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Elliptic Curves in Moduli Space of Stable Bundles

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Dedicated to the memory of Eckart Viehweg

Abstract: Let M be the moduli space of rank 2 stable bundles with fixed determinant of degree 1 on a smooth projective curve C of genus $g \geq 2$. When C is generic, we show that any elliptic curve on M has degree (respect to anti-canonical divisor $-K_M$) at least 6, and we give a complete classification for elliptic curves of degree 6. Moreover, if $g > 4$, we show that any elliptic curve passing through the generic point of M has degree at least 12. We also formulate a conjecture for higher rank.
Keywords: Moduli spaces, Stable bundles, Elliptic curves.

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ be a line bundle of degree d on C. Let $M := \mathcal{SU}_C(r, \mathcal{L})^s$ be the moduli space of stable vector bundles on C of rank r and with fixed determinant $\mathcal{L}$, which is a smooth qusi-projective Fano variety with $\text{Pic}(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2(r, d)\Theta$, where Θ is an ample divisor. Let B be a smooth projective curve of genus b. The degree of a curve $\phi : B \to M$ is defined to be $\deg \phi^*(-K_M)$. It seems quite natural to ask what is the lower bound of degree and to classify the curves of lowest degree.

When $B = \mathbb{P}^1$, we have determined all $\phi : \mathbb{P}^1 \to M$ with lowest degree in [6] and all $\phi : \mathbb{P}^1 \to M$ passing through the generic point of M with lowest degree

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in [9]. In fact, one can construct $\phi : \mathbb{P} \to M$ for various projective spaces $\mathbb{P}$ such that $\phi^*(-K_M) = \mathcal{O}_{\mathbb{P}}(2(r,d))$, and $\phi : \mathbb{P}^{r-1} \to M$ passing through the generic point of M such that $\phi^*(-K_M) = \mathcal{O}_{\mathbb{P}^{r-1}}(2r)$. Then it was proved in [6] and [9] that the images of lines in these projective spaces exhaust all minimal rational curves on M (resp. minimal rational curves passing through generic point of M). Some applications of the results were also pointed out in [6] and [9]. Thus it is natural to ask what are the situation when $b > 0$. This note is a start to study the case of $b = 1$. It may happen that the normalization of $\phi(B)$ is $\mathbb{P}^1$. To avoid this case, we call $\phi : B \to M$ **an essential elliptic curve** of M if the normalization of $\phi(B)$ is an elliptic curve.

It is easy to construct essential elliptic curves of degree $6(r,d)$ on M, and essential elliptic curves of degree $6r$ that pass through the generic point of M. For example, for smooth elliptic curves $B \subset \mathbb{P}$ of degree 3, the morphism $\phi : \mathbb{P} \to M$ defines essential elliptic curves $\phi|_B : B \to M$ of degree $6(r,d)$ (See Example 3.6), which are called **elliptic curves of split type**. For smooth elliptic curves $B \subset \mathbb{P}^{r-1}$ of degree 3 (here we assume $r > 2$), the morphism $\phi : \mathbb{P}^{r-1} \to M$ defines essential elliptic curves $\phi|_B : B \to M$ of degree $6r$ passing through the generic point of M (See Example 3.5), which are called **elliptic curves of Hecke type**. Are they minimal elliptic curves of M (resp. minimal elliptic curves passing through generic point of M)? Do they exhaust all minimal essential elliptic curves on M (See Conjecture 4.8 for detail)?

In this note, we consider the case that $r = 2$ and $d = 1$, then M is a smooth projective Fano manifold of dimension $3g - 3$. When C is generic, we show that any essential elliptic curve $\phi : B \to M$ has degree at least 6, and it must be an **elliptic curve of split type** if it has degree 6 (See Theorem 4.6). When $g > 4$ and C is generic, we show that any essential elliptic curve $\phi : B \to M$ passing through the generic point of M has degree at least 12 (See Theorem 4.7). When C is generic, there is no nontrivial morphism from C to an elliptic curve, which implies that $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$ (in fact, for any line bundle $\mathcal{L}$ on $C \times B$, $\mathcal{L}$ defines a morphism $C \to J^d(B)$, which must be trivial since $J^d(B)$ is isomorphic to an elliptic curve. Thus $\mathcal{L}|_{\{x_1\}\times B} \cong \mathcal{L}|_{\{x_2\}\times B}$ for any $x_1, x_2 \in C$, and there is a line bundle L_2 on B such that $\mathcal{L}' := \mathcal{L} \otimes p_B^* L_2^{-1}$ is trivial at each fiber of $p_C : C \times B \to C$, then $L_2 := p_{C*}\mathcal{L}'$ is a line bundle and $\mathcal{L}' = p_C^* L_2$). It is the condition that we need through the whole paper.

We give a brief description of the article. In Section 2, we show a formula of degree for general case. In Section 3, we show how the general formula implies the known case $B = \mathbb{P}^1$ and construct the examples of essential elliptic curves of degree $6(r, d)$ and $6r$. In Section 4, we prove the main theorems (Theorem 4.6 and Theorem 4.7), which is the special case $r = 2$, $d = 1$ of Conjecture 4.8. Although I believe the conjecture, I leave the case of $r > 2$ to other occasion.

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2. The degree formula of curves in moduli spaces

Let C be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ a line bundle on C of degree d. Let $M = \mathcal{SU}_C(r, \mathcal{L})^s$ be the moduli spaces of stable bundles on C of rank r, with fixed determinant $\mathcal{L}$. It is well-known that $Pic(M) = \mathbb{Z} \cdot \Theta$, where Θ is an ample divisor.

Lemma 2.1. *For any smooth projective curve B of genus b, if*

$$\phi : B \to M$$

is defined by a vector bundle E on $C \times B$, then

$$\deg\phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2 := \Delta(E)$$

Proof. In general, there is no universal bundle on $C \times M$, but there exist vector bundle $\mathcal{E}nd^0$ and projective bundle $\mathcal{P}$ on $C \times M$ such that $\mathcal{E}nd^0|_{C\times\{[V]\}} = \mathcal{E}nd^0(V)$ and $\mathcal{P}|_{C\times\{[V]\}} = \mathbb{P}(V)$ for any $[V] \in M$. Let $\pi : C \times M \to M$ be the projection, then $T_M = R^1\pi_*(\mathcal{E}nd^0)$, which commutes with base changes since $\pi_*(\mathcal{E}nd^0) = 0$.

