Pure and Applied Mathematics Quarterly
Volume 8, Number 1
(*Special Issue: In honor of
F. Thomas Farrell and Lowell E. Jones, Part 1 of 2*)
221—312, 2012

Trees, Ultrametrics, and Noncommutative Geometry

Bruce Hughes

Abstract: Noncommutative geometry is used to study the local geometry of ultrametric spaces and the geometry of trees at infinity. Connes's example of the noncommutative space of Penrose tilings is interpreted as a non-Hausdorff orbit space of a compact, ultrametric space under the action of its local isometry group. This is generalized to compact, locally rigid, ultrametric spaces. The local isometry types and the local similarity types in those spaces can be analyzed using groupoid C^*-algebras.
The concept of a locally rigid action of a countable group Γ on a compact, ultrametric space by local similarities is introduced. It is proved that there is a faithful unitary representation of Γ into the germ groupoid C^*-algebra of the action. The prototypical example is the standard action of Thompson's group V on the ultrametric Cantor set. In this case, the C^*-algebra is the Cuntz algebra $\mathcal{O}_2$ and representations originally due to Birget and Nekrashevych are recovered.
End spaces of trees are sources of ultrametric spaces. Some connections are made between locally rigid, ultrametric spaces and a concept in the theory of tree lattices of Bass and Lubotzky.
Keywords: Ultrametric Cantor sets, groupoids, Cuntz algebra, Penrose tilings, Thompson's group, local similarities, locally rigid action.

Received June 12, 2007.
Supported in part by NSF Grant DMS–0504176.
2000 Math. Subject Class. Primary 46L85, 54E45, 20F65; Secondary 22A22, 05C05, 19K14.

Contents

1 Introduction

This paper applies techniques from noncommutative geometry, in particular, Renault's groupoid C^*-algebras [43], to study various aspects of the local geometry of ultrametric spaces. Since the geometry of infinite trees at infinity and the local geometry of ultrametric spaces are related by the end space functor, results about the large-scale geometry of infinite trees are also obtained.

There are two main motivating examples for this paper. First, Connes [14] used the space of Penrose tilings as an illustration of a noncommutative space. We observe here that the space of Penrose tiles can be interpreted as a certain compact ultrametric space modulo local isometry type. We then address the problem of finding other compact ultrametric spaces for which the noncommutative geometric point of view can be used to study local isometry types. To solve this problem, locally rigid ultrametric spaces are introduced. The results are described in Section 1.1 below.

In addition to the equivalence relation of local isometry type, we also analyze a closely related equivalence relation on the points of a locally rigid ultrametric space, namely local similarity type. Those results are summarized in Section 1.2.

The second main motivating example is Birget's faithful unitary representation of Thompson's group V into the Cuntz algebra $\mathcal{O}_2$ [7], also obtained independently by Nekrashevych [37]. In this paper we derive such a representation from a more general result establishing a faithful unitary representation of any countable group Γ acting locally rigidly by local similarities on a compact ultrametric space X into the C^*-algebra of a groupoid associated to the action of Γ on X. See Section 1.3 for more details.

The concept of a rigid tree appears in the Bass-Lubotzky theory of tree lattices. In Section 12.2 it is observed that this condition is equivalent to the end space of the tree being a locally rigid, ultrametric space.

1.1 Local isometries

The first main motivating example for this paper is the description by Connes [14] of the space of Penrose tilings. Consider the space X of infinite sequences of 0's and 1's, where any 1 must be followed by 0; the topology on X comes from considering X as a subspace of the countable product of the discrete space $\{0, 1\}$:

$$X = \left\{ (x_0, x_1, x_2, \dots) \in \prod_0^\infty \{0,1\} \mid x_i = 1 \text{ implies } x_{i+1} = 0 \right\}.$$

It is known that the set of Penrose tilings is parametrized as a quotient space of X with respect to the equivalence relation R of *tail equivalence* (where $x = (x_i)_{i=0}^\infty$ and $y = (y_i)_{i=0}^\infty$ are tail equivalent if and only if there exists $N \geq 0$ such that $x_i = y_i$ for all $i \geq N$) (see Grünbaum and Shephard [23]). The problem is that X/R is not Hausdorff and, hence, cannot be studied by ordinary topological methods. Nevertheless, Connes shows how to associate to X/R a natural noncommutative C^*-algebra; that is to say, X/R can be viewed as a *noncommutative topological space* (whereas the Gelfand-Naimark Theorem shows that commutative C^*-algebras correspond to locally compact Hausdorff spaces). In fact, the C^*-algebra constructed by Connes for X/R is AF (that is, *approximately finite*, or the norm closure of a direct limit of finite dimensional matrix algebras over $\mathbb{C}$).

We call (X, d) the *Fibonacci space* because it is the end space of the Fibonacci tree (see Figure 1) as will be explained below.

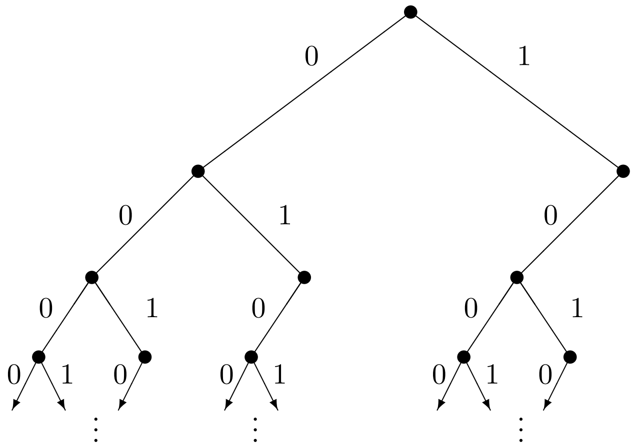

Figure 1: The Fibonacci tree

The point of view of this paper begins with the simple observation that the equivalence relation R on X has a geometric interpretation when X is endowed with the natural metric $d(x, y) = e^{-n}$, where $n = \inf\{i \geq 0 \mid x_i \neq y_i\}$. Namely, xRy if and only if X has the same *local isometry type* at x and y (that is, there exists $\epsilon > 0$ and an isometry $h\colon B(x, \epsilon) \to B(y, \epsilon)$ with $hx = y$).

One may then ask, what is a natural class of metric spaces whose local isometry types are able to be studied by noncommutative geometric methods? It is this question that we seek to answer in the first part of this paper.

A key feature of the metric d on X is that it is an *ultrametric*; that is, d satisfies the strong triangle inequality $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

A important geometric property of the Fibonacci space X is that it is *rigid*; that is, X has no isometries other than the identity. In particular, local isometry types in X are not reflected by global symmetries in X. Furthermore, the rigidity property of X is inherited by balls, so that, in particular, X is *locally rigid.*

In general, we find that it is compact ultrametric spaces satisfying the local rigidity property whose local isometry types can be studied using noncommutative geometric methods. The entry into these methods is through the theory of groupoids and their C^*-algebras.

The equivalence relation R on X is an example of a groupoid. In general, one can define a groupoid $\mathcal{G}_{LI}(X)$ of local isometries on a metric space X. If the groupoid is sufficiently well-behaved, Renault [43] showed how to define the C^*-algebra of the groupoid (generalizing the C^*-algebra of a group). We show that compact, locally rigid, ultrametric spaces have groupoids of local isometries to which Renault's theory can be applied.

The results we are able to obtain in this general situation are summarized in the following theorem.

Theorem 1.1 *If X is a compact, locally rigid, ultrametric space and $\mathcal{G}_{LI}(X)$ is the groupoid of local isometries on X, then*

1. *$\mathcal{G}_{LI}(X)$ is a locally compact, Hausdorff, second countable, étale groupoid;*
2. *the groupoid C^*-algebra $C^*\mathcal{G}_{LI}(X)$ is a unital AF C^*-algebra;*
3. *the topological groupoid $\mathcal{G}_{LI}(X)$, the unital groupoid C^*-algebra $C^*\mathcal{G}_{LI}(X)$, and the unital, partially ordered abelian group $K_0C^*\mathcal{G}_{LI}(X)$ are each invariants of X up to micro-scale equivalence of X;*
4. *there exists a Bratteli diagram $B(X)$ such that $\mathcal{G}_{LI}(X)$ is the path groupoid of $B(X)$;*
5. *$K_0C^*\mathcal{G}_{LI}(X)$ as a unital partially ordered abelian group is isomorphic to the symmetry at infinity group $Sym_\infty(T,v)$ of any rooted, geodesically complete, locally finite simplicial tree (T,v) whose end space is isometric to X.*

To measure local isometry types in X, it might seem more natural to focus on the group $LI(X)$ of local isometries from X to itself, rather than the groupoid $\mathcal{G}_{LI}(X)$. One of the main purposes of this paper is to show that the groupoid approach can be used to study local isometries on a compact, locally rigid ultrametric space. In this case, the groupoid $\mathcal{G}_{LI}(X)$ is an effective replacement of the quotient space $X/LI(X)$, which, in general, need not be Hausdorff. This is quite different from what happens for the isometry group $Isom(X)$, in which case $X/Isom(X)$ is a perfectly well-behaved space.

As with the Fibonacci space and tree, there is in general a well-known relationship between ultrametric spaces and trees. In fact, compact ultrametric spaces are exactly the end spaces of rooted, proper $\mathbb{R}$-trees. Under this correspondence, local isometries of the end space come from *uniform isometries at infinity* of the rooted tree. These are isometries between complements of open balls in the tree centered at the root. It follows that Theorem 1.1 provides an invariant of rooted, geodesically complete, proper $\mathbb{R}$-trees, with locally rigid end spaces, up to uniform isometry at infinity. See Corollary 10.16 for more details.

Elliott [18] proved that a unital AF C^*-algebra is determined up to isomorphism by its K_0 group (as a unital partially ordered abelian group). Consequently, in light of Theorem 1.1(2), knowing $K_0C^*\mathcal{G}_{LI}(X)$ becomes important. For any geodesically complete, locally finite simplicial tree T with root v, a group $Sym_\infty(T, v)$ is introduced in Section 9, called *the group of symmetries at infinity of T*. It is a unital partially ordered abelian group, constructed as a direct limit of a sequence of finitely generated free abelian groups. The direct sequence is elementary to construct from a diagram of the tree. Item (5) in Theorem 1.1 is established by showing that $Sym_\infty(T, v)$ is isomorphic to $K_0C^*\mathcal{G}_{LI}(end(T, v))$ as a unital partially ordered abelian group.

Theorem 1.1 only applies in the locally rigid case; however, there are many compact ultrametric spaces that are not locally rigid. An important example is the end space of the Cantor tree in Figure 2 (see Example 5.7).

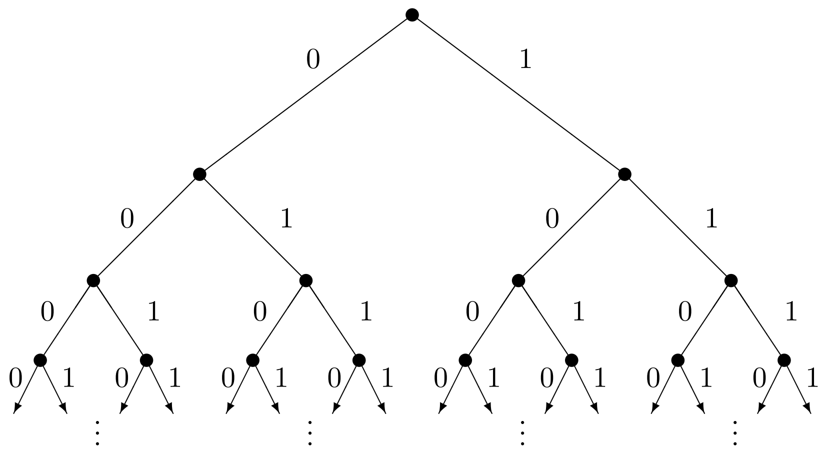

Figure 2: The Cantor tree

To a certain extent noncommutative geometric methods can still be applied to such spaces, and this is discussed in Section 1.3 below. But first we turn to the second main result of this paper.

1.2 Local similarities

In addition to studying local isometry types in compact ultrametric spaces, one may relax this relation and instead study *local similarity types.* When considering end spaces of rooted trees, just as local isometries of the end space correspond to uniform isometries at infinity of the rooted tree, local similarities of the end space correspond to (not necessarily uniform) *isometries at infinity* of the rooted tree. These are isometries between complements of the interiors of finite subtrees of the tree containing the root.

In the case that X is the Fibonacci space, then $x, y \in X$ are *tail equivalent with lag* (i.e., there exists $m, n \geq 0$ such that $x_{m+j} = y_{n+j}$ for all $j \geq 0$) if and only if X has the same local similarity type at x and y (i.e., there exist $\epsilon, \lambda > 0$ and a λ-similarity $h\colon B(x,\epsilon) \to B(y, \lambda\epsilon)$ with $hx = y$) (see [26]).

In analogy with the groupoid of local isometries, one can define a groupoid $\mathcal{G}_{LS}(X)$ of local similarities on a metric space X. We show that compact, locally rigid, ultrametric spaces have groupoids of local similarities to which Renault's theory can be applied. The results in this general situation are summarized in the following theorem.

Theorem 1.2 *If X is a compact, locally rigid, ultrametric space and $\mathcal{G}_{LS}(X)$ is the groupoid of local similarities on X, then*

1. *$\mathcal{G}_{LS}(X)$ is a locally compact, Hausdorff, second countable, étale groupoid, and*
2. *$\mathcal{G}_{LS}(X)$ is invariant up to local similarity of X.*

The contrast between local isometry types (Theorem 1.1) and local similarity types (Theorem 1.2) is already foreshadowed in the C^*-algebra literature. For example, Mingo [34] studied C^*-algebras of spaces of sequences modulo tail equivalence (generalizing Connes [14]). On the other hand, Kumjian, Pask, Raeburn and Renault [31] studied Cuntz-Krieger C^*-algebras of spaces of sequences modulo tail equivalence with lag. Roughly, tail equivalence corresponds to local isometry type, whereas tail equivalence with lag corresponds to local similarity type. For the end space of the Fibonacci tree (and some other trees), the analogy is exact.

1.3 Faithful unitary representations

The second main motivating example for this paper is Birget's faithful unitary representation of Thompson's group V into the Cuntz algebra $\mathcal{O}_2$ [7]. Such a representation was obtained independently by Nekrashevych [37]. This is related to this paper in the following two ways:

1. Thompson's group V is a subgroup of the group of local similarities on the end space Y of the Cantor tree. (References for this are given in Section 12.3.)

2. Renault [43] defined a groupoid O_2, called the *Cuntz groupoid*, and showed that the C^*-algebra of O_2 is the Cuntz algebra $\mathcal{O}_2$. The groupoid O_2 is easily seen to be a groupoid of local similarities on the end space of the Cantor tree.

However, as pointed out above, the end space of the Cantor tree is not locally rigid; therefore, the point of view as developed in Theorems 1.1 and 1.2 does not apply directly.

The key observation needed to overcome the lack of local rigidity is that in items (1) and (2) above, not *all* local similarities of the end space Y of the Cantor tree are needed—just those that *locally preserve the natural total order*. The group Γ of local similarities of Y that are locally order preserving has an important property: it *acts locally rigidly on* Y. (See Section 6.2 for the definition and Section 12.3 for the fact that this action is locally rigid.) This is the key property shared with the full group $LS(X)$ of all local similarities on a compact, locally rigid ultrametric space X.

Thus, we are led to consider subgroups Γ of the group $LS(X)$ of local similarities of an arbitrary compact ultrametric space X that act locally rigidly on X.

In analogy with the groupoids of local isometries and local similarities, one can define a subgroupoid $\mathcal{G}_\Gamma(X)$ of $\mathcal{G}_{LS}(X)$ whenever Γ is a subgroup of the group of local similarities on a metric space X. In the case that X is a compact ultrametric space and the action of Γ on X is locally rigid, we show that Renault's theory can be applied. The results in this general situation are summarized in the following theorem.

Theorem 1.3 *If X is a compact, ultrametric space and Γ is a countable group acting locally rigidly on X by local similarities, then*

1. *the germ groupoid $\mathcal{G}_\Gamma(X)$ of Γ on X is a locally compact, Hausdorff, second countable, étale groupoid;*
2. *if $h \in LS(X)$, then $h^{-1}\Gamma h$ also acts locally rigidly on X by local similarities and $\mathcal{G}_\Gamma(X)$ and $\mathcal{G}_{h^{-1}\Gamma h}(X)$ are isomorphic topological groupoids;*
3. *there is a faithful unitary representation of Γ into $C^*\mathcal{G}_\Gamma(X)$.*

Example 1.4 If Y is the end space of the Cantor tree and Γ is the subgroup of $LS(Y)$ consisting of locally order preserving local similarities, then $\Gamma = V$, Thompson's group, and Γ acts locally rigidly on Y. In this case, $\mathcal{G}_\Gamma(Y) = O_2$, the Cuntz groupoid. Since $C^*\mathcal{G}_\Gamma(Y) = \mathcal{O}_2$, the Cuntz algebra by Renault [43], the representation of Birget [7] and Nekrashevych [37] is a special case of Theorem 1.3.

More generally, Birget and Nekrashevych obtained faithful unitary representations of the Higman–Thompson groups $G_{n,1}$ into the Cuntz algebra $\mathcal{O}_n$ for all $n \geq 2$ ($G_{2,1} = V$). Such representations also follow from the results in this paper—see Section 12.3.

If X is a compact, locally rigid ultrametric space, then $\Gamma = LI(X)$ acts locally rigidly on X and $\mathcal{G}_{LI}(X) = \mathcal{G}_\Gamma(X)$. Thus, Theorems 1.1 and 1.3 imply the following result.

Corollary 1.5 *The local isometry group $LI(X)$ of a compact, locally rigid ultrametric space X has a faithful unitary representation into the AF C^*-algebra $C^*\mathcal{G}_{LI}(X)$.*

It should be mentioned that Berestovskii [5] proved that the isometry group of any compact ultrametric space has a faithful representation into the orthogonal group of a separable, real Hilbert space.

1.4 Guide

We indicate where the proofs of the theorems in the Introduction may be found in the body of the paper. The proof of Theorem 1.1 can be found as follows. Item (1) is proved in Section 6.1; item (2) is proved in Section 7; items (3) and (4) are proved in Section 10; item (5) is given by Corollary 9.3.

The proof of Theorem 1.2 can be found as follows. Item (1) is given by Corollary 6.12. Item (2) is in Section 3.

The proof of Theorem 1.3 can be found as follows. Item (1) is given in Section 6 (see Theorem 6.21). Item (2) is proved in Section 3 (Proposition 3.15) and Section 6 (Proposition 6.23). Item (3) is in Section 11.

For the theory of C^*-algebras of groupoids, see Muhly [36], Paterson [42], and Renault [43]. For some general background on ultrametric spaces, see Khrennikov [29] and Robert [44]. In addition to other references in the body of the paper, see Nikolaev [41] for an exposition of AF C^*-algebras and their K-theory.

Throughout this paper we use the notation $\mathbb{N} = \{1, 2, 3, \dots\}$ for the natural numbers, $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the nonnegative integers, $\mathbb{R}$ for the real numbers, and $\mathbb{C}$ for the complex numbers.

Acknowledgements. I have benefited from conversations with Berndt Brenken, Jon Brown, Alex Kumjian, Igor Nikolaev, Mark Sapir, Jack Spielberg, Andreas Thom, and Guoliang Yu.

2 Local isometry and local similarity groups

If (X, d) is a metric space, $x \in X$ and $\epsilon > 0$, then we use the notation $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$ for the *open ball about x of radius ϵ*, and $\bar{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$ for the *closed ball about x of radius ϵ*.

Definition 2.1 Let (X, d_X) and (Y, d_Y) be metric spaces. A homeomorphism $h\colon X \to Y$ is

1. an *isometry* if $d_Y(hx, hy) = d_X(x, y)$ for all $x, y \in X$.
2. a *similarity* if there exists $\lambda > 0$ such that $d_Y(hx, hy) = \lambda d_X(x, y)$ for all $x, y \in X$. In this case, h is a *λ-similarity*, λ is the *similarity modulus* of h, and we write $sim(h) = \lambda$.
3. a *local isometry* if for every $x \in X$ there exists $\epsilon > 0$ such that h restricts to an isometry $h|\colon B(x, \epsilon) \to B(hx, \epsilon)$.
4. a *local similarity* if for every $x \in X$ there exist $\epsilon > 0$ and $\lambda > 0$ such that h restricts to a λ-similarity $h|\colon B(x, \epsilon) \to B(hx, \lambda\epsilon)$. In this case, λ is the *similarity modulus of h at x* and we write $sim(h, x) = \lambda$.
5. a *uniform local similarity* if there exist $\epsilon > 0$ and $\lambda > 0$ such that for every $x \in X$ the restriction $h|\colon B(x, \epsilon) \to B(hx, \lambda\epsilon)$ is a λ-similarity. In this case, $sim(h, x) = \lambda$ for all $x \in X$.

An important point of this definition is that each of these maps is surjective.[1]

For a local similarity h, the similarity modulus $sim(h, x)$ is uniquely determined by h and x, except in the case x is an isolated point of X. In that case, we will always take $sim(h, x) = 1$.

For a metric space X (with a given metric), $Homeo(X)$ is the group of homeomorphisms from X to X, $Isom(X)$ is the group of isometries from X to X, $LI(X)$ is the group of local isometries from X to X, and $LS(X)$ is the group of local similarities from X to X.

[1]The terminology conflicts slightly with [26], where *similarity*, *local similarity*, and *uniform local similarity* were modified by *equivalence* to indicate that they were homeomorphisms.

Note that there are inclusions of subgroups

$$Isom(X) \leq LI(X) \leq LS(X) \leq Homeo(X).$$

These groups are given the compact-open topology unless otherwise specified.

If X is a compact metric space and $h\colon X \to X$ is a similarity (respectively, a uniform local similarity), then h is an isometry (respectively, a local isometry) (for the second of these statements, see [26]).

3 Local isometry and local similarity groupoids

In this section we define the topological groupoids of local similarities and local isometries of a metric space. Unfortunately, these groupoids are rarely Hausdorff or second countable (see Examples 3.13 and 3.14)—two conditions needed for Renault's machinery [43] to work.[2] We will eventually overcome this problem in Section 6 by either restricting to second countable, locally rigid ultrametric spaces or to certain subgroupoids.

An alternative treatment of local isometry groupoids is in Bridson and Haefliger [10, Part III, Chapter $\mathcal{G}$]; however, beyond the basic definitions, their point of view is quite a bit different from the present paper (in particular, they do not discuss C^*-algebras).

Let (X, d) be a metric space.

Definition 3.1 Let $x_1, x_2 \in X$. A *local similarity germ from x_1 to x_2 in X* is an equivalence class $[g, x_1]$ represented by a λ-similarity $g\colon B(x_1, \epsilon) \to B(x_2, \lambda\epsilon)$ for some $\epsilon > 0$ and $\lambda > 0$ such that $gx_1 = x_2$. Another such similarity $g'\colon B(x_1, \epsilon') \to B(x_2, \lambda'\epsilon')$ is *equivalent* to g if $g|B(x_1, \epsilon'') = g'|B(x_1, \epsilon'')$ for some $\epsilon'' > 0$ with $\epsilon'' \leq \min\{\epsilon, \epsilon'\}$.

If $[g, x]$ is a local similarity germ, then the modulus $sim(g, x)$ is independent of the choice of representative for the equivalence class.

[2] The Hausdorff condition is relaxed a bit in Paterson's approach [42].

Definition 3.2 The *local similarity groupoid* $\mathcal{G}_{LS}(X)$ *of* X is the set of all local similarity germs between pairs of points in X.

The groupoid structures on $\mathcal{G}_{LS}(X)$ are the obvious ones. Thus, the unit space is X and the domain $d\colon\ \mathcal{G}_{LS}(X) \to X$ and range $r\colon\ \mathcal{G}_{LS}(X) \to X$ maps are given by $d([g,x]) = x$ and $r([g,x]) = gx$. If $[g_1,x_1]$ and $[g_2,x_2]$ are local similarity germs from x_1 to x_2 and from x_2 to x_3, respectively, then the composition $[g_2,x_2][g_1,x_1]$ is the local similarity germ from x_1 to x_3 defined by composing g_1 and g_2 after suitably restricting their domains: $[g_2,x_2][g_1,x_1] = [g_2g_1,x_1]$. The inverse is $[g,x]^{-1} = [g^{-1},gx]$.

The topology on $\mathcal{G}_{LS}(X)$ is determined as follows.

Definition 3.3 For every germ $[g,x]$ represented by a λ-similarity $g\colon\ B(x,\epsilon) \to B(gx,\lambda\epsilon)$ and every $y \in B(x,\epsilon)$, there is a λ-similarity $g|\colon\ B(y,\delta) \to B(gy,\lambda\delta)$ where $\delta = \epsilon - d(x,y)$ representing a germ $[g,y]$. Let

$$U(g,x,\epsilon) = \{[g,y] \mid y \in B(x,\epsilon)\} \subseteq \mathcal{G}_{LS}(X).$$

The collection of all such $U(g,x,\epsilon)$ for $[g,x] \in \mathcal{G}_{LS}(X)$ forms a basis for a topology on $\mathcal{G}_{LS}(X)$ called the *germ topology.*

Note that $U(g_1,x_1,\epsilon_1) \cap U(g_2,x_2,\epsilon_2) =$

$$\bigcup\{U(g,x,\epsilon) \mid B(x,\epsilon) \subseteq B(x_1,\epsilon_1) \cap B(x_2,\epsilon_2) \text{ and } g = g_1|B(x,\epsilon) = g_2|B(x,\epsilon)\}.$$

Throughout the rest of this paper, $\mathcal{G}_{LS}(X)$ will always be given the germ topology.

The following result gives a proof of Theorem 1.2(2) in the Introduction.

Proposition 3.4 *A local similarity* $h\colon\ X \to Y$ *of metric spaces induces an isomorphism* $h_*\colon\ \mathcal{G}_{LS}(X) \to \mathcal{G}_{LS}(Y)$ *of topological groupoids.*

Proof. Let $[g,x] \in \mathcal{G}_{LS}(X)$ be a local similarity germ. We may assume that g is defined on $B(x,\epsilon)$ with $\epsilon > 0$ sufficiently small that there exists $\lambda_1 > 0$ such that $h|\colon B(x,\epsilon) \to B(hx,\lambda_1\epsilon)$ is a λ_1-similarity and there exists $\lambda_2 > 0$ such that $h|\colon B(gx,\epsilon) \to B(hgx,\lambda_1\epsilon)$ is a λ_2-similarity. Then $h_*[g,x]\colon = [hgh^{-1}, hx]$ can be seen to define an isomorphism of topological groupoids. $\square$

Lemma 3.5 *$\mathcal{G}_{LS}(X)$ is an étale groupoid. That is, $r\colon \mathcal{G}_{LS}(X) \to X$ is a local homeomorphism. In fact, the collection $\mathcal{G}_{LS}(X)^{op}$ of open subsets A of $\mathcal{G}_{LS}(X)$ such that $d|A$ and $r|A$ are homeomorphisms onto open subsets of X forms a basis for the germ topology* [3] *on $\mathcal{G}_{LS}(X)$.*

Proof. It suffices to observe that for each λ-similarity $g\colon B(x,\epsilon) \to B(gx,\lambda\epsilon)$, $d|\colon U(g,x,\epsilon) \to B(x,\epsilon)$ and $r|\colon U(g,x,\epsilon) \to B(gx,\lambda\epsilon)$ are homeomorphisms. It is clear that these are bijections; that they are homeomorphisms follows from the fact that $U(g,y,\delta) \subseteq U(g,x,\epsilon)$ whenever $y \in B(x,\epsilon)$ and $0 < \delta \leq \epsilon - d(x,y)$. $\square$

Lemma 3.6 *If X is locally compact, then $\mathcal{G}_{LS}(X)$ is locally compact.*

Proof. From the proof of Lemma 3.5, $\mathcal{G}_{LS}(X)$ has a basis of open sets homeomorphic to open balls of X. $\square$

Example 3.7 If (X,d) is a discrete metric space, then $\mathcal{G}_{LS}(X)$ and $X \times X$ are isomorphic as topological groupoids when $X \times X$ is given the pair groupoid structure (e.g. see [48, page 747]).

Definition 3.8 An *open subgroupoid* of $\mathcal{G}_{LS}(X)$ is a subset $\mathcal{G}$ of $\mathcal{G}_{LS}(X)$ such that

1. $\mathcal{G}$ is open in $\mathcal{G}_{LS}(X)$ (as topological spaces),

2. $\mathcal{G}$ is closed under composition,

3. $\mathcal{G}$ contains the unit space X.

[3] Some authors take this as the definition of étale; others refer to it as *r-discreteness*. Note that we are not insisting that our groupoids are locally compact, Hausdorff or second countable.

Definition 3.9 The *local isometry groupoid* $\mathcal{G}_{LI}(X)$ *of* X is the groupoid of all local isometry germs between pairs of points of X; that is,

$$\mathcal{G}_{LI}(X) = \{[g, x] \in \mathcal{G}_{LS}(X) \mid sim(g, x) = 1\}.$$

Definition 3.10 For a subgroup $\Gamma \leq LS(X)$, the *germ groupoid* $\mathcal{G}_\Gamma(X)$ *of* Γ *on* X is the subgroupoid of $\mathcal{G}_{LS}(X)$ given by

$$\mathcal{G}_\Gamma(X) = \{[g, x] \in \mathcal{G}_{LS}(X) \mid g \in \Gamma\}.$$

Remark 3.11 It is clear that both $\mathcal{G}_{LI}(X)$ and $\mathcal{G}_\Gamma(X)$ are open subgroupoids of $\mathcal{G}_{LS}(X)$. In fact, if $g\colon B(x, \epsilon) \to B(gx, \epsilon)$ represents a $[g, x] \in \mathcal{G}_{LI}(X)$, then $[g, x] \in U(g, x, \epsilon) \subseteq \mathcal{G}_{LI}(X)$. Likewise, if $g \in \Gamma$, $x \in X$ and $\epsilon > 0$, then $[g, x] \in U(g, x, \epsilon) \subseteq \mathcal{G}_\Gamma(X)$.

