-1)!\,\rho\big(\sigma - i\,c\,|\nu|^2\big) - i\,(2a)!c\,|\nu|^2\bigg)\\ &= \overline{\varepsilon}\bigg(\frac{(2a)!\big(\sigma - i\,c\,|\nu|^2\big)}{2} - i\,(2a)!c\,|\nu|^2\bigg)\\ &= \frac{(2a)!\,\overline{\varepsilon}\,\big(\sigma - i\,3c\,|\nu|^2\big)}{2},\end{aligned}$$

verifying all twelve conditions of (24).

4.2. **Case 2:** $d > 1$**.** The proof of this case is very similar to the $d = 1$ case, although it is significantly easier, and so we simply give a sketch of it. In this case, a similar determinant calculation shows that $\mathcal{U} \subseteq \{0, 2a\} \cap \mathbb{Z}$. If $\mathcal{U} = \{0\}$, the *exact same* argument as in Subcase (a) shows that H takes the form $H^{a,b,c,d}_{\varepsilon,\rho,0,0}$.

Hence, let us assume $2a \in \mathbb{N}$. Examining (19) in this case, we must satisfy the eight conditions below:

$$\begin{aligned} \lambda^0_0 &= 0, & \lambda^1_0 &= \overline{\varepsilon},\\ \mu^0_0 &= \rho, & \mu^1_0 &= 0,\\ \lambda^0_{2a} &= 0, & \lambda^1_{2a} &= \frac{(2a)!(1+i\,b)\,\overline{\varepsilon}\,\sigma}{2d},\\ \mu^1_{2a} &= 0, & \mu^0_{2a} &= (2a-1)!\,\rho\,\sigma. \end{aligned} \tag{25}$$

The attack is very similar to that in Subcase (b) above.

We begin by computing f_{2a} and g_{2a} explicitly. Since $d > 1$, it follows that

$$S'(0) = \frac{d}{dt}\bigg|_{t=0} S_{a,b}(c\,t^d) = 0.$$

A similar argument using the mapping identity (23) and Proposition 2.5 implies that

$$f_{2a}(z) = \left(\frac{2a\,\mu^0_{2a}}{d\,\lambda^1_0\,\mu^0_0} - \frac{\lambda^1_{2a}}{(\lambda^1_0)^2}\right)z, \qquad g_{2a}(z) = \mu^0_{2a}.$$

In particular, this implies $\lambda^0_{2a} = \mu^1_{2a} = 0$. Moreover, substituting these formulas and their conjugates into (23) and taking derivatives (as in Subcase (b) above) yields

$$\lambda^1_{2a} = \frac{a(1+i\,b)\lambda^1_0\,\mu^0_{2a}}{d\,\mu^0_0}$$

whereas equating $\overline{g_{2a}(0)}$ with μ_{2a}^0 proves that μ_{2a}^0 is real. The reader is invited to show that setting

$$\varepsilon := \overline{\lambda_0^1} \in \mathbb{U}, \quad \rho := \mu_0^0 \in \mathbb{R}^*, \quad \sigma := \frac{\mu_{2a}^0}{(2a-1)!\mu_0^0} \in \mathbb{R}, \quad \nu := 0$$

indeed satisfies the eight conditions of (25), which completes the proof of Case 2, and with it Lemma 4.3 and Theorem 4.1. □

5. The proof of Theorem 1.3

In this final section, we prove prove Theorem 1.3. Let us fix the germ of a hypersurface $(M, 0) \in \mathcal{G}$, and consider its stability group $\mathrm{Aut}(M, 0)$.

Assume first that $\mathrm{Aut}(M, 0)$ is determined by 1-jets. We claim that

$$\mathrm{Aut}(M, 0) = \left\{ H_{\varepsilon,\rho,0,0}^{1,0,0,1} : \varepsilon \in \mathbb{U},\, \rho \in \mathbb{R}^* \right\};$$

that is, we may take $(a, b, c, d) = (1, 0, 0, 1)$ and $\mathcal{P} = \mathbb{U} \times \mathbb{R}^* \times \mathbf{0} \times \mathbf{0}$ in the language of the Theorem. As proof, note that

$$H_{\varepsilon,\rho,0,0}^{1,0,0,1}(z, w) = \big(\varepsilon\, z, \rho\, w\big)$$

is a *global* automorphism of $(M, 0)$ for any unimodular number ε and nonzero real number ρ. Conversely, given an arbitrary $H \in \mathrm{Aut}(M, 0)$, Proposition 2.4 asserts H and $H_{\varepsilon,\rho,0,0}^{1,0,0,1}$ both have the same 1-jet if

$$(\varepsilon, \rho, 0, 0) := \left(\overline{\lambda_0^1}, \mu_0^0, 0, 0\right) \in \mathcal{P},$$

whence it follows they agree as mappings as well.