For any curve $\phi : B \to M$, let $X := C \times B$, $\mathbb{E} = (id \times \phi)^*\mathcal{E}nd^0$ and $\pi : X = C \times B \to B$ still denote the projection. Then $\phi^*T_M = R^1\pi_*\mathbb{E}$. By Riemann-Roch theorem, we have

$$\deg\phi^*(-K_M) = \chi(R^1\pi_*\mathbb{E}) + (r^2 - 1)(g - 1)(b - 1).$$

By using Leray spectral sequence and $\chi(\mathbb{E}) = \deg(ch(\mathbb{E}) \cdot td(T_X))_2$, we have $\chi(R^1\pi_*\mathbb{E}) = -\chi(\mathbb{E}) = c_2(\mathbb{E}) - (r^2 - 1)(g - 1)(b - 1)$, hence

$$\deg\phi^*(-K_M) = c_2(\mathbb{E}).$$

If $\phi : B \to M$ is defined by a vector bundle E on $X = C \times B$, then $\mathbb{E} = \mathcal{E}nd^0(E)$ (cf. the proof of lemma 2.1 in [9]). Thus

$$\deg\phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2.$$

□

Let $f : X \to C$ be the projection. Then, for any vector bundle E on X, there is a relative Harder-Narasimhan filtration (cf Theorem 2.3.2, page 45 in [5])

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i-1}$ $(i = 1, ...\, n)$ are flat over C and its restriction to general fiber $X_p = f^{-1}(p)$ is the Harder-Narasimhan filtration of $E|_{X_p}$. Thus F_i are semi-stable of slop μ_i at generic fiber of $f : X \to B$ with $\mu_1 > \mu_2 > \cdots > \mu_n$. Then we have the following theorem

Theorem 2.2. *For any vector bundle E of rank r on X, let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be the relative Harder-Narasimhan filtration over C with $F_i = E_i/E_{i-1}$ and $\mu_i = \mu(F_i|_{f^{-1}(x)})$ for generic $x \in C$. Let $\mu(E)$ and $\mu(E_i)$ denote the slope of $E|_{\pi^{-1}(b)}$ and $E_i|_{\pi^{-1}(b)}$ for generic $b \in B$. Then, if

$$\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B),$$

we have the following formula

$$\Delta(E) = 2r \left(\begin{array}{l} \sum_{i=1}^{n} \left(c_2(F_i) - \frac{\mathrm{rk}(F_i) - 1}{2\,\mathrm{rk}(F_i)} c_1(F_i)^2 \right) \\ + \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i))\mathrm{rk}(E_i)(\mu_i - \mu_{i+1}) \end{array} \right). \tag{2.1}$$

Proof. It is easy to see that

$$\begin{aligned} 2c_2(E) &= 2\sum_{i=1}^{n} c_2(F_i) + 2\sum_{i=1}^{n} c_1(E_{i-1})c_1(F_i) \\ &= 2\sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - \sum_{i=1}^{n} c_1(F_i)^2. \end{aligned}$$

Thus

$$\Delta(E) = 2r\sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - r\sum_{i=1}^{n} c_1(F_i)^2.$$

Let r_i be the rank of F_i and d_i be the degree of F_i on the generic fiber of $\pi : C \times B \to B$. Then we can write

$$c_1(F_i) = f^*\mathcal{O}_C(d_i) + \pi^*\mathcal{O}_B(r_i\mu_i)$$

where $\mathcal{O}_C(d_i)$ (resp. $\mathcal{O}_B(r_i\mu_i)$) denotes a divisor of degree d_i (resp. degree $r_i\mu_i$) of C (resp. B). Note that

$$c_1(F_i)^2 = 2d_ir_i\mu_i, \qquad c_1(E)^2 = 2d\sum_{i=1}^{n} r_i\mu_i$$

we have

$$\begin{aligned}\Delta(E) &= 2r\left(\sum_{i=1}^{n} c_2(F_i) + \mu(E)\sum_{i=1}^{n} r_i\mu_i - \sum_{i=1}^{n} d_ir_i\mu_i\right)\\ &= 2r\left(\sum_{i=1}^{n}(c_2(F_i) - (r_i-1)d_i\mu_i) + \mu(E)\sum_{i=1}^{n} r_i\mu_i - \sum_{i=1}^{n} d_i\mu_i\right).\end{aligned}$$

Let $\deg(E_i)$ denote the degree of E_i on the generic fiber of

$$\pi : C \times B \to B.$$

Using $d_i = \deg(E_i) - \deg(E_{i-1})$ and $r_i = \mathrm{rk}(E_i) - \mathrm{rk}(E_{i-1})$, we have

$$\mu(E)\sum_{i=1}^{n} r_i\mu_i - \sum_{i=1}^{n} d_i\mu_i = \sum_{i=1}^{n-1}(\mu(E) - \mu(E_i))\mathrm{rk}(E_i)(\mu_i - \mu_{i+1}).$$

Since $d_i\mu_i = c_1(F_i)^2/2r_i$, we get the formula

$$\Delta(E) = 2r\left(\begin{array}{l}\displaystyle\sum_{i=1}^{n}\left(c_2(F_i) - \frac{r_i-1}{2r_i}c_1(F_i)^2\right)\\ \displaystyle\quad + \sum_{i=1}^{n-1}(\mu(E)-\mu(E_i))\mathrm{rk}(E_i)(\mu_i - \mu_{i+1})\end{array}\right).$$

□

Remark 2.3. I do not know if the formula holds without the assumption that $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$. On the other hand, the assumption holds when B is an elliptic curve and C is generic.

