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Arithmetically Gorenstein Schubert Varieties in a Minuscule G/P

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Abstract: In this paper, we give a characterization of arithmetically Gorenstein Schubert varieties in a minuscule G/P. We further give a nice combinatorial description of the arithmetically Gorenstein Schubert varieties in the classical & orthogonal Grassmannians. We also prove the arithmetically-Gorenstein property for $SL(n)/Q, Q$ being any parabolic subgroup, in particular, the arithmetically-Gorenstein property for the flag variety $SL(n)/B$.
Keywords: Schubert varieties, minuscule, Gorenstein.

Introduction

The main goal of this paper is to give a characterization of the arithmetically Gorenstein Schubert varieties in a minuscule G/P.

Let K be the base field which we assume to be algebraically closed of arbitrary characteristic. Let G be a semisimple, simply connected algebraic group over K, T a maximal torus, and B a Borel subgroup containing T. Let W be the Weyl group. For $w \in W$, let e_w be the (T-fixed point) wB in G/B, and $X(w)(=\overline{Be_w} = \overline{BwB}(mod\, B))$, the Schubert variety associated to w; more generally, for a parabolic subgroup $Q \supseteq B$, we denote by $X_Q(w)$ (or just $X(w)$, when there is no room for confusion), the Schubert variety $\overline{BwQ}(mod\, Q)$ associated to the T-fixed point wQ in G/Q. While all Schubert varieties are Cohen-Macaulay (cf.[22]), not all of the Schubert varieties are smooth; one has (thanks to the works of several mathematicians during 1980's & 1990's) a complete classification of smooth Schubert varieties (see [1] for a detailed account on this).

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The Gorenstein property is a geometric property in between Cohen-Macaulayness and smoothness properties (see §1 for the definition of the Gorenstein property). The problem of classifying the Gorenstein Schubert varieties is an open problem. Recently, Woo and Yong (cf. [26]) have given a characterization of the (geometrically) Gorenstein Schubert varieties in the flag variety $SL(n)/B$. In this case (namely, $G = SL(n)$), we have a natural identification of W with the symmetric group S_n. In [26], the authors give a characterization of the Gorenstein Schubert varieties in $SL(n)/B$ in terms of certain "pattern avoidance" for the associated permutations (in the same spirit as the "pattern avoidance" description for smoothness given in [16]). As a consequence, one obtains a combinatorial characterization of the Gorenstein Schubert varieties in the Grassmannian (see §3 for details); it should be remarked that we prove a stronger result than that of [26] (see also [25]), namely, while in [25, 26], the authors give a characterization of Gorenstein Schubert varieties, we give a characterization of the Gorenstein property even for the cones over Schubert varieties (cf. Theorem 3.7). As a consequence of Theorem 3.7, it turns out that a Schubert variety in the Grassmannian is arithmetically Gorenstein (for the Plücker embedding) if and only if it is geometrically Gorenstein.

Let us now take a minuscule maximal parabolic subgroup P in a semi-simple algebraic group G (see §1 for a complete list of minuscule maximal parabolics). One usually refers to the corresponding G/P as a *minuscule* G/P, and the Schubert varieties in G/P as *minuscule Schubert varieties.* If $G = SL(n)$, then every maximal parabolic is minuscule, and G/P is just a Grassmann variety. In this case, a Schubert variety in G/P has a representation by a Young diagram (of suitable width and height). Then by [26] (see also [25]), one obtains that a Schubert variety in G/P is Gorenstein if and only if the outer corners of the associated Young diagram lie on the same anti-diagonal. Now taking G to be $SO(m)$ and P to be the maximal parabolic subgroup associated to the right end root (one of the two right end roots if m is even) in the Dynkin diagram (following the indexing of the Dynkin diagram as in [3]), a Schubert variety in the orthogonal Grassmannian $SO(m)/P$ again has a representation by a (self-dual) Young diagram.

Our first main result (cf. Theorem 5.12) is a generalization of the combinatorial characterization of Gorenstein Schubert varieties in a Grassmannian to the case of an orthogonal Grassmannian. In Theorem 5.12, we show that a Schubert variety in $SO(m)/P$ is arithmetically Gorenstein if and only if the outer corners of the associated Young diagram lie on the same anti-diagonal. We now give a brief sketch of the proof of this result. Given a maximal parabolic subgroup P, let L be the ample generator of $\operatorname{Pic} G/P (\cong \mathbb{Z})$, the isomorphism classes of line bundles on G/P. For the canonical projective embedding

$$X(w) \hookrightarrow G/P \hookrightarrow Proj(H^0(G/P, L))$$

($X(w)$ being a Schubert variety in G/P), let $R(w)$ denote the homogeneous coordinate ring of $X(w)$. Now taking P to be minuscule, using the results of [23], we have that $R(w)$ is a graded Hodge algebra ("Hodge algebra" in the sense of [5]), having a set of algebra generators indexed by $H(w)$, the Bruhat poset of Schubert subvarieties of $X(w)$ (here, "poset" is the abbreviation for a partially ordered set). Also, $R(w)$ is Cohen-Macaulay (by [22, 21], Schubert varieties are arithmetically Cohen-Macaulay and arithmetically normal). These facts together with a result of Stanley (cf.[24], Theorem 4.4) giving a characterization of Gorenstein graded Cohen-Macaulay domains (in terms of the Hilbert series) implies that $R(w)$ is Gorenstein if and only if the poset $J(H(w))$ of join-irreducibles of $H(w)$ is a ranked poset (i.e., all maximal chains in $J(H(w))$ have the same length). We show (cf. §3) that the poset of join-irreducibles of $H(w)$ ($X(w)$ being a Schubert variety in an orthogonal Grassmannian) is a ranked poset if and only if the outer corners of the associated Young diagram lie on the same anti-diagonal. In the remaining minuscule cases, if G is $SP(2n)$ and $P = P_1$ (the maximal parabolic corresponding to the left end root in the Dynkin diagram, the indexing of the Dynkin diagram being as in [3]), then $G/P \cong \mathbb{P}^{2n-1}$; all of the Schubert varieties in G/P are certain linear subspaces, and hence are smooth (and hence Gorenstein and also arithmetically Gorenstein). If G is $SO(2n)$, and $P = P_1$ (the maximal parabolic corresponding to the left end root), then one easily checks that the poset of join-irreducibles of $H(w)$ is a ranked poset for all Schubert varieties $X(w)$ in G/P. Hence all of the Schubert varieties are arithmetically Gorenstein. If G is E_6 or E_7 and P is a minuscule parabolic (using the above criterion for Gorenstein property), we have listed (cf. §6) the arithmetically Gorenstein Schubert varieties.

There is one important non-minuscule case for which also we prove the arithmetic Gorenstein property, namely, the (partial) flag variety $SL(n)/Q, Q$ being any parabolic subgroup. Here again, the line of argument is the same as in the minuscule case, namely, using the results of [17], we exhibit a Hodge algebra structure for the multi-homogeneous co-ordinate ring of $SL(n)/Q$, with a set of algebra generators indexed by certain "Young lattices" (see §7 for details).

Gorenstein Schubert varieties in a minuscule G/P are also studied in [19], wherein the author describes the Gorenstein locus of a Schubert variety in a minuscule G/P, using the combinatorial tool introduced in [20]. It is not immediately clear if our characterization of the arithmetically Gorenstein Schubert varieties as described above could be deduced from the results of [19].

The paper is organized as follows. In §1, we give a resume of the standard monomial basis for Schubert varieties in a minuscule G/P and establish the fact that the homogeneous co-ordinate rings of minuscule Schubert varieties are in fact Hodge algebras. In §2, we recall generalities on distributive lattices. In §3, we present results on the arithmetic Gorenstein property for Schubert varieties in the Grassmannian as well as for Schubert varieties in the minuscule G/P_1 in types C & D (here, P_1 is the maximal parabolic subgroup corresponding to the left-end root in the Dynkin diagram). In §4, we discuss the specific lattice associated to the orthogonal Grassmannian. In §5, we present results for Schubert varieties in an orthogonal Grassmannian. In §6, we give the complete list of arithmetically Gorenstein Schubert varieties in the minuscule G/P's, G being E_6 or E_7. In §7, we present results for the (partial) flag variety $SL(n)/Q$.

1. Schubert Varieties and Hodge Algebras

In this section we recall the standard monomial basis for the homogeneous co-ordinate ring $K[X]$ of a Schubert variety X in a minuscule G/P, and exhibit the Hodge algebra structure for $K[X]$.