So let us assume instead that $\mathrm{Aut}(M, 0)$ is not determined by 1-jets. If we can show that $M \in \mathcal{G}_S$, then this will complete the proof of Theorem 1.3, for not only will this prove the theorem's final statement, but Theorem 4.1 will show that M has desired automorphism group.

To this end, assume M is expressed with θ, and expand θ about $t = 0$ as

$$\theta(t) = \sum_{k=d}^{\infty} \theta_k t^k = \theta_d\, t^d + \theta_{d+1}\, t^{d+1} + \theta_{d+2}\, t^{d+2} + \cdots \tag{26}$$

with $\theta_d \neq 0$. Let $\mathcal{U}$ denote the set given in Definition 2.6.

To prove that $M \in \mathcal{G}_S$, it suffices to find a 3-tuple $(a,b,c) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^*$ such that $M = M^{a,b,c,d}$, where d is as above. We shall prove this result through a sequence of five lemmas.

Lemma 5.1. *The set $\mathcal{U}$ contains an integer $n > 0$.*

Proof. We prove this by contrapositive. If $\mathcal{U}$ consisted of only the integer 0, then Proposition 2.5 implies that any automorphism of M would be uniquely determined by the values

$$\lambda_0^0 = 0, \quad \lambda_0^1 \in \mathbb{U}, \quad \mu_0^0 \in \mathbb{R}^*, \quad \mu_0^1 = 0.$$

Since the only variable parameters λ_0^1 and μ_0^0 depend on first-order derivatives of H, it follows that $\mathrm{Aut}(M,0)$ is determined by 1-jets. $\square$

Recall the definition of the functions υ_j^n given in Definition 2.6.

Lemma 5.2. *At least one of the following is true:*

1. *One of the functions υ_1^n, υ_2^n is a multiple of the other.*
2. *$d = 1$ and υ_3^n vanishes identically.*
3. *$d = 1$ and υ_4^n vanishes identically.*

Proof. Let us first consider the case $d = 1$. Observe that the alternate, linear independence characterization of $\mathcal{U}$ given in Definition 2.6 implies that there exists a linear combination

$$A_1\,\upsilon_1^n(z\chi) + A_2\,\upsilon_2^n(z\chi) + A_3\,z\,\upsilon_3^n(z\chi) + A_4\,\chi\,\upsilon_4^n(z\chi) \equiv 0 \tag{27}$$

with at least one of the coefficients A_j being nonzero.

If $\upsilon_4^n \equiv 0$, then condition (3) of the lemma holds and we're done; otherwise, there exists some derivative of υ_4^n that does not vanish at 0, say the j-th derivative. If we differentiate (27) j times in z, $j+1$ times in χ, and set $(z,\chi) = (0,0)$, we obtain

$$A_4 (j+1)! \left(\frac{d^j}{dt^j} \big(\upsilon_4^n(t)\big) \bigg|_{t=0} \right) = 0,$$

which forces $A_4 = 0$.

Similar reasoning shows that either $\upsilon_3^n \equiv 0$ (and thus condition (2) holds) or $A_3 = 0$. In this latter case, we have the equation

$$A_1\,\upsilon_1^n(z\chi) + A_2\,\upsilon_2^n(z\chi) \equiv 0,$$

with at least one of the A_j nonzero. It follows trivially that one of these functions is a multiple of the other, proving condition (1) holds.

In the event that $d > 1$, Definition 2.6 implies that $n \in \mathcal{U}$ is equivalent to the two functions $\upsilon_1^n(z\chi)$ and $\upsilon_2^n(z\chi)$ being linearly independent. But this just there exists a dependence relation among υ_1^n and υ_2^n, and this is equivalent to condition (1). □

Lemma 5.3. *If condition (1) of Lemma 5.2 holds, then* $M = M^{a,b,c,d}$ *with*

$$a := \frac{n}{2}, \qquad b := -\frac{\theta_{2d}}{n\,{\theta_d}^2}, \qquad c := n\,\theta_d,$$

where the numbers θ_k *are defined by the expansion of* θ *given in* (26).