Theorem 2.4. *For any torsion free sheaf $\mathcal{F}$ on $X = C \times B$, where B is any smooth projective curve, if its restriction to a fiber of $f : X = C \times B \to C$ is semi-stable, then*

$$\Delta(\mathcal{F}) = 2\,\mathrm{rk}(\mathcal{F})\,c_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1)c_1(\mathcal{F})^2 \geq 0.$$

If the determinants $\{\det(\mathcal{F}^{**})_x\}_{x\in C}$ are isomorphic to each other, then $\Delta(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is locally free and satisfies the following

- *All the bundles $\{\mathcal{F}_x := \mathcal{F}|_{\{x\}\times B}\}_{x\in C}$ are semi-stable and s-equivalent to each other, and*
- *the bundles $\{\mathcal{F}_y := \mathcal{F}|_{C\times\{y\}}\}_{y\in B}$ are isomorphic to each other.*

Proof. Since $\Delta(\mathcal{F}) \geq \Delta(\mathcal{F}^{**})$, we can assume that $\mathcal{F}$ is a vector bundle. There is a $x \in C$ such that $\mathcal{F}_x = \mathcal{F}|_{\{x\}\times B}$ is semi-stable, so is $\mathcal{E}nd^0(\mathcal{F})_x = \mathcal{E}nd^0(\mathcal{F}_x)$. Thus, by a theorem of Faltings (cf. Theorem I.2. of [1]), there is a vector bundle V on B such that

$$\mathrm{H}^0(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = \mathrm{H}^1(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = 0,$$

which defines a global section $\vartheta(V)$ of the line bundle

$$\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) = (\det f_!(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V))^{-1}$$

such that $\vartheta(V)(x) \neq 0$ where $\pi : C \times B \to B$ denotes the projection. By Grothendieck-Riemann-Roch theorem,

$$\begin{aligned} c_1(\det f_!(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)) &= f_*(\mathrm{ch}(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)\mathrm{td}(\pi^*T_B))_2 \\ &= -c_2(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) \end{aligned}$$

which means that the line bundle $\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)$ has degree

$$c_2(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) = \mathrm{rk}(V) \cdot c_2(\mathcal{E}nd^0(\mathcal{F})) = \mathrm{rk}(V) \cdot \Delta(\mathcal{F})$$

with a nonzero global section $\vartheta(V)$. Thus $\Delta(\mathcal{F}) \geq 0$.

If $\Delta(\mathcal{F}) = 0$, then $\mathcal{F} = \mathcal{F}^{**}$ must be locally free and $\vartheta(V)(x) \neq 0$ for any $x \in C$, which means that for any $x \in C$, we have

$$\mathrm{H}^0(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = \mathrm{H}^1(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = 0.$$

Then, by the theorem of Faltings (cf. Theorem I.2. of [1]), the bundles

$$\{\,\mathcal{E}nd^0(\mathcal{F})_x\,\}_{x\in C}$$

are all semi-stable. Thus, for any $x \in C$, the bundle $\mathcal{F}_x := \mathcal{F}|_{\{x\}\times B}$ is semi-stable. The bundle $\mathcal{F}$ defines a morphism $\phi_{\mathcal{F}} : C \to \mathcal{U}_B$ from C to the moduli space $\mathcal{U}_B$ of semi-stable bundles on B, the line bundle $\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)$ clearly descends to a line bundle on $\mathcal{U}_B$. If the determinants $\det(\mathcal{F}_x)$ $(x \in C)$ are fixed, then $\mathrm{Pic}(\mathcal{U}_B) \cong \mathbb{Z}$ and $\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)$ descends to ample line bundle (a positive power of anti-canonical bundle of $\mathcal{U}_B$). Thus

$$\deg(\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)) = 0$$

implies that $\mathcal{F}$ defines a constant morphism $\phi_{\mathcal{F}} : C \to \mathcal{U}_B$, which means that all $\{\mathcal{F}_x\}_{x\in C}$ are s-equivalent.

By using a technique of [4] (see Step 5 in the proof of Theorem 4.2 in [4], see also the proof of Theorem I.4 in [1]), we will show

$$\mathcal{F}|_{C\times\{y_1\}} \cong \mathcal{F}|_{C\times\{y_2\}}, \quad \forall\, y_1,\, y_2 \in B.$$

Choose a nontrivial extension $0 \to V \to V' \xrightarrow{q_1} \mathcal{O}_{y_1} \to 0$ on B, let $\mathfrak{Q}$ be the Quot-scheme of rank 0 and degree 1 quotients of V', and

$$0 \to \mathcal{K} \to p_B^* V' \to \mathfrak{T} \to 0$$

be the tautological exact sequence on $B \times \mathfrak{Q}$. Fix a point $x_1 \in C$, then the set $q \in \mathfrak{Q}$ such that $\mathrm{H}^0(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = \mathrm{H}^1(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = 0$ is an open set $U \subset \mathfrak{Q}$ and $U \neq \emptyset$ since $q_1 = (0 \to V \to V' \xrightarrow{q_1} \mathcal{O}_{y_1} \to 0) \in U$.

Let $\Gamma \subset B \times \mathbb{P}(V')$ be the graph of $\mathbb{P}(V') \xrightarrow{p} B$, then

$$p_B^* V' \to p_B^* V'|_{\Gamma} = p^* V' \to \mathcal{O}(1) \to 0$$

induces a quotient $p_B^* V' \to {}_{\Gamma}\mathcal{O}(1) \to 0$ on $B \times \mathbb{P}(V')$, which defines a morphism $\mathbb{P}(V') \to \mathfrak{Q}$. It is easy to see that $\mathbb{P}(V') \to \mathfrak{Q}$ is surjective (in fact, it is a isomorphism). Thus there is an open $B_1 \subset B$ with $y_1 \in B_1$ such that for any $y \in B_1$ there exists an exact sequence

$$0 \to \mathcal{K}_q \to V' \xrightarrow{q} \mathcal{O}_y \to 0 \tag{2.2}$$

such that $\mathrm{H}^0(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = \mathrm{H}^1(\mathcal{F}_{x_1} \otimes \mathcal{K}_q) = 0$, which implies

$$\mathrm{H}^0(\mathcal{F}_x \otimes \mathcal{K}_q) = \mathrm{H}^1(\mathcal{F}_x \otimes \mathcal{K}_q) = 0 \qquad \forall\, x \in C$$

since $\mathcal{F}_x$ is s-equivalent to $\mathcal{F}_{x_1}$ for any $x \in C$. Pull back the exact sequence (2.2) by $\pi : C \times B \to B$ and tensor with $\mathcal{F}$, we have the exact sequence

$$0 \to \mathcal{F} \otimes \pi^* \mathcal{K}_q \to \mathcal{F} \otimes \pi^* V' \to \mathcal{F}_y \to 0. \tag{2.3}$$

Take direct images of (2.2) under $f : C \times B \to C$, we have

$$\mathcal{F}_y \cong f_*(\mathcal{F} \otimes \pi^* V'), \quad \forall\, y \in B_1$$

which implies that all $\{\mathcal{F}_y\}_{y \in B}$ are isomorphic each other. □

We will need the following lemma in the later computation, whose proofs are straightforward computations (see Lemma 1 in Chapter 2 of [2] for the case of rank 1).