Remark 3.12 The unit space is naturally an open subspace of $\mathcal{G}_{LS}(X)$ via the map $\alpha\colon X \to \mathcal{G}_{LS}(X), x \mapsto [\mathrm{id}, x]$, where $\mathrm{id}\colon B(x, \epsilon) \to B(x, \epsilon)$ is the identity for some $\epsilon > 0$. Obviously, α is injective. To see that it is continuous, let $U(g, y, \epsilon)$ be a basis element of $\mathcal{G}_{LS}(X)$ and suppose $\alpha(x) \in U(g, y, \epsilon)$. Then $[\mathrm{id}, x] \in U(g, y, \epsilon)$, so $x \in B(y, \epsilon)$ and $g = \mathrm{id}$ near x. It follows that if z is close enough to x, then $\alpha(z) \in U(g, y, \epsilon)$, thereby verifying continuity of α. To see that α is an open map, note that $\alpha(B(x, \epsilon)) = \cup\{[\mathrm{id}, y] \mid y \in B(x, \epsilon)\} = U(\mathrm{id}, x, \epsilon)$. In particular, $\alpha(X)$ is an open subset of $\mathcal{G}_{LS}(X)$. It need not be the case that $\alpha(X)$ is closed in $\mathcal{G}_{LI}(X)$, but see Remark 6.4 for an instance when it is.

Example 3.13 If X is the end space of the Cantor tree C, then $\mathcal{G}_{LI}(X)$ is not Hausdorff. On the other hand, if Y is the end space of the Fibonacci tree F, then $\mathcal{G}_{LI}(Y)$ is Hausdorff. See Theorem 6.3 and Example 6.5.

Example 3.14 If $X = \{x \in \mathbb{R}^2 \mid ||x|| \leq 1\}$, the closed unit ball in $\mathbb{R}^2$ with the usual metric, then $\mathcal{G}_{LI}(X)$ is not second countable. To see this, for each $0 \leq \theta < 2\pi$, let $g_\theta\colon X \to X$ be counterclockwise rotation through angle θ. Then for every $x \in X$ and $\theta_1 \neq \theta_2$, $[g_{\theta_1}, x] \neq [g_{\theta_2}, x]$. It follows that if $\mathbf{0}$ is the origin in $\mathbb{R}^2$, $\{U(g_\theta, \mathbf{0}, 1) \mid 0 \leq \theta < 2\pi\}$ is an uncountable collection of mutually disjoint, nonempty open subsets of $\mathcal{G}_{LI}(X)$. Hence, $\mathcal{G}_{LI}(X)$ is not second countable.

Proposition 3.15 *If X is a metric space with a subgroup $\Gamma \leq LS(X)$ and $h \in LS(X)$, then $\mathcal{G}_\Gamma(X)$ and $\mathcal{G}_{h^{-1}\Gamma h}(X)$ are isomorphic topological groupoids.*

Proof. According to Proposition 3.4, $(h^{-1})_*\colon \mathcal{G}_{LS}(X) \to \mathcal{G}_{LS}(X)$, $[g,x] \mapsto [h^{-1}gh, h^{-1}x]$, is an isomorphism of topological groups. Clearly, $(h^{-1})_*$ takes the open subgroupoid $\mathcal{G}_\Gamma(X)$ onto $\mathcal{G}_{h^{-1}\Gamma h}(X)$. $\square$

4 Ultrametric spaces

In this section we recall the definition of ultrametric spaces and some of their well-known properties. We then establish some elementary properties which have not appeared previously in the literature. These properties will be useful in studying local isometry and similarity groups and groupoids of ultrametric spaces.

Definition 4.1 If (X,d) is a metric space and $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for all $x,y,z \in X$, then d is an *ultrametric* and (X,d) is an *ultrametric space*.

The following proposition lists some well-known properties of ultrametric spaces. They are readily verified.

Proposition 4.2 (Elementary properties of ultrametric spaces) *The following properties hold in any ultrametric space (X,d).*

1. *If two open balls (or two closed balls) in X intersect, then one contains the other.*
2. **(Egocentricity)** *Every point in an open (or closed) ball is a center of the ball.*
3. *Every open ball is closed, and every closed ball is open.*
4. **(ISB)** *Every triangle in X is isosceles with a short base. That is, if $x_1, x_2, x_3 \in X$, then there exists an i such that $d(x_j,x_k) \leq d(x_i,x_j) = d(x_i,x_k)$ whenever $j \neq i \neq k$.* $\square$

Lemma 4.3 (Isometry Extension) *Suppose X is an ultrametric space, $x \in X$ and $\epsilon > 0$. If $h\colon B(x,\epsilon) \to B(x,\epsilon)$ is an isometry, then $\tilde{h}\colon X \to X$ defined by*

$$\tilde{h} = \begin{cases} h & \text{on } B(x,\epsilon) \\ \text{inclusion} & \text{on } X \setminus B(x,\epsilon) \end{cases}$$

is also an isometry.

Proof. First observe that for all $y, z \in X$, $y, z \in B(x,\epsilon)$ implies $d(y,z) < \epsilon$, and $y \in B(x,\epsilon), z \notin B(x,\epsilon)$ implies $d(y,z) \geq \epsilon$. [The first implication follows immediately from the ultrametric inequality. The second follows because $\epsilon \leq d(x,z) \leq \max\{d(x,y), d(y,z)\}$, and $d(x,y) < \epsilon$; thus, $\epsilon \leq d(y,z)$.] Now to show that $\tilde{h}$ is an isometry, it suffices to let $x_1 \in B(x,\epsilon)$, $x_2 \notin B(x,\epsilon)$ and show $d(x_1,x_2) = d(hx_1,x_2)$. For this note that on one hand,

$$d(x_1,x_2) \leq \max\{d(x_1,hx_1), d(hx_1,x_2)\} = d(hx_1,x_2).$$

And on the other hand,

$$d(hx_1,x_2) \leq \max\{d(hx_1,x_1), d(x_1,x_2)\} = d(x_1,x_2). \quad \square$$

Remark 4.4 Lemma 4.3 need not hold for isometries $h\colon B(x,\epsilon) \to B(y,\epsilon)$. For example, the end space of the Fibonacci tree is rigid, but there are some local isometries (see [26], Prop. 9.5).

Lemma 4.5 (Circular Equidistance) *If (X,d) is an ultrametric space with points w, x, y, z in X such that $d(x,w) \neq d(x,y) = d(x,z)$, then $d(w,y) = d(w,z)$. That is, if there exists a point $x \in X$ an equidistance ℓ to two points y, z then every other point w whose distance from x is different from ℓ is equidistant to y and z. In yet other words, let $\ell > 0$, $x \in X$ and consider the "circle" $C = \{y \in X \mid d(x,y) = \ell\}$. Then any point not on the circle C is equidistant to any two points on the circle C.*

Proof. Let $\ell = d(x,y) = d(x,z)$ and let $r = d(x,w)$. If $r < \ell$, then by the ISB property (Proposition 4.2), since $d(x,y) > d(x,w)$, it must be the case that

$d(y,w) = d(y,x)$. Likewise $d(z,w) = d(z,x)$. Hence, $d(z,w) = d(y,w) = \ell$. If $r > \ell$, then by the ISB property, since $d(w,x) > d(y,x)$, it must be the case that $d(w,y) = d(y,x)$. Likewise, $d(z,w) = d(z,x)$. Hence, $d(z,w) = d(y,w) = r$ □

Lemma 4.6 (Modification of Local Isometry) *If (X,d) is an ultrametric space, $x \in X$, $\epsilon > 0$ and $g\colon B(x,\epsilon) \to B(x,\epsilon)$ is an isometry such that $g(x) = x$ and g is non-trivial arbitrarily close to x (that is, for every $\delta > 0$, $\delta \leq \epsilon$, there exists $y \in B(x,\delta)$ such that $g(y) \neq y$), then there exists an isometry $\tilde{g}\colon B(x,\epsilon) \to B(x,\epsilon)$ such that $\tilde{g}(x) = x$, $\tilde{g}$ is non-trivial arbitrarily close to x, and for every $\delta > 0$, $\delta \leq \epsilon$, there exists $y \in B(x,\delta)$ and $\mu > 0$ such that $\tilde{g}|\colon B(y,\mu) \to B(y,\mu)$ is the identity.*

Proof. Choose a sequence $\{x_i\}_{i=1}^{\infty}$ of distinct points in $B(x,\epsilon)$ converging to x such that

1. for every $i \in \mathbb{N}$, $g(x_i) \neq x_i$, and
2. $d(x,x_1) > d(x,x_2) > d(x,x_3) > \cdots$.

For each $i \in \mathbb{N}$ let $C_i = \{y \in X \mid d(x,y) = d(x,x_i)\}$ and note that $g(C_i) = C_i$. Define

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in \cup_{i=1}^{\infty} C_{2i} \\ x & \text{if } x \notin \cup_{i=1}^{\infty} C_{2i}. \end{cases}$$

Note that the Circular Equidistance Lemma 4.5 implies that $\tilde{g}\colon B(x,\epsilon) \to B(x,\epsilon)$ is an isometry. The rest of the properties are straightforward to verify. □

Lemma 4.7 (Local Isometry Extension) *If (X,d) is an ultrametric space, then $\mathcal{G}_{LI}(X) = \mathcal{G}_{\Gamma}(X)$, where $\Gamma = LI(X)$.*

Proof. Clearly $\mathcal{G}_{\Gamma}(X) \subseteq \mathcal{G}_{LI}(X)$. Now let $g\colon B(x,\epsilon) \to B(gx,\epsilon)$ be an isometry representing $[g,x] \in \mathcal{G}_{LI}(X)$. Define $\tilde{g}\colon X \to X$ by

$$\tilde{g} = \begin{cases} g & \text{on } B(x,\epsilon) \\ g^{-1} & \text{on } B(gx,\epsilon) \setminus B(x,\epsilon) \\ \text{inclusion} & \text{on } X \setminus (B(x,\epsilon) \cup B(gx,\epsilon)) \end{cases}$$

It is easy to verify that $\tilde{g}$ is a local isometry (recall that open balls are closed and $B(x,\epsilon) = B(gx,\epsilon)$ or $B(x,\epsilon) \cap B(gx,\epsilon) = \emptyset$). Thus, $[g,x] = [\tilde{g},x] \in \mathcal{G}_\Gamma(X)$. $\square$

A similar result does not hold for local similarities as the next example shows.

Example 4.8 Let $X = \{z_\infty, z_0, z_1, z_2, \ldots\}$ with ultrametric d given by

$$d(z_i, z_j) = e^{-\min\{i,j\}} \text{ if } i \neq j.$$

Thus, X is the end space of the Sturmian tree—see Example 5.9. Define $g\colon X \to X$ by $gz_\infty = z_\infty$ and $gz_i = z_{i+1}$ for $i = 0, 1, 2, \ldots$. Then $g|\colon B(z_\infty, 1) \to B(z_\infty, e^{-1})$ is an e^{-1}-similarity representing $[g, z_\infty] \in \mathcal{G}_{LI}(X)$. However, there is no local similarity $h\colon X \to X$ with $[h, z_\infty] = [g, z_\infty]$. Hence, $\mathcal{G}_\Gamma(X) \subsetneqq \mathcal{G}_{LS}(X)$, where $\Gamma = LS(X)$.

The groupoids associated to a compact ultrametric space X studied in this paper are of two types. First, there is the full groupoid $\mathcal{G}_{LS}(X)$ of local similarity germs on X. Second, there are the groupoids of the form $\mathcal{G}_\Gamma(X)$ where Γ is a subgroup of $LS(X)$. By Lemma 4.7, this second type includes $\mathcal{G}_{LI}(X)$. Moreover, by Remark 3.11, the groupoids $\mathcal{G}_\Gamma(X)$ are open subgroupoids of $\mathcal{G}_{LS}(X)$.

5 Recollections on trees and their ends

The material in this section is well-known; we collect it here for the convenience of the reader. For more background on $\mathbb{R}$-trees, see Bestvina [6], Chiswell [13], and Morgan and Shalen [35]. For more information and references on end spaces of $\mathbb{R}$-trees, see Hughes [26] and Martínez-Pérez and Morón [33].

5.1 Trees

An $\mathbb{R}$-*tree* is a metric space (T, d) that is uniquely arcwise connected, and for any two points $x, y \in T$ the unique arc from x to y, denoted $[x, y]$, is isometric to the subinterval $[0, d(x, y)]$ of $\mathbb{R}$.

An $\mathbb{R}$-tree is *proper* if every closed metric ball in T is compact.

As an example, let T be a *locally finite simplicial tree*; that is, T is the (geometric realization of) a locally finite, one-dimensional, simply connected, simplicial complex. There is a natural unique metric d on T such that (T, d) is an $\mathbb{R}$-tree, every edge is isometric to the closed unit interval $[0, 1]$, and the distance between distinct vertices v_1, v_2 is the minimum number of edges in a sequence of edges $e_0, e_1, \ldots, e_n$ with $v_1 \in e_0$, $v_2 \in e_n$ and $e_i \cap e_{i+1} \neq \emptyset$ for $0 \leq i \leq n-1$. It follows that (T, d) is a proper $\mathbb{R}$-tree.

Whenever we refer to a *locally finite simplicial tree* T, the metric d on T will be understood to be the natural one just described.

Choose a *root* (i.e., a base vertex) $v \in T$. The rooted tree (T, v) is *geodesically complete* if for every isometric embedding $x\colon [0, t] \to T$, $t > 0$, with $x(0) = v$, extends to an isometric embedding $\tilde{x}\colon [0, \infty) \to T$. Such a map $\tilde{f}$ is a *geodesic ray in T beginning at v*. In other words, T is geodesically complete if every vertex of T, except possibly the root, lies in at least two edges.

5.2 Ends of trees

The *end space* of a rooted $\mathbb{R}$-tree (T, v) is given by

$$end(T, v) = \{x\colon [0, \infty) \to T \mid x(0) = v \text{ and } x \text{ is an isometric embedding}\}.$$

For $x, y \in end(T, v)$, define

$$d_e(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/e^{t_0} & \text{if } x \neq y \text{ and } t_0 = \sup\{t \geq 0 \mid x(t) = y(t)\}. \end{cases}$$

It follows that $(end(T, v), d_e)$ is a complete ultrametric space of diameter ≤ 1. The elements of $end(T, v)$ are called *ends* of (T, v).

Proposition 5.1 *Let (T, v) be a geodesically complete, rooted $\mathbb{R}$-tree. Then T is proper if and only if $end(T, v)$ is compact.*

Proof. First, assume T is proper and show that $end(T, v)$ is totally bounded. Let $\epsilon > 0$ be given; to show that $end(T, v)$ can be covered by a finite number of closed ϵ-balls, we may assume $\epsilon < 1$. Let $r = -\ln \epsilon$. Since $\overline{B}(v, r)$ is compact, so is $\partial\overline{B}(v,r) = \{t \in T \mid d(t,v) = r\}$. We claim that $\partial\overline{B}(v,r)$ is finite. On the contrary assume that there is an infinite set $\{t_i\}_{i=1}^{\infty}$ of distinct points in $\partial\overline{B}(v,r)$. Choose $\{x_i\}_{i=1}^{\infty} \subseteq end(T,v)$ such that $x_i(r) = t_i$ for all $i \geq 1$. Then the sets $x_i([r, r+1])$, $i \geq 1$, are mutually disjoint. Hence, $d(x_i(r+1), x_j(r+1)) \geq 2$ if $i \neq j$. This contradicts the compactness of $\partial\overline{B}(v, r+1)$.

Therefore, write $\partial\overline{B}(v,r) = \{t_i\}_{i=1}^{N}$ and choose $\{x_i\}_{i=1}^{N} \subseteq end(T,v)$ as above (so that $x_i(r) = t_i$). Clearly, $end(T,v) = \cup_{i=1}^{N}\overline{B}(x_i,\epsilon)$.

Conversely, assume that $end(T,v)$ is compact, let $r > 0$ be given, and show that $\overline{B}(v,r)$ is compact in T by showing every sequence in $\overline{B}(v,r)$ has a convergent subsequence. Let $\{t_i\}_{i=1}^{\infty}$ be a sequence in $\overline{B}(v,r)$ and choose $\{x_i\}_{i=1}^{\infty} \subseteq end(T,v)$ such that $t_i = x_i(d(v,t_i))$ for all $i \geq 1$. By passing to a subsequence, we may assume there exists $x_0 \in end(T,v)$ such that $x_i \to x_0$ in $end(T,v)$ as $i \to \infty$. Hence, there exists N such that $d_e(x_0, x_i) \leq e^{-r}$ for all $i \geq N$. That is, $x_0(t) = x_i(t)$ if $0 \leq t \leq r$ and $i \geq N$. In particular, $t_i = x_0(d(v,t_i))$ for all $i \geq N$. So t_i is in the compact subset $x_0([0,r])$ of $\overline{B}(v,r)$ for all $i \geq N$. Thus, $\{t_i\}_{i=1}^{\infty}$ has a convergent subsequence. □

Corollary 5.2 *Let (T,v) be a geodesically complete, rooted $\mathbb{R}$-tree. T is a locally finite simplicial tree if and only if $end(T,v)$ is compact and has distance set*

$$\{t \in \mathbb{R} \mid \text{ there exists } x, y \in end(T,v) \text{ such that } d_e(x,y) = t\}$$

contained in $\{0\} \cup \{e^{-i} \mid i = 0, 1, 2, \dots\}$.

Proof. Necessity follows from Proposition 5.1 and obvious facts about the metric d_e when T is simplicial. Conversely, given the distance set condition, declare all points of T of the form $x(n)$ with $x \in end(T,v)$ and $n \in \{0, 1, 2, \dots\}$ to be vertices; likewise, sets of the form $x([n, n+1])$ are edges. It is easily seen that this makes T into a simplicial tree. Compactness of $end(T,v)$ guarantees local finiteness. □

Remark 5.3 Let (T,v) be a rooted, geodesically complete, locally finite simplicial tree. The ends of (T,v) are in one-to-one correspondence with infinite sequences of distinct edges $e_0, e_1, e_2, \dots$ such that $v \in e_0$ and for $i \geq 1$, $e_{i-1} \cap e_i$ consists of exactly one vertex, say v_i, and the vertices $v, v_1, v_2, \dots$ are distinct.

Remark 5.4 One can verify that proper $\mathbb{R}$-trees are equivalent to the $\mathbb{R}$-trees called *simplicial* in [6] that are additionaly required to be locally finite.

We include the following definition from [26].

Definition 5.5 A *cut set* C for a geodesically complete, rooted $\mathbb{R}$-tree (T, v) is a subset C of T such that $v \notin C$ and for every isometric embedding $\alpha\colon [0, \infty) \to T$ with $\alpha(0) = v$ there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C$. For $v \neq c \in T$, let T_c denote the *subtree of* (T, v) *descending from* c; that is, $T_c = \{x \in T \mid c \in [v, x]\}$. An *isometry at infinity* between geodesically complete, rooted $\mathbb{R}$-trees (T, v) and (S, w) is a triple (f, C_T, C_S) where C_T and C_S are cut sets of T and S, respectively, and $f\colon \cup\{T_c \mid c \in C_T\} \to \cup\{S_c \mid c \in C_S\}$ is a homeomorphism such that

1. $f(C_T) = C_S$, and
2. for every $c \in C_T$, $f|\colon T_c \to S_{f(c)}$ is an isometry.

An isometry at infinity $(f, C_T, C_S)\colon (T, v) \to (S, w)$ is a *uniform isometry at infinity* provided there exist $\epsilon, \delta > 0$ such that $C_T = \partial B(v, \epsilon)$ and $C_S = \partial B(w, \delta)$.

The end space functor. Let $\mathbf{U}_1$ be the category whose objects are compact ultrametric spaces of diameter less than or equal to 1 and whose morphisms are isometries. Let $\begin{Bmatrix} \mathbf{U}_2 \\ \mathbf{U}_3 \end{Bmatrix}$ be the category whose objects are compact ultrametric spaces and whose morphisms are $\begin{Bmatrix} \text{uniform local similarities} \\ \text{local similarities} \end{Bmatrix}$. Let $\begin{Bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \mathbf{T}_3 \end{Bmatrix}$ be the category whose objects are proper, rooted, geodesically complete $\mathbb{R}$-trees and whose morphisms are

$$\begin{Bmatrix} \text{rooted isometries} \\ \text{equivalence classes of uniform isometries at infinity} \\ \text{equivalence classes of isometries at infinity} \end{Bmatrix}.$$

The equivalence classes just referred to are germs-at-infinity (see [26] for precise definitions). The following result follows from Proposition 5.1 and [26].

Proposition 5.6 *The end space functor $\mathcal{E}$ restricts to equivalences of categories $\mathcal{E}\colon \mathbf{T}_i \to \mathbf{U}_i$ for $i = 1, 2, 3$.*

Balls in the ends of simplicial trees. It will be convenient to have the following description of the metric balls in $end(T, v)$, where (T, v) is a rooted, geodesically complete, locally finite simplicial tree. For each $x \in end(T, v)$ and $0 < \epsilon \leq 1$,

$$B(x, \epsilon) = \{y \in end(T, v) \mid d_e(x, y) < \epsilon\} = \{y \in end(T, v) \mid -\ln \epsilon < t_0\}$$

and

$$\overline{B}(x, \epsilon) = \{y \in end(T, v) \mid d_e(x, y) \leq \epsilon\} = \{y \in end(T, v) \mid -\ln \epsilon \leq t_0\},$$

where $t_0 = \sup\{t \geq 0 \mid x(t) = y(t)\}$. Let $\lceil -\ln \epsilon \rceil$ be the smallest positive integer greater than or equal to $-\ln \epsilon$. Then $x(\lceil -\ln \epsilon \rceil)$ is a vertex of T that we denote by $v_{\{x,\epsilon\}}$. Let $T_{\{x,\epsilon\}}$ denote the subtree of T descending from $v_{\{x,\epsilon\}}$; i.e.,

$$T_{\{x,\epsilon\}} = \bigcup \{y(t) \mid y \in end(T, v), y(\lceil -\ln \epsilon \rceil) = v_{\{x,\epsilon\}}, \text{ and } t \geq \lceil -\ln \epsilon \rceil\}.$$

Then $(T_{\{x,\epsilon\}}, v_{\{x,\epsilon\}})$ is itself a rooted, geodesically complete, locally finite simplicial tree. We make the identification

$$\overline{B}(x, \epsilon) = end(T_{\{x,\epsilon\}}, v_{\{x,\epsilon\}}),$$

where $y \in \overline{B}(x, \epsilon)$ is identified with $\tilde{y} \in end(T_{\{x,\epsilon\}}, v_{\{x,\epsilon\}})$ defined by $\tilde{y}(t) = y(t + \lceil -\ln \epsilon \rceil)$ for $t \geq 0$. Conversely, of course, $\tilde{y} \in end(T_{\{x,\epsilon\}}, v_{\{x,\epsilon\}})$ is identified with $y \in \overline{B}(x, \epsilon)$ defined by

$$y(t) = \begin{cases} x(t) & \text{for } 0 \leq t \leq \lceil -\ln \epsilon \rceil \\ \tilde{y}(t - \lceil -\ln \epsilon \rceil) & \text{for } t \geq \lceil -\ln \epsilon \rceil. \end{cases}$$

Likewise, let $\lceil\lceil -\ln \epsilon \rceil\rceil$ be the smallest positive integer greater than $-\ln \epsilon$. Thus, $\lceil -\ln \epsilon \rceil \leq \lceil\lceil -\ln \epsilon \rceil\rceil$, with equality if and only if $-\ln \epsilon$ is an integer. Then $x(\lceil\lceil -\ln \epsilon \rceil\rceil)$ is a vertex of T that we denote by $v_{\langle x,\epsilon \rangle}$. Let $T_{\langle x,\epsilon \rangle}$ denote the subtree of T descending from $v_{\langle x,\epsilon \rangle}$; i.e.,

$$T_{\langle x,\epsilon \rangle} = \bigcup \{y(t) \mid y \in end(T, v), y(\lceil\lceil -\ln \epsilon \rceil\rceil) = v_{\langle x,\epsilon \rangle}, \text{ and } t \geq \lceil\lceil -\ln \epsilon \rceil\rceil\}.$$

Then $(T_{\langle x,\epsilon\rangle}, v_{\langle x,\epsilon\rangle})$ is itself a rooted, geodesically complete, simplicial tree. We make the identification

$$B(x,\epsilon) = end(T_{\langle x,\epsilon\rangle}, v_{\langle x,\epsilon\rangle}),$$

5.3 Examples

In this section we give examples of a few trees and their end spaces. These examples appear again in Section 9.

Example 5.7 The Cantor tree C. The Cantor tree C, also called the infinite binary tree, is a locally finite simplicial tree. It has a root v of valency two (i.e., there exists exactly two edges containing v) and every other vertex is of valency three. If w is a vertex different from v, then the two edges that contain w and are separated from v by w are not labelled identically. Each edge is labelled 0 or 1 so that for every vertex w, at least one edge containing w is labelled 0 and at least one is labelled 1.

Let $end(C) = end(C, v)$ since the root v is understood. An element of $end(C)$, being an infinite sequence of successively adjacent edges in C beginning at v, can be labelled uniquely by an infinite sequence of 0's and 1's. Thus,

$$end(C) = \{(x_0, x_1, x_2, \dots) \mid x_i \in \{0,1\} \text{ for each } i\}$$

and

$$d_e((x_i),(y_i)) = \begin{cases} 0 & \text{if } (x_i) = (y_i) \\ 1/e^n & \text{if } (x_i) \neq (y_i) \text{ and } n = \inf\{i \geq 0 \mid x_i \neq y_i\} \end{cases}.$$

Example 5.8 The Fibonacci tree F. The Fibonacci tree F is a subtree of C with the same root v and labelling scheme. In F, only edges labelled 0 are allowed to follow edges labelled 1 as one moves away from the root. Thus,

$$end(F) = \{(x_0, x_1, x_2, \dots) \in end(C) \mid x_i = 1 \text{ implies } x_{i+1} = 0\}.$$

See [26] for some compaisons of the Cantor and Fibonacci trees.

Example 5.9 The Sturmian tree S. The Sturmian tree S is also a subtree of C with the same root v and labelling scheme. In S, only edges labelled 1 are allowed to follow edges labelled 1 as one moves away from the root. Thus,

$$end(S) = \{(x_0, x_1, x_2, \dots) \in end(C) \mid x_i = 1 \text{ implies } x_{i+1} = 1\}.$$

In particular, $end(S)$ is countably infinite: $end(S) = \{z_\infty, z_0, z_1, z_2, \dots\}$, where $z_\infty = (0,0,0,\dots), z_0 = (1,1,1,\dots), z_1 = (0,1,1,1,\dots), z_2 = (0,0,1,1,\dots),\dots$. The metric is given by $d(z_i, z_j) = e^{-\min\{i,j\}}$ if $i \neq j$.

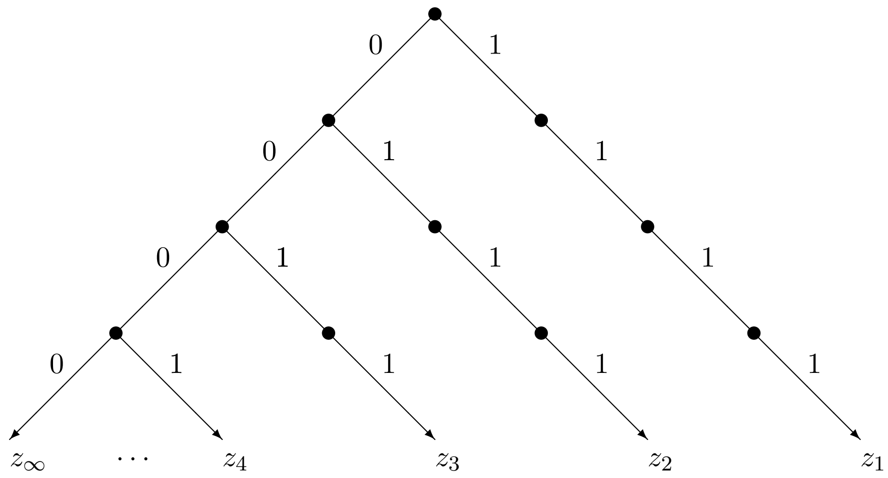

Figure 3: The Sturmian tree

Example 5.10 The n-regular tree R_n. For $n = 1, 2, 3, \dots$, the n-regular tree R_n is the simplicial tree such that every vertex has valency $n+1$. It is homogeneous so that a root can be chosen arbitrarily. It is geodesically complete and locally finite. The edges can be labelled by the integers $0, 1, \dots, n$ so that for each vertex each label appears on exactly one edge containing the vertex. Thus,

$$end(R_n) = \{(x_0, x_1, x_2, \dots) \mid x_i \in \{0, 1, \dots, n\} \text{ and } x_{i+1} \neq x_i \text{ for each } i\}.$$

Example 5.11 The infinite n-ary tree A_n. For $n = 1, 2, 3, \dots$, the infinite n-ary tree A_n is the rooted, geodesically complete, locally finite simplicial tree such that every vertex except the root has valency $n+1$, and the root has valency n. For example, A_2 is the Cantor tree.

Example 5.12 The n-ended tree E_n. For $n = 1, 2, 3, \ldots$, (E_n, v) is the simplicial tree such that the root v has valency n and all other vertices have valency 2. Thus, $end(E_n)$ consists of n points, each a distance 1 from any other.

Example 5.13 The irrational tree T_α. Let α be positive irrational number and $\alpha = [a_0, a_1, a_2, \ldots]$ its continued fraction expansion. Thus, $\{a_i\}_{i=0}^{\infty}$ is a sequence of non-negative integers such that $a_i \geq 1$ if $i \geq 1$ and

$$\alpha \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Consider the compact metric space

$$X_\alpha = \left\{ (x_0, x_1, x_2, \ldots) \in \prod_0^\infty \mathbb{Z}_+ \mid 0 \leq x_i \leq a_i \text{ and } x_i = a_i \text{ implies } x_{i+1} = 0 \right\}$$

with ultrametric

$$d_e((x_i), (y_i)) = \begin{cases} 0 & \text{if } (x_i) = (y_i) \\ 1/e^n & \text{if } (x_i) \neq (y_i) \text{ and } n = \inf\{i \geq 0 \mid x_i \neq y_i\} \ . \end{cases}$$

Let (T_α, v) be the rooted, geodesically complete, locally finite simplicial tree such that $end(T_\alpha, v)$ is isometric to X_α. For example, the golden mean $\frac{1+\sqrt{5}}{2} = [1, 1, 1, \ldots]$ and $T_{\frac{1+\sqrt{5}}{2}} = F$, the Fibonacci tree. The spaces X_α appear in Mingo [34].