Proof. By Proposition 2.1, M is equivalently expressed with S, where $S = q \circ \theta$ and $q : \mathbb{R} \to \mathbb{C}$ is defined by

$$q(t) := \frac{1 + i\,t}{1 - i\,t}.$$

Comparing this with the defining equation (16) for $M^{a,b,c,d}$, to prove this lemma it suffices to prove that

$$S_{\frac{n}{2},b}\big(n\,\theta_d\,t^d\big) \equiv q\big(\theta(t)\big)$$

as power series in t. We do this by showing both these analytic functions solve the same complex initial value problem, namely

$$t\,S'(t) = \frac{2d\,S(t)\big(S(t)^n - 1\big)}{n\big((1 - i\,b)S(t)^n + (1 + i\,b)\big)} \tag{28}$$

subject to the initial conditions

$$\begin{aligned} S(0) &= 1, \\ S'(0) &= S''(0) = \cdots = S^{(p-1)}(0) = 0, \\ S^{(d)}(0) &= i\,2d!\,\theta_d. \end{aligned}$$

Note that the initial conditions may be equivalently expressed as

$$S(t) = 1 + i\,2\,\theta_d\,t^d + O(t^{d+1}). \tag{29}$$

Of course, we must first establish that this initial value problem admits a *unique* solution. To do so, suppose that S is a solution to the differential equation (28) satisfying the given initial conditions (29). Expand S as a power series

$$S(t) = 1 + \sum_{k=d}^{\infty} s_k\, t^k,$$

where the s_k are complex numbers and $s_d = i\,2\,\theta_d$. From this, it follows that

$$S(t)^n = 1 + \sum_{k=d}^{\infty} \big(n s_k + \mathcal{Q}^1_{n,k}(s_d, \dots, s_{k-1})\big) t^k,$$

$$\frac{2S(t)}{(1 - i\,b) S(t)^n + (1 + i\,b)} = 1 + \sum_{k=d}^{\infty} \mathcal{Q}^2_{n,k}(s_d, \dots, s_k) t^k,$$

where each $\mathcal{Q}^j_{n,k}$ is a complex polynomial that does not depend on S. Using these to expand the left- and right-hand sides of the differential equation (28) as power series, we obtain

$$\sum_{k=d}^{\infty} k s_k\, t^k = \frac{d}{n}\Bigg(\sum_{k=d}^{\infty} \big(n s_k + \mathcal{Q}^1_{n,k}(s_d, \dots, s_{k-1})\big) t^k \Bigg) \times \left(1 + \sum_{k=d}^{\infty} \mathcal{Q}^2_{n,k}(s_d, \dots, s_k) t^k \right).$$

If we fix a power $\ell > d$ and compare the coefficients of the t^ℓ terms of both sides, we obtain

$$\ell\, s_\ell = d\, s_\ell + \frac{d}{n} \mathcal{Q}^1_{n,\ell}(s_d, \dots, s_{\ell-1}) + \frac{d}{n} \sum_{k=d}^{\ell-1} \big(n s_k + \mathcal{Q}^1_{n,k}(s_d, \dots, s_{k-1})\big)\, \mathcal{Q}^2_{n,\ell-k}(s_d, \dots, s_{\ell-k}),$$

from which it follows that

$$s_\ell = \frac{d}{n(\ell - d)} \mathcal{R}_{n,\ell}(s_d, \cdots, s_{\ell-1}),$$

where $\mathcal{R}_{n,\ell}$ is another polynomial that does not depend on S. In particular, this shows inductively that if S is a solution to the initial value problem (28), then each coefficient s_k is parameterized (and hence uniquely determined) by the initial condition s_d. That is, solutions to this initial value problem are unique.

It is now straightforward to show that the function $S(t) = S_{\frac{n}{2},b}(n\,\theta_d\, t^d)$ solves it. First, observe that power series expansion of $S_{a,b}$ given in equation (12) shows that $S_{\frac{n}{2},b}(z) = 1 + (i\,2/n)z + O(z^2)$, whence

$$S(t) = S_{\frac{n}{2},b}\big(n\,\theta_d\,t^d\big) = 1 + \frac{i\,2}{n}\big(n\,\theta_d\,t^d\big) + O(t^{2d}) = 1 + i\,2\,\theta_d\,t^d + O(t^{2d}),$$

which implies (29). Moreover, the defining property of $S_{n/2,b}$ given in Lemma 3.1 implies that

$$S(t)^n = 1 + i\,2n\,\theta_d\,t^d\,S(t)^{\frac{n}{2}(1+i\,b)}. \tag{30}$$

Differentiating this equation with respect to t yields

$$\begin{aligned} &n\,S(t)^{n-1}S'(t) \\ &\quad = i\,2dn\,\theta_d\,t^{d-1}S(t)^{n(1+i\,b)/2} + i\,n^2(1+i\,b)\,\theta_d\,t^d\,S(t)^{\frac{n}{2}(1+i\,b)-1}S'(t). \end{aligned}$$