Lemma 2.5. *Let X be a smooth projective surface and $j : D \hookrightarrow X$ be an effective divisor. Then, for any vector bundle V on D, we have*

$$c_1(j_*V) = \mathrm{rk}(V) \cdot D$$
$$c_2(j_*V) = \frac{\mathrm{rk}(V)(\mathrm{rk}(V)+1)}{2} D^2 - j_* c_1(V).$$

Recall that $X_t = f^{-1}(t)$ denotes the fiber of $f : X \to C$ and for any vector bundle $\mathcal{F}$ on X, $\mathcal{F}_t$ denote the restrictions of $\mathcal{F}$ to X_t.

Lemma 2.6. *Let $\mathcal{F}_t \to W \to 0$ be a locally free quotient and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to {}_{X_t}W \to 0$$

be the elementary transformation of $\mathcal{F}$ along W at $X_t \subset X$. Then

$$\Delta(\mathcal{F}) = \Delta(\mathcal{F}') + 2r(\mu(\mathcal{F}_t) - \mu(W))\mathrm{rk}(W).$$

3. Mminimal rational curves and examples of elliptic curves on moduli spaces

When $B = \mathbb{P}^1$, the condition $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$ always hold and any morphism $B \to M$ is defined by a vector bundle on $C \times B$ (cf. Lemma 2.1 of [9]).

Recall that given two nonnegative integers k, ℓ, a vector bundle W of rank r and degree d on C is (k, ℓ)-stable, if, for each proper subbundle W' of W, we have

$$\frac{\deg(W') + k}{\mathrm{rk}(W')} < \frac{\deg(W) + k - \ell}{r}.$$

The usual stability is equivalent to $(0,0)$-stability. The (k, ℓ)-stability is an open condition. The proofs of following lemmas are easy and elementary (cf. [7]).

Lemma 3.1. *If $g \geq 3$, M contains $(0,1)$-stable and $(0,1)$-stable bundles. M contains a $(1,1)$-stable bundle W unless $g = 3$, d, r both even.*

Lemma 3.2. *Let $0 \to V \to W \to \mathcal{O}_p \to 0$ be an exact sequence, where $\mathcal{O}_p$ is the 1-dimensional skyscraper sheaf at $p \in C$. If W is (k,ℓ)-stable, then V is $(k, \ell - 1)$-stable.*

A curve $B \to M$ defined by E on $C \times B$ passing through the generic point of M satisfies that $E_y := E|_{C\times\{y\}}$ is $(1,1)$-stable for generic $y \in B$. Thus in the formula (2.1) of Theorem 2.2 we have

$$(\mu(E) - \mu(E_i))\mathrm{rk}(E_i) > 1. \tag{3.1}$$

On the other hand, any semi-stable bundle on $B = \mathbb{P}^1$ must have integer slope. By the formula (2.1) in Theorem 2.2, we have

$$\Delta(E) > 2r$$

if E is not semi-stable on the generic fiber of $f : X = C \times \mathbb{P}^1 \to C$.

When E is semi-stable on the generic fiber of $f : X \to C$, by tensor E with a line bundle, we can assume that E is trivial on the generic fiber of $f : X \to C$. Thus $\Delta(E) = 2rc_2(E) \geq 2r$ and there must be a fiber $X_t = f^{-1}(t)$ such that $E_t = E|_{X_t}$ is not semi-stable by Theorem 2.4. If $\Delta(E) = 2r$, by Lemma 2.6, we must have $\mathrm{rk}(W) = 1$, $\mu(W) = -1$ and $\Delta(\mathcal{F}') = 0$ in Lemma 2.6. Thus $\Delta(E) = 2r$ if and only if E satisfies

$$0 \to f^*V \to E \to {}_{X_t}\mathcal{O}_{\mathbb{P}^1}(-1) \to 0$$

which defines a so called Hecke curve. Therefore we get the main theorem in [9].

Theorem 3.3. *If $g \geq 3$, then any rational curve of M passing through the generic point of M has at least degree $2r$ with respect to $-K_M$. It has degree $2r$ if and only if it is a Hecke curve unless $g = 3$, $r = 2$ and $(2,d) = 2$.*

At the end of this section, we give some examples of elliptic curves on M. Let us recall the construction of Hecke curves. Let $\mathcal{U}_C(r, d-1)$ be the moduli space of stable bundles of rank r and degree $d - 1$. Let

$$\mathfrak{D} \subset \mathcal{U}_C(r, d-1)$$

be the open set of $(1,0)$-stable bundles. Let $C\times\mathfrak{O}\xrightarrow{\psi}J^d(C)$ be defined as $\psi(x,V)=\mathcal{O}_C(x)\otimes\det(V)$ and

$$\mathscr{R}_C:=\psi^{-1}(\mathcal{L})\subset C\times\mathfrak{O},$$

which consists of the points (x,V) such that V are $(1,0)$-stable bundles on C with $\det(V)=\mathcal{L}(-x)$. There exists a projective bundle

$$p:\mathscr{P}\to\mathscr{R}_C$$

such that for any $(x,V)\in\mathscr{R}_C$ we have $p^{-1}(x,V)=\mathbb{P}(V_x^\vee)$. Let

$$V_x^\vee\otimes\mathcal{O}_{\mathbb{P}(V_x^\vee)}\to\mathcal{O}_{\mathbb{P}(V_x^\vee)}(1)\to 0$$

be the universal quotient, $f:C\times\mathbb{P}(V_x^\vee)\to C$ be the projection, and

$$0\to\mathscr{E}^\vee\to f^*V^\vee\to {}_{\{x\}\times\mathbb{P}(V_x^\vee)}\mathcal{O}_{\mathbb{P}(V_x^\vee)}(1)\to 0$$

where $\mathscr{E}^\vee$ is defined to the kernel of the surjection. Taking dual, we have

$$0\to f^*V\to\mathscr{E}\to {}_{\{x\}\times\mathbb{P}(V_x^\vee)}\mathcal{O}_{\mathbb{P}(V_x^\vee)}(-1)\to 0, \tag{3.2}$$

which, at any $\xi=(V_x^\vee\to\Lambda\to 0)\in\mathbb{P}(V_x^\vee)$, gives an exact sequence

$$0\to V\xrightarrow{\iota}\mathscr{E}_\xi\to\mathcal{O}_x\to 0$$

on C such that $\ker(\iota_x)=\Lambda^\vee\subset V_x$. V being $(1,0)$-stable implies the stability of $\mathscr{E}_\xi$. Thus (3.2) defines

$$\Psi_{(x,V)}:\ \mathbb{P}(V_x^\vee)=p^{-1}(x,V)\to M. \tag{3.3}$$

Definition 3.4. The images (under $\{\Psi_{(x,V)}\}_{(x,V)\in\mathscr{R}_C}$) of lines in the fibres of $p:\mathscr{P}\to\mathscr{R}_C$ are the so called **Hecke curves** in M. The images (under $\{\Psi_{(x,V)}\}_{(x,V)\in\mathscr{R}_C}$) of elliptic curves in the fibres of

$$p:\mathscr{P}\to\mathscr{R}_C$$

are called **elliptic curves of Hecke type**.