6 Local rigidity, locally rigid actions, and Hausdorffness

6.1 Locally rigid ultrametric spaces

The main goal of this section is to characterize when the groupoid of local isometries on an ultrametric space is Hausdorff. The answer is in terms of a local rigidity condition on the ultrametric space. We also discuss the second countability of the local isometry groupoid of locally rigid ultrametric spaces.

Definition 6.1 A metric space (X,d) is *locally rigid* if for every $x \in X$ there exists $\epsilon_x > 0$ such that any isometry $h\colon B(x,\epsilon_x) \to B(x,\epsilon_x)$ is the identity.

Lemma 6.2 *An ultrametric space X is locally rigid if and only if for every $x \in X$ there exists $\epsilon_x > 0$ such that for any $0 < \epsilon \leq \epsilon_x$, every isometry $h\colon B(x,\epsilon) \to B(x,\epsilon)$ is the identity.*

Proof. This follows immediately from Lemma 4.3. □

Theorem 6.3 *For an ultrametric space (X,d) the following are equivalent.*

1. *For every $x \in X$, $\epsilon > 0$ and isometry $g\colon B(x,\epsilon) \to B(x,\epsilon)$ such that $g(x) = x$ there exists $\delta = \delta(\epsilon, x, g) > 0$ such that $g|\colon B(x,\delta) \to B(x,\delta)$ is the identity.*
2. *For every $x \in X$ there exists $\epsilon_x > 0$ such that if $g\colon B(x,\epsilon_x) \to B(x,\epsilon_x)$ is an isometry with $g(x) = x$, then g is the identity.*
3. *X is locally rigid.*
4. *$\mathcal{G}_{LI}(X)$ is Hausdorff.*

Proof. (1) implies (2). Suppose on the contrary that X satisfies (1) but not (2). Then there is a sequence of "circles" C_i (in the sense of Lemma 4.5), $i \in \mathbb{N}$, about some $x \in X$ of decreasing diameter and non-trivial isometries $g_i\colon C_i \to C_i$. These can be pieced together to give a non-trivial isometry $g\colon B(x,\epsilon) \to B(x,\epsilon)$ which is non-trivial on each ball about x.

(2) implies (3). Suppose on the contrary that there exists $x \in X$ without the property in Definition 6.1. Property (2) implies that there exists $\epsilon_0 > 0$ such that if $0 < \epsilon \leq \epsilon_0$ and $g\colon B(x,\epsilon) \to B(x,\epsilon)$ is an isometry with $gx = x$, then g is the identity (this uses Lemma 4.3 in a manner similar to how it is used in Lemma 6.2). Choose $0 < \epsilon_2 < \epsilon_1 < \epsilon_0$ so that for $i = 1,2$ there exists an isometry $h_i\colon X \to X$ such that $h_i B(x,\epsilon_i) = B(x,\epsilon_i)$, $h_i|(X \setminus B(x,\epsilon_i)) =$ inclusion (this uses Lemma 4.3 again) and h_i is not the identity. It follows that $h_i x \neq x$ for $i = 1,2$. By choosing

ϵ_1 and h_1 before ϵ_2 and h_2, we may assume that $\epsilon_2 < d(x, h_1x)$. Consider the composition

$$g\colon\ B(h_1x,\epsilon_2) \xrightarrow{h_1^{-1}} B(x,\epsilon_2) \xrightarrow{h_2} B(x,\epsilon_2) \xrightarrow{h_1|} B(h_1x,\epsilon_2).$$

This isometry can be extended (by Lemma 4.3) to an isometry $\tilde{g}\colon X \to X$ such that $\tilde{g}|(X \setminus B(h_1x,\epsilon_2))$ is the inclusion. Now $\tilde{g}B(x,\epsilon_1) = B(x,\epsilon_1)$ and $\tilde{g}x = x$ (because $x \notin B(h_1x,\epsilon_2)$. Thus, $\tilde{g}$ is the identity. Since $\tilde{g}h_1x = h_1h_2x$ we have $h_1x = h_1h_2x$ and $x = h_2x$, a contradiction.

(3) implies (4). Let $[g_1,x_1] \neq [g_2,x_2]$ in $\mathcal{G}_{LI}(X)$. If $d(x_1,x_2) = \epsilon > 0$, then $U(g_1,x_1,\epsilon) \cap U(g_2,x_2,\epsilon) = \emptyset$.

If $d(g_1x_1,g_2x_2) = \epsilon > 0$, then choose $\epsilon_i > 0$ such that $g_iB(x_i,\epsilon_i) \subseteq B(g_ix_i,\epsilon)$ for $i = 1,2$ and observe that $U(g_1,x_1,\epsilon_1) \cap U(g_2,x_2,\epsilon_2) = \emptyset$. Finally suppose $x_1 = x_2$ and $g_1x_1 = g_2x_2$. Choose $\epsilon > 0$ so that g_i is defined on $B(x_i,\epsilon)$ for $i = 1,2$ and so that $\epsilon \leq \epsilon_x$ where $\epsilon_x > 0$ comes from Lemma 6.2. Then $h = g_2^{-1}g_1\colon B(x_1,\epsilon) \to B(x_1,\epsilon)$ is an isometry so h is the identity. Hence $[g_1,x_1] = [g_2,x_2]$.

(4) implies (1). Let $x \in X$, $\epsilon > 0$ and $g\colon B(x,\epsilon) \to B(x,\epsilon)$ be an isometry such that $gx = x$. Suppose on the contrary that g does not equal the identity on a sufficiently small ball about x. Lemma 4.6 gives another isometry $\tilde{g}\colon B(x,\epsilon) \to B(x,\epsilon)$ such that $\tilde{g}x = x$, $\tilde{g}$ is non-trivial arbitrarily close to x and there exist points y arbitrarily close to x such that $\tilde{g}$ is the identity on sufficiently small balls about y. It follows that $U(\tilde{g},x,\epsilon_1) \cap U(\mathrm{id},x,\epsilon_2) \neq \emptyset$ for all $\epsilon_1,\epsilon_2 > 0$, contradicting the Hausdorff property of $\mathcal{G}_{LI}(X)$. □

Remark 6.4 If X is a locally rigid ultrametric space, then X is an open and closed subset of $\mathcal{G}_{LI}(X)$. To see this, recall that the embedding $\alpha\colon X \to \mathcal{G}_{LI}(X)$ is given by $\alpha(x) = [\mathrm{id},x]$. If $[g,x] \in \mathcal{G}_{LI}(X)$ is not in the image of α, then $[g,x] \neq [\mathrm{id},x]$. Local rigidity, in particular Theorem 6.3 (2), implies that $gx \neq x$. If $0 < \epsilon < \frac{1}{2}d(x,gx)$, then $U(g,x,\epsilon) \cap \alpha(X) = \emptyset$. This shows that $\alpha(X)$ is closed in $\mathcal{G}_{LI}(X)$. It is open by Remark 3.12.

Example 6.5 1. The end spaces of the following trees are not locally rigid: the Cantor tree C, the n regular tree R_n ($n \geq 2$), and the n-ary tree A_n

$(n \geq 2)$.

2. The end spaces of the following trees are locally rigid: the Fibonacci tree F and the Sturmian tree S.

3. The end space of the irrational tree T_α is locally rigid if and only if α is equivalent to the golden mean $\frac{1+\sqrt{5}}{2}$ under the action of $SL(2,\mathbb{Z})$ by fractional linear transformations (because this condition is equivalent to the continued fraction expansion of α eventually ending in all 1's).

Lemma 6.6 *If X is a compact ultrametric space, then X is locally rigid if and only if there exists $\epsilon_X > 0$ such that for any $0 < \epsilon \leq \epsilon_X$ and $x \in X$, every isometry $h\colon B(x,\epsilon) \to B(x,\epsilon)$ is the identity.*

Proof. Assume X is locally rigid. For each $x \in X$ let $\epsilon_x > 0$ be given by Definition 6.1 and let ϵ_X be a Lebesgue number for $\{B(x,\epsilon_x) \mid x \in X\}$. Then if $0 < \epsilon \leq \epsilon_X$ and $x \in X$, there exists $y \in X$ such that $B(x,\epsilon) \subseteq B(x,\epsilon_X) \subseteq B(y,\epsilon_y)$. Thus, $\epsilon \leq \epsilon_y$ and $B(x,\epsilon) = B(y,\epsilon)$, so any isometry $h\colon B(x,\epsilon) \to B(x,\epsilon)$ is the identity.

The converse is obvious. □

Proposition 6.7 *If X is a second countable, locally rigid ultrametric space, then $\mathcal{G}_{LI}(X)$ is second countable.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a countable dense subset of X. For each i, choose the least positive integer i_0 such that if $j \geq i_0$ and $g\colon B(x_i,1/j) \to B(x_i,1/j)$ is an isometry, then $g = \mathrm{id}_{B(x_i,1/j)}$. Now suppose $j \geq i_0$ and $g,h\colon B(x_i,1/j) \to X$ are two different isometric embeddings. Then $B(gx_i,1/j) \neq B(hx_i,1/j)$ (for otherwise $g^{-1}h\colon B(x_i,1/j) \to B(x_i,1/j)$ would be a nontrivial isometry) and, hence, $B(gx_i,1/j) \cap B(hx_i,1/j) = \emptyset$. It follows that for each i and each $j \geq i_0$ there are at most countably many distinct isometric embeddings, say $g_{(i,j,k)}\colon B(x_i,1/j) \to X$, $1 \leq k < N_{(i,j)}$, where $N_{(i,j)}$ is either a positive integer or ∞. The proof will be complete once we show that

$$\mathcal{B} = \{U(g_{(i,j,k)}, x_i, 1/j) \mid 1 \leq i < \infty,\ i_0 \leq j < \infty,\ 1 \leq k < N_{(i,j)}\}$$

is a countable basis for $\mathcal{G}_{LI}(X)$. Given $U(g,x,\epsilon)$, we show that $U(g,x,\epsilon)$ is a union of elements of $\mathcal{B}$. If $y \in B(x,\epsilon)$, let $0 < 1/n \leq \epsilon$ be chosen such that any self-isometry of $B(y,1/n)$ is the identity. Then there exists $x_i \in B(y,1/n)$. Thus, $B(x_i,1/n) = B(y,1/n)$ and by the choice of i_0, $i_0 \leq n$. It follows that $U(g,x_i,1/n) \in \mathcal{B}$ and $[g,y] \in U(g,x_i,1/n) \subseteq U(g,x,\epsilon)$. Finally, note that if $U(g_{(i,j,k)},x_i,1/j), U(g_{(i',j',k')},x_{i'},1/j') \in \mathcal{B}$, $j' \geq j$ and their intersection is nonempty, then $B(x_{i'},1/j') \subseteq B(x_i,1/j)$ and $g_{(i',j',k')} = g_{(i,j,k)}|B(x_{i'},1/j')$ (because they must agree somewhere, hence, they agree everywhere on their common domain). Hence, the intersection is $U(g_{(i',j',k')},x_{i'},1/j')$. $\square$

The following two corollaries follow from Theorem 6.3, Proposition 6.7, Lemmas 3.5 and 3.6, and Remark 3.11.

Corollary 6.8 *If X is a locally compact, second countable, locally rigid ultrametric space, then $\mathcal{G}_{LI}(X)$ is a locally compact, second countable, Hausdorff étale groupoid.*

Corollary 6.9 *If X is a compact, locally rigid ultrametric space, then $\mathcal{G}_{LI}(X)$ is a locally compact, second countable, Hausdorff étale groupoid.*

Finally, we establish the following two results that complete the proof of Theorem 1.2(1) of the Introduction.

Lemma 6.10 *If X is a compact, locally rigid ultrametric space, then $\mathcal{G}_{LS}(X)$ is second countable.*

Proof. Use the first part of the proof of Proposition 6.7 to find a countable basis $\{B_i\}_{i=1}^{\infty}$ of X by open balls, each of which admits only countably many distinct isometric embeddings into X. Assume that for every $\epsilon > 0$, there are only finitely many i's with $diam(B_i) > \epsilon$. Because the distance set of X is countable (see [5] and the proof of Proposition 10.7), there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of positive numbers such that if $g\colon\ B_i \to g(B_i)$ is a similarity onto some open ball in X and $x \in B_i$, then $sim(g,x) = \lambda_j$ for some j; i.e., g is a λ_j-similarity (the λ_j's are all ratios of distances in X).

Now if $g\colon B_i \to g(B_i)$ and $h\colon B_i \to h(B_i)$ are two λ_j-similarities such that $g(B_i) \cap h(B_i) \neq \emptyset$, then $g(B_i) = h(B_i)$ and $h^{-1}g\colon B_i \to B_i$ is an isometry. If the radius of B_i is sufficiently small, then local rigidity implies $g = h$. Hence, there exist only countably many distinct similarities of B_i onto open balls of X, say $g_{i,k}$, where $1 \leq k < N(i)$ and $N(i) \leq \infty$.

Choose $x_i \in B_i$ for all i. The proof will be complete once we show that

$$\mathcal{B} = \{U(g_{(i,k)}, x_i, diam(B_i)) \mid 1 \leq i < \infty,\ 1 \leq k < N(i)\}$$

is a countable basis for $\mathcal{G}_{LS}(X)$. Given a basis element $U(g, x, \epsilon)$ of $\mathcal{G}_{LS}(X)$, $B(x, \epsilon)$ can be written as a union of B_i's. If $B_i \subseteq B(x, \epsilon)$, then $g|B_i = g_{i,k}$ for some k. It follows that $U(g, x, \epsilon)$ is a union of elements of $\mathcal{B}$. □

Lemma 6.11 *If X is a compact, locally rigid ultrametric space, then $\mathcal{G}_{LS}(X)$ is Hausdorff.*

Proof. This is very similar to the proof of Theorem 6.3 above and Theorem 6.15 below. Let $[g_1, x_1] \neq [g_2, x_2]$ in $\mathcal{G}_{LS}(X)$. It is easy to reduce to the case that $x_1 = x_2$ and $g_1x_1 = g_2x_2$. If $sim(g_1, x_1) = sim(g_2, x_2)$, then $g_2^{-1}g_1\colon B(x_1, \epsilon) \to B(x_1, \epsilon)$ is an isometry for some $\epsilon > 0$. Local rigidity implies that $g_2^{-1}g_1$ is the identity; hence, $[g_1, x_1] = [g_2, x_2]$. If $sim(g_1, x_1) \neq sim(g_2, x_2)$, then $U(g_1, x_1, \epsilon) \cap U(g_2, x_2, \epsilon) = \emptyset$ for some sufficiently small $\epsilon > 0$. □

Corollary 6.12 *If X is a compact, locally rigid ultrametric space, then $\mathcal{G}_{LS}(X)$ is a locally compact, Hausdorff, second countable, étale groupoid.*

Proof. This follows from Lemmas 3.5, 3.6, 6.10, and 6.11. □

6.2 Locally rigid actions

Definition 6.13 Let X be a metric space with a subgroup $\Gamma \leq LS(X)$. The action of Γ on X is *locally rigid* if for every $x \in X$ and for every $g \in \Gamma_x$ such that $sim(g, x) = 1$, there exists $\epsilon > 0$ such that $g \in \Gamma_y$ for all $y \in B(x, \epsilon)$.

Note that if Γ acts locally rigidly on X and H is a subgroup of Γ, then H also acts locally rigidly on X.

Lemma 6.14 *Let X be a metric space with a subgroup $\Gamma \leq LS(X)$. The following are equivalent:*

1. *The action of Γ on X is locally rigid.*
2. *For every $x \in X$ and for every $g, h \in \Gamma_x$ such that $sim(g, x) = sim(h, x)$, there exists $\epsilon > 0$ such that $gy = hy$ for every $y \in B(x, \epsilon)$.*
3. *For every $x \in X$ and for every $g, h \in \Gamma$ such that $gx = hx$ and $sim(g, x) = sim(h, x)$, there exists $\epsilon > 0$ such that $gy = hy$ for every $y \in B(x, \epsilon)$.*

Proof. (1) implies (2): Let $x \in X$ and $g, h \in \Gamma_x$ such that $sim(g, x) = sim(h, x)$ be given. Then $h^{-1}g \in \Gamma_x$ and $sim(h^{-1}g, x) = 1$. Since the action of Γ on X is assumed to be locally rigid, there exists $\epsilon > 0$ such that $h^{-1}g \in \Gamma_y$ for all $y \in B(x, \epsilon)$; i.e., $gy = hy$ for all $y \in B(x, \epsilon)$.

(2) implies (3): Let $x \in X$ and $g, h \in \Gamma$ such that $gx = hx$ and $sim(g, x) = sim(h, x)$ be given. Then $h^{-1}g \in \Gamma_x$ and $sim(h^{-1}g, x) = 1 = sim(\mathrm{id}_X, x)$. Hence, there exists $\epsilon > 0$ such that $h^{-1}gy = y$ for all $y \in B(x, \epsilon)$; i.e., $gy = hy$ for all $y \in B(x, \epsilon)$.

(3) implies (1): Let $x \in X$ and $g \in \Gamma_x$ such that $sim(g, x) = 1$ be given. Since $sim(g, x) = sim(\mathrm{id}_X, x)$, the result is obvious. □

Theorem 6.15 *Let X be an ultrametric space.*

1. *$\mathcal{G}_{LS}(X)$ is Hausdorff if and only if for every $x \in X$ and for every $[g, x], [h, x] \in \mathcal{G}_{LS}(X)$ such that $gx = hx$ and $sim(g, x) = sim(h, x)$, it follows that $[g, x] = [h, x]$.*
2. *If $\Gamma \leq LS(X)$ and Γ acts locally rigidly on X, then $\mathcal{G}_\Gamma(X)$ is Hausdorff.*

Proof. (1) Assume first that $\mathcal{G}_{LS}(X)$ is Hausdorff, and let $x \in X$ and $[g, x], [h, x] \in \mathcal{G}_{LS}(X)$ such that $gx = hx$ and $sim(g, x) = sim(h, x)$ be given. Suppose on the contrary that $[g, x] \neq [h, x]$. Choose $\epsilon > 0$ so that $h^{-1}g$ is an isometry on $B(x, \epsilon)$. Since $[h^{-1}g, x] \neq [\mathrm{id}_X, x]$, $h^{-1}g$ is non-trivial arbitrarily close. Hence, Lemma 4.6 implies that there exists an isometry $\tilde{g}\colon B(x, \epsilon) \to B(x, \epsilon)$ such that $\tilde{g}(x) \in \Gamma_x$, $\tilde{g}$ is non-trivial arbitrarily close to x, and for every $\delta > 0$, $\delta \leq \epsilon$, there exists $y \in B(x, \delta)$ and $\mu > 0$ such that $\tilde{g}|\colon B(y, \mu) \to B(y, \mu)$ is the identity. It follows that $[\tilde{g}, x]$ and $\mathrm{id}_X, x]$ can not be separated by open sets in $\mathcal{G}_{LS}(X)$.

Conversely, let $[g, x], [h, y] \in \mathcal{G}_{LS}(X)$ be given. If $x \neq y$, choose $\epsilon > 0$ with $\epsilon \leq d(x, y)$; it is easy to see that $U(gx, \epsilon) \cap U(h, y, \epsilon) = \emptyset$. If $gx \neq hy$, choose $\epsilon > 0$ such that $g(B(x, \epsilon)) \cap h(B(y, \epsilon)) = \emptyset$; it is easy to see that $U(gx, \epsilon) \cap U(h, y, \epsilon) = \emptyset$. If $sim(g, x) \neq sim(h, y)$ choose $\epsilon > 0$ such that $sim(g, z) = sim(g, x)$ for all $z \in B(x, \epsilon)$ and $sim(h, z) = sim(h, y)$ for all $z \in B(y, \epsilon)$; it is easy to see that $U(gx, \epsilon) \cap U(h, y, \epsilon) = \emptyset$. [In each of these three cases, ϵ must be chosen so small that the germs are represented on ϵ-balls.] Thus, we are left with the case that $x = y$, $gx = hx$ and $sim(g, x) = sim(h, x)$. Since the assumption in this case is that $[g, x] = [h, x]$, there is nothing to separate.

(2) Let $[g, x], [h, y] \in \mathcal{G}_\Gamma(X)$ be given. As in the proof just given, it is easy to reduce to the case that $x = y$, $gx = hx$ and $sim(g, x) = sim(h, x)$. The assumption that Γ is acting locally rigidly implies $h^{-1}g = \mathrm{id}$ near x; that is, $[g, x] = [h, x]$. $\square$

The converse of Theorem 6.15(2) need not hold as the next example shows.

Example 6.16 Let $X = \{x_\infty, x_{a0}, x_{a1}, x_{a2}, \ldots \mid a \in \{0, 1\}\}$ with ultrametric d given by $d(x_{ai}, x_{aj}) = e^{-\min\{i,j\}}$ if $i \neq j$ and $a \in \{0, 1\}$, and $d(x_\infty, x_{0i}) = d(x_\infty, x_{1i}) = d(x_{0i}, x_{1i}) = e^{-i}$ for $i = 0, 1, 2, \ldots$. The space X is the end space of the tree in Figure 4. Define $g\colon X \to X$ by $gx_\infty = x_\infty$ and $gx_{ai} = x_{|a-1|i}$ for $i = 0, 1, 2, \ldots$ and $a \in \{0, 1\}$. Let Γ be the subgroup of $LS(X)$ generated by g (thus, Γ is cyclic of order 2). Note that Γ does not act locally rigidly on X even though $\mathcal{G}_\Gamma(X)$ is Hausdorff. This example also shows that finite subgroups need not act locally rigidly.

Theorem 6.17 *If X is an ultrametric space, then the following are equivalent:*

1. *X is locally rigid.*

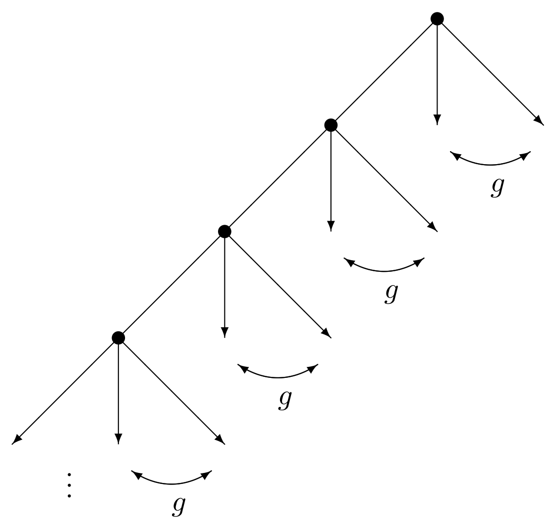

Figure 4: A finite subgroup of $LS(X)$ not acting locally rigidly. See Example 6.16

2. *$LS(X)$ acts locally rigidly on X.*

3. *Every subgroup Γ of $LS(X)$ acts locally rigidly on X.*

4. *$LI(X)$ acts locally rigidly on X.*

5. *There exists a group Γ such that $Isom(X) \leq \Gamma \leq LS(X)$ and Γ acts locally rigidly on X.*

6. *$Isom(X)$ acts locally rigidly on X.*

7. *$\mathcal{G}_{LI}(X)$ is Hausdorff.*

Proof. (1) implies (2): Let $x \in X$ and $g \in LS(X)$ such that $gx = x$ and $sim(g, x) = 1$. Since X is locally rigid, there exists $\epsilon_x > 0$ such that if $0 < \epsilon \leq \epsilon_x$ and $h\colon B(x, \epsilon) \to B(x, \epsilon)$ is an isometry, then $h = \text{id}$. Now choose $\epsilon > 0$ such that $\epsilon \leq \epsilon_x$ and $g|\colon B(x, \epsilon) \to B(x, \epsilon)$ is an isometry. Thus, $g \in \Gamma_y$ for all $y \in B(x, \epsilon)$.

That (2) implies (3) implies (4) implies (5) implies (6) is obvious from the comment made above that subgroups of groups acting locally rigidly also act locally rigidly.

(6) implies (1): Suppose on the contrary that X is not locally rigid. Using Theorem 6.3 (2), there exist $x \in X$ and a sequence $\epsilon_1 > \epsilon_2 > \epsilon_3 \cdots > 0$ such that $\lim_{i\to\infty} \epsilon_i = 0$ together with isometries $h_i \colon B(x,\epsilon_i) \to B(x,\epsilon_i)$ and $y_i \in B(x,\epsilon_i) \setminus B(x,\epsilon_{i+1})$ such that $h_i y_i \neq y_i$. Define $h \colon X \to X$ by

$$h(z) = \begin{cases} z & \text{if } z = x \text{ or } z \notin B(x,\epsilon_1) \\ h_i z & \text{if } z \in B(x,\epsilon_i) \setminus B(x,\epsilon_{i+1}) \end{cases}$$

Then $h \in Isom(X)$, $hx = x$ and h is non-trivial arbitrarily close to x, contradicting the assumption that $Isom(X)$ acts locally rigidly.

Finally, (1) and (7) are equivalent by Theorem 6.3. □

We now discuss countability properties of the germ groupoid.

Lemma 6.18 *If X is a second countable ultrametric space with a countable subgroup $\Gamma \leq LS(X)$, then the groupoid $\mathcal{G}_\Gamma(X)$ is second countable.*

Proof. Let $\Gamma = \{g_i\}_{i=1}^\infty$ and let $\{x_j\}_{j=1}^\infty$ be a countable dense subset of X. It follows that $\{U(g_i, x_j, 1/k) \mid i,j,k \in \{1,2,3,\dots\}\}$ is a countable basis for $\mathcal{G}_\Gamma(X)$. For given any basis element $U(h,x,\epsilon)$ with $h \in \Gamma$, $x \in X$ and $\epsilon > 0$, and any germ $[h,y] \in U(h,x,\epsilon)$, simply choose i,j,k such that $g_i = h$, $1/k < \epsilon$ and $x_j \in B(y, 1/k)$. Then $[h,y] \in U(g_i,x_j,1/k) \subseteq U(h,x,\epsilon)$. □

Example 6.19 There exists a compact ultrametric space X and an uncountable subgroup Γ of $Isom(X)$ such that Γ acts locally rigidly on X and $\mathcal{G}_\Gamma(X)$ is not second countable. Let X be the end space of the Cantor tree (see Example 5.7). For $s \in \{0,1\}$, let $\bar{s} = |s-1|$. For $x = (x_1,x_2,x_3,\dots) \in X$, let $\overline{x} = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \dots)$. For $x = (x_1,x_2,x_3,\dots) \in X$, define $\alpha_x \colon X \to X$ as follows. First, $\alpha_x(x) = \bar{x}$. Second, if $x \neq y = (y_1,y_2,y_3,\dots) \in X$, let $n = \min\{i \mid x_i \neq y_i\}$ and $\alpha_x(y) = (\overline{y_1}, \overline{y_2}, \dots, \overline{y_n}, y_{n+1}, y_{n+2}, \dots)$.

We show now that each α_x is an isometry. For suppose $y,z \in X$ with $y \neq z \neq x \neq y$ and let $\ell = \min\{i \mid x_i \neq y_i\}$, $m = \min\{i \mid x_i \neq z_i\}$, and $n = \min\{i \mid y_i \neq z_i\}$. We may write

$$y = (x_1, \dots, x_{\ell-1}, \overline{x_\ell}, y_{\ell+1}, y_{\ell+2}, \dots)$$

and

$$\alpha_x(y) = (\overline{x_1}, \dots, \overline{x_{\ell-1}}, x_\ell, y_{\ell+1}, y_{\ell+2}, \dots).$$

It follows that $d(x, y) = d(\alpha_x(x), \alpha_x(y))$. To see that $d(y, z) = d(\alpha_x(y), \alpha_x(z))$, assume without loss of generality that $\ell \leq m$. We may write

$$z = (x_1, \dots, x_{m-1}, \overline{x_m}, z_{m+1}, z_{m+2}, \dots)$$

and

$$\alpha_x(z) = (\overline{x_1}, \dots, \overline{x_{m-1}}, x_m, z_{m+1}, z_{m+2}, \dots).$$

It follows that $n \geq \ell$. If $n = \ell$, then

$$z = (x_1, \dots, x_{\ell-1}, x_\ell, \dots, x_{m-1}, \overline{x_m}, z_{m+1}, z_{m+2}, \dots)$$

and

$$\alpha_x(z) = (\overline{x_1}, \dots, \overline{x_\ell}, \dots \overline{x_{m-1}}, x_m, z_{m+1}, z_{m+2}, \dots)$$

from which it follows that $d(y, z) = d(\alpha_x(y), \alpha_x(z))$. If $n > \ell$, then $m = \ell$ (from the ultrametric property) and we may write

$$z = (x_1, \dots, x_{\ell-1}, \overline{x_\ell}, y_{\ell+1}, \dots, y_{n-1}, \overline{y_n}, z_{n+1}, z_{n+2}, \dots)$$

and

$$\alpha_x(z) = (\overline{x_1}, \dots, \overline{x_\ell - 1}, x_\ell, y_{\ell+1}, \dots, y_{n-1}, \overline{y_n}, z_{n+1}, z_{n+2}, \dots)$$

from which it follows that $d(y, z) = d(\alpha_x(y), \alpha_x(z))$. Finally, to see that α_x is bijective, note that the inverse of α_x is given by $\alpha_x^{-1} = \alpha_{\overline{x}}$.

Let Γ be the subgroup of $Isom(X)$ generated by $\{\alpha_x \mid x \in X\}$. Clearly, Γ is uncountable.