Multiplying both sides of this equation by $t\,S(t)$ and solving for $t\,S'(t)$ yields

$$t\,S'(t) = \frac{d\,S(t)\big(i\,2n\,\theta_d\,t^d\,S^{n(1+i\,b)/2}\big)}{n\big(S(t)^n - i\,n\,\theta_d(1+i\,b)t^d\,S^{n(1+i\,b)/2}\big)}$$

Substituting in equation (30), this equation may be rewritten

$$\begin{aligned} t\,S'(t) &= \frac{d\,S(t)\big(S(t)^n - 1\big)}{n\left(S(t)^n - (1+i\,)\dfrac{S(t)^n-1}{2}\right)} \\ &= \frac{2d\,S(t)\big(S(t)^n - 1\big)}{n\big(2S(t)^n - (1+i\,b)(S(t)^n - 1)\big)} \\ &= \frac{2d\,S(t)\big(S(t)^n - 1\big)}{n\big((1-i\,b)S(t)^n + (1+i\,b)\big)} \end{aligned}$$

proving that $S(t) = S_{\frac{n}{2},b}\big(\frac{1}{2}n\,\theta_d\,t^d\big)$ satisfies (28).

Finally, we must show that

$$S(t) = q\big(\theta(t)\big) = \frac{1+i\,\theta(t)}{1-i\,\theta(t)}$$

also solves the same initial value problem. A straightforward power series calculation shows that

$$S(t) = 1 + i\,2\,\theta_d\,t^d + O(t^{d+1}),$$

so the initial conditions (29) are met. Expanding further, we also find

$$S(t)^n = 1 + i\,2n\left(\sum_{j=d}^{2d-1}\theta_j\,t^j\right) + i\,2n(\theta_{2d} + i\,n\,{\theta_d}^2)t^{2d} + O(t^{2d+1}).$$

Substituting this into the formulas for the υ_j^n given in Definition 2.6, we find

$$\upsilon_1^n(t) = i\,2dn\,{\theta_d}^2\,t^{2d} + O(t^{2d+1})$$

$$\upsilon_2^n(t) = -\sum_{j=d+1}^{2d-1} \frac{i\,2(j-d)n}{d}\theta_j\,t^j - 2n(n\,{\theta_d}^2 + i\,\theta_{2d})\,t^{2d} + O(t^{2d+1})$$

Note that this implies that neither υ_1^n nor υ_2^n vanish identically, so the assumption of the lemma asserts that

$$\upsilon_2^n(t) \equiv A\,\upsilon_1^n(t) \tag{31}$$

for some nonzero A. Moreover, examining the t^{2d} terms of both power series, it follows that

$$A = \frac{-2n(n\,{\theta_d}^2 + i\,\theta_{2d})}{i\,2dn\,{\theta_d}^2} = \frac{i\,n\,{\theta_d}^2 - \theta_{2d}}{d\,{\theta_d}^2} = \frac{n}{d}(i+b),$$

where the last equality follows from the definition of b in Lemma 5.3. Substituting this value of A together with the explicit formulas for the υ_j^n back into the dependence relation (31) yields

$$\begin{aligned}
&\frac{n}{d}(i+b)\,t\,\theta'(t)\left(\left(\frac{1+i\,\theta(t)}{1-i\,\theta(t)}\right)^n - 1\right)\\
&\qquad = \big(1+\theta(t)^2\big)\left(\left(\frac{1+i\,\theta(t)}{1-i\,\theta(t)}\right)^n - 1\right) - \frac{i\,2n}{d}\,t\,\theta'(t).
\end{aligned}$$

Since S and θ are related by the two equations (5) in Proposition 2.1, we can convert this into an equation involving only S as

$$\begin{aligned}
&\frac{n}{d}(i+b)\,t\left(\frac{-i\,2\,S'(t)}{(1+S(t))^2}\right)\big(S(t)^n - 1\big)\\
&\qquad = \left(1 - \frac{(1-S(t))^2}{(1+S(t))^2}\right)\big(S(t)^n - 1\big) - \frac{i\,2n}{d}\,t\left(\frac{-i\;S'(t)}{(1+S(t))^2}\right)
\end{aligned}$$