It is known (cf. [7, Lemma 5.9]) that the morphisms in (3.3) are closed immersions. By a straightforward computation, we have

$$\Psi^*_{(x,V)}(-K_M)=\mathcal{O}_{\mathbb{P}(V_x^\vee)}(2r). \tag{3.4}$$

For any point $[W]\in M$ and $(W_x\to\mathbb{C}\to 0)\in\mathbb{P}(W_x)$, where W is $(1,1)$-stable, we define a $(1,0)$-stable bundle V by

$$0\to V\xrightarrow{\alpha}W\to {}_x\mathbb{C}\to 0.$$

Then the images of $p^{-1}(x,V) = \mathbb{P}(V_x^\vee)$ are projective spaces that pass through $[W] \in M$, and the images of lines $\ell \subset \mathbb{P}(V_x^\vee)$ that pass through $[\ker(\alpha_x)] \in \mathbb{P}(V_x^\vee)$ are Hecke curves passing through $[W] \in M$.

Example 3.5. When $g \geq 4$ and $r > 2$, for generic $[W] \in M$, the images of smooth elliptic curves $B \subset \mathbb{P}(V_x^\vee)$ with degree 3 and $[\ker(\alpha_x)] \in B$ are smooth elliptic curves on M that pass through $[W] \in M$, which have degree $6r$ by (3.4).

If we do not require the curve $\phi : B \to M$ passing through generic point of M, we may construct rational curves and elliptic curves with smaller degree. Let us recall the Construction 2.3 from [6].

For any given r and d, let r_1, r_2 be positive integers and d_1, d_2 be integers that satisfy the equalities $r_1 + r_2 = r$, $d_1 + d_2 = d$ and

$$r_1 \frac{d}{(r,d)} - d_1 \frac{r}{(r,d)} = 1, \quad d_2 \frac{r}{(r,d)} - r_2 \frac{d}{(r,d)} = 1.$$

Let $\mathcal{U}_C(r_1,d_1)$ (resp. $\mathcal{U}_C(r_2,d_2)$) be the moduli space of stable vector bundles with rank r_1 (resp. r_2) and degree d_1 (resp. d_2). Then, since $(r_1,d_1) = 1$ and $(r_2,d_2) = 1$, there are universal vector bundles $\mathcal{V}_1$, $\mathcal{V}_2$ on $C \times \mathcal{U}_C(r_1,d_1)$ and $C \times \mathcal{U}_C(r_2,d_2)$ respectively. Consider

$$\mathcal{U}_C(r_1,d_1) \times \mathcal{U}_C(r_2,d_2) \xrightarrow{\det(\bullet)\times\det(\bullet)} J_C^{d_1} \times J_C^{d_2} \xrightarrow{(\bullet)\otimes(\bullet)} J_C^d,$$

let $\mathcal{R}(r_1,d_1)$ be its fiber at $[\mathcal{L}] \in J_C^d$. The pullback of $\mathcal{V}_1$, $\mathcal{V}_2$ by the projection $C \times \mathcal{R}(r_1,d_1) \to C \times \mathcal{U}_C(r_i,d_i)$ $(i = 1,2)$ is still denoted by $\mathcal{V}_1$, $\mathcal{V}_2$ respectively. Let $p : C \times \mathcal{R}(r_1,d_1) \to \mathcal{R}(r_1,d_1)$ and

$$\mathcal{G} = R^1 p_*(\mathcal{V}_2^\vee \otimes \mathcal{V}_1),$$

which is locally free of rank $r_1 r_2(g-1) + (r,d)$. Let

$$q : P(r_1,d_1) = \mathbb{P}(\mathcal{G}) \to \mathcal{R}(r_1,d_1)$$

be the projective bundle parametrzing 1-dimensional subspaces of $\mathcal{G}_t$ $(t \in \mathcal{R}(r_1,d_1))$ and $f : C \times P(r_1,d_1) \to C$, $\pi : C \times P(r_1,d_1) \to P(r_1,d_1)$ be the projections. Then there is a universal extension

$$0 \to (id \times q)^* \mathcal{V}_1 \otimes \pi^* \mathcal{O}_{P(r_1,d_1)}(1) \to \mathcal{E} \to (id \times q)^* \mathcal{V}_2 \to 0 \tag{3.5}$$

on $C \times P(r_1,d_1)$ such that for any $x = ([V_1],[V_2],[e]) \in P(r_1,d_1)$, where $[V_i] \in \mathcal{U}_C(r_i,d_i)$ with $\det(V_1) \otimes \det(V_2) = \mathcal{L}$ and $[e] \subset \mathrm{H}^1(C, V_2^\vee \otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}|_{C\times\{x\}}$ is the isomorphism class of vector bundles V given by extensions

$$0 \to V_1 \to V \to V_2 \to 0$$

that are defined by vectors on the line $[e] \subset \mathrm{H}^1(C, V_2^\vee \otimes V_1)$. Then V must be stable by [6, Lemma 2.2], and the sequence (3.5) defines

$$\Phi : P(r_1, d_1) \to \mathcal{SU}_C(r, \mathcal{L})^s = M.$$

On each fiber $q^{-1}(\xi) = \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))$ at $\xi = (V_1, V_2)$, the morphisms

$$\Phi_\xi := \Phi|_{q^{-1}(\xi)} : q^{-1}(\xi) = \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1)) \to M \tag{3.6}$$

is birational and $\Phi_\xi^*(-K_M) = \mathcal{O}_{\mathbb{P}(\mathrm{H}^1(V_2^\vee\otimes V_1))}(2(r,d))$ by [6, Lemma 2.4].