We will now show that Γ acts locally rigidly on X. Let $x \in X$ and $\alpha \in \Gamma$ be given. Write $\alpha = \alpha_k \circ \cdots \alpha_1$, where for each $1 \leq j \leq k$ there exists $x^j \in X$ such that $\alpha_j = \alpha_{x^j}$. Let $a^0 = x$ and $a^j = \alpha_j \circ \cdots \alpha_1(x)$ for $1 \leq j \leq k$. Let $J = \{j \in \{1, \dots, k\} \mid x^j \neq a^{j-1}\}$ and $J' = \{j \in \{1, \dots, k\} \mid j \notin J\}$. For each $1 \leq j \leq k$, let

$$P_j = \begin{cases} \min\{i \mid x_i^j \neq a_i^{j-1}\} & \text{if } j \in J \\ \infty & \text{if } j \in J'. \end{cases}$$

Let

$$P = \begin{cases} \max\{P_j \mid j \in J\} & \text{if } J \neq \emptyset \\ 1 & \text{if } J = \emptyset. \end{cases}$$

Now let $y \in X$ be any point such that $y \neq x$ and such that if

$$N = \max\{i \mid x_i \neq y_i\},$$

then $N > P$. Suppose $\alpha(x) = x$. We will prove that Γ acts locally rigidly by showing $\alpha(y) = y$. We may write

$$y = (x_1, \dots, x_{N-1}, \overline{x_N}, y_{N+1}, y_{N+2}, \dots).$$

Let $b^0 = y$ and $b^j = \alpha_j \circ \cdots \circ \alpha_1(y)$ for $1 \leq j \leq k$. Since Γ acts by isometries on X, we have $a_i^j = b_i^j$, for $1 \leq i \leq N-1$, and $a_N^j \neq b_N^j$, whenever $0 \leq j \leq k$. In particular, $(\alpha y)_i = y_i$ for $1 \leq i \leq N$. We will therefore be done once we establish the

Claim 6.20 *For each $j \in \{0, \dots, k\}$, $b_i^j = y_i$ for $i \geq N+1$.*

Proof. This is true for $j = 0$, so we proceed by induction, assuming $j > 0$ and $b_i^{j-1} = y_i$ for $i \geq N+1$.

Case 1. $j \in J$. Recall $P_j = \min\{i \mid x_i^j \neq a_i^{j-1}\} \leq N-1$. We may write

$$\begin{aligned} b^{j-1} &= (a_1^{j-1}, \dots, a_{N-1}^{j-1}, \overline{a_N^{j-1}}, y_{N+1}, y_{N+2}, y_{N+3}, \dots) \\ &= (x_1^j, \dots, x_{P_j-1}^j, \overline{x_{P_j}^j}, a_{P_j+1}^{j-1}, \dots, a_{N-1}^{j-1}, \overline{a_N^{j-1}}, y_{N+1}, y_{N+2}, y_{N+3}, \dots). \end{aligned}$$

Thus,

$$b^j = \alpha_j(b^{j-1}) = (\overline{x_1^j}, \dots, \overline{x_{P_j-1}^j}, x_{P_j}^j, a_{P_j+1}^{j-1}, \dots, a_{N-1}^{j-1}, \overline{a_N^{j-1}}, y_{N+1}, y_{N+2}, y_{N+3}, \dots),$$

and there is agreement where claimed.

Case 2. $j \in J'$. In this case $x^j = a^{j-1}$ and we may write

$$\begin{aligned} b^{j-1} &= (a_1^{j-1}, \dots, a_{N-1}^{j-1}, \overline{a_N^{j-1}}, y_{N+1}, y_{N+2}, y_{N+3}, \dots) \\ &= (x_1^j, \dots, x_{N-1}^j, \overline{x_N^j}, y_{N+1}, y_{N+2}, y_{N+3}, \dots). \end{aligned}$$

Thus,

$$b^j = \alpha_j(b^{j-1}) = (\overline{x_1^j}, \dots, \overline{x_{N-1}^j}, x_N^j, y_{N+1}, y_{N+2}, y_{N+3}, \dots),$$

and there is agreement where claimed.

This completes the proof of the claim and also the assertion that Γ acts locally rigidly on X. $\square$

We now show that $\mathcal{G}_\Gamma(X)$ is not second countable. Note that $\{U(\alpha_x, x, 1) \mid x \in X\}$ is an uncountable collection of open subsets of $\mathcal{G}_\Gamma(X)$ and the germ $[\alpha_x, x] \in U(\alpha_x, x, 1)$ for every $x \in X$. However, if $x \neq y \in X$, then $\alpha_y(y) = \overline{y}$ while $\alpha_x(y) \neq \overline{y}$. Thus, $[\alpha_y, y] \notin U(\alpha_x, x, 1)$. This implies that $\mathcal{G}_\Gamma(X)$ has no countable basis.

Finally, note that Γ does not act freely on X. For example, let $x^1 = (1,0,1,0,1,0,\dots), x^2 = (1,1,0,0,0,0,\dots), x^3 = (1,0,0,0,0,0,\dots)$ and $p = (0,0,0,0,\dots)$. Then $\alpha = \alpha_{x^3}^2 \circ \alpha_{x^2} \circ \alpha_{x^1} \in \Gamma$, $\alpha(p) = p$ and $\alpha(x^1) \neq x^1$. Hence, $\alpha \neq 1$ but it fixes the point p.

This completes the discussion of the example.

Finally, we give a proof of Theorem 1.3(1) from the Introduction.

Theorem 6.21 *If X is a compact ultrametric space with a countable subgroup $\Gamma \leq LS(X)$ acting locally rigidly on X, then $\mathcal{G}_\Gamma(X)$ is a locally compact, Hausdorff, second countable, étale groupoid.*

Proof. This follows from Lemma 3.5, Lemma 3.6, Remark 3.11, Theorem 6.15(2), and Lemma 6.18. $\square$

Remark 6.22 If in the hypothesis of Theorem 6.21, "compact" is replaced by "locally compact and second countable," then the conculsion still holds with the same proof.

Proposition 6.23 *If X is a metric space with a subgroup $\Gamma \leq LS(X)$ acting locally rigidly on X and $h \in LS(X)$, then $h^{-1}\Gamma h$ also acts locally rigidly on X.*

Proof. Let $x \in X$ and $g \in (h^{-1}\Gamma h)_x$ such that $sim(g, x) = 1$ be given. We need to show that there exists $\epsilon > 0$ such that $g|\colon B(x, \epsilon) \to X$ is the inclusion. Note that $hgh^{-1} \in \Gamma_{hx}$. To see that $sim(hgh^{-1}, hx) = 1$, choose $\delta > 0$ and $\lambda > 0$ such that $h|\colon B(x, \delta) \to B(hx, \lambda\delta)$ is a λ-similarity. Then $h^{-1}|\colon B(hx, \lambda\delta) \to B(x, \delta)$ is a λ^{-1}-similarity. Assume that δ is small enough that $g|\colon B(x, \delta) \to B(x, \delta)$ is a 1-similarity. Then $hgh^{-1}|\colon B(hx, \lambda\delta) \to B(hx, \lambda\delta)$ is a 1-similarity.

Since $hgh^{-1} \in \Gamma$, it follows that there exists $\epsilon > 0$ such that $hgh^{-1}|\colon B(hx, \epsilon) \to X$ is the inclusion, from which it follows that $g|\colon B(x, \epsilon) \to X$ is the inclusion. $\square$

7 The approximating groupoids

This section contains a proof that $\mathcal{G}_{LI}(X)$ is an AF groupoid if X is a compact, locally rigid ultrametric space.

Throughout this section, let (X, d) denote an ultrametric space.

Definition 7.1 The *pseudogroup* $\mathcal{P}_{LI}(X)$ *of local isometries on* X is the set of all isometries between open subsets of X. That is, an element of $\mathcal{P}_{LI}(X)$ consists of open subsets U, V of X and an isometry $g\colon U \to V$. [4]

Definition 7.2 Let $\epsilon > 0$. The ϵ*−local isometry groupoid* $\mathcal{G}^{\epsilon}_{LI}(X)$ *of* X is the subset of $\mathcal{P}_{LI}(X) \times X$ given by

$$\mathcal{G}^{\epsilon}_{LI}(X) = \{(g, x) \in \mathcal{P}_{LI}(X) \times X \mid g\colon B(x, \epsilon) \to B(gx, \epsilon)\}.$$

Thus, $(g, x) \in \mathcal{G}^{\epsilon}_{LI}(X)$ means g is an isometry from $B(x, \epsilon)$ onto $B(gx, \epsilon)$.

The groupoid structures on $\mathcal{G}^{\epsilon}_{LI}(X)$ are the obvious ones. Thus, the unit space is X; the domain $d\colon \mathcal{G}^{\epsilon}_{LI}(X) \to X$ and range $r\colon \mathcal{G}^{\epsilon}_{LI}(X) \to X$ maps are given by $d(g, x) = x$ and $r(g, x) = gx$. If (g_1, x_1) and (g_2, x_2) are in $\mathcal{G}^{\epsilon}_{LI}(X)$, then the composition is defined by $(g_2, x_2)(g_1, x_1) = (g_2 g_1, x_1)$ provided $x_2 = g_1 x_1$.[5] The inverse is $(g, x)^{-1} = (g^{-1}, gx)$.

[4]Of course, $\mathcal{P}_{LI}(X)$ has the structure of a pseudogroup, but we do not explicitly use it.

[5]Thus, $[\mathcal{G}^{\epsilon}_{LI}(X)]^2 = \{((g_2, x_2), (g_1, x_1)) \in \mathcal{G}^{\epsilon}_{LI}(X) \times \mathcal{G}^{\epsilon}_{LI}(X) \mid x_2 = g_1 x_1\}$.

A basis for a topology on $\mathcal{G}^{\epsilon}_{LI}(X)$ consists of all sets $U_\delta(g,x)$ where $(g,x) \in \mathcal{G}^{\epsilon}_{LI}(X)$, $0 < \delta \leq \epsilon$ and

$$U_\delta(g,x) = \{(h,y) \in \mathcal{G}^{\epsilon}_{LI}(X) \mid d(x,y) < \delta \text{ and } d(gz,hz) < \delta \text{ for all } z \in B(x,\epsilon)\}.$$

Proposition 7.3 *If X is an ultrametric space and $\epsilon > 0$, then $\mathcal{G}^{\epsilon}_{LI}(X)$ is a Hausdorff topological groupoid. Moreover, the domain and range maps are open.*

Proof. To see that the collection of all $U_\delta(g,x)$ forms a basis, first note that the collection certainly covers $\mathcal{G}^{\epsilon}_{LI}(X)$. And if $(g,x) \in U_{\delta_1}(g_1,x_1) \cap U_{\delta_2}(g_2,x_2)$, let $\delta = \min\{\delta_1,\delta_2\}$ and observe that $U_\delta(g,x) \subseteq U_{\delta_1}(g_1,x_1) \cap U_{\delta_2}(g_2,x_2)$.

To see that the resulting topology is Hausdorff, let $(g_1,x_1) \neq (g_2,x_2)$ in $\mathcal{G}^{\epsilon}_{LI}(X)$ and choose $0 < \delta \leq \epsilon$ such that

$$\delta < \begin{cases} d(x_1,x_2) & \text{if } x_1 \neq x_2 \\ \sup\{d(g_1 z, g_2 z) \mid z \in B(x,\epsilon)\} & \text{if } x_1 = x_2 \end{cases}$$

and observe that $U_\delta(g_1,x_1) \cap U_\delta(g_2,x_2) = \emptyset$.

To see that $d, r \colon \mathcal{G}^{\epsilon}_{LI}(X) \to X$ are continuous, let $x \in X$ and $0 < \delta \leq \epsilon$. Then one can check that $d^{-1}(B(x,\delta)) = \cup\{U_\delta(h,y) \mid d(x,y) < \delta\}$ and

$$r^{-1}(B(x,\delta)) = \cup\{U_\delta(h,h^{-1}y) \mid d(x,y) < \delta\}.$$

To see that d, r are open, let $(h,y) \in \mathcal{G}^{\epsilon}_{LI}(X)$ and $0 < \delta \leq \epsilon$. Then one can check that $d(U_\delta(h,y)) = B(y,\delta)$ and $r(U_\delta(h,y)) = B(hy,\delta)$.

That multiplication $m \colon [\mathcal{G}^{\epsilon}_{LI}(X)]^2 \to \mathcal{G}^{\epsilon}_{LI}(X)$ is continuous follows from the following fact: if $(h,y) \in U_\delta(g,x)$ and $m((hk^{-1},ky),(k,y)) = (h,y)$, then

$$[U_\delta(hk^{-1},ky) \times U_\delta(k,y)] \cap [\mathcal{G}^{\epsilon}_{LI}(X)]^2 \subseteq m^{-1}(U_\delta(g,x)).$$

Finally, inversion is continuous because $[U_\delta(g,x)]^{-1} = U_\delta(g^{-1},gx)$. $\square$

Theorem 7.4 *If X is a compact, locally rigid ultrametric space, then there exists $\epsilon_X > 0$ such that for every $0 < \epsilon \leq \epsilon_X$:*

1. $\mathcal{G}^{\epsilon}_{LI}(X)$ *is a Hausdorff, locally compact, étale groupoid,*

2. $\mathcal{G}^{\epsilon}_{LI}(X)$ *is an elementary groupoid in the sense of Renault [43].*

Proof. Let ϵ_X be given by Lemma 6.6.

(1) For $0 < \epsilon \leq \epsilon_X$, we already know that $\mathcal{G}^{\epsilon}_{LI}(X)$ is Hausdorff (Proposition 7.3). To say that it is étale means $r\colon \mathcal{G}^{\epsilon}_{LI}(X) \to X$ is a local homeomorphism. To verify this it suffices to let $(g, x) \in \mathcal{G}^{\epsilon}_{LI}(X)$ and show that $r|\colon U_\delta(g, x) \to B(gx, \delta)$ is injective whenever $0 < \delta \leq \epsilon$ (because the proof of Proposition 7.3 shows $r|$ is continuous, open and surjective). To this end let $(h_i, y_i) \in U_\delta(g, x)$ for $i = 1, 2$ such that $r(h_1, y_1) = h_1 y_1 = h_2 y_2 = r(h_2, y_2)$. Then $h_i\colon B(y_i, \epsilon) \to B(h_i y_i, \epsilon)$ is an isometry for $i = 1, 2$. Hence, $h = h_2^{-1} h_1\colon B(y_1, \epsilon) \to B(y_2, \epsilon) = B(y_1, \epsilon)$ is an isometry. The choice of ϵ_X implies h is the identity, so $h_1 = h_2$ from which it also follows that $y_1 = y_2$. Finally, note that this also implies that $\mathcal{G}^{\epsilon}_{LI}(X)$ is locally compact, being locally homeomorphic to the compact space X.

(2) Let $0 < \epsilon \leq \epsilon_X$. According to Renault [43, page 123] we need to show that $\mathcal{G}^{\epsilon}_{LI}(X)$ is the disjoint union of a sequence of elementary groupoids G_i of type n_i (the definitions will be recalled below). In fact, we will show that the sequence is finite, say $1 \leq i \leq i_\epsilon$. Let $\mathcal{B}_\epsilon$ be the collection of all open ϵ-balls in X. Since X is compact ultrametric, $\mathcal{B}_\epsilon$ is a finite collection and any two distinct members of $\mathcal{B}_\epsilon$ are disjoint. By the choice of ϵ_X, if $B_1, B_2 \in \mathcal{B}_\epsilon$ then either there exists a unique isometry $B_1 \to B_2$ or, B_1 and B_2 are not isometric. Thus, we may express $\mathcal{B}_\epsilon$ as a finite disjoint union $\cup_{i=1}^{i_\epsilon} \mathcal{B}_i$ such that if $1 \leq i, j \leq i_\epsilon$, $B_1 \in \mathcal{B}_i$, $B_2 \in \mathcal{B}_j$, then B_1 and B_2 are isometric if and only if $i = j$; moreover, if $i = j$, then there exists a unique isometry $B_1 \to B_2$.

For $1 \leq i \leq i_\epsilon$, let $G_i = \{(g, x) \in \mathcal{G}^{\epsilon}_{LI}(X) \mid B(x, \epsilon) \in \mathcal{B}_i\}$ and let n_i equal the cardinality of $\mathcal{B}_i$. Clearly, $\mathcal{G}^{\epsilon}_{LI}(X) = \cup_{i=1}^{i_\epsilon} G_i$ and the G_i's are mutually disjoint subgroupoids of $\mathcal{G}^{\epsilon}_{LI}(X)$. It remains to show that each G_i is elementary of type n_i. Given i choose $x_i \in X$ such that $B(x_i, \epsilon) \in \mathcal{B}_i$. Let $\hat{G}_i = \{gx_i \mid (g, x_i) \in G_i\} \subseteq X$. Note that if $(g, x_i), (h, x_i) \in G_i$ and $gx_i = hx_i$, then $g = h$.

Now $\hat{G}_i \times \hat{G}_i$ has a natural groupoid structure with set of composable pairs $[\hat{G}_i \times \hat{G}_i]^2 = \{((g_1 x_i, g_2 x_i), (g_3 x_i, g_4 x_i)) \in \hat{G}_i \times \hat{G}_i \times \hat{G}_i \times \hat{G}_i \mid g_2 x_i = g_3 x_i\}$, unit space $(\hat{G}_i \times \hat{G}_i)^0 = \hat{G}_i \subseteq X$, $d\colon \hat{G}_i \times \hat{G}_i \to \hat{G}_i$ and $r\colon \hat{G}_i \times \hat{G}_i \to \hat{G}_i$ given by

$d(gx_i, hx_i) = hx_i$ and $r(gx_i, hx_i) = gx_i$ and multiplication

$$(g_1x_i, g_2x_i) \cdot (g_2x_i, g_3x_i) = (g_1x_i, g_3x_i).$$

Clearly, $(r, d)\colon \hat{G}_i \times \hat{G}_i \to \hat{G}_i \times \hat{G}_i$ is bijective (in fact, it is the identity). This is what it means to be a transitive principal groupoid on n_i elements [43, page 6].

Now give $B(x_i, \epsilon)$ the trivial groupoid structure (that is, $B(x_i, \epsilon)$ is the unit space and (x, y) is composable if and only if $x = y$). Since X is compact, $B(x_i, \epsilon)$ is a second countable metric space. Note that G_i is isomorphic to the product $B(x_i, \epsilon) \times (\hat{G}_i \times \hat{G}_i)$ via $G_i \to B(x_i, \epsilon) \times (\hat{G}_i \times \hat{G}_i)$; $(g, x) \mapsto (h^{-1}x, ghx_i, hx_i)$, where $h\colon B(x_i, \epsilon) \to B(x, \epsilon)$ is the unique isometry. This means that G_i is an elementary groupoid of type n_i. $\square$

Remark 7.5 Under the hypothesis and notation of Theorem 7.4, note that the topology of $\mathcal{G}^{\epsilon}_{LI}(X)$ is second countable, being a finite union $\cup_{i=1}^{i_\epsilon} G_i$ and each G_i is homeomorphic to a product of $B(x_i, \epsilon)$ and a finite set. In particular, $\mathcal{G}^{\epsilon}_{LI}(X)$ is compact. In fact, since $B(x_i, \epsilon)$ is closed in X, hence compact, $\mathcal{G}^{\epsilon}_{LI}(X)$ has a countable basis of compact open sets. In fact, there is a countable basis of compact open $\mathcal{G}^{\epsilon}_{LI}(X)$-sets in the sense of Renault [43, page 10]. To see this, let $g\colon B(x_i, \epsilon) \to B(gx_i, \epsilon)$ and $h\colon B(x_i, \epsilon) \to B(hx_i, \epsilon)$ be isometries, and let $A(g, h) = \{(gh^{-1}, hy) \mid y \in B(x_i, \epsilon)\}$. These sets correspond to the images of $B(x_i, \epsilon)$ under the constructions giving a basis of compact open sets for $\mathcal{G}^{\epsilon}_{LI}(X)$. Since $d(gh^{-1}, hy) = hy$ and $r(gh^{-1}, hy) = gy$, d, r restricted to $A(g, h)$ are injective as required. This property of $\mathcal{G}^{\epsilon}_{LI}(X)$ is important when applying Renault's results on AF groupoids and AF algebras (see [43, page 130]).

Theorem 7.6 *If X is a compact, locally rigid ultrametric space, then:*

1. *$\mathcal{G}_{LI}(X)$ is an AF groupoid in the sense of Renault [43],*
2. *The groupoid C^*-algebra $C^*\mathcal{G}_{LI}(X)$ is a unital AF C^*-algebra.*

Proof. (1) First note that the unit space X of $\mathcal{G}_{LI}(X)$ is totally disconnected (since it is ultrametric). Thus, we only need to show that $\mathcal{G}_{LI}(X)$ is the inductive limit of a sequence of elementary groupoids (see [43, pages 122–123]). For this choose a sequence $\epsilon_X > \epsilon_1 > \epsilon_2 > \cdots$ such that $\lim_{i\to\infty} \epsilon_i = 0$ where ϵ_X is given by Lemma 6.6. We will observe that

$$\mathcal{G}_{LI}(X) = \lim_{\to} \mathcal{G}_{LI}^{\epsilon_i}(X).$$

First note that $\mathcal{G}_{LI}^{\epsilon_i}(X)$ is an open subgroupoid of $\mathcal{G}_{LI}(X)$ via the embedding $(g, x) \mapsto [g, x]$. For $0 < \delta \le \epsilon_i$, the embedding takes $U_\delta(g, x)$ onto $U(g, x, \epsilon_i)$. Likewise, $\mathcal{G}_{LI}^{\epsilon_i}(X)$ is an open subgroupoid of $\mathcal{G}_{LI}^{\epsilon_{i+1}}(X)$ via the embedding $(g, x) \mapsto (g|B(x, \epsilon_{i+1}), x)$. Finally observe that $\mathcal{G}_{LI}(X) = \cup_{i=1}^{\infty} \mathcal{G}_{LI}^{\epsilon_i}(X)$.

(2) That the groupoid C^*-algebra is AF follows from (1) and Renault [43, 1.15, page 134]. It is unital because X is compact. □

It should be mentioned that Renault proved that every AF C^*-algebra is the C^*-algebra of an AF groupoid and the groupoid is unique up to isomorphism [43, 1.15, page 134].

8 Trees, Bratteli diagrams and path groupoids

Let (T, v) be a rooted, geodesically complete, locally finite simplicial tree. The purpose of this section is to define a Bratteli diagram $B(T, v)$ associated with (T, v) and to prove that $\mathcal{G}_{LI}(X)$ is isomorphic to the path groupoid of $B(T, v)$, provided $X = \mathit{end}(T, v)$ is locally rigid.

8.1 Recollections on Bratteli diagrams

The material in this section is well-known. See Blackadar [8], Bratteli [9], Davidson [15], Effros [16], Elliott [18], Exel and Renault [20], Giordano, Putnam, and Skau [21], and Herman, Putnam and Skau [24] for more details. In particular, the discussion below relies heavily on the expositions in [21] and [24].

We begin with the definition of a Bratteli diagram, which, for us, comes equipped with a distinguished initial vertex.

Definition 8.1 A directed graph $D = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V}$, edge set $\mathcal{E}$, initial map $s\colon \mathcal{E} \to \mathcal{V}$, and terminal map $r\colon \mathcal{E} \to \mathcal{V}$ is a *Bratteli diagram* if

1. $\mathcal{V}$ is given as the union $\mathcal{V} = \bigcup_{n=0}^{\infty} \mathcal{V}_n$ of mutually disjoint, finite, nonempty sets $\mathcal{V}_n$,

2. for each edge $e \in \mathcal{E}$, if the initial vertex $s(e) \in \mathcal{V}_n$, then the terminal vertex $r(e) \in \mathcal{V}_{n+1}$,

3. for each vertex $v \in \mathcal{V}$ there are at most finitely many edges $e \in \mathcal{E}$ with $s(e) = v$,

4. $\mathcal{V}_0$ consists of a single vertex v_0,

5. every vertex is the initial vertex of some edge,

6. every vertex except v_0 is the terminal vertex of some edge.

For example, let (T, v) be a rooted, geodesically complete, locally finite, simplicial tree. By specifying the root, T is naturally a directed graph (edges are directed away from the root). Thus, T is a Bratteli diagram, where $\mathcal{V}_n$ consists of those vertices a distance n from v (with respect to the metric discussed in Section 5).

We now recall the construction of two invariants associated to a Bratteli diagram $D = (\mathcal{V}, \mathcal{E})$, namely, the unital dimension group $(G(D), G_+(D), [1])$ and the unital AF C^*-algebra $AF(D)$. Both of these invariants depend on a sequence of matrices, which we now define.

For each $i = 0, 1, 2, \ldots$, let $m_i = |\mathcal{V}_i|$, the cardinality of $\mathcal{V}_i$, and write $\mathcal{V}_i = \{v_1^i, \ldots, v_{m_i}^i\}$ (in particular, $v_0 = v_1^0$). For $i = 0, 1, 2, \ldots$, $1 \le k \le m_{i+1}$, and $1 \le \ell \le m_i$, let

$$a_{k\ell}^i = |\{e \in \mathcal{E} \mid s(e) = v_\ell^i \text{ and } r(e) = v_k^{i+1}\}|,$$

the number of edges from the ℓ^{th} vertex at the i^{th} level to the k^{th} vertex at the $(i+1)^{st}$ level. Thus, $A_i \colon = [a_{k\ell}^i]$ is an $(m_{i+1} \times m_i)$-matrix with nonnegative integral entries. Moreover, no column and no row of A_i consists entirely of zeroes.

The direct limit $G(D)$ of the sequence

$$\mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{m_1} \xrightarrow{A_1} \mathbb{Z}^{m_2} \xrightarrow{A_2} \mathbb{Z}^{m_3} \xrightarrow{A_3} \cdots \mathbb{Z}^{m_i} \xrightarrow{A_i} \mathbb{Z}^{m_{i+1}} \longrightarrow \cdots$$

is a partially ordered abelian group with positive cone $G_+(D)$ given by the direct limit of

$$\mathbb{Z}_+ \xrightarrow{A_0} \mathbb{Z}_+^{m_1} \xrightarrow{A_1} \mathbb{Z}_+^{m_2} \xrightarrow{A_2} \mathbb{Z}_+^{m_3} \xrightarrow{A_3} \cdots \mathbb{Z}_+^{m_i} \xrightarrow{A_i} \mathbb{Z}_+^{m_{i+1}} \longrightarrow \cdots .$$

(Here $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.)

Definition 8.2 The pair $(G(D), G_+(D))$ is the *dimension group associated to the Bratteli diagram D*. The class $[1] \in G(D)$ of the unit $1 \in \mathbb{Z}$ is an order unit[6] and the triple $(G(D), G_+(D), [1])$ is the *unital dimension group associated to the Bratteli diagram D*.

The second invariant associated to a Bratteli diagram D by using the sequence of matrices A_i is a direct limit of finite dimensional C^*-algebras (i.e., finite direct sums of matrix algebras over $\mathbb{C}$) defined as follows.

In general, let $\mathbb{M}_r$ denote the C^*-algebra of $(r \times r)$-matrices over $\mathbb{C}$. For each $v \in \mathcal{V}$, let $k(v)$ be the number of directed paths in D from v_0 to v. Let $C_0 = \mathbb{C} = \mathbb{M}_1$ and, for $i = 1, 2, 3, \dots$, let

$$C_i = \bigoplus_{j=1}^{m_i} \mathbb{M}_{k(v_j^i)}.$$

The matrices $A_i := [a_{k\ell}^i]$ defined above may be considered to be matrices of multiplicities determining unital C^*-algebra homomorphisms, also denoted A_i, $A_i \colon C_i \to C_{i+1}$ for each $i = 0, 1, 2, \dots$. Hence, there is a direct sequence

$$\mathbb{C} = C_0 \xrightarrow{A_0} C_1 \xrightarrow{A_1} C_2 \xrightarrow{A_2} C_3 \xrightarrow{A_3} \cdots C_i \xrightarrow{A_i} C_{i+1} \longrightarrow \cdots .$$

Let $\mathrm{AF}(D)$ denote the C^*-direct limit of the sequence just described. It is a unital AF algebra with unit $[1]$, the class of $1 \in \mathbb{C}$.

The two invariants of a Bratteli diagram defined above are related via K-theory. It is well-known that the K_0 group of the C^*-algebra $\mathrm{AF}(D)$ is $G(D)$; in fact, the unital, partially ordered abelian groups, $(K_0(\mathrm{AF}(D)), K_0(\mathrm{AF}(D))_+, [1])$ and $(G(D), G_+(D), [1])$, are isomorphic (see [15]).

[6] An element u of the positive cone G_+ of a partially ordered abelian group G is an *order unit* if for every $x \in G$ there exists $n \in \mathbb{N}$ such that $x \leq nu$.

Finally, we recall the equivalence relation on Bratteli diagrams that are classified by these invariants.

Definition 8.3 A *telescoping* of a Bratteli diagram $D = (\mathcal{V}, \mathcal{E})$ to a Bratteli diagram $D' = (\mathcal{V}', \mathcal{E}')$ consists of a subsequence $0 = m_0 < m_1 < m_2 < \cdots$ of $\mathbb{Z}_+$ such that

1. $\mathcal{V}'_n = \mathcal{V}_{m_n}$ for all $n = 0, 1, 2, \ldots$, and
2. if $n \in \mathbb{Z}_+$, $x \in \mathcal{V}'_n$, and $y \in \mathcal{V}'_{n+1}$, then the number of edges in D' from x to y is exactly the number of directed paths in D from x to y.

Two Bratteli diagrams $D = (\mathcal{V}, \mathcal{E})$ and $D' = (\mathcal{V}', \mathcal{E}')$ are *isomorphic* if there exists an isomorphism $\varphi\colon D \to D'$ of directed graphs such that $\varphi(\mathcal{V}_n) = \mathcal{V}'_n$ for all $n = 0, 1, 2, \ldots$. They are *equivalent* if they are equivalent under the equivalence relation generated by isomorphism and telescoping.

Two partially ordered abelian groups (G, G_+) and (G', G'_+) are *isomorphic* is there is a group isomorphism $\varphi\colon G \to G'$ such that $\varphi(G_+) = G'_+$. If in addition $u \in G$ and $u' \in G'$ are given order units and $\varphi(u) = u'$, then the unital partially ordered abelian groups (G, G_+, u) and (G', G'_+, u') are *isomorphic.*

Theorem 8.4 (Bratteli, Elliott) *For two Bratteli diagrams D, D', the following are equivalent:*

1. *D and D' are equivalent Bratteli diagrams.*
2. *$(G(D), G_+(D), [1])$ and $(G(D'), G_+(D'), [1])$ are isomorphic unital partially ordered abelian groups.*
3. *$(\mathrm{AF}(D), [1])$ and $(\mathrm{AF}(D'), [1])$ are isomorphic unital C^*-algebras.*

Moreover, $(G(D), G_+(D))$ and $(G(D'), G_+(D'))$ are isomorphic partially ordered abelian groups if and only if $\mathrm{AF}(D)$ and $\mathrm{AF}(D')$ are stably isomorphic C^*-algebras.