Let G, T, B, W etc., be as in the introduction. Let R be the root system of G relative to T; let R^+ (resp. $S = \{\alpha_1, \cdots, \alpha_l\}$) be the set of positive (resp. simple) roots in R relative to B (here, l is the rank of G). Let $\{\omega_i, 1 \leq i \leq l\}$ be the fundamental weights. Let $X(T)$ be the character group of T; let $(,)$ a W-invariant inner product on $X(T) \otimes \mathbb{R}$. Let P be a maximal parabolic subgroup of G with ω as the associated fundamental weight; then it is well-known (see [10] for example) that $\operatorname{Pic} G/P$ (the isomorphism classes of line bundles on G/P) is isomorphic to $\mathbb{Z}$, and we shall denote the ample generator of $\operatorname{Pic} G/P$ by L_ω. Then $\operatorname{Pic} G/B$ (which is free abelian of rank l) has a $\mathbb{Z}$-basis given by $\{L_{\omega_i}, 1 \leq i \leq l\}$. Thus a line bundle $L \in \operatorname{Pic} G/B$ may be written as $L = L(\lambda)$, for a unique

$\lambda := \sum_{i=1}^{l} a_i\omega_i, a_i \in \mathbb{Z}$. Let W_P be the Weyl group of P (note that W_P is the subgroup of W generated by $\{s_\alpha \mid \alpha \in S_P\}$). Let $W^P = W/W_P$. We have that the Schubert varieties of G/P are indexed by W^P, and thus W^P can be given the partial order induced by the inclusion of Schubert varieties. For generalities on semisimple algebraic groups, we refer the reader to [2, 10].

1.1. **Extremal weight vectors** u_w, p_w**:** Let $U(\mathfrak{g}_\mathbb{C})$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ (here, $\mathfrak{g}_\mathbb{C}$ is the Lie algebra of $G_\mathbb{C}$ where $G_\mathbb{C}$ is the group over $\mathbb{C}$ with the same root data as G), $U^+(\mathfrak{g}_\mathbb{C})$ the subalgebra of $U(\mathfrak{g}_\mathbb{C})$ generated by $\{X_\alpha,\ \alpha \in R^+\}$. Let $U^+_\mathbb{Z}(\mathfrak{g}_\mathbb{C})$ be the Kostant $\mathbb{Z}$-form (cf. [12]) of $U^+(\mathfrak{g}_\mathbb{C})$, namely, the $\mathbb{Z}$-subalgebra of $U^+(\mathfrak{g}_\mathbb{C})$ generated by

$$\left\{\frac{X_\alpha^n}{n!},\ \alpha \in R^+,\ n \in \mathbb{Z}_+\right\}$$

Let λ be a dominant integral weight (i.e., $\lambda = \sum_{i=1}^{l} a_i\omega_i, a_i \in \mathbb{Z}_+$), and let $V(\lambda)$ be the irreducible $G_\mathbb{C}$-module (over $\mathbb{C}$) with highest weight λ. Let us fix a highest weight vector u in $V(\lambda)$. For $w \in W$, fix a representative n_w for w in $N_T(G)$ (the normalizer of T in G), and set $u_w = n_w \cdot u$, and $V_{w,\mathbb{Z}} = U^+_\mathbb{Z}(\mathfrak{g})\, u_w$; note that u_w is a T-weight vector of weight $w(\lambda)$ (and is unique up to scalars). One usually refers to $w(\lambda)$ as the *extremal weight* in $V(\lambda)$ corresponding to w, and u_w as an *extremal weight vector*. For any field K, let $V_{w,\lambda} = V_{w,\mathbb{Z}} \otimes K,\ w \in W$. Then $V_K(\lambda) := V_{w_0,\lambda} (= V_{w_0,\mathbb{Z}} \otimes K, w_0$ being the element of largest length in W), is the *Weyl module* corresponding to λ; for $w \in W$, $V_{w,\lambda}$ is the *Demazure module* corresponding to w (and λ). Note that for $\tau, w \in W, u_\tau \in V_{w,\lambda} \Leftrightarrow w \geq \tau$. (This is assuming λ to be regular, i.e., $\lambda = \sum a_i\omega_i, a_i \neq 0, 1 \leq i \leq l$. If λ is not regular, then one should replace W by a set of representatives of $W/W_\lambda, W_\lambda$ being the stabilizer of λ in W). Recall the following theorem:

Theorem 1.2.

$$(*) \qquad H^0(G/B, L(\lambda)) \cong V_K(\lambda)^*,\ \ H^0(X(w), L(\lambda)) \cong V^*_{w,\lambda}.$$

See (for example) [10] for a proof.

The Vectors $\{p_w, w \in W\}$: In view of equation $(*)$, we have $\{-w(\lambda),\ w \in W\}$ are weights in the T-module $H^0(G/B, L(\lambda))$ of multiplicity 1. Let us fix a generator e for the one-dimensional weight space in $H^0(G/B, L(\lambda))$ of weight $-\lambda$ (note that e is a lowest weight vector). For $w \in W$, as above fix a representative

n_w for w in $N_T(G)$, and set $p_w = n_w \cdot e$. Note that $H^0\,(G/B, L\,(\lambda)) = V_K\,(\iota\,(\lambda))$, where $\iota := -w_0$ (in $\mathrm{Aut}\,R$) is the *Weyl involution*, and p_w, $w \in W$ are simply the extremal weight vectors in $V_K\,(\iota\,(\lambda))$. Note also that the weight of p_w is $-w(\lambda)$. Further, we have the following

Proposition 1.3. *Let notations be as above. Let $Q = W_\lambda$. For $\tau, w \in W/W_Q$, we have*

$$(**) \qquad p_\tau|_{X_Q(w)} \neq 0 \Leftrightarrow w \geq \tau$$

1.4. **Minuscule Weights and Lattices.**

Definition 1.5. A fundamental weight ω is called *minuscule* if $\langle \omega, \beta \rangle \; (= \frac{2(\omega,\beta)}{(\beta,\beta)}) \leq 1$ for all $\beta \in R^+$; the maximal parabolic subgroup associated to ω is called a *minuscule parabolic subgroup*.

Following the indexing of the simple roots as in [3], we have the complete list of minuscule weights for each type:

Type $\mathbf{A_n}$: Every fundamental weight is minuscule
Type $\mathbf{B_n}$: ω_n
Type $\mathbf{C_n}$: ω_1
Type $\mathbf{D_n}$: $\omega_1, \omega_{n-1}, \omega_n$
Type $\mathbf{E_6}$: ω_1, ω_6
Type $\mathbf{E_7}$: ω_7.

There are no minuscule weights in types $\mathbf{E_8}, \mathbf{F_4},$ or $\mathbf{G_2}$.

Definition 1.6. A Schubert variety in a minuscule G/P will be called a *minuscule Schubert variety*.

1.7. **Standard monomial basis for minuscule Schubert varieties:** Given a maximal parabolic subgroup P, let $L := L_\omega$ (the ample generator of $\mathrm{Pic}\,G/P$); for $w \in W/W_P$, we shall denote the restriction of L to $X(w)$ also by just L. For the canonical projective embedding

$$X(w) \hookrightarrow G/P \hookrightarrow Proj(H^0(G/P, L))$$

($X(w)$ being a Schubert variety in G/P), let $R(w)$ denote the homogeneous coordinate ring of $X(w)$. We have a natural inclusion

$$R(w) \hookrightarrow \bigoplus_{m \in \mathbb{Z}_+} H^0(X(w), L^m)$$

In the case of a minuscule G/P, we have (cf. [23]) that the extremal weight vectors $\{p_w, w \in W/W_P\}$ form a K-basis for $H^0(G/P, L)$. Further, for $\tau, w \in W/W_P$, we have (cf. $(**)$)

$$p_\tau|_{X(w)} \neq 0 \Leftrightarrow w \geq \tau$$

Hence we obtain that $\{p_\tau, \tau \leq w\}$ is a basis for $H^0(X(w), L)$. In fact, a basis has been constructed for $H^0(X(w), L^m)$ in terms of standard monomials on $X(w)$ (cf. [23]). We now recall this basis:

Definition 1.8. A monomial $p_{\tau_1}p_{\tau_2}\cdots p_{\tau_m}, \tau_i \in W/W_P, 1 \leq i \leq m$ is *standard on* $X(w)$ if $w \geq \tau_1 \geq \ldots \geq \tau_m$.

Proposition 1.9 (cf. [23])**.**

(1) *Standard monomials on* $X(w)$ *of degree* m *form a basis for* $H^0(X(w), L^m)$, $m \in \mathbb{Z}^+$.
(2) $R(w)_m = H^0(X(w), L^m), m \in \mathbb{Z}^+$, *and thus* $R(w)_m$ *has a basis consisting of standard monomials of degree* m.