Example 3.6. The images of lines $\ell \subset \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))$ are rational curves of degree $2(r,d)$ on M, which is clearly of the minimal degree since $-K_M = 2(r,d)\Theta$. For smooth elliptic curves $B \subset \mathbb{P}(\mathrm{H}^1(V_2^\vee \otimes V_1))$ of degree 3, the images of $\Phi_\xi : B \to M$ are of degree $6(r,d)$. For any smooth elliptic curve $B \subset q^{-1}(\xi)$ ($\forall\ \xi \in \mathcal{R}(r_1, d_1)$), the images of $\Phi_\xi : B \to M$ are called **elliptic curves of split type**.

4. Minimal elliptic curves on moduli spaces

In this section, we consider the moduli space M of rank 2 stable bundles on C with a fixed determinant $\mathcal{L}$ of degree 1. We also assume that the curve C is generic in the sense that C admits no surjective morphism to an elliptic curve. With this assumption, we know that $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$ for any elliptic curve B.

For a morphism $\phi : B \to M$, it may happen that the normalization of $\phi(B)$ is a rational curve. To avoid this case, we make the following definition

Definition 4.1. $\phi : B \to M$ is called an essential elliptic curve of M if the normalization of $\phi(B)$ is an elliptic curve.

Let $\phi : B \to M$ be a morphism defined by a vector bundle E on $X = C \times B$ (Such E exists for any ϕ since M has a universal family and it is determined up to tensoring by a pull-back of line bundle on B). In this section, B will always denote an elliptic curve.

Proposition 4.2. *Let $\phi : B \to M$ be an essential elliptic curve of M defined by a vector bundle E. If E is not semi-stable on the generic fiber of $f : X \to C$, then*

$$\Delta(E) \geq 6.$$

If $g = g(C) \geq 4$ and the curve $\phi : B \to M$ passes through the generic point of M, then

$$\Delta(E) > 12.$$

Proof. Let $0 \to E_1 \to E \to F_2 \to 0$ be the relative Harder-Narasimhan filtration over C. Then we have exact sequence

$$0 \to E_1|_{X_t} \to E|_{X_t} \to F_2|_{X_t} \to 0$$

on each fiber $X_t = \{t\} \times B$ of $f : X \to C$ since E_1, F_2 are flat over C. Thus E_1 is locally free (cf. Lemma 1.27 of [8]) and

$$\Delta(E) = 4c_2(F_2) + 4(\mu(E) - \mu(E_1))(\mu_1 - \mu_2) \tag{4.1}$$

where $\mu_1 = \deg(E_1|_{X_t})$, $\mu_2 = \deg(F_2|_{X_t})$ for $t \in C$ (cf. Theorem 2.2).

That $0 \to E_1 \to E \to F_2 \to 0$ is the relative Harder-Narasimhan filtration over C means for almost all $t \in C$ the exact sequences

$$0 \to E_1|_{X_t} \to E|_{X_t} \to F_2|_{X_t} \to 0$$

are the Harder-Narasimhan filtration of $E|_{X_t}$, which in particular means that F_2 is locally free over $f^{-1}(C \setminus T)$ where $T \subset C$ is a finite set. Thus

$$0 \to E_1|_{C\times\{y\}} \to E|_{C\times\{y\}} \to F_2|_{C\times\{y\}} \to 0\ , \quad \forall\, y \in B \tag{4.2}$$

are exact sequences, which imply that F_2 is also B-flat.

If $c_2(F_2) = 0$, then F_2 is a line bundle and there are line bundles V_1, V_2 on C such that

$$E_1 = f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1), \quad F_2 = f^*V_2 \otimes \pi^*\mathcal{O}(\mu_2)$$

where $\mathcal{O}(\mu_i)$ denote line bundles on B of degree μ_i. Replace E by $E \otimes \pi^*\mathcal{O}(-\mu_2)$, we can assume that E satisfies

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \to E \to f^*V_2 \to 0. \tag{4.3}$$

Let $d_i = \deg(V_i)$, $J = \{(L_1, L_2) \in J_C^{d_1} \times J_C^{d_2} \,|\, L_1 \otimes L_2 = \mathcal{L}\}$, and let $\mathcal{V}_i$ be the pullback of universal line bundle on $C \times J_C^{d_i}$ (under the morphism $C \times J \to$

$C \times J_C^{d_i}$). Then $\mathcal{G} := \mathrm{R}^1 p_{J*}(\mathcal{V}_2^{-1} \otimes \mathcal{V}_1)$ is locally free of rank $d_2 - d_1 + g - 1$, where $p_J : C \times p_J \to J$ is the projection. Let

$$q : P = \mathbb{P}(\mathcal{G}) \to J$$

be the projective bundle parametrizing 1-dimensional subspaces of $\mathcal{G}_t$ for any point $t \in J$. Then there is an universal extension

$$0 \to (id \times q)^* \mathcal{V}_1 \otimes \pi^* \mathcal{O}_P(1) \to \mathcal{E} \to (id \times q)^* \mathcal{V}_2 \to 0 \tag{4.4}$$

on $C \times P$, where $\pi : C \times P \to P$ denotes the projection. For any $x = ([V_1], [V_2], [e]) \in P$, the bundle $\mathcal{E}|_{C \times \{x\}}$ is the isomorphism class of vector bundles V given by extensions

$$0 \to V_1 \to V \to V_2 \to 0$$

which are defined by vectors on the line $[e] \subset \mathrm{H}^1(C, V_2^{-1} \otimes V_1)$. Thus the exact sequence (4.3) induces a morphism

$$\psi : B \to \mathbb{P}^{d_2 - d_1 + g - 2} = q^{-1}(V_1, V_2) \subset P \tag{4.5}$$

such that $\mathcal{O}(\mu_1 - \mu_2) = \psi^* \mathcal{O}_P(1)$ and $\phi : B \to M$ factors through $\psi : B \to \psi(B) \subset \mathbb{P}^{d_2 - d_1 + g - 2}$, which implies that the normalization of $\psi(B)$ is an elliptic curve. Hence $\mu_1 - \mu_2 \geq 3$ and $\Delta(E) \geq 6$ by (4.1). If $\phi : B \to M$ passes through the generic point, then $\mu(E) - \mu(E_1) > 1$ and $\Delta(E) > 12$.