The equivalence of the first two conditions is due to Bratteli [9]; the equivalence of the second two, as well as the final statement, is due to Elliott [18].

8.2 The Bratteli diagram $B(T, v)$ associated to a tree (T, v)

Let (T, v) be a rooted, geodesically complete, locally finite simplicial tree. As mentioned above, the choice of root v gives an orientation to each edge of T: the edges point away from the root. Thus, (T, v) is a connected, directed graph (in fact, a Bratteli diagram). Let $s(e)$ denote the initial, and $r(e)$ the terminal, vertex of the edge e.

For notation, let $vert(T)$ be the set of vertices of T and let V_i be the set of vertices at level i. Thus,

$$V_i = \{w \in vert(T) \mid \text{the minimal simplicial path from } v \text{ to } w \text{ has length } i\}.$$

For each $w \in vert(T)$, $w \neq v$, let T_w denote the subtree of T descending from w.[7] We let T_w be rooted at w. If $w \in V_i$, then we say (T_w, w) *is a level i rooted subtree of* (T, v).

In turn, a *level one subtree of a level i rooted subtree* (T_w, w) *of* (T, v) is a level $(i+1)$ subtree (T_u, u) of (T, v) that is also a subtree of (T_w, w) (i.e., u is a vertex of T_w). For each $i = 0, 1, 2, \dots$ let m_i be the number of rooted isometry classes of level i subtrees of (T, v) and let $T^i_1, T^i_2, \dots T^i_{m_i}$ be a complete set of representatives of the rooted isometry classes. Note that $m_0 = 1$ and $T^0_1 = T$. Thus, for each level i subtree S of T there exists a unique integer l such that $1 \leq l \leq m_i$ and T^i_l is rooted isometric to S. Call these chosen subtrees the *admissible* ones.

For each $i \geq 1$ and for each level i subtree S of T, choose a rooted isometry

$$\alpha(T^i_l, S) \colon\, T^i_l \to S$$

where l is the unique integer such that $1 \leq l \leq m_i$ and T^i_l is rooted isometric to S. In choosing these isometries, insist that

$$\alpha(T^i_l, T^i_l) = \mathrm{id}_{T^i_l} \text{ for each } i \geq 1 \text{ and } 1 \leq l \leq m_i.$$

Call these chosen rooted isometries the *admissible* ones.

Define an equivalence relation $\sim$ on T as follows. For two distinct vertices w_1, w_2 of T, we have $w_1 \sim w_2$ if and only if there exists $i \geq 1$ such that $w_1, w_2 \in V_i$

[7] Thus, T_w contains all vertices and edges of T that are in directed paths beginning at w.

and T_{w_1} is rooted isometric to T_{w_2}. For two distinct edges e_1, e_2 of T, we have $e_1 \sim e_2$ if and only if each of the following hold:

1. there exists $i \geq 1$ such that $s(e_1), s(e_2) \in V_i$,
2. $T_{s(e_1)}$ is rooted isometric to $T_{s(e_2)}$ (in particular, $s(e_1) \sim s(e_2)$),
3. if l is the unique integer with $1 \leq l \leq m_i$ such that T_l^i is rooted isometric to $T_{s(e_1)}$ (which, of course, also implies T_l^i is rooted isometric to $T_{s(e_2)}$), then
$$\alpha(T_l^i, T_{s(e_1)})^{-1}(e_1) = \alpha(T_l^i, T_{s(e_2)})^{-1}(e_2)$$
as edges of T_l^i.

Let $B(T, v) = T/\sim$, which has the structure of a connected directed graph. Level i vertices of $B(T, v)$ are equivalence classes of level i vertices of (T, v), and edges of $B(T, v)$ are equivalence classes of edges of (T, v) with the induced orientation. There is an initial vertex of $B(T, v)$, namely the class $[v]$ (which consists only of v). Thus, $B(T, v)$ is a Bratteli diagram and is called *the Bratteli diagram associated to* (T, v).

Note that the quotient map $\kappa\colon\ T \to B(T, v)$ is a morphism of directed graphs.

Proposition 8.5 *The Bratteli diagram $B(T, v)$ is well-defined up to isomorphism.*

Proof. We must show that if other choices of level i subtrees $T_1^i, T_2^i, \ldots T_{m_i}^i$ and admissible isometries $\alpha(T_l^i, S)$ are made, then the resulting Bratteli diagram is isomorphic to $B(T, v)$. The vertex set $\mathcal{V}$ of $B(T, v)$, and its expression as $\mathcal{V} = \bigcup_{n=0}^{\infty} \mathcal{V}_n$, is obviously independent of the choices.

Thus, it remains to show that if w_1, w_2 are vertices of T such that $w_1 \in V_i$ and $w_2 \in V_{i+1}$ for some i, then the number of edges from $[w_1]$ to $[w_2]$ in $B(T, v)$ is independent of the choices. For this, note that the number of edges in $B(T, v)$ beginning at $[w_1]$ equals the number of edges in T_{w_1} beginning at w_1, and $e \mapsto \kappa(e)$, where e is an edge in T_{w_1} beginning at w_1, gives the bijection. Now observe that for such an edge e in T_{w_1} from w_1 to some w_3, its image $\kappa(e)$ ends at $[w_2]$ if and only if T_{w_2} is rooted isometric to T_{w_3}. $\square$

Remark 8.6 We give here an explicit description of the vertices $\mathcal{V}$ and edges $\mathcal{E}$ of the Bratteli diagram $B(T, v)$. For $i = 0, 1, 2, \ldots,$ the level i vertices can be written as a set of equivalence classes $\mathcal{V}_i = \{[v_1^i], \ldots, [v_{m_i}^i]\}$, where v_ℓ^i is the root of T_ℓ^i $(1 \leq \ell \leq m_i)$. The number of edges in $B(T, v)$ from $[v_\ell^i]$ to $[v_k^{i+1}]$, where $i = 0, 1, 2, \ldots,$ $1 \leq \ell \leq m_i$, and $1 \leq k \leq m_{i+1}$, is nonzero if and only if there exists $w \in [v_k^{i+1}]$ such that $w \in T_\ell^i$. When such a vertex w exists, the number of edges from $[v_\ell^i]$ to $[v_k^{i+1}]$ is the number of level 1 subtrees of T_ℓ^i that are rooted isometric to T_k^{i+1}.

Example 8.7 If (T, v) denotes the Cantor tree, the Fibonacci tree, the Sturmian tree, the 2-regular tree R_2, or the 3-ary tree A_3 as defined in Section 5.3, the corresponding Bratteli diagram $B(T, v)$ is pictured in Figures 5 through 9. In each case, the initial vertex appears on the far left.

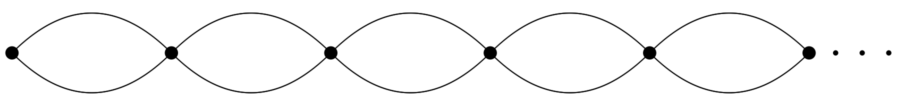

Figure 5: Bratteli diagram of the Cantor tree C

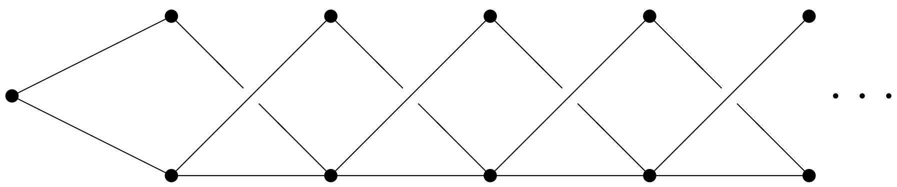

Figure 6: Bratteli diagram of the Fibonacci tree F

8.3 Recollections on path groupoids

In this section, we recall the construction of the groupoid of infinite directed paths beginning at a distinguished vertex of a directed graph. The main properties of this groupoid, due to Renault [43], are summarized in Theorem 8.8 below. For

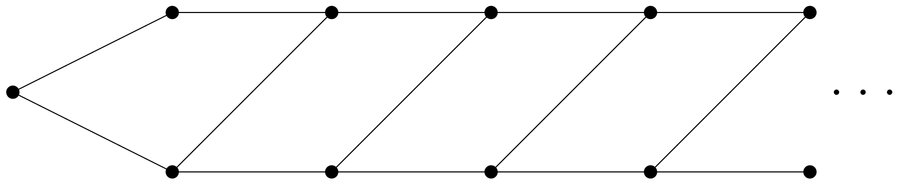

Figure 7: Bratteli diagram of the Sturmian tree S

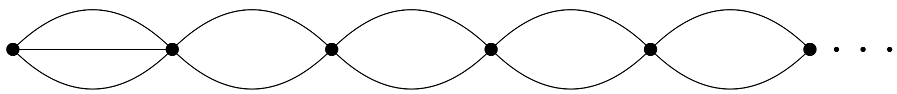

Figure 8: Bratteli diagram of the 2-regular tree R_2

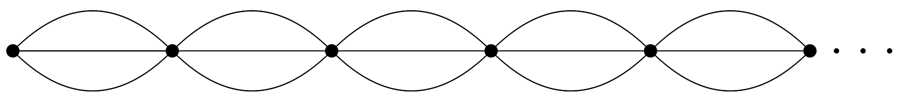

Figure 9: Bratteli diagram of the 3-ary tree A_3

more details, see Kumjian, Pask, Raeburn, and Renault [31], Paterson [42], and Renault [43].

We will only need the path groupoid of Bratteli diagrams, but it is just as easy to recall the definitions for arbitrary directed graphs.

Let $D = (\mathcal{V}, \mathcal{E})$ be a directed graph with vertex set $\mathcal{V}$, edge set $\mathcal{E}$, initial map $s\colon \mathcal{E} \to \mathcal{V}$, terminal map $r\colon \mathcal{E} \to \mathcal{V}$, and distinguished vertex v_0. Assume that for each vertex $v \in \mathcal{V}$ there are at most finitely many edges $e \in \mathcal{E}$ with initial vertex $s(e) = v$ (thus, D is *row finite*).

A *path in D beginning at v_0* is an infinite sequence $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ of edges such that $s(\alpha_0) = v_0$ and for each $n \geq 0$, $r(\alpha_n) = s(\alpha_{n+1})$. Note that by convention, our paths are infinite.

The *path groupoid* $\mathcal{PG}(D, v_0)$ is the set of all pairs (α, β) of paths in D beginning at v_0 such that α and β are *tail equivalent*, i.e., there exists $n \geq 0$ such that $\alpha_k = \beta_k$ for all $k \geq n$. Tail equivalence of α and β is denoted by $\alpha \sim \beta$. The unit space is $\mathcal{P} = \mathcal{P}(D, v_0)$, the set of all paths in D beginning at v_0. The domain $d\colon \mathcal{PG}(D, v_0) \to \mathcal{P}$ and range $r\colon \mathcal{PG}(D, v_0) \to \mathcal{P}$ maps are given by $d(\alpha, \beta) = \alpha$ and $r(\alpha, \beta) = \beta$. Pairs $(\gamma, \delta), (\alpha, \beta) \in \mathcal{PG}(D, v_0)$ are composable if and only if $\beta = \gamma$, in which case $(\gamma, \delta) \cdot (\alpha, \beta) = (\alpha, \delta)$.[8]

Observe that $\mathcal{P}$ has a natural topology; namely, consider $\mathcal{P}$ as a subspace of the countably infinite product $\prod_0^\infty \mathcal{E}$ where $\mathcal{E}$ is given the discrete topology and the product has the product topology. This makes $\mathcal{P}$ a compact, totally disconnected metric space.[9]

For example, let (T, v) be a rooted, geodesically complete, locally finite, simplicial tree considered as a directed graph as in Section 8.1. Then $\mathcal{P}(T, v) = end(T, v)$ as topological spaces. In addition, we may form the Bratteli diagram $B(T, v)$ with distinguished vertex $[v]$ associated to the tree (T, v). In this case,

[8]Thus, $\mathcal{PG}(D, v_0)$ is the groupoid associated to the equivalence relation of tail equivalence on $\mathcal{P}$.

[9]The topology on $\mathcal{P}$ is metrized by the ultrametric

$$d(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ e^{-n}, \ \text{where } n = \min\{j \mid \alpha_j \neq \beta_j\} & \text{otherwise.} \end{cases}$$

we write $\mathcal{P}(B(T,v),[v]) = \mathcal{P}(B(T,v))$ and $\mathcal{PG}(B(T,v),[v]) = \mathcal{PG}(B(T,v))$.

Returning to the general discussion, we want to put a topology on $\mathcal{PG}(D, v_0)$ so that it is a locally compact groupoid with unit space $\mathcal{P}$. It is *not* the subspace topology from $\mathcal{P}\times\mathcal{P}$ that we want, because that would not, in general, be locally compact. Instead, we procede as follows. For each $n \geq 0$, define an equivalence relation $\sim_n$ on $\mathcal{P}$ by $\alpha \sim_n \beta$ if and only if $\alpha_k = \beta_k$ for all $k \geq n$. Let

$$\mathcal{R}_n = \{(\alpha,\beta) \in \mathcal{P}\times\mathcal{P} \mid \alpha \sim_n \beta\}.$$

Thus, $\mathcal{PG}(D, v_0) = \bigcup_{n=0}^{\infty} \mathcal{R}_n$. Let each $\mathcal{R}_n$ have the subspace topology from $\mathcal{P}\times\mathcal{P}$. Each $\mathcal{R}_n$ is easily seen to be closed in $\mathcal{P}\times\mathcal{P}$; hence, $\mathcal{R}_n$ is compact. Finally, give $\mathcal{PG}(D, v_0)$ the inductive (direct) limit topology.[10] Note that for each $n \geq 0$, the quotient space $\mathcal{P}/\sim_n$ is Hausdorff. In the terminology of Exel and Lopes [19] each $\sim_n$ is a *proper equivalence relation* and tail equivalence $\sim$ is an *approximately proper equivalence relation.*

For a Bratteli diagram, there is the following result about the path groupoid.

Theorem 8.8 (Renault) *Let D be a Bratteli diagram.*

1. *The path groupoid $\mathcal{PG}(D, v_0)$ is a locally compact, Hausdorff, second countable, étale, AF groupoid.*

2. *The groupoid C^*-algebra $C^*(\mathcal{PG}(D, v_0))$ is isomorphic to $AF(D)$ as a unital C^*-algebra.*

These results can be found in Renault [43]; see Exel and Renault [20] for a recent alternative treatment. The groupoid C^*-algebra in the second statement is defined in [43].

The first statement in Theorem 8.8 above, combined with Theorem 8.9 below, gives another way of establishing the first two statements in Theorem 1.1.

[10] $U \subseteq \mathcal{PG}(D, v_0)$ is open if and only if $U \cap \mathcal{R}_n$ is open for all $n \geq 0$.

8.4 Theorems on path groupoids of Bratteli diagrams

The main result of this section is the following theorem. It concerns locally rigid end spaces of trees and relates their groupoids of local isometries to the path groupoids of the Bratteli diagram associated to the tree.

Theorem 8.9 *Let (T, v) be a rooted, geodesically complete, locally finite, simplicial tree. If $end(T, v) = X$ is locally rigid, then the quotient map $\kappa\colon T \to B(T, v)$ induces an isomorphism of groupoids*

$$\kappa_*\colon \mathcal{G}_{LI}(X) \to \mathcal{PG}(B(T, v)).$$

Proof. We first show that the quotient map $\kappa\colon T \to B(T, v)$ induces a homeomorphism $\kappa_\#\colon end(T, v) = X \to \mathcal{P}(B(T, v)) = \mathcal{P}$ between unit spaces of the groupoids $\mathcal{G}_{LI}(X)$ and $\mathcal{PG}(B(T, v))$. This part of the proof does not use the local rigidity hypothesis.

Define $\kappa_\#$ as follows. Represent $x \in end(T, v)$ (which is a geodesic ray $x\colon [0, \infty) \to T$ with $x(0) = v$) by an infinite sequence of edges $(x_0, x_1, x_2, \dots)$ of T. That is, $x_i = x([i, i+1])$ for all $i = 0, 1, 2, \dots$. Then set $\kappa_\# x = (\kappa x_0, \kappa x_1, \kappa x_2, \dots) \in \mathcal{P}$.

Note the following simple fact about the map κ.

Fact 8.10 *If e_1, e_2 are edges of T such that $e_1 \neq e_2$ and $s(e_1) = s(e_2)$, then $\kappa(e_1) \neq \kappa(e_2)$ as edges of $B(T, v)$.*

This is true because otherwise $\alpha(T^i_l, T_{s(e_1)})^{-1}(e_1) = \alpha(T^i_l, T_{s(e_1)})^{-1}(e_2)$, where l is the integer such that T^l_i is rooted isometric to $T_{s(e_1)} = T_{s(e_2)}$, contradicting the fact that $\alpha(T^i_l, T_{s(e_1)})$ is an isometry.

From this fact it follows that κ is a local homeomorphism in the sense that for all $t \in T$ there exists an open neighborhood U_t of t in T such that $\kappa|\colon U_t \to \kappa(U_t)$ is a homeomorphism. (However, $\kappa(U_t)$ need not be open in $B(T, v)$ because there might be edges $e_1 \neq e_2$ in T which go to edges in $B(T, v)$ with the same terminal vertices $r(\kappa(e_1)) = r(\kappa(e_2))$.)

To see that $\kappa_\#$ is surjective, suppose $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \mathcal{P}$. Choose an edge e_0 in T beginning at v such that $\kappa(e_0) = \alpha_0$. Then $\kappa(r(e_0)) = r(\alpha_0) = s(\alpha_1)$. By the local homeomorphism property of κ mentioned above, there exists an edge e_1 in T beginning at $r(e_0)$ such that $\kappa(e_1) = \alpha_1$. Continue this process to construct $x = (e_0, e_1, e_2, \dots) \in end(T, v)$ such that $\kappa_\# x = \alpha$.

To see that $\kappa_\#$ is injective, suppose $x \neq y$ in $end(T, v)$, and let $t_0 = \sup\{t \geq 0 \mid x(t) = y(t)\}$. Then $e_1 = x([t_0, t_0 + 1])$ and $e_2 = y([t_0, t_0 + 1])$ are distinct edges of T with $s(e_1) = s(e_2)$. It follows from Fact 8.10 that $\kappa(e_1) \neq \kappa(e_2)$. It follows that $\kappa_\# x \neq \kappa_\# y$.

Moreover,

$$\begin{aligned} t_0 &= \min\{j \in \{0, 1, 2, \dots\} \mid x(j+1) \neq y(j+1)\} \\ &= \min\{j \in \{0, 1, 2, \dots\} \mid \kappa(x([j, j+1])) \neq \kappa(y([j, j+1]))\}. \end{aligned}$$

Since $d_e(x, y) = e^{-t_0}$, it follows that $\kappa_\#$ is an isometry with respect to the natural metric on $\mathcal{P}$. This completes the proof that $\kappa_\#$ is a homeomorphism.

In order to define $\kappa_*\colon \mathcal{G}_{LI}(X) \to \mathcal{PG}(B(T, v))$, choose $\epsilon_X > 0$ by Lemma 6.6. This uses the local rigidity assumption on X. Represent a given groupoid element $[g, x] \in \mathcal{G}_{LI}(X)$ by an isometry $g\colon B(x, \epsilon) \to B(gx, \epsilon)$ with $0 < \epsilon \leq \epsilon_X$.

Claim 8.11 *$\kappa_\# x$ and $\kappa_\# gx$ are tail equivalent.*

Proof of Claim. Since

$$B(x, \epsilon) = end(T_{\langle x,\epsilon\rangle}, v_{\langle x,\epsilon\rangle}) \text{ and } B(gx, \epsilon) = end(T_{\langle gx,\epsilon\rangle}, v_{\langle gx,\epsilon\rangle}),$$

(see Section 5) the isometry g induces a rooted isometry

$$\tilde{g}\colon (T_{\langle x,\epsilon\rangle}, v_{\langle x,\epsilon\rangle}) \to (T_{\langle gx,\epsilon\rangle}, v_{\langle gx,\epsilon\rangle}).$$

This map $\tilde{g}$ is defined on edges as follows: if e is an edge of $T_{\langle x,\epsilon\rangle}$, choose $y \in end(T, v)$ such that $e = y([i, i+1])$ for some i. Then $y \in B(x, \epsilon)$ and so $gy \in B(gx, \epsilon)$. Thus, $(gy)([i, i+1])$ is an edge of $T_{\langle gx,\epsilon\rangle}$ and we set $\tilde{g}e = (gy)([i, i+1])$. Isometries $B(x, \epsilon) \to B(gx, \epsilon)$ correspond bijectively to rooted isometries $(T_{\langle x,\epsilon\rangle}, v_{\langle x,\epsilon\rangle}) \to (T_{\langle gx,\epsilon\rangle}, v_{\langle gx,\epsilon\rangle})$ (e.g., see [26]). Since $\epsilon \leq \epsilon_X$, g is the unique isometry from $B(x,\epsilon)$ to $B(gx,\epsilon)$. Thus, $\tilde{g}$ is the unique rooted isometry. It follows that if T_l^i is the admissible level i subtree of T that is rooted isometric to $T_{\langle x,\epsilon\rangle}$, and $\alpha_1\colon T_l^i \to T_{\langle x,\epsilon\rangle}$ and $\alpha_2\colon T_l^i \to T_{\langle gx,\epsilon\rangle}$, then $\tilde{g}\alpha_1 = \alpha_2$. From the definition of $\tilde{g}$, it follows that if $i \geq \lceil -\ln\epsilon\rceil$, then $\tilde{g}(x([i,i+1])) = (gx)([i,i+1])$. Thus, $x([i,i+1]) \sim (gx)([i,i+1])$ for $i \geq \lceil -\ln\epsilon\rceil$. Thus, $\kappa_\# x$ and $\kappa_\# gx$ are tail equivalent. □

Thus, define
$$\kappa_*([g,x]) = (\kappa_\# x, \kappa_\# gx) \in \mathcal{P}\times\mathcal{P}.$$
The claim implies that $(\kappa_\# x, \kappa_\# gx) \in \mathcal{PG}(B(T,v))$.

Note that $\kappa_*([g,x])$ is well-defined in the sense that it does not depend on the isometry representing $[g,x]$, only on the germ of the isometry at x (in fact, a feature of local rigidity is that $\kappa_*([g,x])$ only depends on x and gx).

Note also that the diagram
$$\begin{tikzcd} X \arrow[r, "\kappa_\#"] \arrow[d, "\alpha"'] & \mathcal{P} \arrow[d, "\Delta"] \\ \mathcal{G}_{LI}(X) \arrow[r, "\kappa_*"] & \mathcal{PG}(B(T,v)) \end{tikzcd}$$
commutes, where the vertical maps are the natural inclusions of unit spaces (α is given in Remark 3.12 and Δ is the diagonal map $\Delta(\beta,\beta)$).

It is equally obvious that the diagrams
$$\begin{tikzcd} \mathcal{G}_{LI}(X) \arrow[r, "\kappa_*"] \arrow[d, "d"'] & \mathcal{PG}(B(T,v)) \arrow[d, "d"] \\ X \arrow[r, "\kappa_\#"] & \mathcal{P} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathcal{G}_{LI}(X) \arrow[r, "\kappa_*"] \arrow[d, "r"'] & \mathcal{PG}(B(T,v)) \arrow[d, "r"] \\ X \arrow[r, "\kappa_\#"] & \mathcal{P} \end{tikzcd}$$
commute.

To see that κ_* is multiplicative, suppose $[g_1,x_1],[g_2,x_2] \in \mathcal{G}_{LI}(X)$ with $x_2 = g_1x_1$. Then $\kappa_*([g_2,x_2]\cdot[g_1,x_1]) = \kappa_*([g_2g_1,x_1]) = (\kappa_\# x_1, \kappa_\# g_2g_1x_1) = (\kappa_\# g_1x_1, \kappa_\# g_2g_1x_1)\cdot(\kappa_\# x_1, \kappa_\# g_1x_1) = \kappa_*([g_2,g_1x_1])\cdot\kappa_*([g_1,x_1])$.

It only remains to show that $\kappa_*\colon \mathcal{G}_{LI}(X) \to \mathcal{PG}(B(T,v))$ is a homeomorphism. Let $\epsilon_X > 0$ be given by Lemma 6.6 and choose a positive integer $N \geq -\ln\epsilon_X$. For $i = 0,1,2,3,\dots$, let $\epsilon_i = e^{-(N+i)}$.

As in the proof of Theorem 7.6, $\mathcal{G}_{LI}(X)$ is the union of approximating groupoids, $\mathcal{G}_{LI}(X) = \cup_{i=0}^{\infty} \mathcal{G}_{LI}^{\epsilon_i}(X)$. Elements of $\mathcal{G}_{LI}^{\epsilon_i}(X)$ are written (g, x), but when considered in $\mathcal{G}_{LI}(X)$, are written $[g, x]$.

Claim 8.12 *For every $i \geq 0$, κ_* restricts to a homeomorphism*

$$\kappa_*| \colon \mathcal{G}_{LI}^{\epsilon_i}(X) \to \mathcal{R}_{N+i}.$$

Proof. If $(g, x) \in \mathcal{G}_{LI}^{\epsilon_i}(X)$, then $g\colon B(x, \epsilon_i) \to B(gx, \epsilon_i)$ is an isometry. We first need to observe that $(\kappa_\# x, \kappa_\# gx) \in \mathcal{R}_{N+i}$. According to the proof of Claim 8.11, $x([j, j+1]) \sim (gx)([j, j+1])$ for $j \geq \lceil -\ln \epsilon_i \rceil$. Since $-\ln \epsilon_i = N + i$, we have the desired observation. This shows $\kappa_*(g, x) \in \mathcal{R}_{N+i}$.

To see that $\kappa_*|$ is continuous, let $(g, x) \in \mathcal{G}_{LI}^{\epsilon_i}(X)$ and let $\epsilon > 0$ be given. We may assume $\epsilon \leq \epsilon_i$. Recall from Section 7 that (g, x) has an open neighborhood $U_\epsilon(g, x)$ in $\mathcal{G}_{LI}^{\epsilon_i}(X)$ given by

$$U_\epsilon(g, x) = \{(h, y) \in \mathcal{G}_{LI}^{\epsilon_i}(X) \mid d(x, y) < \epsilon \text{ and } d(gz, hz) < \epsilon \text{ for every } z \in B(x, \epsilon_i\}.$$

Let $(h, y) \in U_\epsilon(g, x)$. Then $d(x, y) < \epsilon$ and $d(gy, hy) < \epsilon$. Since g is an isometry, $d(gx, gy) < \epsilon$. Hence, $d(x, y) < \epsilon$ and $d(gx, hy) < 2\epsilon$. Since it was shown above that $\kappa_\#$ is an isometry, we have $d(\kappa_\# x, \kappa_\# y) < \epsilon$ and $d(\kappa_\# gx, \kappa_\# hy) < 2\epsilon$ in $\mathcal{P}$. Thus, the distance between $\kappa_*(g, x) = (\kappa_\# x, \kappa_\# gx)$ and $\kappa_*(h, y) = (\kappa_\# y, \kappa_\# hy)$ in $\mathcal{P} \times \mathcal{P}$ is small if ϵ is small enough. Since $\mathcal{R}_{N+i}$ is topologized as a subspace of $\mathcal{P} \times \mathcal{P}$, this verifies the continuity of $\kappa_*|$.

To see that $\kappa_*|$ is injective, suppose $(g, x), (h, y) \in \mathcal{G}_{LI}^{\epsilon_i}(X)$. Since $\kappa_*(g, x) = (\kappa_\# x, \kappa_\# gx)$, $\kappa_*(h, y) = (\kappa_\# y, \kappa_\# hy)$, and $\kappa_\#$ is injective, it follows that $\kappa_*(g, x) = \kappa_*(h, y)$ implies that $x = y$ and $gx = hy$. By the choice of ϵ_X, it follows that $g = h$.

To see that $\kappa_*|$ is surjective, let $(\alpha, \beta) \in \mathcal{R}_{N+i}$ be given. Choose $x, y \in X$ such that $\kappa_\# x = \alpha$ and $\kappa_\# y = \beta$. It suffices to show there exists an isometry $g\colon B(x, \epsilon_i) \to B(y, \epsilon_i)$ such that $gx = y$; for then, $(g, x) \in \mathcal{G}_{LI}^{\epsilon_i}(X)$ and $\kappa_*(g, x) = (\alpha, \beta)$. Since $(\alpha, \beta) \in \mathcal{R}_{N+i}$, it follows that $\alpha_k = \beta_k$ for all $k \geq N + i$. Denote the sequence of edges of x by $(x_0, x_1, x_2, \dots)$ and those of y by $(y_0, y_1, y_2, \dots)$. That is, $x_k = x[k, k+1]$ and $y_k = y[k, k+1]$, where $x, y\colon [0, \infty) \to T$.

If $x_k = y+k$ for all k, then we are done because we can take $g = \mathrm{id}$. Therefore, assume this is not the case and let $M = \min\{k \mid x_k \neq y_k\}$.

Since $x_M \neq y_M$ and $s(x_M) = x(M) = y(M) = s(y_M)$, Fact 8.10 implies $\alpha_M = \kappa(x_M) \neq \kappa(y_M) = \beta_M$. Thus, $M < N + i$.

Since $\alpha_{N+i} = \beta_{N+i}$, it follows that the tree $T_{x(N+i)}$ is rooted isometric to $T_{y(N+i)}$. Let ℓ be the unique integer with $1 \leq \ell \leq m_{N+i}$ such that T_ℓ^{N+i} is rooted isometric to $T_{x(N+i)}$ and $T_{y(N+i)}$. Consider the following rooted isometry defined as a composition of an admissible isometry and the inverse of an admissible isometry:

$$\tilde{g} := \alpha(T_\ell^{N+i}, T_{y(N+i)}) \circ \alpha(T_\ell^{N+i}, T_{x(N+i)})^{-1} \colon T_{x(N+i)} \to T_{y(N+i)}.$$

Thus, $\tilde{g}(x_{N+i}) = y_{N+i}$.