As an immediate consequence, we have the following

Lemma 1.10 (cf. [23])**.** *Suppose* $\tau, \phi \in W/W^P$ *are such that* τ, ϕ *are not comparable. Writing*

$$p_\tau p_\phi = \sum c_{\alpha,\beta} p_\alpha p_\beta, c_{\alpha,\beta} \in K \tag{†}$$

where the right hand side is a sum of standard monomials, we have that any $\alpha >$ *both* τ *and* ϕ*, and any* $\beta <$ *both* τ *and* ϕ*.*

Proof. Restricting (†) to $X(\alpha)$, the right hand side is a non-zero sum of standard monomials on $X(\alpha)$ (each term restricted to $X(\alpha)$ is either zero or remains standard on $X(\alpha)$, and the term $p_\alpha p_\beta$ is non-zero on $X(\alpha)$). Linear independence of standard monomials implies that $p_\tau p_\phi|_{X(\alpha)}$ is non-zero. In particular, we obtain $\alpha \geq$ both τ and ϕ; in fact, we have that $\alpha >$ both τ and ϕ (note that $\alpha = \tau$ would imply $\tau \geq \phi$ which is not true). In a similar way, by considering $w_0\tau$, $w_0\phi$, where w_0 is the element of maximal length in W, we obtain that any $\beta <$ both τ and ϕ. □

1.11. **ASL:.** In this subsection, we recall the definition (cf. [5, 8]) of *Hodge algebras* or also known as *algebras with straightening laws*, abbreviated ASL.

Let H be a finite poset and N be the set of non-negative integers. A *monomial* $\mathcal{M}$ on H is a map from H to N. The *support* of $\mathcal{M}$ is the set $\mathrm{Supp}(\mathcal{M}) = \{x \in H \mid \mathcal{M}(x) \neq 0\}$; $\mathcal{M}$ is *standard* if $\mathrm{Supp}(\mathcal{M})$ is a chain in H (a chain is a totally ordered subset, see Definition 2.2 below).

If $\mathcal{R}$ is a commutative ring, and we are given an injection $\varphi : H \hookrightarrow \mathcal{R}$, then to each monomial $\mathcal{M}$ on H we may associate

$$\varphi(\mathcal{M}) := \prod_{x \in H} \varphi(x)^{\mathcal{M}(x)} \in \mathcal{R}.$$

Definition 1.12. Let $\mathcal{R}$ be a commutative K-algebra. Suppose that H is a finite poset with an injection $\varphi : H \hookrightarrow \mathcal{R}$. Then we call $\mathcal{R}$ a *Hodge algebra* or also an *algebra with straightening laws* (abbreviated ASL) on H over K if the following conditions are satisfied:

ASL-1 The set of standard monomials is a basis of the algebra $\mathcal{R}$ as a vector space over K.

ASL-2 If τ and ϕ in H are incomparable and if

$$\tau\phi = \sum_i a_i\, \gamma_{i1}\gamma_{i2}\cdots\gamma_{it_i},$$

(where $0 \neq a_i \in K$ and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots \gamma_{it_i}$) is the unique expression for $\tau\phi \in \mathcal{R}$ as a linear combination of distinct standard monomials (guaranteed by ASL-1), then $\gamma_{i1} \leq \tau, \phi$ for every i.

1.13. **ASL structure for homogeneous co-ordinate rings of minuscule Schubert varieties:** Let $X(w)$ be a minuscule Schubert variety, and let $R(w)$ be its homogeneous co-ordinate ring (cf. 1.7). Let $H(w)$ be the Bruhat poset of Schubert subvarieties of $X(w)$ (the partial order being given by inclusion). Then in view of Proposition 1.9 and Lemma 1.10, we obtain the following

Proposition 1.14. *The homogeneous coordinate ring $R(w)$ for a Schubert variety $X(w)$ in a minuscule G/P is an ASL (on $H(w)$ over K).*

1.15. **Cohen-Macaulay & Gorenstein properties:** Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $k = R\,/\,\mathfrak{m}$.

Definition 1.16. The local ring $(R, \mathfrak{m})$ is *Cohen-Macaulay* if

$\mathrm{Ext}^i_R(k, R) = 0$, for $i < \dim R$; it is *Gorenstein* if in addition, we have, $\mathrm{Ext}^{\dim R}_R(k, R) = 1$.

Definition 1.17. An algebraic variety X is *Cohen-Macaulay at a point* (resp. *Gorenstein at a point*) $x \in X$, if the stalk $\mathcal{O}_{X,x}$ is Cohen-Macaulay (resp. Gorenstein); X is *Cohen-Macaulay* (resp. *Gorenstein*), if it is Cohen-Macaulay (resp. Gorenstein) at all $x \in X$.

Definition 1.18. A projective variety $X = \mathrm{Proj}\, S$ is *arithmetically Cohen-Macaulay* (resp. *arithmetically Gorenstein*), if $\widehat{X}(= \mathrm{Spec}\, S)$, the cone over X, is Cohen-Macaulay (resp. Gorenstein).

Remark 1.19. Note that the cone $\widehat{X}$ is Cohen-Macaulay (resp. Gorenstein), if and only if it is so at its vertex. Also note that if $\widehat{X}$ is Cohen-Macaulay (resp. Gorenstein), then so is X.

2. Distributive Lattices

Let $(\mathcal{L}, \leq)$ be a finite partially ordered set. We shall suppose that $\mathcal{L}$ is *bounded*, i.e., it has a unique maximal, and a unique minimal element, denoted $\widehat{1}$ and $\widehat{0}$ respectively. For $\mu, \lambda \in \mathcal{L}, \mu \leq \lambda$, we shall denote

$$[\mu, \lambda] := \{\tau \in \mathcal{L}, \mu \leq \tau \leq \lambda\}$$

We shall refer to $[\mu, \lambda]$ as the *interval from μ to λ*.

Definition 2.1. The ordered pair (λ, μ) is called a *cover* (and we also say that λ *covers* μ or μ is *covered* by λ) if $[\mu, \lambda] = \{\mu, \lambda\}$.

Definition 2.2. A *chain* is a totally ordered subset of a poset. We say that a poset is *ranked* if all maximal chains have the same cardinality.

Definition 2.3. A *lattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the *join*, respectively the *meet* of x and y, defined by:

$$x \vee y \geq x,\ x \vee y \geq y,\ \text{and if } z \geq x \text{ and } z \geq y,\ \text{then } z \geq x \vee y,$$
$$x \wedge y \leq x,\ x \wedge y \leq y,\ \text{and if } z \leq x \text{ and } z \leq y,\ \text{then } z \leq x \wedge y.$$

Clearly, the operation $\vee$ (resp. $\wedge$) is commutative and associative.

Definition 2.4. A lattice is called *distributive* if the following identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{2}$$

Definition 2.5. Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$ if $x, y \in \mathcal{L}'$ implies $x \wedge y \in \mathcal{L}'$, $x \vee y \in \mathcal{L}'$.

Definition 2.6. An element z of a lattice $\mathcal{L}$ is called *join-irreducible* (respectively *meet-irreducible*) if $z = x \vee y$ (respectively $z = x \wedge y$) implies $z = x$ or $z = y$. The set of join-irreducible (respectively meet-irreducible) elements of $\mathcal{L}$ is denoted by $J(\mathcal{L})$ (respectively $M(\mathcal{L})$), or just by J (respectively M) if no confusion is possible.

Definition 2.7. A subset I of a poset P is called an *ideal* of P if for all x, $y \in P$,

$$x \in I \text{ and } y \leq x \text{ imply } y \in I.$$

Theorem 2.8 (Birkhoff)**.** *Let $\mathcal{L}$ be a distributive lattice with $\hat{0}$, and P the poset of its nonzero join-irreducible elements. Then $\mathcal{L}$ is isomorphic to the lattice of ideals of P, by means of the lattice isomorphism*

$$\alpha \mapsto I_\alpha := \{\tau \in P \mid \tau \leq \alpha\}, \qquad \alpha \in \mathcal{L}.$$

The following Lemma is easily checked.

Lemma 2.9. *With the notations as above, we have*

(*a*) $J = \{\tau \in \mathcal{L} \mid$ *there exists at most one cover of the form* $(\tau, \lambda)\}$.

(*b*) $M = \{\tau \in \mathcal{L} \mid$ *there exists at most one cover of the form* $(\lambda, \tau)\}$.

Lemma 2.10 (cf. [14])**.** *Let (τ, λ) be a cover in $\mathcal{L}$. Then I_τ equals $I_\lambda \dot{\cup} \{\beta\}$ for some $\beta \in J(\mathcal{L})$.*

Give $\mathbb{N} \times \mathbb{N}$ the lattice structure

$$(\alpha_1, \alpha_2) \wedge (\beta_1, \beta_2) = (\delta_1, \delta_2),\ (\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) = (\gamma_1, \gamma_2),$$

where $\delta_i = \min\{\alpha_i, \beta_i\}$, $\gamma_i = \max\{\alpha_i, \beta_i\}$.

Definition 2.11. Let J be a finite, distributive sublattice of $\mathbb{N} \times \mathbb{N}$, such that if α covers β in J, then α covers β in $\mathbb{N} \times \mathbb{N}$ as well. Then we say J is a *grid lattice*.