If $c_2(F_2) \neq 0$, F_2 is not locally free, which implies that there is a $y_0 \in B$ such that $F_2|_{C \times \{y_0\}}$ has torsion $\tau(F_2|_{C \times \{y_0\}}) \neq 0$ since F_2 is B-flat (cf. Lemma 1.27 of [8]). Let

$$0 \to \tau(F_2|_{C \times \{y_0\}}) \to F_2|_{C \times \{y_0\}} \to F_2^0 \to 0. \tag{4.6}$$

Then F_2^0 being a quotient line bundle of $E|_{C \times \{y_0\}}$ implies

$$\deg(F_2^0) > \mu(E|_{C \times \{y_0\}}) = \frac{1}{2}$$

since $E|_{C \times \{y_0\}}$ is stable. By sequences (4.2) and (4.6), we have

$$\mu(E_1) = \deg(E_1|_{C \times \{y_0\}}) = 1 - \deg(F_2^0) - \dim \tau(F_2|_{C \times \{y_0\}}) \leq -1$$

which, by the formula (4.1), implies that

$$\Delta(E) \geq 4c_2(F_2) + 4(\frac{1}{2} + 1)(\mu_1 - \mu_2) \geq 10.$$

When $\phi : B \to M$ passes through a generic point, in order to show $\Delta(E) > 12$, we note that $c_2(F_2) \neq 0$ and F_2 being C-flat also imply that there exists a $t_0 \in C$

such that $F_2|_{X_{t_0}}$ has torsion $\tau(F_2|_{X_{t_0}}) \neq 0$. Let $0 \to \tau(F_2|_{X_{t_0}}) \to F_2|_{X_{t_0}} \to \mathcal{Q} \to 0$ and $E' = \ker(E \to {}_{X_{t_0}}\mathcal{Q})$, then

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{Q} \to 0$$

which, for any $y \in B$, induces exact sequence

$$0 \to E'|_{C\times\{y\}} \to E|_{C\times\{y\}} \to {}_{(t_0,y)}\mathcal{Q} \to 0. \tag{4.7}$$

Thus all $E'_y := E'|_{C\times\{y\}}$ are semi-stable of degree 0. If $\phi : B \to M$ passes through a generic point, then there is a $y_0 \in B$ such that E_{y_0} is $(1,1)$-stable on $X_{y_0} = C \times \{y_0\}$, thus E'_{y_0} is stable by (4.7) and Lemma 3.2. This implies that $\Delta(E') > 0$. Otherwise $\{E'_y\}_{y\in B}$ are s-equivalent to each other by applying Theorem 2.4 to $\pi : X \to B$, which implies $E' = f^*V \otimes \pi^*L$ for a stable bundle V on C and a line bundle L on B. Then $E_t = E'_t = L \oplus L$ for any $t \neq t_0$, which is a contradiction since E is not semi-stable on the generic fiber of $f : X \to C$.

To compute $\Delta(E')$, consider the Harder-Narasimhan filtration

$$0 \to E'_1 \to E' \to F'_2 \to 0$$

over C, let $\mu'_1 = \deg(E'_1|_{X_t})$, $\mu'_2 = \deg(F'_2|_{X_t})$ for $t \in C$, then

$$\Delta(E') = 4c_2(F'_2) + 4(\mu(E') - \mu(E'_1))(\mu'_1 - \mu'_2) \geq 8.$$

To see it, we can assume $c_2(F'_2) = 0$, then there are line bundles V'_i on C and line bundles $\mathcal{O}(\mu'_i)$ on B of degree μ'_i such that

$$0 \to f^*V'_1 \otimes \pi^*\mathcal{O}(\mu'_1 - \mu'_2) \to E' \otimes \pi^*\mathcal{O}(-\mu'_2) \to f^*V'_2 \to 0$$

which defines a morphism $\psi : B \to \mathbb{P}$ to a projective space such that $\mathcal{O}(\mu'_1 - \mu'_2) = \psi^*\mathcal{O}_{\mathbb{P}}(1)$. Thus $\mu'_1 - \mu'_2 \geq 2$ and $\Delta(E') \geq 8$. Then

$$\Delta(E) = \Delta(E') + 4(\mu(E|_{X_{t_0}}) - \mu(\mathcal{Q})) \geq \Delta(E') + 6 \geq 14.$$

□

Now we consider the case that E is semi-stable on the generic fiber of $f : X \to C$. We can assume $0 \leq \deg(E|_{X_t}) \leq 1$ on $X_t = f^{-1}(t)$.

Proposition 4.3. *When E is semi-stable of degree 1 on the generic fiber of $f : X \to C$, we have $\Delta(E) \geq 10$. If $g > 4$ and $\phi : B \to M$ passes through the generic point, then $\Delta(E) \geq 14$.*

Proof. In the case of rank 2 and degree 1, semi-stability is equivalent to stability. If all $\{E_t = E|_{X_t}\}_{t\in C}$ are semi-stable (note that their determinants are fixed), then all $\{E_t\}_{t\in C}$ are isomorphic to each other since the moduli space of stable rank 2 bundle with a fixed determinant of degree 1 on an elliptic curve has dimension 0. Thus $\Delta(E) > 0$ if and only if there exists $t_1 \in C$ such that $E_{t_1} = E|_{X_{t_1}}$ is not semi-stable.

Let $E_{t_1} \to \mathcal{O}(\mu_1) \to 0$ be the quotient of minimal degree and

$$0 \to E^{(1)} \to E \to {}_{X_{t_1}}\mathcal{O}(\mu_1) \to 0$$

be the elementary transformation of E along $\mathcal{O}(\mu_1)$ at X_{t_1}. If $E^{(i)}$ is defined and $\Delta(E^{(i)}) > 0$, let $t_{i+1} \in C$ such that $E^{(i)}_{t_{i+1}} = E^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $E^{(i)}_{t_{i+1}} \to \mathcal{O}(\mu_{i+1}) \to 0$ be the quotient of minimal degree, then we define $E^{(i+1)}$ to be the elementary transformation of $E^{(i)}$ along $\mathcal{O}(\mu_{i+1})$ at $X_{t_{i+1}}$, namely $E^{(i+1)}$ satisfies the exact sequence

$$0 \to E^{(i+1)} \to E^{(i)} \to {}_{X_{t_{i+1}}}\mathcal{O}(\mu_{i+1}) \to 0. \tag{4.8}$$