It follows that $\tilde{g}$ induces an isometry $g\colon B(x, \epsilon_i) \to B(y, \epsilon_i)$. (According to Section 5, there is an isometry between closed balls, but it restricts to an isometry between open balls.) To show $gx = y$, it suffices to show that $\tilde{g}(x_k) = y_k$ for all $k \geq N + i$. Assume to the contrary that this is not the case, and let $K = \min\{k \geq N + i \mid \tilde{g}(x + k) \neq y + k\}$. Thus, $K > N + i$.

Proceed as above: since $\alpha_K = \beta_K$, it follows that $T_{x(K)}$ is rooted isometric to $T_{y(K)}$. Moreover, there exists a rooted isometry $\tilde{g}_K\colon T_{x(K)} \to T_{y(K)}$ such that $\tilde{g}_K(x_K) = y_K$, and $\tilde{g}_K$ induces an isometry $g_K\colon \overline{B}(x, e^{-K}) \to \overline{B}(y, e^{-K})$. It follows that $g(\overline{B}(x, e^{-K})) = \overline{B}(y, e^{-K})$ (because $g(x) \in \overline{B}(y, e^{-K})$). Hence, $g_K^{-1} \circ g|\colon \overline{B}(x, e^{-K}) \to \overline{B}(x, e^{-K})$ is an isometry with $g_K^{-1} \circ g(x) \neq x$. This contradicts the choice of ϵ_X. For this, we need to observe that $\overline{B}(x, e^{-K}) = B(x, \eta)$ if $e^{-K} < \eta < e^{-(K+1)}$, and any such η satisfies $\eta < e^{-(K+1)} \leq \epsilon_X$. Hence, $gx = y$ and $\kappa_*|$ is surjective.

Since $\mathcal{G}_{LI}^{\epsilon_i}(X)$ is compact Hausdorff by Remark 7.5, this shows that $\kappa_*|$ is a homeomorphism and completes the proof of Claim 8.12. $\square$

Note that the following diagram commutes, where the vertical arrows are inclusion maps:

$$\begin{array}{ccc} \mathcal{G}_{LI}^{\epsilon_i}(X) & \xrightarrow{\kappa_*|} & \mathcal{R}_{N+i} \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{G}_{LI}^{\epsilon_{i+1}}(X) & \xrightarrow{\kappa_*|} & \mathcal{R}_{N+i+1} \end{array}$$

Recall from the proof of Theorem 7.6 that $\mathcal{G}_{LI}(X)$ is the inductive limit (as $i \to \infty$) of the left-hand vertical maps. By definition, $\mathcal{R}$ is the inductive limit of the right-hand vertical maps. Hence, κ_* is a homeomorphism.

This completes the proof of Theorem 8.9. □

8.5 Summary of Section 8

Let (T, v) be a rooted, geodesically complete, locally finite simplicial tree and let $X = end(T, v)$. By examining isomorphic subtrees of T rooted at the same level of T, we defined a Bratteli diagram $B(T, v)$ that is a quotient of T, $\kappa\colon\ T \to B(T, v)$.

For any Bratteli diagram (in fact, for any rooted directed graph) there is a well-known construction of a groupoid, called the path groupoid, based on tail equivalence of infinite directed paths beginning at the distinguished vertex of the diagram. In our case, we denote the path groupoid of $B(T, v)$ by $\mathcal{PG}(B(T, v))$. This groupoid satisfies sufficient conditions so that Renault's theory can be applied to obtain a unital AF C^*-algebra $C^*\mathcal{PG}(B(T, v))$.

On the other hand, Bratteli showed how to construct a unital AF C^*-algebra from any Bratteli diagram. For the Bratteli diagram $B(T, v)$, this algebra is denoted by $\mathrm{AF}(B(T, v))$.

It is well-known that Bratteli's construction and Renault's theory lead to isomorphic unital C^*-algebras. In particular, there is an isomorphism

$$C^*\mathcal{PG}(B(T, v)) \cong \mathrm{AF}(B(T, v))$$

of unital C^*-algebras.

A unital partially ordered abelian group is obtained from (T, v) in two ways.

First, we take the unital, ordered K_0-group of a unital C^*-algebra and get

$$(K_0(C^*\mathcal{PG}(B(T,v))), K_0(C^*\mathcal{PG}(B(T,v)))_+, [1]).$$

Second, there is the unital dimension group associated to a Bratteli diagram. In particular, we get $(G(B(T,v)), G_+(B(T,v)), [1])$. Since this is the unital ordered K_0-group of $\mathrm{AF}(B(T,v))$, these two constructions lead to isomorphic unital partially ordered abelian groups. In particular, there is an isomorphism

$$(K_0(C^*\mathcal{PG}(B(T,v))), K_0(C^*\mathcal{PG}(B(T,v)))_+, [1]) \cong (G(B(T,v)), G_+(B(T,v)), [1])$$

of unital partially ordered abelian groups.

These constructions are summarized in the following diagram:

$$\begin{aligned}
(T,v) &\rightsquigarrow B(T,v)\\
&\rightsquigarrow \mathcal{PG}(B(T,v))\\
&\rightsquigarrow C^*\mathcal{PG}(B(T,v)) \cong \mathrm{AF}(B(T,v))\\
&\rightsquigarrow (K_0(C^*\mathcal{PG}(B(T,v))), K_0(C^*\mathcal{PG}(B(T,v)))_+, [1])\\
&\cong (K_0(\mathrm{AF}(B(T,v)), (K_0(\mathrm{AF}(B(T,v))_+, [1])\\
&\cong (G(B(T,v)), G_+(B(T,v)), [1])
\end{aligned}$$

Under the assumption that $X = end(T,v)$ is locally rigid, there was another route that led to groupoids, unital AF C^*-algebras, and unital partially ordered abelian groups. Namely, we formed the groupoid $\mathcal{G}_{LI}(X)$ of local isometries on X and verified sufficient conditions so that Renault's theory produces a unital AF C^*-algebra $C^*\mathcal{G}_{LI}(X)$. We can take the K_0-group of that C^*-algebra and get a unital partially ordered abelian group. This route is summarized by the following diagram:

$$\begin{aligned}
(T,v) &\rightsquigarrow end(T,v) = X\\
&\rightsquigarrow \mathcal{G}_{LI}(X)\\
&\rightsquigarrow C^*\mathcal{G}_{LI}(X)\\
&\rightsquigarrow (K_0(C^*\mathcal{G}_{LI}(X)), K_0(C^*\mathcal{G}_{LI}(X))_+, [1])
\end{aligned}$$

The main isomorphisms established in this section in the locally rigid case are summarized in the following corollary.

Corollary 8.13 *If (T, v) is a rooted, geodesically complete, locally finite simplicial tree such that $X = end(T, v)$ locally rigid, then*

1. *there is an isomorphism of topological groupoids $\mathcal{G}_{LI}(X) \cong \mathcal{PG}(B(T, v))$,*
2. *there are isomorphisms of unital C^*-algebras*

$$\begin{aligned} C^*\mathcal{G}_{LI}(X) &\cong \mathrm{AF}(B(T, v)) \\ &\cong C^*\mathcal{PG}(B(T, v)), \end{aligned}$$

3. *there are isomorphisms of unital partially ordered abelian groups*

$$\begin{aligned} &(K_0(C^*\mathcal{G}_{LI}(X)), K_0(C^*\mathcal{G}_{LI}(X))_+, [1]) \\ &\quad\cong (K_0(C^*\mathcal{PG}(B(T, v))), K_0(C^*\mathcal{PG}(B(T, v)))_+, [1]) \\ &\quad\cong (K_0(\mathrm{AF}(B(T, v)), (K_0(\mathrm{AF}(B(T, v))_+, [1]) \\ &\quad\cong (G(B(T, v)), G_+(B(T, v)), [1]). \end{aligned}$$

In particular, $B(T, v)$ is the Bratteli diagram for the unital AF algebra $C^*\mathcal{G}_{LI}(X)$.

Here is one last consequence of the results of this section.

Corollary 8.14 *If (T, v) and (S, w) are rooted, geodesically complete, locally finite simplicial trees such that $X = end(T, v)$ and $Y = end(S, w)$ are locally rigid, then the Bratteli diagrams $B(T, v)$ and $B(S, w)$ are equivalent if and only if $K_0C^*\mathcal{G}_{LI}(X)$ and $K_0C^*\mathcal{G}_{LI}(Y)$ are isomorphic as unital partially ordered abelian groups.*

Proof. By Theorem 8.4, $B(T, v)$ and $B(S, w)$ are equivalent if and only if $G(B(T, v))$ and $G(B(S, w))$ are isomorphic as unital partially ordered abelian groups. The result now follows from the isomorphisms above. □

9 The symmetry at infinity group

Let (T,v) be a rooted, geodesically complete, locally finite simplicial tree. In Section 8 we defined a Bratteli diagram $B(T,v)$ associated to (T,v) by identifying infinite subtrees of T that occur at the same level. The Bratteli diagram $B(T,v)$ leads to three isomorphic unital partially ordered abelian groups (as is the case for all Bratteli diagrams)[11] :

$$\begin{aligned}(K_0(C^*\mathcal{PG}(B(T,v))), &K_0(C^*\mathcal{PG}(B(T,v)))_+, [1])\\ &\cong (K_0(\mathrm{AF}(B(T,v)), (K_0(\mathrm{AF}(B(T,v))_+, [1])\\ &\cong (G(B(T,v)), G_+(B(T,v)), [1])\end{aligned}$$

Of these, $(G(B(T,v)), G_+(B(T,v)), [1])$ is obviously the most directly defined.

We want to point out in this section that this unital partially ordered abelian group can be defined even more directly from the tree (T,v) without passing to the Bratteli diagram $B(T,v)$. Of course, this is implicit in Section 8, but we want to make clear just how elementary the idea is. Some examples are included at the end of this section.

Since the group is defined directly from the tree (T,v) and it measures the symmetries at infinity of (T,v)—that is, the number of isometric infinite subtrees of (T,v)—we denote it by $Sym_\infty(T,v)$.

We use the notation from Section 8.2. Recall that for each $i = 0,1,2,\dots$, m_i is the number of rooted isometry classes of level i subtrees of (T,v). We choose level i subtrees $T^i_1, T^i_2, \dots T^i_{m_i}$ that form a complete set of representatives of the rooted isometry classes. In particular, $m_0 = 1$ and $T^0_1 = T$.

For $0 \le i$, $1 \le k \le m_{i+1}$ and $1 \le \ell \le m_i$, let $a^i_{k\ell}$ be the number of level one subtrees of T^i_ℓ that are rooted isometric to T^{i+1}_k. The matrix $A_i = [a^i_{k\ell}]$ is an $(m_{i+1} \times m_i)-$ matrix with nonnegative integral entries. The indexing of the rows and columns is indicated here:

[11] If $X = end(T,v)$ is locally rigid, then $(K_0(C^*\mathcal{G}_{LI}(X)), K_0(C^*\mathcal{G}_{LI}(X))_+, [1])$ is a fourth group isomorphic to these.

$$A_i = \begin{array}{cc} \begin{array}{ccccc} T_1^i & \cdots & T_\ell^i & \cdots & T_{m_i}^i \end{array} & \\ \begin{bmatrix} a_{11}^i & \cdots & a_{1\ell}^i & \cdots & a_{1m_i}^i \\ \vdots & & \vdots & & \vdots \\ a_{k1}^i & \cdots & a_{k\ell}^i & \cdots & a_{km_i}^i \\ \vdots & & \vdots & & \vdots \\ a_{m_{i+1}1}^i & \cdots & a_{m_{i+1}\ell}^i & \cdots & a_{m_{i+1}m_i}^i \end{bmatrix} & \begin{array}{c} T_1^{i+1} \\ \vdots \\ T_k^{i+1} \\ \vdots \\ T_{m_i+1}^{i+1} \end{array} \end{array}$$

The simplicial cone of $\mathbb{Z}^n$ is $\mathbb{Z}_+^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid x_i \geq 0\}$. It is a subsemigroup of $\mathbb{Z}^n$ and the resulting partial order on $\mathbb{Z}^n$ is called the *simplicial ordering.*

Definition 9.1 For a rooted, locally finite simplicial tree (T, v), using the notation above, the *symmetry at infinity group* $Sym_\infty(T, v)$ is the unital partially ordered abelian group given by the direct limit

$$Sym_\infty(T,v) = \varinjlim(\mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{m_1} \xrightarrow{A_1} \mathbb{Z}^{m_2} \xrightarrow{A_2} \mathbb{Z}^{m_3} \xrightarrow{A_3} \cdots \mathbb{Z}^{m_i} \xrightarrow{A_i} \cdots).$$

The *positive cone of* $Sym_\infty(T, v)$ is the subsemigroup given by the direct limit

$$Sym_\infty(T,v)_+ = \varinjlim(\mathbb{Z}_+ \xrightarrow{A_0} \mathbb{Z}_+^{m_1} \xrightarrow{A_1} \mathbb{Z}_+^{m_2} \xrightarrow{A_2} \mathbb{Z}_+^{m_3} \xrightarrow{A_3} \cdots \mathbb{Z}_+^{m_i} \xrightarrow{A_i} \cdots).$$

The order unit $[1]$ of $Sym_\infty(T, v)$ is the class of $1 \in \mathbb{Z}$ in the direct limit.

Thus, we simply use $Sym_\infty(T, v)$ to denote the unital partially ordered abelian group given by the triple $(Sym_\infty(T, v), Sym_\infty(T, v)_+, [1])$.

Proposition 9.2 *If (T, v) is a rooted, geodesically complete, locally finite simplicial tree, then $Sym_\infty(T, v)$ is isomorphic to the unital dimension group of the Bratteli diagram $B(T, v)$; that is, there is an isomorphism of unital partially ordered abelian groups*

$$Sym_\infty(T, v) \cong (G(B(T, v)), G_+(B(T, v)), [1]).$$

Proof. We continue to use the notation from Section 8.2. Recall that the Bratteli diagram $B(T,v) = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \cup_{i=0}^{\infty} \mathcal{V}_i$ and $\mathcal{V}_i = \{[v_1^i], \ldots, [v_{m_i}^i]\}$ where v_ℓ^i is the root of T_ℓ^i. It follows from Remark 8.6 that the number $a_{k\ell}^i$ defined above is exactly the number of edges in $\mathcal{E}$ from $[v_\ell^i]$ to $[v_k^{i+1}]$. This gives the required isomorphism of unital partially ordered abelian groups. □

The following corollary is Theorem 1.15 of the Introduction.

Corollary 9.3 *If X is a locally rigid, compact ultrametric space and $X = end(T,v)$, where (T,v) is a rooted, geodesically complete, locally finite simplicial tree, then $Sym_\infty(T,v)$ is isomorphic to $K_0 C^* \mathcal{G}_{LI}(X)$ as a unital partially ordered abelian group.*

Proof. This follows from Proposition 9.2 and Corollary 8.13. □

We now give the symmetry at infinity groups of the trees in Section 5.3. All of the calculations are elementary and are well-known (perhaps in other contexts). For each of the trees, the natural root is denoted v.

Example 9.4 (The Cantor tree C) $Sym_\infty(C,v)$ is isomorphic to the additive group of dyadic rationals $\mathbb{Z}[\frac{1}{2}] = \{\frac{m}{2^i} \mid m, i \in \mathbb{Z}\} \subseteq \mathbb{Q}$ with positive cone $Sym_\infty(C,v)_+$ corresponding to the nonnegative dyadic rationals $\mathbb{Z}[\frac{1}{2}]_+$ and order unit $1 \in \mathbb{Z}[\frac{1}{2}]$. The direct sequence is

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots .$$

Example 9.5 (The Fibonacci tree F) $Sym_\infty(F,v)$ is isomorphic to the two-dimensional integral lattice $\mathbb{Z}^2$ with positive cone $Sym_\infty(F,v)_+$ corresponding to $\{(x,y) \in \mathbb{Z}^2 \mid \tau x + y \geq 0\}$ where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean. The order unit is $(1,1) \in \mathbb{Z}^2$. Thus, $Sym_\infty(F,v) \cong (\mathbb{Z}+\tau\mathbb{Z}, (\mathbb{Z}+\tau\mathbb{Z}) \cap \mathbb{R}_+, \tau)$. The direct sequence is

$$\mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]} \mathbb{Z}^2 \xrightarrow{\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]} \cdots .$$

Example 9.6 (The Sturmian tree S) $Sym_\infty(S,v)$ is isomorphic to the two-dimensional integral lattice $\mathbb{Z}^2$ with positive cone $Sym_\infty(S,v)_+$ corresponding to

$\{(x,y) \in \mathbb{Z}^2 \mid x, y \geq 0 \text{ or } x > 0\}$. This is the lexiographic order of $\mathbb{Z}^2$. The order unit is $(1,1) \in \mathbb{Z}^2$. The direct sequence is

$$\mathbb{Z} \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&0\\1&1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&0\\1&1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&0\\1&1\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}1&0\\1&1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}1&0\\1&1\end{bmatrix}} \cdots .$$

Example 9.7 (The n-regular tree R_n) $Sym_\infty(R_n, v)$ is isomorphic to the additive group $\mathbb{Z}[\frac{1}{n}] = \{\frac{m}{n^i} \mid m, i \in \mathbb{Z}\} \subseteq \mathbb{Q}$ with positive cone $Sym_\infty(R_n, v)_+$ corresponding to the nonnegative elements of $\mathbb{Z}[\frac{1}{n}]_+$ and order unit $n+1 \in \mathbb{Z}[\frac{1}{n}]$. The direct sequence is

$$\mathbb{Z} \xrightarrow{n+1} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \cdots .$$

Example 9.8 (The n-ary tree A_n) $Sym_\infty(A_n, v)$ is isomorphic to $\mathbb{Z}[\frac{1}{n}]$ with positive cone $Sym_\infty(A_n, v)_+$ corresponding to $\mathbb{Z}[\frac{1}{n}]_+$ and order unit $1 \in \mathbb{Z}[\frac{1}{n}]$. The direct sequence is

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{n} \cdots .$$

Example 9.9 (The n-ended tree E_n) $Sym_\infty(E_n, v)$ is isomorphic to $\mathbb{Z}$ with $Sym_\infty(E_n, v)_+$ corresponding to the nonnegative integers $\mathbb{Z}_+$ and order unit $n \in \mathbb{Z}$. The direct sequence is

$$\mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{1} \cdots .$$

Example 9.10 (The irrational tree T_α) Let $\alpha = [a_0, a_1, a_2, \dots]$ be the continued fraction expansion of the positive irrational number α. $Sym_\infty(T_\alpha, v)$ is isomorphic to the two-dimensional integral lattice $\mathbb{Z}^2$ with positive cone $Sym_\infty(T_\alpha, v)_+$ corresponding to $\{(x,y) \in \mathbb{Z}^2 \mid \alpha x + y \geq 0\}$. The order unit is $(a_0, 1) \in \mathbb{Z}^2$. Thus, $Sym_\infty(T_\alpha, v) \cong (\mathbb{Z} + \alpha\mathbb{Z}, (\mathbb{Z} + \alpha\mathbb{Z}) \cap \mathbb{R}_+, a_0\alpha)$. The direct sequence is

$$\mathbb{Z} \xrightarrow{\begin{bmatrix}a_0\\1\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}a_1&1\\1&0\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}a_2&1\\1&0\end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix}a_3&1\\1&0\end{bmatrix}} \cdots .$$

For this calculation, see Effros and Shen [17].

10 Scalings and micro-scalings of ultrametrics

In this section we associate to any compact, ultrametric space X a Bratteli diagram $B(X)$. This is accomplished by taking any rooted, geodesically complete, locally finite simplicial tree (T, v) with X scale equivalent (defined below) to $end(T, v)$ and defining $B(X) = B(T, v)$. Along with scale equivalence we also introduce the notion of micro-scale equivalence of metric spaces and prove that the equivalence class of $B(X)$ as a Bratteli diagram only depends on X up to micro-scale equivalence.

Definition 10.1 Let d and d' be metrics on a set X.

1. d' is a *scaling* of d if there exists a homeomorphism $\lambda\colon [0, \infty) \to [0, \infty)$ such that $\lambda d = d'$. In this case, d and d' are said to be *scale equivalent*.
2. d' is a *micro-scaling* of d if there exist $\epsilon > 0$ and a homeomorphism $\lambda\colon [0, \infty) \to [0, \infty)$ such that $\lambda d(x, y) = d'(x, y)$ whenever $x, y \in X$ and $\min\{d(x, y), d'(x, y)\} < \epsilon$. In this case, d and d' are said to be *micro-scale equivalent*.

Note that scale equivalent or micro-scale equivalent metrics are topologically equivalent. In addition, scale equivalence and micro-scale equivalence are equivalence relations on the set of all metrics on X. Moreover, if (X, d) is an ultrametric space, then part of the conditions in the definition are unnecessary in that whenever $\lambda\colon [0, \infty) \to [0, \infty)$ is a homeomorphism, λd is also an ultrametric.

Definition 10.2 Let $h\colon X \to Y$ be a bijection between metric spaces (X, d_X) and (Y, d_Y).

1. $h\colon X \to Y$ is a *scale equivalence* if d_X and $h^* d_Y$ are scale equivalent.
2. $h\colon X \to Y$ is a *micro-scale equivalence* if d_X and $h^* d_Y$ are micro-scale equivalent.

Here $h^* d_Y$ denotes the pull-back metric, $h^* d_Y(x, y) = d_Y(hx, hy)$. Note that scale equivalences and micro-scale equivalences are necessarily homeomorphisms.

Proposition 10.3 *If $h\colon X \to Y$ is a uniform local similarity between compact metric spaces (X, d_X) and (Y, d_Y), then h is a micro-scale equivalence.*

Proof. Since X is compact, there exist $\epsilon > 0$ and $\hat{\lambda} > 0$ such that $h|\colon B(x,\epsilon) \to B(hx, \hat{\lambda}\epsilon)$ is a $\hat{\lambda}$-similarity for all $x \in X$. Thus, the homeomorphism $\lambda\colon [0,\infty) \to [0,\infty)$, defined by $\lambda(t) = \hat{\lambda}t$, shows that d_X and h^*d_Y are micro-scale equivalent (because $d_X(x,y) < \epsilon$ implies $d_Y(hx,hy) = \hat{\lambda}d_X(x,y) = (\lambda \circ d_X)(x,y)$). $\square$

In particular, local isometries between compact metric spaces are micro-scale equivalences.

Proposition 10.4 *If (X, d_X) and (Y, d_Y) are micro-scale equivalent ultrametric spaces and X is locally rigid, then Y is also locally rigid.*

Proof. Let $h\colon X \to Y$ be a bijection, $\lambda\colon [0,\infty) \to [0,\infty)$ a homeomorphism, and $\epsilon > 0$ such that $\lambda d_X(x,y) = d_Y(hx,hy)$ whenever $\min\{d_X(x,y), d_Y(hx,hy)\} < \epsilon$ and $x, y \in X$.

If $0 < \delta \leq \min\{\epsilon, \lambda^{-1}(\epsilon)\}$, then $h|\colon B(x,\delta) \to B(hx, \lambda(\delta))$ is a homeomorphism. If $0 < \mu \leq \min\{\epsilon, \lambda(\epsilon), \lambda^{-1}(\epsilon)\}$, and $g\colon B(y,\mu) \to B(y,\mu)$ is an isometry between balls in Y, then

$$B(h^{-1}y, \lambda^{-1}(\mu)) \xrightarrow{h|} B(y,\mu) \xrightarrow{g} B(y,\mu) \xrightarrow{h|^{-1}} B(h^{-1}y, \lambda^{-1}(\mu))$$

is an isometry between balls in X.

Suppose $y \in Y$ is given and let $x = h^{-1}y$. Local rigidity of the ultrametric space X implies there exists $\epsilon_x > 0$ such that for any $0 < \nu \leq \epsilon_x$, every isometry $B(x,\nu) \to B(x,\nu)$ is the identity (see Lemma 6.2).

Let $\epsilon_y = \min\{\lambda(\epsilon_x), \epsilon, \lambda(\epsilon), \lambda^{-1}(\epsilon)\}$ and suppose $g\colon B(y,\epsilon_y) \to B(y,\epsilon_y)$ is an isometry. It follows that the composition

$$B(x, \lambda^{-1}(\epsilon_y)) \xrightarrow{h|} B(y,\epsilon_y) \xrightarrow{g} B(y,\epsilon_y) \xrightarrow{h|^{-1}} B(x, \lambda^{-1}(\epsilon_y))$$

is an isometry; hence, it is the identity. Thus, g is the identity and Y is locally rigid. $\square$

Proposition 10.5 *A micro-scale equivalence $h\colon X \to Y$ of metric spaces induces an isomorphism $h_*\colon \mathcal{G}_{LI}(X) \to \mathcal{G}_{LI}(Y)$ of topological groupoids.*

Proof. Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a homeomorphism and $\epsilon > 0$ such that $\lambda d_X(x,y) = d_Y(hx,hy)$ whenever $\min\{d_X(x,y), d_Y(hx,hy)\} < \epsilon$ and $x, y \in X$.

If $0 < \delta \leq \min\{\epsilon, \lambda^{-1}(\epsilon)\}$, then $h|\colon B(x,\delta) \to B(hx,\lambda(\delta))$ is a homeomorphism. If $0 < \mu \leq \min\{\epsilon, \lambda(\epsilon), \lambda^{-1}(\epsilon)\}$, and $g\colon B(x,\mu) \to B(y,\mu)$ is an isometry between balls in X, then

$$B(hx,\lambda(\mu)) \xrightarrow{h|^{-1}} B(x,\mu) \xrightarrow{g} B(y,\mu) \xrightarrow{h} B(hy,\lambda(\mu))$$

is an isometry between balls in Y.

Thus, there is a function $h_*\colon \mathcal{G}_{LI}(X) \to \mathcal{G}_{LI}(Y)$ defined by $h_*[g,x] = [hgh^{-1}, hx]$ for each local isometry germ $[g,x] \in \mathcal{G}_{LI}(X)$ such that the domain of g has sufficiently small radius, which can be shown to be an isomorphism of groupoids. $\square$

Note that a micro-scale equivalence need not induce an isomorphism between local similarity groupoids.

The following result follows immediately from the preceding proposition. Note that we have already established that, under the hypothesis of the corollary, that $\mathcal{G}_{LI}(X)$ satisfies the conditions required to apply Renault's theory of groupoid C^*-algebras (see Theorem 6.21).

Corollary 10.6 *If (X,d) is a compact, locally rigid ultrametric space, then the unital C^*-algebra $C^*\mathcal{G}_{LI}(X)$ and the unital partially ordered abelian group $K_0C^*\mathcal{G}_{LI}(X)$ are invariants of X up to micro-scale equivalence of X.*

Note that the preceding proposition and corollary imply Theorem 1.1(3) of the Introduction.

We now begin to establish Theorem 1.1(4) of the Introduction.

Proposition 10.7 *If (X,d) is a compact ultrametric metric space, then there exists a rooted, geodesically complete, locally finite simplicial tree (T,v) such that X is scale equivalent to end(T,v).*

Proof. It is well-known that there exists a finite or infinite sequence $t_0 > t_1 > t_2 > \cdots > 0$ such that $\{d(x,y) \mid x,y \in X\} = \{0, t_0, t_1, t_2, \dots\}$. Moreover, the sequence is finite if and only if X is finite, and if X is infinite, then $\lim_{i\to\infty} t_i = 0$. See [5]. Let $\lambda\colon [0,\infty) \to [0,\infty)$ be a homeomorphism such that $\lambda(t_i) = e^{-i}$ for all i.

We may assume that X has more than one point, for otherwise the proof is trivial; hence, $(X, \lambda d)$ has diameter 1. In [26], there is constructed a rooted, geodesically complete $\mathbb{R}$-tree (T,v) such that $(X,\lambda d)$ is isometric to *end*(T,v). It follows that (X,d) is scale equivalent to *end*(T,v).

From Proposition 5.1 we know that T must be a proper $\mathbb{R}$-tree. It only remains to observe from the construction in [26], that T is in fact a locally finite simplicial tree. This is because the set of distances in $(X,\lambda d)$ is contained in $\{0, 1, e^{-1}, e^{-2}, \dots\}$. See Corollary 5.2. □

The existence of the tree in the preceding proposition allows us to make the following definition.

Definition 10.8 If X is a compact ultrametric space, then *the Bratteli diagram $B(X)$ associated to X* is defined to be the Bratteli diagram $B(T,v)$ associated to a rooted, geodesically complete, locally finite simplicial tree (T,v) such that X is scale equivalent to *end*(T,v).

The equivalence class of the Bratteli diagram $B(X)$ in the preceding definition is well-defined as the next result shows.

Theorem 10.9 *Let (T,v) and (S,w) be rooted, geodesically complete, locally finite simplicial trees. If end(T,v) and end(S,w) are micro-scale equivalent, then $B(T,v)$ and $B(S,w)$ are equivalent Bratteli diagrams.*

Proof. Let $X = \mathit{end}(T,v)$ and $Y = \mathit{end}(S,w)$. Since X and Y are micro-scale equivalent, there are homeomorphisms $h\colon X \to Y$ and $\lambda\colon [0,\infty) \to [0,\infty)$ and $\epsilon > 0$ such that $\lambda d_X(x,y) = d_Y(hx,hy)$ whenever $\min\{d_X(x,y), d_Y(hx,hy)\} < \epsilon$ and $x,y \in X$. We assume $0 < \epsilon < 1$. Fix $M > \max\{-\ln\epsilon, -\ln\lambda^{-1}(1)\}$. Define $\hat{h}\colon T \setminus \bar{B}(v,M) \to S \setminus \bar{B}(w, -\ln\lambda(e^{-M}))$ as follows. Let $x \in X$ and $M < t < \infty$; thus, $x(t) \in T \setminus \bar{B}(v,M)$. Set $\hat{h}(x(t)) = (h(x))(-\ln\lambda(e^{-t}))$.