Remark 2.12 (cf [9])**.** Let P be a maximal parabolic subgroup; if P is minuscule then W/W_P is a distributive lattice; further, for any $\tau \in W/W_P$, the Bruhat poset $H(\tau)$ of Schubert subvarieties of $X(\tau)$ is a distributive sublattice of W/W_P.

Definition 2.13. For P a minuscule parabolic subgroup, we call $\mathcal{L} = W/W_P$ a *minuscule lattice.*

It is shown in [4] that the poset of join-irreducibles of W/W_P (P being a minuscule maximal parabolic subgroup) is a grid lattice.

3. A characterization of the Gorenstein property

Let $\mathcal{L}$ be a distributive lattice, and $\mathcal{R}$ a graded ASL domain on $\mathcal{L}$ over a field K. Further, let

$$\deg(\alpha) + \deg(\beta) = \deg(\alpha \vee \beta) + \deg(\alpha \wedge \beta)$$

for all $\alpha, \beta \in \mathcal{L}$. We have the following characterization of the Gorenstein property for $\mathcal{R}$:

Theorem 3.1 (cf. §3 of [8])**.** *$\mathcal{R}$ is Gorenstein if and only if $J(\mathcal{L})$ is a ranked poset.*

Remark 3.2. The above theorem was first a result of Stanley (cf. [24]), later generalized by Buchweitz.

Arithmetically Gorenstein Schubert varieties in a Grassmannian: Using the above Theorem and Proposition 1.14, we shall give a characterization of the Gorenstein property for the cones over Schubert varieties in a Grassmannian; as a byproduct, we obtain an alternate proof of the result of [25, 26] on Gorenstein Schubert varieties in a Grassmannian.

Fix two positive integers $d, n, d < n$; let $G_{d,n}$ be the Grassmann variety consisting of the d-dimensional subspaces of K^n. Let $G = SL_n(K)$, and P_d be the maximal parabolic subgroup in G:

$$P_d = \left\{ \begin{pmatrix} * & * \\ 0_{n-d \times d} & * \end{pmatrix} \in G \right\}$$

We have an identification $G_{d,n} = G/P_d$; further, W/W_{P_d} may be identified with

$$I_{d,n} = \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \leq i_1 < \ldots < i_d \leq n\}$$

(see, for example [13] for details). For $\tau \in I_{d,n}$, let $X(\tau)$ be the associated Schubert variety in $G_{d,n}$. For the Plücker embedding

$$X(\tau) \hookrightarrow G_{d,n} \hookrightarrow \text{Proj}\,(\Lambda^d K^n)^*$$

let $R(\tau)$ be the homogeneous co-ordinate ring of $X(\tau)$. Then by Proposition 1.14 and Remark 2.12, we have that $R(\tau)$ is a graded domain which is an ASL on $H(\tau)$ over K, $H(\tau)$ being as in Remark 2.12. Hence by Theorem 3.1, $X(\tau)$ is arithmetically Gorenstein if and only if $J(\tau)$ (the poset of join-irreducibles in $H(\tau)$) is a ranked poset.

Proposition 3.3. *Let $\tau \in I_{d,n}$. Then $X(\tau)$ is arithmetically Gorenstein if and only if τ consists of intervals $I_1, I_2, \cdots, I_s$ where*

$$I_t = [x_t, y_t], 1 \leq t \leq s,\ x_{t+1} - y_t = y_t + 2 - x_t,\ 1 \leq t \leq s-1$$

(Here, $[x_t, y_t]$ denotes the set $\{x_t, x_{t+1}, \cdots, y_{t-1}, y_t\}$)

The proof is similar to (and simpler than) that of Lemma 5.6 in §5.

3.4. **Outer-corner description.** (cf. [26]) To a $\tau \in I_{d,n}$, we associate a Young diagram (or also a partition) λ^τ as follows: Let $\tau = (\tau_1, \cdots, \tau_d)$ (as a d-tuple). Set

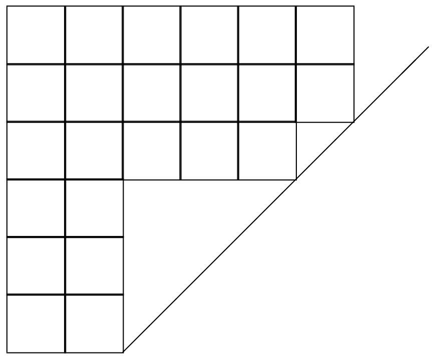
$$\lambda_r^\tau = \tau_r - r,\ \forall\ 1 \leq r \leq d.$$

Thus we write $\lambda^\tau = (\lambda_d^\tau, \cdots, \lambda_1^\tau)$; when there is no room for confusion, we drop the superscript and just write λ.

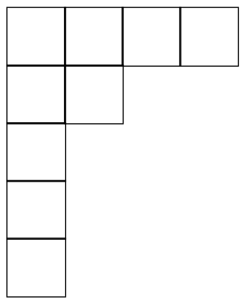
Let us write λ as a Young diagram, and place the bottom left corner of the first row (λ_1) at $(0,0)$ on the grid; then each block is a square of unit 1. Thus our diagram will be d-units high, and λ_d-units wide.

Definition 3.5. The partition λ satisfies the *outer corner condition* if all of the outer corners lie on a line of slope 1; we also refer to this as "the outer corners of λ lie on the same antidiagonal" (same terminology as in [26]).

Example 3.6. Let $n = 14, d = 6, \tau = (3,4,5,9,11,12)$, and thus $\lambda^\tau = (6,6,5,2,2,2)$. We write λ^τ as a diagram:



We can see that λ^{τ} satisfies the outer corner condition. Now let $\tau' = (1, 3, 4, 5, 7, 10)$, thus $\lambda^{\tau'} = (4, 2, 1, 1, 1, 0)$:



We can see that $\lambda^{\tau'}$ does not satisfy the outer corner condition.

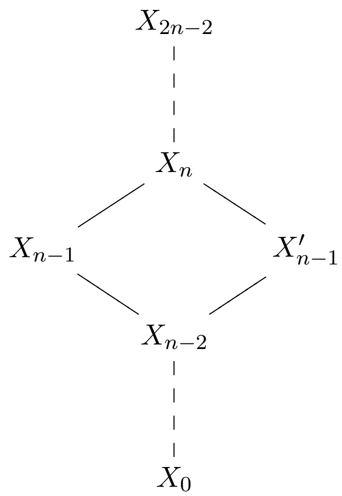
As an immediate consequence of Theorem 3.1, Proposition 3.3 and Definition 3.5, we obtain the following

Theorem 3.7 (cf. [26])**.** *Let $\tau \in I_{d,n}$. Then $X(\tau)$ is arithmetically Gorenstein if and only if the outer corners of λ^{τ} lie on the same anti-diagonal.*

Remark 3.8. Note that the above Theorem gives a stronger result than that of [25, 26], namely, while in [25, 26], one has a characterization of Gorenstein Schubert varieties, the above Theorem gives a characterization for Gorenstein property even for the cones over Schubert varieties. As a by-product, we obtain that a Schubert variety in the Grassmannian is arithmetically Gorenstein (for the Plücker embedding) if and only if it is geometrically Gorenstein

3.9. **The minuscule G/P_1 in types C, D:.** Using Theorem 3.1 and Proposition 1.14, the Gorenstein property for $R(w)$ is immediate if G is either $SP(2n)$ or $SO(2n)$, and $P = P_1$ (the maximal minuscule parabolic subgroup corresponding to the left end root of the Dynkin diagram, following the indexing of the Dynkin diagram as in [3]). In the former case, $G/P \cong \mathbb{P}^{2n-1}$; further, all of the Schubert varieties in G/P are certain linear subspaces, and hence are smooth (and hence Gorenstein and also arithmetically Gorenstein).

In the latter case, we have an identification of G/P with a certain quadric Q in $\mathbb{P}^{2n-1}$: denoting the projective co-ordinates on $\mathbb{P}^{2n-1}$ by $x_1, \cdots, x_n, y_1, \cdots, y_n$, Q is the quadric defined by $\sum_{1 \le i \le n} x_i y_{n+1-i} = 0$. We have, $\dim G/P = 2n-2$; further, for $0 \le i \le 2n-2$, there is precisely one Schubert variety in dimension $i \ne n-1$, and two Schubert varieties in dimension $n-1$:



In the above diagrammatic representation, the suffix denotes the dimension of the corresponding Schubert variety (note that X_{2n-2} is just G/P). Further, for $n \le i \le 2n-3$, X_i is obtained by intersecting X_{i+1} with the hyperplane $x_{2n-2-i} = 0$, and for $0 \le i \le n-2$, X_i is obtained by intersecting X_{i+1} with the hyperplane $y_{n-1-i} = 0$. Also, $X_{n-1} \cup X'_{n-1}$ is obtained by intersecting X_n with the hyperplane $x_{n-1} = 0$; thus, we have

$$X_{n-1} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \mid x_1 = \cdots = x_n = 0\},$$
$$X'_{n-1} = \{(x_1, \cdots, x_n, y_1, \cdots, y_n) \mid x_1 = \cdots = x_{n-1} = y_1 = 0\}$$

Note that X_{n-2} is also obtained by intersecting X'_{n-1} with the hyperplane $x_n = 0$ (for more details, see [15]). From this it easily follows that the poset of join-irreducibles in $H(w)$ is a ranked poset for all $w \in W/W_P$.