Multiplying both sides of this equation by $(1+S(t))^2$ and solving for $S'(t)$ yields

$$\begin{aligned}
S'(t) &= \frac{d\big((1+S(t))^2 - (1-S(t))^2\big)\big(S(t)^n - 1\big)}{-i\,2n\,t\big(i\,2 + (i+b)\big(S(t)^n - 1\big)\big)}\\
&= \frac{4d\,S(t)\big(S(t)^n - 1\big)}{2n\,t\big((1-i\,b)S(t)^n + (1+i\,b)\big)}
\end{aligned}$$

which is equivalent to the differential equation (28), completing the proof of the lemma.

□

Lemma 5.4. *If condition (2) of Lemma 5.2 holds, then $M = M^{n,0,c,1}$ with $c = 2n\,\theta_1$.*

Proof. Recall that condition (2) states that both $d = 1$ and $\upsilon_3^n \equiv 0$. The proof is similar to that of Lemma 5.3, with a few simplifications. As in the previous lemma, it suffices to show that $q\big(\theta(t)\big) \equiv S_{n,0}\big(2n\,\theta_1\,t\big)$, which we do by showing both solve the complex initial value problem

$$S'(t) = \frac{i\,2\,\theta_1\,S(t)^{n+1}}{1 + i\,2n\,\theta_1\,t\,S(t)^n}, \qquad S(0) = 1, \tag{32}$$

(That this problem has a unique power series solution, is a straightforward calculation left to the reader.)

On one hand, it is easy to show that the function $S(t) = S_{n,0}\big(2n\,\theta_1\,t\big)$ solves equation (32). Lemma 3.1 implies that $S_{n,0}(0) = 1$ and

$$S(t)^{2n} = 1 + i\,4n\,\theta_1\,t\,S(t)^n. \tag{33}$$

Differentiating this equation with respect to t yields

$$2n\,S(t)^{2n-1}S'(t) = i\,4n\,\theta_1\,S(t)^n + i\,4n^2\,\theta_1\,t\,S(t)^{n-1}S'(t).$$

Multiplying both sides of this equation by $\frac{1}{2n}S(t)$ and solving for $S'(t)$ yields

$$S'(t) = \frac{i\,2\,\theta_1\,S(t)^{n+1}}{S(t)^{2n} - i\,2n\,\theta_1\,t\,S(t)^n} \tag{34}$$

Using equation (33), this equation may be rewritten

$$S'(t) = \frac{i\,2\,\theta_1\,S(t)^{n+1}}{\big(1 + i\,4n\,\theta_1\,t\,S(t)^n\big) - i\,2n\,\theta_1\,t\,S(t)^n} = \frac{i\,2\,\theta_1\,S(t)^{n+1}}{1 + i\,2n\,\theta_1\,t\,S(t)^n},$$

proving that $S(t) = S_{n,0}\big(2nc_1\,t\big)$ satisfies the desired initial value problem (32).

On the other hand, to show that $S(t) = q\big(\theta(t)\big)$ is also a solution, we use the assumption that $\upsilon_3^n \equiv 0$. Using the formula for υ_3^n given in Definition 2.6, we find

$$0 = \frac{d^2}{dt^2}\big(\upsilon_3^n(t)\big)\bigg|_{t=0} = -2\,{\theta_1}^3 - 4n^2\,{\theta_1}^3 - i\,4n\,\theta_1\,\theta_2 - \frac{8\,{\theta_2}^2}{\theta_1} + 6\,\theta_3,$$

which implies

$$\theta_3 = \frac{1}{3\,\theta_1}\big((1 + 2n^2)\,{\theta_1}^4 + 4\,{\theta_2}^2 - i\,2n\,{\theta_1}^2\,\theta_2\big).$$

Since $\theta(t)$ is real-valued for real t, it follows that θ_3 must be real, whence $\theta_2 = 0$ necessarily. Making this substitution into υ_3^n gives

$$0 = \upsilon_3^n(t) = \theta'(t) - \left(\frac{1 + i\,\theta(t)}{1 - i\theta(t)}\right)^n \big(\theta_1(1 + \theta(t)^2) - i\,2n\,\theta_1\,t\,\theta'(t)\big).$$