Claim 10.10 *$\hat{h}$ is a homeomorphism.*

Proof of Claim. We begin by showing that $\hat{h}$ is well-defined. For $x, y \in X$ and $M < t < \infty$ such that $x(t) = y(t)$, we must show that $(h(x))(-\ln\lambda(e^{-t})) = (h(y))(-\ln\lambda(e^{-t}))$. Assuming $x \neq y$, $d_X(x,y) = e^{-t_0}$, where $t_0 = \sup\{s \geq 0 \mid x(s) = y(s)\}$. Thus, $t_0 \geq t \geq -\ln\epsilon$, which implies $d_X(x,y) \leq e^{-t} < \epsilon$. Hence, $\lambda(e^{-t_0}) = \lambda d_X(x,y) = d_Y(hx,hy) = e^{-t_1}$, where $t_1 = \sup\{s \geq 0 \mid (hx)(s) = (hy)(s)\}$. It follows that $t_1 = -\ln\lambda(e^{-t_0})$. Since $t \leq t_0$, we have $-\ln\lambda(e^{-t}) \leq -\ln\lambda(e^{-t_0}) = t_1$. The definition of t_1 and this inequality imply that $(h(x))(-\ln\lambda(e^{-t})) = (h(y))(-\ln\lambda(e^{-t}))$. Therefore, $\hat{h}$ is well-defined.

To see that $\hat{h}$ is bijective, define $g\colon S \setminus \bar{B}(w, -\ln\lambda(e^{-M})) \to T \setminus \bar{B}(v, M)$ by $g(y(s)) = (h^{-1}(y))(-\ln\lambda^{-1}(e^{-s}))$ for $y \in Y$ and $-\ln\lambda(e^{-M}) < s < \infty$. It can be checked that g is well-defined and $g = (\hat{h})^{-1}$.

We now proceed to show that $\hat{h}$ is continuous. Suppose first that $x \in X$ and $M < s < t < \infty$, so that in T, $d(x(s), x(t)) = t - s$. In S, $d(\hat{h}(x(s)), \hat{(}x(t))) = -\ln\lambda(e^{-t}) + \ln\lambda(e^{-s}) = \ln\left(\frac{\lambda(e^{-s})}{\lambda(e^{-t})}\right)$. The continuity of $\hat{h}$ on $x((M,\infty))$ follows from this.

Now suppose $x, y \in X$, $x \neq y$ and let $d_X(x,y) = e^{-t_0}$. Further suppose $M < t_0 < s \leq t < \infty$. In T, $d(x(s), y(t)) = s + t - 2t_0$. In S, $d(\hat{h}(x(s)), \hat{h}(y(t)) = -\ln\lambda(e^{-s}) - \ln\lambda(e^{-t}) + 2\ln\lambda(e^{-t_0}) = \ln\left(\frac{[\lambda(e^{-t_0})]^2}{\lambda(e^{-s})\lambda(e^{-t})}\right)$. If $d(x(s), y(t))$ is small, then s and t are both close to t_0; hence, $d(\hat{h}(x(s)), \hat{h}(y(t))$ is small.

This now establishes that $\hat{h}$ is continuous on connected components of $T \setminus \bar{B}(v, M)$; thus, $\hat{h}$ is continuous. Likewise, g is continuous and $\hat{h}$ is a homeomorphism. This completes the proof of the claim. □

Suppose $n > M$ is an integer and $x, y \in X$. Let T_x and T_y be the rooted subtrees of T descending from $x(n)$ and $y(n)$, respectively. Let S_x and S_y be the rooted subtrees of S descending from $\hat{h}(x(n))$ and $\hat{h}(y(n))$, respectively.

Claim 10.11 *$(T_x, x(n))$ and $(T_y, y(n))$ are rooted isometric if and only if $(S_x, \hat{h}(x(n)))$ and $(S_y, \hat{h}(y(n)))$ are rooted isometric.*

Proof of Claim. We suppress the roots of the subtrees from the notation. Suppose T_x and T_y are rooted isometric. Then $end(T_x)$ and $end(T_y)$ are isometric when these end spaces are given the end space metric as recalled in Section 5.2 (see Proposition 5.6). However, we want to give $end(T_x)$ and $end(T_y)$ the metrics they inherit as subspaces of $end(T,v)$—that is, under the identifications $end(T_x) = \bar{B}(x, e^{-n}) \subseteq end(T,v)$, and $end(T_y) = \bar{B}(y, e^{-n}) \subseteq end(T,v)$. Since the pairs of possible metrics differ by a factor of e^{-n}, $end(T_x)$ and $end(T_y)$ remain isometric with the subspace metrics. Let $j\colon \bar{B}(x, e^{-n}) \to \bar{B}(y, e^{-n})$ be an isometry. Then $\hat{h}j(\hat{h}|)^{-1}\colon \bar{B}(h(x), \lambda(e^{-n})) \to \bar{B}(h(y), \lambda(e^{-n}))$ is also an isometry. This means that $end(S_x)$ and $end(S_y)$ are isometric as subspaces of $end(S,w)$. As above, we conclude that S_x and S_y are rooted isometric. Similar reasoning gives the converse. This completes the proof of the claim. □

We can now complete the proof that $B(T,v)$ and $B(S,w)$ are equivalent. We will show that $Sym_\infty(T,v) \cong Sym_\infty(S,w)$. Since these groups are isomorphic to the unital dimension groups of $B(T,v)$ and $B(S,w)$, respectively, (by Proposition 9.2), it follows from Bratteli's Theorem 8.4 that $B(T,v)$ and $B(S,w)$ are isomorphic.

Let $D_X = \{t \in \mathbb{R} \mid \text{ there exists } x, y \in X \text{ such that } d_X(x,y) = t\}$ and $D_Y = \{t \in \mathbb{R} \mid \text{ there exists } x, y \in Y \text{ such that } d_Y(x,y) = t\}$, the distance sets of X and Y, respectively. Write $D_X = \{0 < \cdots < \nu_{i+1} < \nu_i < \cdots < \nu_0 \le 1\}$. Then $D_Y = \{0 < \cdots < \lambda(\nu_{i+1}) < \lambda(\nu_i) < \cdots < \lambda(\nu_0) \le 1\}$. For each $i = 0, 1, 2, \ldots$, let $L_i = -\ln \nu_i$ and $M_i = -\ln \lambda(\nu_i)$. Note that $0 \le L_0 < L_1 < L_2 < \cdots$ and $0 \le M_0 < M_1 < M_2 < \cdots$. The L_i's and M_i's correspond to the levels in the trees T and S, respectively, where nontrivial branching occurs.

For each $i = 0, 1, 2, \ldots$, let μ_i be the number of rooted isometry classes of level L_i subtrees of (T,v) and let $\tau_1^i, \tau_2^i, \ldots \tau_{\mu_i}^i$ be a complete set of representatives of the rooted isometry classes of level L_i subtrees.

According to Claim 10.11, μ_i is also the number of rooted isometry classes of level M_i subtrees of (S,w). Moreover, $\hat{h}(\tau_1^i), \hat{h}(\tau_2^i), \ldots, \hat{h}(\tau_{\mu_i}^i)$ is a complete set of representatives of the rooted isometry classes of level M_i subtrees of (S,w).

For $0 \le i$, $1 \le k \le \mu_{i+1}$ and $1 \le \ell \le \mu_i$, let $\alpha_{k\ell}^i$ be the number of level $(L_{i+1} - L_i)$ subtrees of τ_ℓ^i that are rooted isometric to τ_k^{i+1}. The matrix $\alpha_i = [\alpha_{k\ell}^i]$ is a $(\mu_{i+1} \times \mu_i)-$ matrix with nonnegative integral entries.

Using Claim 10.11 again, it follows that $a^i_{k\ell}$ is also the number of level $(M_{i+1} - M_i)$ subtrees of $\hat{h}(\tau^i_\ell)$ that are rooted isometric to $\hat{h}(\tau^{i+1}_k)$.

We claim that there is an isomorphism of unital partially ordered abelian groups:

$$Sym_\infty(T, v) \cong \lim_{\rightarrow}(\mathbb{Z} \xrightarrow{\alpha_0} \mathbb{Z}^{\mu_1} \xrightarrow{\alpha_1} \mathbb{Z}^{\mu_2} \xrightarrow{\alpha_2} \mathbb{Z}^{\mu_3} \xrightarrow{\alpha_3} \cdots \mathbb{Z}^{\mu_i} \xrightarrow{\alpha_i} \cdots).$$

It will follow by a similar argument that $Sym_\infty(S, w)$ is also isomorphic to this direct limit, finishing the proof.

We use the notation of Section 9. In particular, we have the matrices $A_i = [a^i_{k\ell}]$ for $i = 0, 1, 2, \ldots$, $1 \leq \ell \leq m_i$, and $1leqk \leq m_{i+1}$. Note that $\mu_i = m_{L_i}$ and we can take $\tau^i_\ell = T^{L_i}_\ell$. In order to show that $Sym_\infty(T, v) \cong \lim_\rightarrow \alpha_i$, it suffices to show that $\alpha_i = A_{(L_{i+1}-1)} \cdots A_{(L_i+1)} A_{L_i}$ for $i, j = 0, 1, 2, \ldots$. Hence, the following claim completes the proof.

Claim 10.12 *For $1 \leq \ell \leq m_i$, $1 \leq k \leq m_{i+j+1}$, and $i, j = 0, 1, 2, \ldots$, the $k\ell$-entry of the product $A_{i+j} \cdots A_i$ is the number of level j subtrees of T^i_ℓ that are rooted isometric to T^{i+j+1}_k.*

Proof of Claim. The proof is by induction on j. The statement is obviously true for $j = 0$; so assume $j > 0$ and the statement is true for $j - 1$. Let $B = A_{i+j-1} \cdots A_i$ and denote its entries by $B = [b_{pq}]$. By the inductive assumption, $b_{p\ell}$ is the number of level $j-1$ subtrees of T^i_ℓ that are rooted isometric to T^{i+j}_p. The entries of the matrix $A_{i+j} = [a^{i+j}_{kp}]$ have the following interpretation by definition: a^{i+j}_{kp} is the number of level 1 subtrees of T^{i+j}_p that are rooted isometric to T^{i+j+1}_k. Hence, the number of level j subtrees of T^i_ℓ that are rooted isometric to T^{i+j+1}_k is given by $\sum_{p=1}^{m_{i+j}} a_{kp} b_{p\ell}$; that is, the $k\ell$-entry of $A_{i+j} \cdots A_i$. $\square$

This completes the proof of the theorem. $\square$

The converse of the preceding theorem is not true, as the following example shows.

Example 10.13 There are two rooted, geodesically complete, locally finite simplicial trees, (T, v) and (S, w), such that $B(T, v)$ and $B(S, w)$ are equivalent

Bratteli diagrams, but $X = end(T, v)$ and $Y = end(S, w)$ are not micro-scale equivalent. The trees are pictured in Figure 10. Elements $x \in X$ are sequences $x = (x_0, x_1, x_2, \dots)$ such that $x_i \in \begin{cases} \{0,1\} & \text{if } i \text{ is even,} \\ \{0,1,2\} & \text{if } i \text{ is odd.} \end{cases}$ Elements $y \in Y$ are sequences $y = (y_0, y_1, y_2, \dots)$ such that $y_i \in \begin{cases} \{0,1\} & \text{if } i \text{ is odd,} \\ \{0,1,2\} & \text{if } i \text{ is even.} \end{cases}$ Suppose X and Y are micro-scale equivalent. Then there are homeomorphisms $h\colon X \to Y$ and $\lambda\colon [0,\infty) \to [0,\infty)$ and $0 < \epsilon \le 1$ such that $\lambda d_X(x,y) = d_Y(hx, hy)$ whenever $\min\{d_X(x,y), d_Y(hx,hy)\} < \epsilon$ and $x, y \in X$. There exists integers $i_0 > -\ln\epsilon$ and $c \le i_0$ such that $\lambda(e^{-i}) = e^{c-i}$ for all $i \ge i_0$. For each $i = 0, 1, 2, \dots$, let $\alpha_i = \begin{cases} 2\cdot(3\cdot 2)^{\frac{i}{2}} & \text{if } i \text{ is even,} \\ (2\cdot 3)^{\frac{i+1}{2}} & \text{if } i \text{ is odd} \end{cases}$ and $\beta_i = \begin{cases} 3\cdot(2\cdot 3)^{\frac{i}{2}} & \text{if } i \text{ is even,} \\ (3\cdot 2)^{\frac{i+1}{2}} & \text{if } i \text{ is odd.} \end{cases}$ The number $\begin{cases} \alpha_i \\ \beta_i \end{cases}$ is the maximum number of distinct points of $\begin{cases} X \\ Y \end{cases}$ whose distances from each other are e^{-i}. Clearly, $\alpha_i = \beta_{i-c}$ for all $i \ge i_0$. In particular, $\alpha_{i_0} = \beta_{i_0-c}$ and $\alpha_{i_0+1} = \beta_{i_0+1-c}$. This implies $c = 0$ and both i_0 and i_0+1 are odd—a contradiction; hence, X and Y are not micro-scale equivalent. The Bratteli diagrams $B(T,v)$ and $B(S,w)$ both telescope to the Bratteli diagram $D = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}_i = \{v_i\}$, a single vertex and six edges from v_i to v_{i+1} for each $i = 0, 1, 2, \dots$.

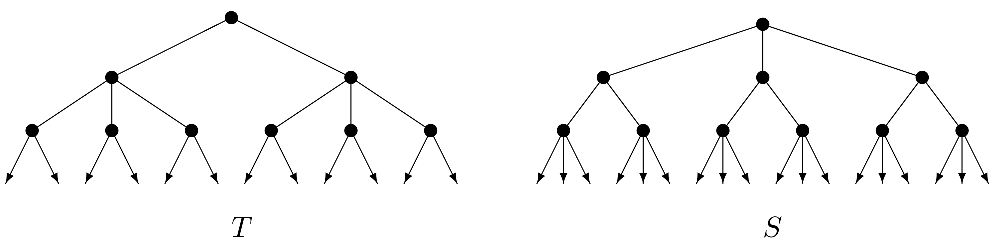

Figure 10: The trees of Example 10.13 with $B(T,v)$ and $B(S,w)$ equivalent.

The following result is a restatement of Theorem 1.1(4) in the Introduction.

Theorem 10.14 *If X is a compact, locally rigid ultrametric space, then there exists a Bratteli diagram $B(X)$ such that $\mathcal{G}_{LI}(X)$ is isomorphic to the path groupoid of $B(X)$.*

Proof. Proposition 10.7 gives a rooted, geodesically complete, locally finite simplicial tree (T, v) such that X is scale equivalent to $Y := end(T, v)$. We have defined $B(X) := B(T, v)$; hence, we have equality of the path groupoids $\mathcal{PG}(B(X)) \cong \mathcal{PG}(B(T, v))$. Now Corollary 8.13 implies there is an isomorphism of topological groupoids $\mathcal{PG}(B(T, v)) \cong \mathcal{G}_{LI}(Y)$. Since X and Y are scale equivalent, Proposition 10.5 implies there is an isomorphism of topological groupoids $\mathcal{G}_{LI}(X) \cong \mathcal{G}_{LI}(Y)$. Thus, $\mathcal{G}_{LI}(X) \cong \mathcal{PG}(B(X))$ as required. □

We now reinterpret our results on invariants for ultrametric spaces as invariants for trees.

Corollary 10.15 *Let (T, v) and (S, w) be rooted, geodesically complete, locally finite simplicial trees. If (T, v) and (S, w) are uniformly isometric at infinity, then*

1. *$B(T, v)$ and $B(S, w)$ are equivalent Bratteli diagrams, and*
2. *$Sym_\infty(T, v)$ and $Sym_\infty(S, w)$ are isomorphic partially ordered abelian groups.*

If, in addition, either $X := end(T, v)$ or $Y := end(S, w)$ is locally rigid, then so is the other, and

3. *$(K_0 C^* \mathcal{G}_{LI}(X), K_0 C^* \mathcal{G}_{LI}(X)_+, [1])$ and $(K_0 C^* \mathcal{G}_{LI}(Y), K_0 C^* \mathcal{G}_{LI}(Y)_+, [1])$ are isomorphic unital partially ordered abelian groups.*

Proof. The first statement follows from Propositions 5.6 and 10.3 and Theorem 10.9. The second statement follows from the first, Proposition 9.2, and Theorem 8.4. The final statement follows from the first, Proposition 10.4, and Corollary 8.13. □

Corollary 10.16 *Let (T, v) and (S, w) be rooted, geodesically complete, proper $\mathbb{R}$-trees that are uniformly isometric at infinity. If either $X := end(T, v)$ or $Y := end(S, w)$ is locally rigid, then so is the other, and*

$$(K_0 C^* \mathcal{G}_{LI}(X), K_0 C^* \mathcal{G}_{LI}(X)_+, [1]) \text{ and } (K_0 C^* \mathcal{G}_{LI}(Y), K_0 C^* \mathcal{G}_{LI}(Y)_+, [1])$$

are isomorphic unital partially ordered abelian groups.

Proof. This follows from Propositions 5.6, 10.3, 10.4, and 10.7 and Corollary 10.6. □

11 Faithful unitary representations

The goal of this section is to prove the following theorem, which is the third part of Theorem 1.3.

Theorem 11.1 *If X is a compact ultrametric space with a countable subgroup $\Gamma \leq LS(X)$ acting locally rigidly on X, then there is a faithful unitary representation of Γ into $C^*\mathcal{G}_\Gamma(X)$.*

It follows from Theorem 1.3(i) (Corollary 6.12) that under the hypothesis of Theorem 11.1 Renault's theory [43] can be applied so that $C^*\mathcal{G}_\Gamma(X)$ is defined.

Of course, by a faithful unitary representation of Γ into $C^*\mathcal{G}_\Gamma(X)$, we mean an injective homomorphism of Γ into the multiplicative group of unitary elements of $C^*\mathcal{G}_\Gamma(X)$. This is proved by establishing, in Corollary 11.4 below, an injective homomorphism $\rho\colon \Gamma \to C_c(\mathcal{G}_\Gamma(X))$ into the unitary group of the convolution algebra of $\mathcal{G}_\Gamma(X)$. Since the C^*-algebra of $\mathcal{G}_\Gamma(X)$ is a completion of the convolution algebra of $\mathcal{G}_\Gamma(X)$, Theorem 11.1 follows.

We begin by fixing notation for the convolution $*$-algebras of groups and groupoids. For more details, see Muhly [36], Paterson [42] and Renault [43].

If Γ is a discrete group, then $C_c(\Gamma)$ denotes the convolution $*$-algebra of Γ, otherwise known as the complex algebra $\mathbb{C}\Gamma$:

$$C_c(\Gamma) := \{f\colon \Gamma \to \mathbb{C} \mid f \text{ has finite support}\}.$$

Multiplication and involution on this complex vector space are given by

$$(f * g)(\gamma) := \sum_{\beta\in\Gamma} f(\beta)g(\beta^{-1}\gamma) \text{ and } f^*(\gamma) := \overline{f(\gamma^{-1})},$$

where $\overline{\cdot}$ denotes complex conjugation.

Now if $\mathcal{G}$ is a locally compact, Hausdorff étale groupoid, then

$$C_c(\mathcal{G}) := \{f\colon \mathcal{G} \to \mathbb{C} \mid f \text{ is continuous and has compact support}\}.$$

For each u in the unit space of $\mathcal{G}$, $r^{-1}(u) := \mathcal{G}^u$ is discrete. Thus, each element of $C_c(\mathcal{G})$ restricts to a function on each $\mathcal{G}^u$ with finite support. Therefore, multiplication and involution on the complex vector space $C_c(\mathcal{G})$ may be defined by

$$(f * g)(y) := \sum_{x \in \mathcal{G}^{r(y)}} f(x)g(x^{-1}y) \text{ and } f^*(x) := \overline{f(x^{-1})}.$$

Thus, $C_c(\mathcal{G})$ is a topological $*$-algebra.

If the unit space $X = \{xx^{-1} \mid x \in \mathcal{G}\} = \{r(x) \mid x \in \mathcal{G}\}$ is compact, then the algebra $C_c(\mathcal{G})$ has a unit 1, namely, the characteristic function χ_X.[12] In that case, the *unitary group of* $C_c(\mathcal{G})$ is the multiplicative group

$$\{f \in C_c(\mathcal{G}) \mid f^*f = 1 = ff^*\}.$$

For the remainder of this section, let X be a (nonempty) compact ultrametric space with a subgroup $\Gamma \leq LS(X)$ acting locally rigidly on X. Even though Γ need not be a discrete subgroup of $LS(X)$, we will endow Γ with the discrete topology. Denote the identity of Γ by e; that is, $e = \mathrm{id}_X$.

For each $\gamma \in \Gamma$, let

$$A_\gamma := \{[\gamma, x] \mid x \in X\} \subseteq \mathcal{G}_\Gamma(X).$$

Lemma 11.2 *For each $\gamma \in \Gamma$, A_γ is compact and open in $\mathcal{G}_\Gamma(X)$.*

Proof. Clearly, $A_\gamma = \cup_{x\in X} U(\gamma, x, 1)$. This shows A_γ is open. To see that A_γ is compact, note that $E\colon X \to A_\gamma$, defined by $E(x) = [\gamma, x]$, is continuous. This is because $E^{-1}U(\gamma, x, \epsilon) = B(x, \epsilon)$ for each $x \in X$ and $\epsilon > 0$. $\square$

Thus, for each $\gamma \in \Gamma$, the characteristic function χ_{A_γ} of A_γ is in $C_c(\mathcal{G}_\Gamma(X))$.

[12] Do not confuse 1 with the function on $\mathcal{G}$ that is identically 1, i.e., $\chi_{\mathcal{G}}$. Since $\mathcal{G}$ need not be compact, $\chi_{\mathcal{G}}$ need not be in $C_c(\mathcal{G})$. On the other hand, $0 \in C_c(\mathcal{G})$ is the function that is identically 0.

Proposition 11.3 *If $\gamma, \gamma_1, \gamma_2 \in \Gamma$, then*

1. $\chi^*_{A_\gamma} = \chi_{A_{\gamma^{-1}}}$
2. $\chi_{A_{\gamma_1\gamma_2}} = \chi_{A_{\gamma_1}} * \chi_{A_{\gamma_2}}$
3. $A_e = X$, *the unit space*
4. $\chi_{A_\gamma} = 1$ *if and only if* $\gamma = e$
5. $A_\gamma \neq \emptyset$ *(so that* $\chi_{A_\gamma} \neq 0$*).*

Proof. (1) First note that for $x \in X$ and $\beta, \gamma \in \Gamma$, $[\beta, x] = [\gamma^{-1}, x]$ if and only if $[\beta^{-1}, \beta x] = [\gamma, \beta x]$. Thus,

$$\chi^*_{A_\gamma}([\beta, x]) = \overline{\chi_{A_\gamma}([\beta, x]^{-1})} = \chi_{A_\gamma}([\beta, x]^{-1}) = \chi_{A_\gamma}([\beta^{-1}, \beta x])$$

$$= \begin{cases} 1 \text{ if } [\beta^{-1}, \beta x] = [\gamma, \beta x] \\ 0 \text{ otherwise} \end{cases} = \begin{cases} 1 \text{ if } [\beta, x] = [\gamma^{-1}, x] \\ 0 \text{ otherwise} \end{cases}$$

$$= \chi_{A_{\gamma^{-1}}}([\beta, x]).$$

(2) Let $[\gamma, x] \in \mathcal{G}_\Gamma(X)$ be given. Then

$$\chi_{A_{\gamma_1\gamma_2}}[\gamma, x] = \begin{cases} 1 \text{ if } [\gamma, x] = [\gamma_1\gamma_2, x] \\ 0 \text{ otherwise.} \end{cases}$$

On the other hand,

$$\chi_{A_{\gamma_1}} * \chi_{A_{\gamma_2}}[\gamma, x] =$$

$$\sum_{[\beta, y] \in r^{-1}(r[\gamma, x])} \chi_{A_{\gamma_1}}[\beta, y] \cdot \chi_{A_{\gamma_2}}([\beta, y]^{-1}[\gamma, x]) = \sum_{\beta y = \gamma x} \chi_{A_{\gamma_1}}[\beta, y] \cdot \chi_{A_{\gamma_2}}[\beta^{-1}\gamma, x].$$

Since

$$\chi_{A_{\gamma_1}}[\beta, y] = \begin{cases} 1 \text{ if } [\beta, y] = [\gamma_1, y] \\ 0 \text{ otherwise,} \end{cases}$$

we have

$$\chi_{A_{\gamma_1}} * \chi_{A_{\gamma_2}}[\gamma, x] = \sum_S \chi_{A_{\gamma_2}}[\beta^{-1}\gamma, x],$$

where $S = \{[\beta, y] \in \mathcal{G}_\Gamma(X) \mid \beta y = \gamma x \text{ and } [\beta, y] = [\gamma_1, y]\}$. If $[\beta, y] \in S$, then $\gamma x = \beta y = \gamma_1 y$; thus, $y = \gamma_1^{-1}\gamma x$ and $[\beta, y] = [\gamma_1, \gamma_1^{-1}\gamma x]$. It follows that $S = \{[\gamma_1, \gamma_1^{-1}\gamma x]\}$; in other words, the sum has only one term and

$$\chi_{A_{\gamma_1}} * \chi_{A_{\gamma_2}}[\gamma, x] = \chi_{A_{\gamma_2}}[\gamma_1^{-1}\gamma, x] = \begin{cases} 1 \text{ if } [\gamma_1^{-1}\gamma, x] = [\gamma_2, x] \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} 1 \text{ if } [\gamma, x] = [\gamma_1\gamma_2, x] \\ 0 \text{ otherwise} \end{cases} = \chi_{A_{\gamma_1\gamma_2}}[\gamma, x].$$

(3) This follows from the description of the unit space in Remark 3.12.

(4) If $\gamma = e$, then A_e is the unit space by 3. Thus, χ_{A_γ} is the unit of $C_c(\mathcal{G}_\Gamma(X))$ by the general remarks made above. Conversely, if $\chi_{A_\gamma} = 1$, then $\chi_{A_\gamma} = \chi_{A_e}$ and $A_\gamma = A_e$. Thus, for each $x \in X$, $[\gamma, x] = [\mathrm{id}_X, x]$; hence, $\gamma = e$.

(5) is obvious. □

Corollary 11.4 *$\rho\colon \Gamma \to C_c(\mathcal{G}_\Gamma(X))$, defined by $\rho(\gamma) = \chi_{A_\gamma}$, is an injective homomorphism into the unitary group of $C_c(\mathcal{G}_\Gamma(X))$.*

Proof. That ρ is a homomorphism follows from 11.3 (2). The image of ρ lies in the unitary group by 11.3 (1), (2), (4). The injectivity of ρ follows from 11.3 (4). □

This completes the proof of Theorem 11.1.

Example 11.5 Let $X = \{x_1, \ldots, x_n\}$ be the finite ultrametric space with $d(x_i, x_j) = 1$ if $i \neq j$. Then $\Gamma := LS(X) = Isom(X)$ is the symmetric group S_n on n elements and Γ acts locally rigidly on X. The groupoid $\mathcal{G}_\Gamma(X)$ is the transitive principle groupoid on n elements (that is, the trivial groupoid $X \times X$) and $C_c(\mathcal{G}_\Gamma(X)) = \mathbb{M}_n(\mathbb{C})$. The homomorphism $\rho\colon \Gamma \to \mathbb{M}_n(\mathbb{C})$ defined above is the representation of S_n by permutation matrices.

12 Miscellanea

12.1 Isometries of trees

In this section we show that a group acting by isometries on a tree sometimes leads to a group of local similarities acting locally rigidly on the end space of the tree. For the general theory of isometries on trees, see for example Alperin and Bass [1], Bestvina [6], Chiswell [13], Morgan and Shalen [35], and Serre [46].

Let (T, v) be a geodesically complete, rooted, locally finite, simplicial tree, let $X = end(T, v)$, and let $Isom(T)$ denote the group of isometries on T.[13] In particular, X is compact ultrametric.

Description of a group homomorphism $\epsilon\colon Isom(T) \to LS(X)$. We will use the notation and terminology from [26]. Let $\gamma\colon T \to T$ be an isometry and let $r = 1 + d(v, \gamma v)$. Then $\partial B(v, r)$ and $\partial B(\gamma v, r)$ are cut sets for (T, v) (cf. [26, Example 3.2]). Let $x \in X$; thus, $x\colon [0, \infty) \to T$ is an isometric embedding with $x(0) = v$. Let $\hat{x}\colon [0, ||\gamma(x(r))||] \to X$ be the unique isometric embedding such that $\hat{x}(0) = v$ and $\hat{x}(||\gamma(x(r)||) = \gamma(x(r))$ (here $||y|| := d(v, y)$ for all $y \in T$).

Define $\gamma_*(x)\colon [0, \infty) \to T$ by

$$\gamma_*(x)(t) = \begin{cases} \hat{x}(t) & \text{if } 0 \leq t \leq ||\gamma(x(r))|| \\ \gamma \circ x(t - ||\gamma(x(r))|| + r) & \text{if } ||\gamma(x(r))|| \leq t. \end{cases}$$

Then $\gamma_*(x) \in X$, $\gamma_*\colon X \to X$ is in $LS(X)$, and $\epsilon\colon Isom(T) \to LS(X)$ defined by $\epsilon(\gamma) = \gamma_*$ is a group homomorphism (cf. [26, Section 5]).[14]

It follows that for $x \in X$ and $\gamma \in Isom(T)$, $\epsilon(\gamma)(x) = x$ if and only if there exists $t_1, t_2 \geq 0$ such that $x([t_1, \infty)) = \gamma x([t_2, \infty))$. From this the following *key property of* ϵ can be verified: if $x \in X$, $t_0 \geq 0$, and $\gamma \in Isom(T)$ such that $\gamma \in \Gamma_{x(t_0)}$ and $\epsilon(\gamma) \in \epsilon(\Gamma)_x$, then $\gamma \in \Gamma_{x(t)}$ for all $t \geq t_0$.

[13] Some of the facts and constructions in this section hold in the more general context of $\mathbb{R}$-trees.

[14] Of course, it is quite well-known that ϵ is a homomorphism from $Isom(T)$ into the group of homeomorphisms of X; we are just pointing out here that when the end space X is given the natural metric described herein, that the image of ϵ lies in $LS(X)$.