In the next two sections we present results for the orthogonal Grassmannian.

4. The Lattice of Schubert Varieties in Orthogonal Grassmannian

Let G be the special orthogonal group $SO(m)$, and let P be the maximal parabolic subgroup corresponding to the right end root (resp. right end roots if m is even) in the Dynkin diagram (following the indexing of the Dynkin diagram as in [3]). Then G/P is the *orthogonal Grassmannian.* Since P is minuscule, the Bruhat-poset of Schubert varieties in G/P is a minuscule lattice, and hence a distributive lattice (cf. Remark 2.12). Further, we have isomorphisms of the following minuscule lattices:

$$\mathbf{B_{n-1}}\,(\omega_{n-1}) \cong \mathbf{D_n}\,(\omega_{n-1}) \cong \mathbf{D_n}\,(\omega_n)$$

Hence for the rest of this section, we shall suppose that $G = SO(2n)$ and $P = P_n$ (the maximal parabolic subgroup with ω_n as the associated fundamental weight). Also, we shall denote the corresponding minuscule lattice by $\mathcal{L}$.

Description of $\mathcal{L}$ (cf. §2 of [18]): For two integers $r, s \in \mathbb{N}, r \leq s$, let $I_{r,s} = \{(i_1, \ldots, i_r) \mid 1 \leq i_1 < \ldots < i_r \leq s\}$, with partial order $\leq$ given by

$$(i_1, \ldots, i_s) \leq (j_1, \ldots, j_s) \Leftrightarrow i_1 \leq j_1, \ldots, i_s \leq j_s.$$

We have an identification (cf. [18]) of $\mathcal{L}$ as a sublattice of $I_{n,2n}$:

$$\mathcal{L} = \left\{ (i_1, \ldots, i_n) \in I_{n,2n} \,|\, \begin{array}{l} \text{for } 1 \leq j \leq n, \text{ precisely one of} \\ \{j, 2n+1-j\} \in \{i_1, \ldots, i_n\} \end{array} \right\}.$$

Notation. Whenever we refer to a segment $(i, i+1, \ldots, j)$ such that $i > j$, the segment is understood to be empty. Also, we will use the notation $[i, j]$ to represent the string of consecutive integers $\{i, i+1, \ldots, j-1, j\}$.

Lemma 4.1. $J\,(\mathcal{L}) = \{(\,[1, j], [n+1-i, n], [n+1+i, 2n-j]\,) \mid 0 \leq j \leq n, 0 \leq i \leq n-j-1\}$. *Note that when $j = n$, we have the minimal element in $\mathcal{L}$, namely,* $(1, \ldots, n)$.

Proof. For $w = (w_1, \ldots, w_n) \in \mathcal{L}$, let $1 \leq d \leq n$ be such that $w_d \leq n$ and $w_{d+1} \geq n+1$. Notice that w is determined by $(w_1, \ldots, w_d)$. In this way, we can project $\mathcal{L}$ onto $\bigcup_{1 \leq d \leq n} I_{d,n}$. This map is bijective, and order is preserved within each $I_{d,n}$. Further, if w does not project onto a join irreducible element in $I_{d,n}$, then w is not join irreducible in $\mathcal{L}$.

Therefore, we check to see if join irreducible elements in $I_{d,n}$ are join irreducible in $\mathcal{L}$. We have (from §8 of [7]),

$$J(I_{d,n}) = \{(1, \ldots, j, j+i+1, \ldots, d+i) \mid 0 \le j \le d,\ 1 \le i \le n-d\}.$$

Note that the element $(1, \ldots, j, j+i+1, \ldots, d+i) \in I_{d,n}$ corresponds to $([1,j], [j+i+1, d+i], [n+1, 2n-d-i], [2n-j-i+1, 2n-j]) \in \mathcal{L}$. In the case when $i \neq n-d$, this element covers two distinct elements,

$$([1,j], j+i, [j+i+2, d+i], [n+1, 2n-d-i], 2n-j-i, [2n-j-i+2, 2n-j]),$$

$$([1,j], [j+i+1, d+i], n, [n+2, 2n-d-i], [2n-j-i+1, 2n-j]).$$

Thus we only need to consider elements of the form $(1, \ldots, j, n-d+j+1, \ldots n, n+d-j+1, \ldots, 2n-j)$, for all d, and these elements clearly cover only one element each. The result follows. □

Remark 4.2. By [4], we may identify $J(\mathcal{L})$ with a grid lattice (Definition 2.11). This implies that $J(\mathcal{L})$ is a distributive lattice, and thus is ranked.

$J(\mathcal{L})$ being a ranked poset, each element of $J(\mathcal{L})$ has a well-defined level: let $level_w$ be the number of elements in a maximal chain from $\hat{0}$ to w minus one.

Lemma 4.3. *Let $w \in J(\mathcal{L})$, $w = (1, \ldots, j, n+1-i, \ldots, n, n+1+i, \ldots, 2n-j)$. Then $level_w = 2n-2j-i-1$.*

Proof. We give $J(\mathcal{L})$ a grid lattice structure. Namely, we send w to $(n-j, n-j-i) \in \mathbb{N} \times \mathbb{N}$. For book-keeping purposes, let $\hat{0} = (1, \ldots, n) \mapsto (0,1)$. Therefore, in this sublattice of $\mathbb{N} \times \mathbb{N}$, $level_{(a,b)} = a+b-1$, as we can see by taking the maximal chain from $(0,1)$ to $(a,1)$, and then from $(a,1)$ to (a,b). Therefore, $level_w = n-j+n-j-i-1$. The result follows. □

5. Arithmetically Gorenstein Property for Schubert varieties in an orthogonal Grassmannian

For $\beta \in \mathcal{L}$, let $\mathcal{L}_\beta$ be the sublattice $[\hat{0}, \beta]$; then $R(\beta)$ (cf. §1.7) is an ASL on $\mathcal{L}_\beta$. Further, $R(\beta)$ is a Cohen-Macaulay normal domain (cf.[22, 21]). Setting $\mathcal{R} = R(\beta)$, we have $\deg(\alpha) = 1$ for all $\alpha \in \mathcal{L}_\beta$.

As stated in the previous section, we know $\beta = (\beta_1, \ldots, \beta_n)$ is determined by $(\beta_1, \ldots, \beta_d)$ where $\beta_d \le n$ and $\beta_{d+1} \ge n+1$. For now, we will be primarily concerned with $(\beta_1, \ldots, \beta_d)$. We will now break β into its segments, using the notation $[i, j]$ to represent the string of consecutive integers $\{i, i+1, \ldots, j-1, j\}$. Thus we denote

$$(\beta_1, \ldots, \beta_d) = \left([1, j_0], [\beta_{(j_0)+1}, \beta_{j_1}], [\beta_{(j_1)+1}, \beta_{j_2}], \ldots, [\beta_{(j_{s-1})+1}, \beta_{j_s}]\right),$$

where $\{j_0, \ldots, j_s\}$ is a subset of $\{0, \ldots, d\}$, (we may have $j_0 = 0$, but we must have $j_s = d$), and $\beta_{(j_i)+1} - \beta_{j_i} \ge 2$ for $0 \le i \le s-1$.

We may write β as the join of non-comparable join irreducibles:

$$(*) \qquad \beta = w_1 \vee \ldots \vee w_s \, (\vee w_0),$$

where the w_i's are join-irreducible and mutually non-comparable. Specifically,

$$w_i = \left([1, j_{i-1}], [\beta_{(j_{i-1})+1}, n]\right), \ 1 \le i \le s,$$

and

$$w_0 = ([1, j_s]).$$

We only give those integers in w_i less than or equal to n. Note that in the case where $\beta_{j_s} = n$, w_0 is unnecessary, in fact $w_0 \le w_s$, which is why we have listed w_0 in parentheses in $(*)$.

Example 5.1. Let $n = 5$, and $\beta = (2, 4, 6, 8, 10)$; we have

$$(2, 4, 6, 8, 10) = (2, 3, 4, 5, 10) \vee (1, 4, 5, 8, 9) \vee (1, 2, 6, 7, 8).$$

Remark 5.2. With notation as above, $J(\mathcal{L}_\beta) = \bigcup_{0 \le i \le s} [\hat{0}, w_i]$. Then as a result of Theorem 3.1, we have that the Schubert variety $X(\beta)$ is arithmetically Gorenstein if and only if $level_{w_i} = level_{w_j}$ for all $0 \le i, j \le s$.