As before, we use equations (5) to convert this into an equation involving S alone to obtain

$$\frac{-i\,2\,S'(t)}{\big(1 + S(t)\big)^2} = S(t)^n \left(\theta_1\left(1 - \frac{(1 + S(t))^2}{(1 - S(t))^2}\right) - i\,2n\,\theta_1\,t\left(\frac{-i\,2\,S'(t)}{\big(1 + S(t)\big)^2}\right)\right).$$

Multiplying both sides of this equation by $(1 + S(t))^2$ and solving for $S'(t)$ yields

$$\begin{aligned} S'(t) &= \frac{\theta_1\big((1 + S(t))^2 - (1 - S(t))^2\big)S(t)^n}{-i\,2\big(1 + i\,2n\,\theta_1\,t\,S(t)^n\big)} \\ &= \frac{4\,\theta_1\,S(t)^{n+1}}{-i\,2\big(1 + i\,2n\,\theta_1\,t\,S(t)^n\big)}, \end{aligned}$$

which is equivalent to the differential equation (32), as desired. □

Lemma 5.5. *If condition (3) of Lemma 5.2 holds, then* $M = M^{n,0,c,1}$ *with* $c = 2n\,\theta_1$.

Proof. The proof is remarkably similar to that of Lemma 5.4, so we simply give a sketch of it. The lemma is proved if we show both $q\big(\theta(t)\big)$ and $S_{n,0}\big(2n\,\theta_1\,t\big)$ are solutions to the complex initial value problem

$$S'(t) = \frac{-2\,\theta_1\,S(t)}{2n\,\theta_1\,t + i\,S(t)^n}, \qquad S(0) = 1. \tag{35}$$

We have already established the function $S(t) = S_{n,0}\big(2n\,\theta_1\,t\big)$ satisfies the differential equation (34), whence

$$S'(t) = \frac{i\,S(t)^n\big(2\,\theta_1\,S(t)\big)}{i\,S(t)^n\big(-i\,S(t)^{2n} + 2n\,\theta_1\,t\big)} = \frac{-2\,\theta_1\,S(t)}{2n\,\theta_1\,t + i\,S(t)^n},$$

proving that $S(t) = S_{n,0}\big(n\,\theta_1\,t\big)$ satisfies the differential equation (35).

For the function $S(t) = q\big(\theta(t)\big)$, the assumption that $\upsilon_4^n \equiv 0$ implies

$$0 = \frac{d^2}{dt^2}\big(\upsilon_4^n(t)\big)\Big|_{t=0} = 2\,{\theta_1}^2 + 4n^2\,{\theta_1}^2 - i\,4n\,\theta_2 + \frac{8\,{\theta_2}^2}{{\theta_1}^2} - \frac{6\,\theta_3}{\theta_1}.$$

The reality of θ once again forces $\theta_2 = 0$, which implies

$$0 \equiv \upsilon_4^n(t) = 1 + \theta(t)^2 + \frac{\theta'(t)}{\theta_1}\left(i\,2n\,\theta_1\,t - q\big(\theta(t)\big)^n\right).$$

Converting this into S and solving for $S'(t)$ yields

$$S'(t) = \frac{\theta_1\big((1+S(t))^2 - (1-S(t))^2\big)}{-i\,2\big(S(t)^n - i\,2n\,\theta_1\,t\big)} = \frac{4\,\theta_1\,S(t)}{-2\big(2n\,\theta_1\,t + i\,S(t)^n\big)},$$

which is equivalent to (35). □

This completes the proof of Theorem 1.3.

Note that an unexpected consequence of Lemmas 5.4 and 5.5 is that when $d = 1$, $\upsilon_3^n \equiv 0$ if and only if $\upsilon_4^n \equiv 0$, since both of these conditions is equivalent to M being of the form $M^{n,0,c,1}$ for the same value of c. Thus, the last two consequences of Lemma 5.2 are in fact equivalent. Based on Theorem 4.1, this corresponds to the stability group involving both parameters σ and ν. Moreover, if $d = 1$ and $\upsilon_4^n \equiv \upsilon_3^n \equiv 0$ for some n, then by Lemma 5.3 it follows that $\upsilon_2^{2n} = A\,\upsilon_1^{2n}$, so either of the last two consequences implies the first (though for a different value of n).

Hence, if the stability group of a hypersurface $M \in \mathcal{G}$ is not determined by 1-jets, then it is always true that $\upsilon_2^n = A\,\upsilon_1^n$ for some value of n, and this condition corresponds to the stability group of M involving the parameter σ.

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