Remark 12.1 There exists a commuting diagram of groups and group homomorphisms:

$$\begin{tikzcd} Isom(T,v) \arrow[r, "\epsilon|", "\cong"'] \arrow[d] & Isom(X) \arrow[d] \\ Isom(T) \arrow[r, "\epsilon"] & LS(X) \end{tikzcd}$$

The vertical arrows are inclusions of subgroups. The top horizontal map $\epsilon|$ is easily seen to be an isomorphism (cf. Proposition 5.6 and [26, Corollary 8.7]).

Example 12.2 $\epsilon\colon Isom(T) \to LS(X)$ need not be surjective. There are many local similarities of the end space of the Sturmian tree T (see Example 5.9), but T has no non-trivial isometries. In general, ϵ is rarely surjective.

Example 12.3 Let X be the space in Example 6.16 and let (T, v) be the geodesically complete, rooted tree with $X = end(T, v)$. The subgroup $\Gamma \cong \mathbb{Z}/2$ of $LS(X)$ defined in 6.16 is the isomorphic image of a subgroup $\widehat{\Gamma}$ of $Isom(T, v)$ under ϵ. Note that $\widehat{\Gamma}_x$ is finite for each $x \in T$ even though Γ does not act locally rigidly on X.

Theorem 12.4 *Let Γ be a subgroup of $Isom(T)$. The group $\epsilon(\Gamma)$ acts locally rigidly on X if and only if for every $x \in X$ and for every $\gamma \in \Gamma$ such that there exists $t_0 \geq 0$ with $\gamma \in \Gamma_{x(t)}$ for all $t \geq t_0$, there exists $t_1 \geq t_0$ so that if $y \in X$ and $y(t_1) = x(t_1)$, then $\gamma \in \Gamma_{y(t)}$ for all $t \geq t_1$.*

Proof. Assume first that $\epsilon(\Gamma)$ acts locally rigidly. Let $x \in X$, $t_0 \geq 0$, and $\gamma \in \cap\{\Gamma_{x(t)} \mid t \geq t_0\}$ be given. Clearly, $\epsilon(\gamma) \in \epsilon(\Gamma)_x$ and $sim(\epsilon(\gamma), x) = 1$. By the definition of a locally rigid action, there exists $\delta > 0$ such that $\epsilon(\gamma) \in \epsilon(\Gamma)_y$ for all $y \in B(x, \delta)$. Choose $t_1 \geq t_0$ such that if $y \in X$ and $y(t_1) = x(t_1)$, then $y \in B(x, \delta)$. For such a y, $\gamma \in \Gamma_{y(t_1)}$ and $\epsilon(\gamma) \in \epsilon(\Gamma)_y$. Hence, it follows from the key property of ϵ mentioned above that $\gamma \in \Gamma_{y(t)}$ for all $t \geq t_1$.

For the converse, first observe that there is a bijection from X to the set of *open ends of T in the sense of* [1] given by $x \mapsto [x([0, \infty))]$, the open end determined by the image of x. Moreover, for $\gamma \in Isom(T)$ and $x \in X$, $\epsilon(\gamma) \in \epsilon(\Gamma)_x$ if and only if γ fixes the open end of T determined by x. Also recall (e.g.

from [1]) that for $\gamma \in Isom(T)$, $\ell(\gamma) := \min\{d(t, \gamma(t)) \mid t \in T\}$ and $A_\gamma := \{t \in T \mid d(t, \gamma(t)) = \ell(\gamma)\}$.

Now let $x \in X$ and $\gamma \in \Gamma$ be given such that $\epsilon(\gamma) \in \epsilon(\Gamma)_x$ and $sim(\epsilon(\gamma), x) = 1$. We must show that $\epsilon(\gamma)$ fixes all points of X sufficiently close to x. Since γ fixes the open end of T determined by x, it follows that $x(t) \in A_\gamma$ for sufficiently large t (see [1, Corollary 6.17]).

If γ is elliptic (i.e., fixes some point of T), then A_γ is the fixed point set of γ. Thus, $\gamma \in \Gamma_{x(t)}$ for all sufficiently large t, say for $t \geq t_0$. Let $t_1 \geq t_0$ be given by the hypothesis. Then $\gamma \in \Gamma_{y(t)}$ whenever $y \in X$, $t \geq t_1$, and $y(t_1) = x(t_1)$. That is, γ fixes the open end determined by such a y and hence, $\epsilon(\gamma)$ fixes y for y sufficiently close to x.

On the other hand, if γ is hyperbolic (i.e., has no fixed point), it follows that A_γ is isometric to $\mathbb{R}$ (see [1]). The condition $sim(\epsilon(\gamma), x) = 1$ implies $A_\gamma = T$ and x is isolated in X. $\square$

Remark 12.5 The homomorphism $\epsilon : Isom(T) \to LS(X)$ is an injection if and only if T is not isometric to $\mathbb{R}$. If T is isometric to $\mathbb{R}$, one sees that $Isom(T) \cong \mathbb{R} \rtimes \mathbb{Z}/2$ and $LS(X) \cong \mathbb{Z}/2$. In particular, ϵ is not injective. Conversely, if ϵ is not injective, then there exists a non-trivial $\gamma \in \Gamma$ such that $\epsilon(\gamma)$ is the identity on X. Using the notation in the proof of Theorem 12.4, it follows that every point of X determines an open end of A_γ (see [1, Corollary 6.17]). If γ is hyperbolic, then A_γ is isometric to $\mathbb{R}$ and $\gamma | A_\gamma$ is a translation; it follows that $A_\gamma = T$. If γ is elliptic, then A_γ is the fixed point set of γ; it follows that $A_\gamma = T$, a contradiction.

The following two corollaries follow immediately from Theorem 12.4.

Corollary 12.6 *Let Γ be a subgroup of $Isom(T)$. If $\{t \in T \mid \Gamma_t \neq \{1\}\}$ is a bounded subset of T, then $\epsilon(\Gamma)$ acts locally rigidly on X.*

Corollary 12.7 *If $\Gamma \leq Isom(T)$ acts freely on T, then $\epsilon(\Gamma)$ acts locally rigidly on X.*

As an example, let Γ be a finitely generated free group with a free set of generators $S = \{s_1, s_2, s_3, \ldots, s_n\}$ and identity element e. Recall that the Cayley graph $\mathrm{Cay}(\Gamma, S)$ is a geodesically complete, rooted (at e), locally finite, simplicial tree . Moreover, Γ acts freely by isometries on $\mathrm{Cay}(\Gamma, S)$.

The following result follows from the previous corollary.

Corollary 12.8 $\epsilon(\Gamma)$ *acts locally rigidly on end*$(\mathrm{Cay}(\Gamma, S), e)$.

12.2 Local rigidity and tree lattices

We now characterize local rigidity of an ultrametric space X in terms of the isometry group of X. This will be used below when we discuss a connection with the Bass-Lubotzky theory of tree lattices. We will use the majorant topology on the isometry group.[15] This coincides with what is sometimes called the fine Whitney topology and makes the isometry group into a topological group (see [30, Essay I, appendix C]). Of course, when X is compact the majorant topology agrees with the compact-open topology and the uniform topology (the usual sup-metric).

Theorem 12.9 *An ultrametric space* (X, d) *is locally rigid if and only if the topological group* $Isom(X)$ *is discrete in the majorant topology.*

Proof. Assume first that $Isom(X)$ is discrete in the majorant topology. If X is not locally rigid, then there exists $x \in X$ and a decreasing sequence $\{\epsilon_i\}_{i=1}^{\infty}$ of positive numbers converging to 0 and non-trivial isometries $h_i\colon B(x, \epsilon_i) \to B(x, \epsilon_i)$, $i = 1, 2, 3, \ldots$. By Lemma 4.3 we can extend each h_i to a non-trivial isometry $\tilde{h}_i\colon X \to X$ such that $\tilde{h}_i$ is the inclusion on the complement of $B(x, \epsilon_i)$. It follows that $\tilde{h}_i$ converges to the identity on X in $Isom(X)$ as $i \to \infty$ in the majorant topology, contradicting the discreteness of $Isom(X)$.

---

[15] Let $\mathcal{U}$ be an open cover of the metric space X, $h \in Isom(X)$ and define

$$N(h, \mathcal{U}) = \{g \in Isom(X) \mid \text{for each } x \in X \text{ there exists } U \in \mathcal{U} \text{ such that } g(x), h(x) \in U\}.$$

For fixed h the sets $N(h, \mathcal{U})$ form a neighborhood basis for h and the resulting topology is called the *majorant topology*. Alternatively, consider continuous functions $\epsilon\colon X \to (0, \infty]$. Then the sets

$$N_\epsilon(h) = \{g \in Isom(X) \mid d(g(x), h(x) < \epsilon(h(x)) \text{ for all } x \in X\}$$

also form a neighborhood basis for h in the majorant topology.

Conversely, assume that X is locally rigid. For each $x \in X$, let $\epsilon_x > 0$ be given by Definition 6.1. Let $\mathcal{U} = \{B(x, \epsilon_x) \mid x \in X\}$ and suppose $g, h\colon X \to X$ are two isometries that are $\mathcal{U}$-close. If $y \in X$, then there exists $x \in X$ such that $g(y), h(y) \in B(x, \epsilon_x)$. It follows that $y \in g^{-1}B(x, \epsilon_x) \cap h^{-1}B(x, \epsilon_x)$. Moreover, $g^{-1}B(x, \epsilon_x) = B(g^{-1}x, \epsilon_x)$ and $h^{-1}B(x, \epsilon_x) = B(h^{-1}x, \epsilon_x)$. Hence, $g^{-1}B(x, \epsilon_x) = h^{-1}B(x, \epsilon_x)$ (by Proposition 4.2(1)). Thus, $hg^{-1}|\colon B(x, \epsilon_x) \to B(x, \epsilon_x)$ is an isometry and it follows that $hg^{-1}|$ is the identity (by the choice of ϵ_x); in particular, $g(y) = h(y)$. Thus, any two $\mathcal{U}$-close isometries on X are equal and $Isom(X)$ is discrete in the majorant topology. □

Corollary 12.10 *A compact ultrametric space X is locally rigid if and only if $Isom(X)$ is finite.*

Proof. If X is compact, then so is $Isom(X)$. Thus, $Isom(X)$ is finite if and only if it is discrete. □

Example 12.11 We give an example of a noncompact ultrametric space Y that is locally rigid, but $Isom(Y)$ is not discrete in the compact-open topology (or, the uniform topology). Let X be the space of Example 6.16 and let $Y = X \setminus \{x_\infty\}$. For each $i \in \mathbb{N}$, define an isometry $h_i\colon Y \to Y$ by

$$h_i(z) = \begin{cases} x_{|a-1|i} \text{ if } z = x_{ai} \text{ for } i \in \{0, 1\} \\ z \qquad\quad \text{otherwise.} \end{cases}$$

Then h_i converges to the identity in the compact-open topology (hence, also in the uniform topology). See Figure 11.

We now turn to some connections with some concepts encountered in the theory of tree lattices as developed by Bass and Lubotzky [3]. Let T be a locally finite simplicial tree. The locally compact group $Aut(T)$ of simplicial autohomeomorphisms of T is a subgroup of the group $Isom(T)$ of isometries of T onto T.[16,17] If $v \in T$ is a vertex and $Aut(T, v) \subseteq Aut(T)$ and $Isom(T, v) \subseteq Isom(T)$

[16] In fact, $Aut(T) = Isom(T)$ if and only if T is not isometric to $\mathbb{R}$.

[17] The topology on $Aut(T)$ is the compact-open topology, so that two simplicial autohomeomorphisms are close if they agree on a large finite subtree. What is important about this topology is that discrete subgroups of $Aut(T)$ are precisely the subgroups whose vertex stabilizers are finite.

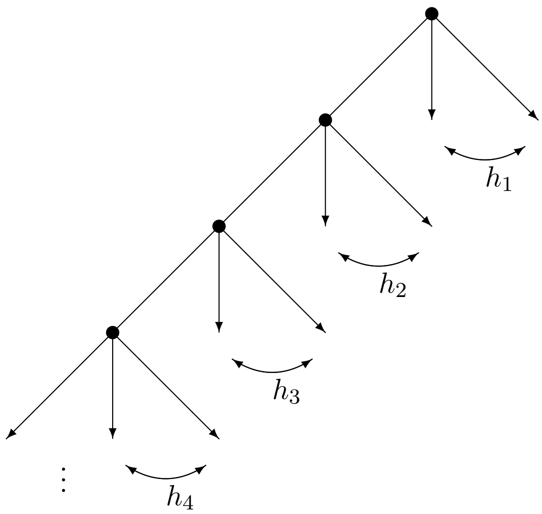

Figure 11: A non-discrete isometry group. See Example 12.11

are the subgroups of automorphisms fixing v, then $Aut(T, v) = Isom(T, v)$.

Fix a vertex $v \in T$, the root, and assume now that (T, v) is geodesically complete. The end space $end(T, v)$ of (T, v) is a compact ultrametric space and the natural function $Aut(T, v) = Isom(T, v) \to Isom(end(T, v))$ is an isomorphism (see Remark 12.1).

Recall the following definitions from [3] and [4]:

Definition 12.12 1. A locally finite simplicial tree T is *rigid* if $Aut(T)$ is discrete.

2. A finite, connected simplicial graph K is *π-rigid* if $\pi_1(K) = Aut(\tilde{K})$, where $\tilde{K}$ is the locally finite simplicial tree that is the universal cover of K.

It follows that T is rigid if and only if $Aut(T, v)$ is finite for all vertices $v \in T$, if and only if $Aut(T, v)$ is finite for some vertex $v \in T$.

Proposition 12.13 *If (T, v) is a geodesically complete, rooted locally finite simplicial tree, then T is rigid if and only if the compact ultrametric space $end(T, v)$ is locally rigid.*

Proof. By the remarks above, T is rigid if and only if the group $Isom(end(T, v))$ is finite. Hence, the result follows from Corollary 12.10. □

It follows that there is a rich source of examples of compact, locally rigid ultrametric spaces. In fact, Bass and Tits [4, page 185] assert that a randomly constructed locally finite tree will have no non-trivial automorphisms. One therefore expects that almost all compact ultrametric spaces are locally rigid.

Bass and Kulkarni [2] and Bass and Tits [4] have provided examples of π-rigid graphs K without terminal vertices. The universal covering trees $\tilde{K}$ of these graphs are rigid (because in this case $Aut(\tilde{K})$ acts freely on $\tilde{K}$) and geodesically complete (with respect to any vertex). Hence, these trees have end spaces that are compact, locally rigid ultrametric spaces.

12.3 R. J. Thompson's groups and their descendants

In this section we see how groups defined, generalized, and developed by Brown, Higman, Thompson, Neretin, Röver, and others can be interpreted as groups of local similarities on compact ultrametric spaces.

For the groups F, T, and V of Thompson [47], the connection with the current paper arises from their description via *reduced tree diagrams* in Cannon, Floyd, and Parry [12] based on work of K. Brown [11]. Higman [25] generalized V to a family of infinite finitely presented groups $G_{n,r}$ $(n = 2, 3, 4, \ldots,\ r = 1, 2, 3, \ldots)$ and Brown [11] extended this to families $F_{n,r} \leq T_{n,r} \leq G_{n,r}$ with $F_{2,1} = F$, $T_{2,1} = T$, and $G_{2,1} = V$. Brown, based on earlier work of Jónsson and Tarski [27], used trees to describe these groups. In particular, it is clear from Brown's work that each of these groups can be realized as subgroups of groups of local similarities on end spaces of trees.

Röver [45], with his notion of *almost automorphisms of trees*, further developed the viewpoint of Brown and described the groups $G_{n,1}$ as subgroups of homeomorphism groups of end spaces of trees (these homeomorphisms are local similarities). See also Greenberg and Serigiescu [22] for an instance of tree diagrams inducing groups of homeomorphisms on end spaces.

Neretin [38], [39], [40] introduced p-adic analogues of the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle, called groups of *spheromorphisms* and, later, *hierarchomorphisms*, that are also groups of homeomorphisms of end spaces of trees. As is the case with Röver's groups, Neretin's groups are subgroups of local similarities on the end space of a tree. For more on Neretin's groups, see Kapoudjian [28] and Lavrenyuk and Sushchansky [32].

To indicate in a bit more detail how these groups are related to the current paper, we need to introduce some more terminology.

Let (T, v) be a rooted, geodesically complete, locally finite simplicial tree. For $i \in \mathbb{Z}_+$, V_i denotes the set of vertices of T a distance i from v (the *vertices at level* i) as in Section 8.2, and $\mathcal{E}_i$ denotes the set of edges of T with one vertex in V_i and the other in V_{i+1} (the *edges at level* i). For a vertex $w \in V_i$, let $\mathcal{E}_w \colon = \{E \in \mathcal{E}_i \mid w \in E\}$.

An *order* of (T, v) consists of a total order on $\mathcal{E}_w$ for each vertex $w \in T$, and (T, v) is *ordered* if it comes with an order.

Note that an order of (T, v) induces a total order on V_i for each $i = 0, 1, 2, \dots$ defined inductively as follows. If v_1 and v_2 are distinct vertices in V_{i+1}, let E_1, E_2 be the unique edges in $\mathcal{E}_i$ with $v_1 \in E_1$, $v_2 \in E_2$, and let $w_1 \in E_1$, $w_2 \in E_2$ be the vertices of these edges in V_i. If $w_1 = w_2$, then define $v_1 < v_2$ if and only if $E_1 < E_2$; if $w_1 \neq w_2$, define $v_1 < v_2$ if and only if $w_1 < w_2$.

Note that every (T, v) can be ordered. Furthermore, an order on the infinite n-ary tree (A_n, v) is often implicitly used: the set of vertices immediately below a given vertex is identified with the ordered set $\{0, 1, 2, \dots n-1\}$.

If (T, v) is ordered, then there is an induced (dictionary) total order on $X = end(T, v)$. This is because X is identified with the set of all sequences $(v_0, v_1, v_2, \dots)$ such that $v_i \in V_i$ for each i. Thus, we can speak of *order preserving* maps $X \to X$. Moreover, a map $h \colon X \to X$ is *locally order preserving* if for each $x \in X$ there exists $\epsilon > 0$ such that $h| \colon B(x, \epsilon) \to X$ is order preserving.

There also is an induced total order on any collection of disjoint balls in X (this uses the description of balls in Section 5.2). In particular, such collections have a cyclic order. Therefore, we say that a local similarity $h \colon X \to X$ is *cyclic order preserving* if there exists $\epsilon > 0$ such that for every $x \in X$ there exists

$\lambda_x > 0$ so that $h|: B(x,\epsilon) \to B(hx, \lambda_x \epsilon)$ is a similarity, and the induced function $\{B(x,\epsilon) \mid x \in X\} \to \{B(x, \lambda_x \epsilon) \mid x \in X\}$, $B(x,\epsilon) \mapsto B(x, \lambda_x \epsilon)$, preserves the cyclic order.

For the rest of this section, let (T, v) be an ordered, rooted, geodesically complete, locally finite simplicial tree, and let $X = end(T, v)$. We denote various subgroups of $LS(X)$ as follows:

$$\begin{aligned}
LS^{o.p.}(X) &= \{h \in LS(X) \mid h \text{ is order preserving}\} \\
LS_{l.o.p.}(X) &= \{h \in LS(X) \mid h \text{ is locally order preserving}\} \\
LS^{o.p.}_{l.o.p.}(X) &= LS^{o.p.}(X) \cap LS_{l.o.p.}(X) \\
LS^{c.o.p.}(X) &= \{h \in LS(X) \mid h \text{ is cyclic order preserving}\} \\
LS^{c.o.p.}_{l.o.p.}(X) &= LS^{c.o.p.}(X) \cap LS_{l.o.p.}(X)
\end{aligned}$$

Although Röver [45] focused on spherically homogeneous trees, it is clear that his definition of the almost automorphism group $\mathrm{AAut}(T, v)$ can be made for any (T, v) as above and that $\mathrm{AAut}(T, v) = LS_{l.o.p}(X)$.

Likewise, Neretin's [40] group $Hier(T, \Gamma)$ of *hierarchomorphisms*, where $\Gamma \leq Isom(T)$, is a subgroup of $LS(X)$.

Example 12.14 Let $X_n = end(A_n, v)$, the end space of the infinite n-ary tree. Then, $LS_{l.o.p}(X_n) = G_{n,1}$, the Higman–Thompson group. In particular, there are the following interpretations of Thompson's groups:

$$\begin{aligned}
LS_{l.o.p}(X_2) &= G_{2,1} \;=\; V \\
LS^{o.p.}_{l.o.p.}(X_2) &= F \\
LS^{c.o.p.}_{l.o.p.}(X_2) &= T
\end{aligned}$$

Proposition 12.15 *If (T, v) is an ordered, rooted, geodesically complete, locally finite simplicial tree and $X = end(T, v)$, then $LS_{l.o.p.}(X)$ acts locally rigidly on X.*

Proof. Let $h \in LS_{l.o.p}(X)$ and $x \in X$ such that $hx = x$ and $sim(h, x) = 1$. Thus, there exists $\epsilon > 0$ such that $h|: B(x,\epsilon) \to B(x,\epsilon)$ is an order preserving isometry.

This implies that $h|$ is the identity (clearly, using the notation of Section 5.2, h fixes the vertex $\langle x, \epsilon\rangle \in T$ and each edge of $\mathcal{E}_{\langle x,\epsilon\rangle}$; continuing by induction, h fixes each vertex of $T_{\langle x,\epsilon\rangle}$). $\square$

Corollary 12.16 (Birget [7], Nekrashevych [37]) *For each $n = 2, 3, 4, \ldots,$ there exists a faithful unitary representation of the Higman–Thompson group $G_{n,1}$ into the Cuntz algebra $\mathcal{O}_n$.*

Proof. By Proposition 12.15, $\Gamma = LS_{l.o.p}(X_n)$ acts locally rigidly on $X_n = end(A_n, v)$. The groupoid $\mathcal{G}_\Gamma(X_n) = O_n$, the Cuntz groupoid, and $C^*\mathcal{G}_\Gamma(X_n) = \mathcal{O}_n$, the Cuntz algebra, by Renault [43]. Hence, the corollary follows from Theorem 1.3(2) (i.e., Theorem 11.1). $\square$

12.4 Symbolic dynamics

This brief section contains a preliminary comparison between the concept of local isometry in ultrametric spaces and the concept of tail equivalence studied in symbolic dynamics.

Let A be an $n \times n$ matrix with entries $A_{ij} \in \{0, 1\}$, $1 \le i, j \le n$. Let X_A be the one-sided subshift of finite type (i.e., the one-sided topological Markov shift) with transition matrix A. Thus,

$$X_A = \{(x_i)_{i=1}^{\infty} \mid \text{for each } i,\ x_i \in \{0, 1, \ldots, n-1\} \text{ and } A_{x_i, x_{i+1}} = 1\}.$$

Define a metric on X_A by

$$d(x, y) = \begin{cases} e^{-k} & \text{if } x_i = y_i \text{ for } 1 \le i < k \text{ and } x_k \ne y_k \\ 0 & \text{if } x = y. \end{cases}$$

As is well known, (X_A, d) is a compact ultrametric space.

Proposition 12.17 *If $x, y \in X_A$ and x and y are tail equivalent, then there exist $\epsilon > 0$ and an isometry $h\colon B(x, \epsilon) \to B(y, \epsilon)$ such that $hx = y$.*

Proof. Suppose $k \in \mathbb{N}$ and $x_i = y_i$ for all $i \geq k$. Define $h\colon B(x, e^{-(k+1)}) \to B(y, e^{-(k+1)})$ by

$$h(z)_i = \begin{cases} y_i \text{ if } 1 \leq i \leq k-1 \\ z_i \text{ if } k \leq i, \end{cases} \qquad \text{for each } z \in B(x, e^{-(k+1)}).$$

It is easy to check that h is the desired isometry. □

Example 12.18 The converse of Proposition 12.17 does not hold in general. For example, if $A = \begin{bmatrix} 1\,1 \\ 1\,1 \end{bmatrix}$, then X_A is the end space of the Cantor tree and is isometrically homogeneous, i.e., the isometry group of X_A acts transitively. Another example is provided by the matrix $A = \begin{bmatrix} 1\,1\,1 \\ 1\,1\,1 \\ 1\,0\,0 \end{bmatrix}$. In this case, X_A is not isometrically homogeneous, but equal local isometry type of points need not imply tail equivalence.

References

[1] Roger Alperin and Hyman Bass. Length functions of group actions on Λ-trees. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, volume 111 of *Ann. of Math. Stud.*, pages 265–378. Princeton Univ. Press, Princeton, NJ, 1987.

[2] Hyman Bass and Ravi Kulkarni. Uniform tree lattices. *J. Amer. Math. Soc.*, 3(4):843–902, 1990.

[3] Hyman Bass and Alexander Lubotzky. *Tree lattices*, volume 176 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001. With appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits.

[4] Hyman Bass and J. Tits. Discreteness criteria for tree automorphism groups. Appendix to [3].

[5] V. N. Berestovskii. Ultrametric spaces. In *Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999)*, pages 47–72. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[6] Mladen Bestvina. ℝ-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002.

[7] Jean-Camille Birget. The groups of Richard Thompson and complexity. *Internat. J. Algebra Comput.*, 14(5-6):569–626, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.

[8] Bruce Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications.* Cambridge University Press, Cambridge, second edition, 1998.

[9] Ola Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.

[10] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[11] Kenneth S. Brown. Finiteness properties of groups. *J. Pure Appl. Algebra*, 44(1-3):45–75, 1987. Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985).

[12] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[13] Ian Chiswell. *Introduction to Λ-trees.* World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[14] Alain Connes. *Noncommutative geometry.* Academic Press Inc., San Diego, CA, 1994.

[15] Kenneth R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs.* American Mathematical Society, Providence, RI, 1996.

[16] Edward G. Effros. *Dimensions and C^*-algebras*, volume 46 of *CBMS Regional Conference Series in Mathematics.* Conference Board of the Mathematical Sciences, Washington, D.C., 1981.

[17] Edward G. Effros and Chao Liang Shen. Approximately finite C^*-algebras and continued fractions. *Indiana Univ. Math. J.*, 29(2):191–204, 1980.

[18] George A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.

[19] R. Exel and A. Lopes. C^*-algebras, approximately proper equivalence relations and thermodynamic formalism. *Ergodic Theory Dynam. Systems*, 24(4):1051–1082, 2004.

[20] R. Exel and J. Renault. AF-algebras and the tail-equivalence relation on Bratteli diagrams. *Proc. Amer. Math. Soc.*, 134(1):193–206 (electronic), 2006.

[21] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.*, 469:51–111, 1995.

[22] Peter Greenberg and Vlad Sergiescu. An acyclic extension of the braid group. *Comment. Math. Helv.*, 66(1):109–138, 1991.

[23] Branko Grünbaum and G. C. Shephard. *Tilings and patterns.* W. H. Freeman and Company, New York, 1987.

[24] Richard H. Herman, Ian F. Putnam, and Christian F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.*, 3(6):827–864, 1992.

[25] Graham Higman. *Finitely presented infinite simple groups.* Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. Notes on Pure Mathematics, No. 8 (1974).

[26] Bruce Hughes. Trees and ultrametric spaces: a categorical equivalence. *Adv. Math.*, 189(1):148–191, 2004.

[27] Bjarni Jónsson and Alfred Tarski. On two properties of free algebras. *Math. Scand.*, 9:95–101, 1961.

[28] Christophe Kapoudjian. Simplicity of Neretin's group of spheromorphisms. *Ann. Inst. Fourier (Grenoble)*, 49(4):1225–1240, 1999.

[29] Andrei Khrennikov. *Information dynamics in cognitive, psychological, social and anomalous phenomena*, volume 138 of *Fundamental Theories of Physics.* Kluwer Academic Publishers Group, Dordrecht, 2004.

[30] Robion C. Kirby and Laurence C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J., 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[31] Alex Kumjian, David Pask, Iain Raeburn, and Jean Renault. Graphs, groupoids, and Cuntz-Krieger algebras. *J. Funct. Anal.*, 144(2):505–541, 1997.

[32] Yaroslav V. Lavrenyuk and Vitalii I. Sushchansky. Automorphisms of homogeneous symmetric groups and hierarchomorphisms of rooted trees. *Algebra Discrete Math.*, (4):33–49, 2003.

[33] Álvaro Martínez-Pérez and Manuel A. Morón. Uniformly contimuous maps between ends of $\mathbb{R}$-trees. arXiv:0704.3151v2 [math.GT].

[34] James A. Mingo. C^*-algebras associated with one-dimensional almost periodic tilings. *Comm. Math. Phys.*, 183(2):307–337, 1997.

[35] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.

[36] Paul S. Muhly. *Coordinates in operator algebra.* 2002.

[37] Volodymyr V. Nekrashevych. Cuntz-Pimsner algebras of group actions. *J. Operator Theory*, 52(2):223–249, 2004.

[38] Yu. A. Neretin. Unitary representations of the groups of diffeomorphisms of the p-adic projective line. *Funktsional. Anal. i Prilozhen.*, 18(4):92–93, 1984.

[39] Yu. A. Neretin. Combinatorial analogues of the group of diffeomorphisms of the circle. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(5):1072–1085, 1992.

[40] Yu. A. Neretin. Groups of hierarchomorphisms of trees and related Hilbert spaces. *J. Funct. Anal.*, 200(2):505–535, 2003.

[41] Igor Nikolaev. *Foliations on surfaces*, volume 41 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2001. With a foreword by Idel Bronshteyn and Chapter 16 by B. Piccoli.

[42] Alan L. T. Paterson. *Groupoids, inverse semigroups, and their operator algebras*, volume 170 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[43] Jean Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[44] Alain M. Robert. *A course in p-adic analysis*, volume 198 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[45] Claas E. Röver. Abstract commensurators of groups acting on rooted trees. *Geom. Dedicata*, 94:45–61, 2002.

[46] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[47] Richard J. Thompson. Embeddings into finitely generated simple groups which preserve the word problem. In *Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976)*, volume 95 of *Stud. Logic Foundations Math.*, pages 401–441. North-Holland, Amsterdam, 1980.

[48] Alan Weinstein. Groupoids: unifying internal and external symmetry. A tour through some examples. *Notices Amer. Math. Soc.*, 43(7):744–752, 1996.

Bruce Hughes
Vanderbilt University
E-mail: bruce.hughes@vanderbilt.edu