Remark 5.3.

$$\begin{aligned} level_{w_i} &= n - 2j_{i-1} + \beta_{(j_{i-1})+1} - 2, \text{ for } 1 \le i \le s; \\ level_{w_0} &= 2n - 2j_s - 1. \end{aligned}$$

Remark 5.4. Note that $\beta_{(j_i)+1} = \beta_{(j_{i+1})} - (j_{i+1} - j_i - 1)$, by construction.

Definition 5.5. We say that an element $\beta \in \mathcal{L}$ satisfies *condition A*, if β satisfies

$$\begin{aligned} \beta_{(j_i)+1} - \beta_{j_i} &= j_i - j_{i-1} + 1, \ \forall\ 1 \le i \le s-1, \text{ and} \\ n - \beta_{j_s} &= j_s - j_{s-1}, \text{ if } \beta_{j_s} \ne n. \end{aligned}$$

Lemma 5.6. *$X(\beta)$ is arithmetically Gorenstein if and only if β satisfies condition A.*

Proof. **Case 1:** Let $\beta_{j_s} = n$.

Let i be such that $1 \leq i \leq s-1$, and for convenience of notation, let $k = i-1$.

Let $X(\beta)$ be arithmetically Gorenstein. Then, by Remarks 5.2 and 5.3, we have

$$n - 2j_i + \beta_{(j_i)+1} - 2 = n - 2j_k + \beta_{(j_k)+1} - 2.$$

Thus $\beta_{(j_i)+1} - \beta_{(j_k)+1} = 2j_i - 2j_k$; this together with Remark 5.4 (applied to $\beta_{(j_k)+1}$) implies that

$$\beta_{(j_i)+1} - (\beta_{j_i} - j_i + j_k + 1) = 2j_i - 2j_k.$$

Therefore $\beta_{(j_i)+1} - \beta_{j_i} = j_i - j_k + 1$, and thus β satisfies Condition A.

Now let β satisfy Condition A. Hypothesis implies that $\beta_{(j_i)+1} - \beta_{j_i} = j_i - j_k + 1$ for every i such that $1 \leq i \leq s-1$, where $k = i-1$. From Remark 5.4, $\beta_{j_i} = \beta_{(j_k)+1} + j_i - j_k - 1$, therefore

$$\beta_{(j_i)+1} - \beta_{(j_k)+1} = 2j_i - 2j_k; \Rightarrow \beta_{(j_i)+1} - 2j_i = \beta_{(j_k)+1} - 2j_k.$$

Thus we get that $level_{w_{i+1}} = level_{w_{k+1}}$, and since this is true for any i, we get $level_{w_i} = level_{w_j}$ for all $1 \leq i, j \leq s$. Therefore $X(\beta)$ is arithmetically Gorenstein.

This completes the proof for Case 1.

Case 2: Let $\beta_{j_s} \neq n$ (making w_0 relevant).

Let $X(\beta)$ be arithmetically Gorenstein. As in Case 1, we have, $\beta_{(j_i)+1} - \beta_{j_i} = j_i - j_k + 1, 1 \leq i \leq s-1$ (where $(k = i-1)$). It remains to show that $n - \beta_{j_s} = j_s - j_{s-1}$. Hypothesis implies that $level_{w_0} = level_{w_s}$; i.e.,

$$2n - 2j_s - 1 = n - 2j_{s-1} + \beta_{(j_{s-1})+1} - 2.$$

We apply Remark 5.4 (for $i = s-1$), and obtain

$$2n - 2j_s - 1 = n - 2j_{s-1} + \beta_{j_s} - (j_s - j_{s-1} - 1) - 2.$$

Therefore $n - \beta_{j_s} = j_s - j_{s-1}$, as desired.

Now let β satisfy Condition A. Hypothesis implies that

$$\beta_{j_{i+1}} - \beta_{j_i} = j_i - j_k + 1, 1 \leq i \leq s-1 (k = i-1),\ n - \beta_{j_s} = j_s - j_{s-1}$$

We must show that $X(\beta)$ is arithmetically Gorenstein, equivalently, we must show that $level_{w_i} = level_{w_j}, 0 \leq i, j \leq s$. As in Case1, hypothesis implies that $level_{w_i} = level_{w_j}, 1 \leq i, j \leq s$. We shall now show that $level_{w_0} = level_{w_s}$, thus completing the proof in this Case. In view of Remark 5.3, it suffices to check that $2n - 2j_s - 1 = n - 2j_{s-1} + \beta_{(j_{s-1})+1} - 2$. We begin with Remark 5.4:

$$\begin{aligned} \beta_{j_s} - \beta_{j_{s-1}+1} &= j_s - j_{s-1} - 1,\ \Rightarrow \\ n - \beta_{j_{s-1}+1} &= (n - \beta_{j_s}) + j_s - j_{s-1} - 1 \end{aligned}$$

By hypothesis, this implies

$$n - \beta_{j_{s-1}+1} = (j_s - j_{s-1}) + j_s - j_{s-1} - 1$$

Hence, it follows that $n - 2j_s = -2j_{s-1} + \beta_{j_{s-1}+1} - 1$.

This completes the proof for Case 2. □

5.7. **Outer-corner condition:** As in §3.4, for $\beta := (\beta_1, \cdots, \beta_n) \in \mathcal{L}$ we associate a Young diagram $\lambda^\beta = \left(\lambda_n^\beta, \ldots, \lambda_1^\beta\right)$, where

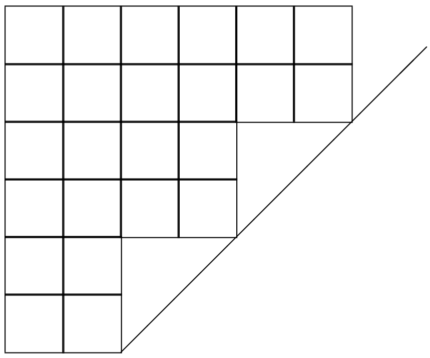
$$\lambda_i^\beta = \beta_i - i,\ \forall\ 1 \leq i \leq n$$

When there is no room for confusion, we drop the superscript and just write λ.

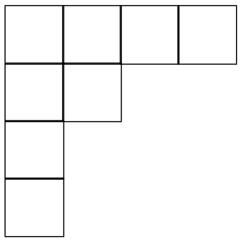
As in §3.4, we place the bottom left corner of the first row (λ_1) at $(0,0)$ on the grid; then each block is a square of unit 1. Thus our diagram will be n-units high, and λ_n-units wide.

Definition 5.8. We follow the terminology "outer corner condition" for λ as in Definition 3.5, namely if all of the outer corners of λ lie on a line of slope 1.

Example 5.9. Let $n = 8$, $\beta = (1, 2, 5, 6, 9, 10, 13, 14)$, and thus $\lambda^\beta = (6, 6, 4, 4, 2, 2, 0, 0)$. We write λ^β as a diagram:



We can see that λ^β satisfies the outer corner condition. Now let $\beta' = (1, 2, 3, 4, 6, 7, 9, 12)$, thus $\lambda^{\beta'} = (4, 2, 1, 1, 0, 0, 0, 0)$:



We can see that $\lambda^{\beta'}$ does not satisfy the outer corner condition.

We now return to a general $\beta \in \mathcal{L}$; we preserve the notation from §5. From our definition of $\{j_0, \dots j_s\} \subseteq \{0, \dots, d\}$, we have $\lambda_{(j_i)+1} = \dots = \lambda_{(j_{i+1})}$. Note that $\lambda_1 = \dots = \lambda_{j_0} = 0$, so the first outer corner takes place at the bottom right of the row assigned to $\lambda_{(j_0)+1}$; specifically, the point $(\beta_{j_0+1} - j_0 - 1, j_0)$. Similarly, the first s outer corners are at the points

$$\left(\lambda_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right) \;\; 1 \leq i \leq s.$$

If $\beta_{j_s} \neq n$, we have the $s+1$ outer corner at $(n - j_s, j_s)$.

Note that the diagram is necessarily "self dual," (cf. [18]); recall that the dual of the partition λ is given by λ', where $\lambda'_i = \#\{\lambda_j \mid \lambda_j \geq i\}$, i.e. the rows of λ' are given by the columns of λ. Thus, for λ derived from $\beta \in \mathcal{L}$, we have $\lambda = \lambda'$. In the case of the grid, this implies that the diagram reflects over the line between points $(0, n)$ and $(n, 0)$. Thus, if there is an outer corner at the point (i, j), there will be an outer corner at the point $(n - j, n - i)$.

Note that in the case where $\beta_{j_s} \neq n$, the $(s+1)^{th}$ outer corner corresponds to itself, it actually lies on the line from $(0,n)$ to $(n,0)$. Therefore, the first s outer corners fall below the line, and correspond to another set of s outer corners above the line. Therefore we get $2s$ outer corners given in pairs: for $1 \leq i \leq s$,

$$\left(\beta_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right), \ \left(n - j_{i-1}, n - \beta_{(j_{i-1})+1} + j_{i-1} + 1\right);$$

with one more outer corner whenever $\beta_{j_s} \neq n$: $(n - j_s, j_s)$.

Lemma 5.10. *If the first $s+1$ outer corners of λ satisfy the outer corner condition, then λ satisfies the outer corner condition.*

Proof. Note that two corners (x_1, y_1) and (x_2, y_2) fall on a line of slope 1 whenever $y_1 - x_1 = y_2 - x_2$. Note that this always holds true for the pair $\left(\beta_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right)$, $\left(n - j_{i-1}, n - \beta_{(j_{i-1})+1} + j_{i-1} + 1\right)$. Therefore, each of the first s outer corners lies on a line of slope 1 with its dual counterpart. If we now assume that the first $s+1$ outer corners are on a line of slope 1, (including the special case if $\beta_{j_s} \neq n$), then all the outer corners satisfy the outer corner condition. The result follows. □

Lemma 5.11. *The lattice point $\beta \in \mathcal{L}$ satisfies condition A if and only if the partition λ^β satisfies the outer corner condition.*

Proof. We begin with the outer corners for i and $i+1$:

$$\left(\beta_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right), \ \left(\beta_{(j_i)+1} - j_i - 1, j_i\right).$$

If we begin letting $\beta_{j_s} = n$, by Lemma 5.10, the outer corner condition is satisfied if and only if for every $1 \leq i \leq s-1$,

$$\begin{aligned} &\beta_{(j_{i-1})+1} - 2j_{i-1} - 1 = \beta_{j_i+1} - 2j_i - 1, \Leftrightarrow \\ &\beta_{j_i} + 1 - \beta_{(j_{i-1})+1} = 2j_i - 2j_{i-1}. \end{aligned}$$

By Remark 5.4, this is if and only if

$$\begin{aligned} &\beta_{j_i+1} - (\beta_{j_i} - j_i + j_{i-1} + 1) = 2j_i - 2j_{i-1} \Leftrightarrow \\ &\beta_{j_i+1} - \beta_{j_i} = j_i - j_{i-1} + 1, \end{aligned}$$

which holds if and only if β satisfies condition A.

Now, if $\beta_{j_s} \neq n$, we must show that the $(s+1)$-th outer corner is on line with the others if and only if $n - \beta_{j_s} = j_s - j_{s-1}$. We have that the corners

$\left(\beta_{(j_{s-1})+1} - j_{s-1} - 1, j_{s-1}\right)$, $(n - j_s, j_s)$ are on line if and only if

$$n - 2j_s = \beta_{(j_{s-1})+1} - 2j_{s-1} - 1$$

By Remark 5.4, we have $\beta_{(j_{s-1})+1} = \beta_{j_s} - j_s - j_{s-1} - 1$, thus

$$\begin{aligned} &n - 2j_s = \beta_{j_s} - j_s + j_{s-1} + 1 - 2j_{s-1} - 1 \Leftrightarrow \\ &n - \beta_{j_s} = j_s - j_{s-1}. \end{aligned}$$

The result follows. □

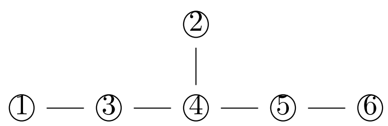
The Lemmas from §4, §5 (together with Theorem 3.1 and Proposition 1.14) lead to the characterization of the arithmetically Gorenstein property for Schubert varieties in the orthogonal Grassmannian:

Theorem 5.12. *The Schubert variety $X(\beta)$ in the orthogonal Grassmannian G/P is arithmetically Gorenstein if and only if λ^β satisfies the outer corner condition.*

6. The Exceptional Groups

In this section, we list the arithmetically Gorenstein minuscule Schubert varieties in types E_6, E_7. We shall index the simple roots as in [3]:

6.1. $\mathbf{E}_6$. In the root system of type $\mathbf{E}_6$, we have two minuscule weights: ω_1 and ω_6.



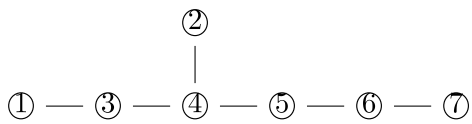
For the results below, we let P be the minuscule parabolic subgroup associated to the root ω_1. Similar results can be found for P associated to ω_6 simply by performing the appropriate permutation of simple roots.

We list the Gorenstein, as well as the non-Gorenstein Schubert varieties in G/P. For a simple root $\alpha_i, 1 \leq i \leq 6$, s_i will denote the reflection with respect to α_i. For convenience of notation, we will denote an element $s_4s_3s_1W_P \in W/W_P$ by just 431; in fact, we will denote the associated Schubert variety also by just 431 (since there is no room for confusion).

All results can be checked by constructing the Bruhat poset of Schubert varieties, and using Theorem 3.1.

Gorenstein	Non - Gorenstein
e (identity element)	652431
1	63452431
31	163452431
431	5613452431
5431	245613452431
65431	
2431	
52431	
452431	
3452431	
6452431	
13452431	
54652431	
563452431	
4563452431	
24563452431	
45613452431	
345613452431	
2345613452431	
42345613452431	
542345613452431	
6542345613452431	

6.2. $\mathbf{E}_7$**.** In the root system of type $\mathbf{E}_7$, we have one minuscule weight: ω_7. We use the indexing of simple roots found in [3]:



For the results below, we let P be the minuscule parabolic subgroup associated to the root ω_7. As with the $\mathbf{E}_6$ case, we categorize the Schubert varieties in G/P as Gorenstein or non-Gorenstein.

Gorenstein	Non - Gorenstein
e (identity element)	1234567
7	541234567
67	6541234567
567	76541234567
4567	65341234567
24567	765341234567
34567	7645341234567
234567	25645341234567
134567	6725645341234567
4234567	7425645341234567
41234567	73425645341234567
54234567	673425645341234567
654234567	173425645341234567
7654234567	1673425645341234567
341234567	15673425645341234567
5341234567	1245673425645341234567
45341234567	
245341234567	
645341234567	
5645341234567	
25645341234567	
72645341234567	
75645341234567	
675645341234567	
725645341234567	
425645341234567	
3425645341234567	
67425645341234567	
13425645341234567	
567425645341234567	

Gorenstein	Non - Gorenstein
5673425645341234567	
45673425645341234567	
245673425645341234567	
145673425645341234567	
3145673425645341234567	
31245673425645341234567	
431245673425645341234567	
5431245673425645341234567	
65431245673425645341234567	
765431245673425645341234567	

7. Arithmetic Gorenstein property for the flag variety

In this section, we present results for the (partial) flag varieties $SL(n)/Q$.

Let us denote by X the flag variety $SL(n)/B$ (here, we take B to be the Borel subgroup in $SL(n)$ consisting of upper triangular matrices). The Picard group of X is free abelian of rank $\ell := n-1$, having the line bundles $L_d :=$ (the ample generator of $\operatorname{Pic} G/P_d$, P_d being the maximal parabolic subgroup associated to the fundamental weight ω_d) as a $\mathbb{Z}$-basis. For λ in the weight lattice, we shall denote the associated line bundle on X by $L(\lambda)$. We have (see [10] for example), $H^0(X, L(\lambda))$ is non-zero if and only if λ is dominant (i.e., λ is a non-negative integral combination of the fundamental weights). Also, G/P_d is simply the Grassmannian $G_{d,n}$ of the d-dimensional subspaces of K^n; further, for $\tau \in W/W_{P_d}$, the extremal weight vectors p_τ's in $H^0(G_{d,n}, L(\omega_d))$ are simply the Plücker co-ordinates, and they give a basis for $H^0(G_{d,n}, L(\omega_d))$ (for details see [13] for example). Let us denote the multi-homogeneous co-ordinate ring of X by R; we have (cf.[11]) that R is a Cohen-Macaulay normal domain. (Note that R is just the K-algebra generated by the extremal weight vectors in $H^0(G_{d,n}, L(\omega_d)), 1 \leq d \leq n-1$). We have a canonical inclusion

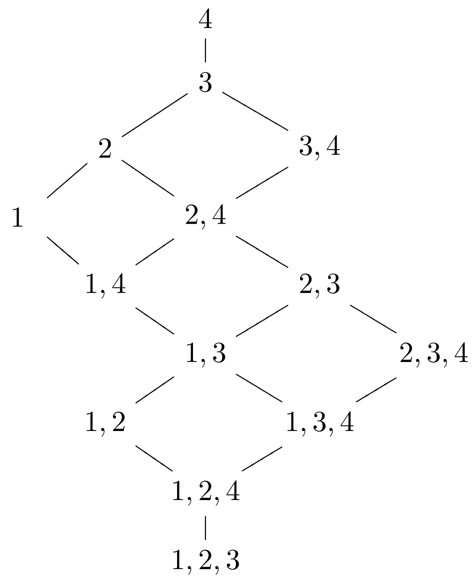
$$R \hookrightarrow \bigoplus_{\{\lambda \text{ dominant}\}} H^0(X, L(\lambda))$$

As in §3, let $I_{d,n} = \{(1 \leq i_1 < i_2 < \cdots < i_d \leq n)\}$, and let

$$H_n = \bigcup_{d=1}^{n-1} I_{d,n}$$

We define a partial order on H_n as follows. Let τ, $w \in H_n$, $\tau = (i_1, \cdots, i_r)$, $w = (j_1, \cdots, j_s)$. Define $\tau \geq w$ if $r \leq s$ and $i_1 \geq j_1, \cdots, i_r \geq j_r$. With this partial order, H_n is in fact a distributive lattice (see [6] for a proof).

Example 7.1. For SL_4/B, the distributive lattice H_4 is given below.



Remark 7.2. In the literature, the distributive lattice H_n is called a *Young lattice.*

As noted above, R has a set of K-algebra generators consisting of $\{p_\tau, \tau \in H_n\}$.

Definition 7.3. Define a monomial $p_{\tau_1} \cdots p_{\tau_m}, \tau_i \in H_n, 1 \leq i \leq m$ of multi-degree $(m_1, \cdots, m_{n-1})$ (where $m_d = \#\{i \,|\, \tau_i \in I_{d,n}\}$) to be *standard on* X if $\tau_1 \geq \cdots \geq \tau_m$.

Let $\lambda = \sum_{d=1}^{n-1} m_d \omega_d$. By the results of [17], we have the following

Proposition 7.4.

(1) $R = \underset{\{\lambda \text{ dominant}\}}{\oplus} H^0(X, L(\lambda))$

(2) *Standard monomials on* X *of multi-degree* $(m_1, \cdots, m_{n-1})$ *form a basis for* $H^0(X, L(\lambda))$.

(3) *Let* τ, ϕ *be two non-comparable elements of* H_n. *Then in the expression*

$$p_\tau p_\phi = \sum c_{\alpha,\beta} p_\alpha p_\beta, c_{\alpha,\beta} \in K$$

where the right hand side is a sum of standard monomials, we have that any $\alpha >$ both τ and ϕ, and any $\beta <$ both τ and ϕ.

Theorem 7.5. *The multicone* $\operatorname{Spec} R$ *is Gorenstein.*

Proof. By the above Proposition, we have that R is a Cohen-Maucaulay, normal ASL over H_n. Hence the result will follow from Theorem 3.1 once we check that $J(H_n)$ (the poset of join-irreducibles in H_n) is a ranked poset. It is easily seen that $J(H_n)$ consists of d-tuples (where $1 \le d \le n-1$) of the form $([1, r], [n+1+r-d, n])$, where $0 \le r \le d$. Via the map $([1, r], [n+1+r-d, n]) \mapsto (n-d, n+1-r)$, we get an identification of $J(H_n)$ with the ranked poset $I_{2,n+1} \setminus \{(n, n+1)\}$. The result now follows.

□

More generally, for a parabolic subgroup $Q \supset B$ with $S_Q = S \setminus \{\alpha_{d_1}, \cdots, \alpha_{d_r}\}$ (S being the set of simple roots), we denote

$$H_Q = \bigcup_{t=1}^{r} I_{d_t,n}$$

Considering H_Q as a subposet of H_n, we have that H_Q is a distributive lattice. The Picard group of G/Q is free abelian of rank r, having the line bundles $L_{d_t} :=$ the ample generator of $\operatorname{Pic} G/P_{d_t}$ as a $\mathbb{Z}$-basis. We define standard monomials in the p_τ's, $\tau \in H_Q$ as in Definition 7.3. Let R_Q be the multi-homogeneous co-ordinate ring of G/Q. (R_Q is just the K-algebra generated by the extremal weight vectors in $H^0(G_{d_t,n}), 1 \le t \le r$). Let $\lambda = \sum_{t=1}^{r} m_t\omega_{d_t}, m_t \in \mathbb{Z}_+$; we shall refer to such a λ as *Q-dominant.* We have a canonical inclusion

$$R_Q \hookrightarrow \underset{\{\lambda, Q-\text{dominant}\}}{\oplus} H^0(G/Q, L(\lambda))$$

We have (by [17]) the following

Proposition 7.6.

(1) $R_Q = \underset{\{\lambda, Q-\text{dominant}\}}{\oplus} H^0(G/Q, L(\lambda))$

(2) *Standard monomials on G/Q of multi-degree* $(m_1, \cdots, m_r)$ *form a basis for* $H^0(X, L(\lambda))$.

(3) *Let τ, ϕ be two non-comparable elements of H_Q. Then in the expression*

$$p_\tau p_\phi = \sum c_{\alpha,\beta} p_\alpha p_\beta, c_{\alpha,\beta} \in K$$

where the right hand side is a sum of standard monomials, we have that any $\alpha >$ both τ and ϕ, and any $\beta <$ both τ and ϕ.

Theorem 7.7. *The multicone Spec R_Q is Gorenstein.*

Proof is similar to that of Theorem 7.5.

References

[1] S. Billey and V. Lakshmibai, *Singular Loci of Svhubert Varieties*, Birkhäuser, Progress in Mathematics, no: 182.

[2] A. Borel, *Linear algebraic groups*, second ed., GTM, no. 126, Springer-Verlag, 1991.

[3] N. Bourbaki, *Groupes et algèbres de Lie chapitres 4, 5 et 6*, Hermann, Paris, 1968.

[4] J. Brown and V. Lakshmibai, *Singular loci of Bruhat-Hibi toric varieties*, J. Algebra **319** (2008), no. 11, 4759-4779.

[5] C. De Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*, Astérisque **91** (1982).

[6] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transformation Groups, vol 1, no:3 (1996), 215-248.

[7] N. Gonciulea and V. Lakshmibai, *Schubert varieties, toric varieties and ladder determinantal varieties*, Ann. Inst. Fourier **47** (1997), no. 4, 1013–1064.

[8] T. Hibi, *Distributive lattices, affine semigroup rings, and algebras with straightening laws*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math., no. 11, Elsevier, 1987, pp. 93–109.

[9] H. Hiller, *Geometry of Coxeter groups*, Research Notes in Mathematics, vol. 54, Pitman Adv. pub. prog., 1982.

[10] J.C. Jantzen, *Representations of Algebraic Groups*, Academic Press (1987).

[11] G. Kempf and A. Ramanathan, *Multicones over Schubert varieties*, Inv. Math. **87** (1987), 353-363.

[12] B. Kostant, *Groups over $\mathbb{Z}$, Algebraic Groups and Discontinuous Subgroups*, Proc. Symp. Pure Math., AMS, **9** (1966).

[13] V. Lakshmibai and N. Gonciulea, *Flag Varieties*, Hermann (2001).

[14] V. Lakshmibai and H. Mukherjee, *Singular loci of Hibi toric varieties*, submitted to J. Ramanujan Math. Soc.

[15] V. Lakshmibai, C. Musili and C.S. Seshadri, *Cohomology of line bundles on G/B*, Ann. E.N.S., **7** (1974), 89–137.

[16] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$*, Proc. Indian Acad. Sci. (Math. Sci.), **100** (1990), 45–52.

[17] V. Lakshmibai and C.S. Seshadri, *Geometry of G/P—V*, J. Alg., **100** (1986), 462–557.

[18] V. Lakshmibai and J. Weyman, *Multiplicities of points on a Schubert variety in a minuscule G/P*, Adv. in Math. **84** (1990), no. 2, 179–208.

[19] N. Perrin, *The Gorenstein locus of minuscule Schubert varieties*, Adv. Math. **220** (2009), no. 2, 505–522.

[20] N. Perrin, *Small resolutions of minuscule Schubert varieties*, Compos. Math. **143** (2007), no. 5, 1255–1312.

[21] S. Ramanan and A. Ramanathan, *Projective normality of Flag varieties and Schubert varieties*, Inv. Math. **79** (1985), 217–224.

[22] A. Ramanathan, *Schubert varieties are arithmetically Cohen–Macualay*, Inv. Math., **80** (1985), 283–294.

[23] C.S. Seshadri, *Geometry of G/P–I*, pp. 207–239, Springer, Bombay, 1978, published by Tata Institute.

[24] R.P. Stanley, *Hilbert functions of graded algebras*, Adv. in Math. **28** (1978), 57–83.

[25] T. Svanes, *Coherent cohomology on Schubert subschemes of flag schemes and applications*, Adv. Math. **14** (1974), 369–453.

[26] A. Woo and A. Yong, *When is a Schubert variety Gorenstein?*, Advances in Mathematics **207** (2006), no. 1, 205–220.

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