Let s be the minimal integer such that $\Delta(E^{(s)}) = 0$. Then

$$\Delta(E) = 2\cdot s - 4\sum_{i=1}^{s}\mu_i \tag{4.9}$$

where $\mu_i \leq 0$ $(i = 1\,,\,2\,,\ldots,s)$. Since $\Delta(E^{(s)}) = 0$, all $\{E^{(s)}_t\}_{t\in C}$ are stable bundles of degree 1, thus $\mathrm{H}^1(B, E^{(s)}_t) \cong \mathrm{H}^0(B, (E^{(s)}_t)^\vee) = 0$, which implies $R^1f_*E^{(s)} = 0$. Taking direct images of (4.8), we have

$$0 \to f_*E^{(s)} \to f_*E^{(s-1)} \to {}_{t_s}\mathrm{H}^0(\mathcal{O}(\mu_s)) \to 0 \tag{4.10}$$

and $\deg(f_*E^{(i+1)}) \leq \deg(f_*E^{(i)})$, which imply

$$\deg(f_*E^{(s)}) \leq \deg(f_*E) - \dim \mathrm{H}^0(\mathcal{O}(\mu_s)). \tag{4.11}$$

Restrict (4.8) to a fiber $X_y = \pi^{-1}(y)$, we have exact sequence

$$0 \to E^{(i+1)}_y \to E^{(i)}_y \to {}_{(t_{i+1},y)}\mathbb{C} \to 0,$$

which implies that

$$\deg(E^{(s)}_y) = \deg(E_y) - s = 1 - s. \tag{4.12}$$

On the other hand, by Theorem 2.4, $\Delta(E^{(s)}) = 0$ implies that there exist a stable rank 2 vector bundle V of degree 1 on B and a line bundle L on C such that $E^{(s)} = \pi^*V \otimes f^*L$. It is easy to see

$$\deg(E_y^{(s)}) = 2\deg(L) = 2\deg(f_*E^{(s)}).$$

Thus, combining (4.11) and (4.12), we have the inequality

$$s \geq 1 - 2\deg(f_*E) + 2\dim \mathrm{H}^0(\mathcal{O}(\mu_s)). \tag{4.13}$$

We claim that $\deg(f_*E) \leq -1$. To show it, consider

$$0 \to \mathcal{F}' := f^*(f_*E) \to E \to \mathcal{F} \to 0 \tag{4.14}$$

where $\mathcal{F}$ is locally free on $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that E_t ($t \in T$) is not semi-stable. Thus, for any $y \in B$, the sequence

$$0 \to \mathcal{F}'_y \to E_y \to \mathcal{F}_y \to 0 \tag{4.15}$$

is still exact, which implies that $\mathcal{F}$ is B-flat (cf. Lemma 2.1.4 of [5]). The sequence (4.15) already implies $\deg(f_*E) = \deg(\mathcal{F}'_y) \leq 0$ since E_y is stable of degree 1. Thus $\mathcal{F}$ can not be locally free since

$$4 \cdot c_2(\mathcal{F}) = \Delta(E) - 4 \cdot \deg(f_*E) + 2 > 0.$$

To see this computation, using (4.14) and noting that f_*E is a line bundle, we have $c_2(\mathcal{F}) = c_2(E) - c_1(\mathcal{F}') \cdot c_1(E) = c_2(E) - \deg(f_*E)$, thus $4 \cdot c_2(\mathcal{F}) = \Delta(E) - 4 \cdot \deg(f_*E) + 2$ since $c_1(E)^2 = 2$ (here we use $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$). Then there is at least a $y_0 \in B$ such that $\mathcal{F}_{y_0}$ has torsion, otherwise $\mathcal{F}$ is locally free (cf. Lemma 1.27 of [8]). The stability of E_{y_0} implies that $\mathcal{F}_{y_0}/torsion$ has degree at least 1. Thus $\deg(\mathcal{F}_{y_0}) \geq 2$ and

$$\deg(f_*E) = \deg(\mathcal{F}'_{y_0}) \leq -1,$$

which proves the claim. The claim implies $s \geq 3 + 2\dim \mathrm{H}^0(\mathcal{O}(\mu_s))$. Therefore, if $\mu_s < 0$, we have $\Delta(E) \geq 2 \cdot s + 4 \geq 10$ by (4.9). If $\mu_s = 0$, by tensoring E with $\pi^*\mathcal{O}(\mu_s)^{-1}$, we may assume $\dim \mathrm{H}^0(\mathcal{O}(\mu_s)) = 1$, then $s \geq 5$ and $\Delta(E) \geq 10$.

If $\phi : B \to M$ passes through the generic point of M, we claim that $\deg(f_*E) \leq -2$, which implies $\Delta(E) \geq 14$. To prove the claim, assume $\deg(f_*E) = -1$, we will show that $\phi(B)$ lies in a given divisor. Note that $\mathcal{F}_y$ must be locally free of degree 2 for generic $y \in B$ (if $\mathcal{F}_y$ has nontrivial torsion, then E_y has a quotient line bundle of degree at most 1, which is impossible since E_y is $(1,1)$-stable for generic $y \in B$). Thus E_y satisfies $0 \to \xi \to E_y \to \xi^{-1} \otimes \mathcal{L} \to 0$ where ξ is a line bundle of degree -1 on C. The locus of such bundles has dimension at most $g + h^1(\xi^2 \otimes \mathcal{L}^{-1}) - 1 = 2g + 1 < \dim(M)$ when $g > 4$. We are done. □

Now we consider the case that E is semi-stable of degree 0 on the generic fiber of $f : X \to C$. If E is semi-stable on every fiber of $f : X \to C$, then E induces a non-trivial morphism

$$\varphi_E : C \to \mathbb{P}^1$$

(cf. Theorem 1.5 of [3]) such that $\varphi_E^* \mathcal{O}_{\mathbb{P}^1}(1) = \Theta(E) = (\det f_! E)^{-1}$, which has degree $c_2(E)$ by Grothendieck-Riemann-Roch theorem. Thus

$$\Delta(E) = 4 \cdot c_2(E) = 4 \cdot \deg(\varphi_E) \geq 8. \tag{4.16}$$

If there is a $t_0 \in C$ such that $E_{t_0} = E|_{X_{t_0}}$ is not semi-stable on $X_{t_0} = f^{-1}(t_0)$, let $E_{t_0} \to \mathcal{O}(\mu) \to 0$ be the quotient line bundle of minimal degree μ and $E' =$ kernel $(E \to {}_{X_{t_0}}\mathcal{O}(\mu) \to 0)$, then we have

Lemma 4.4. *If $\Delta(E') = 0$, then there is a semi-stable vector bundle V on C and a line bundle L of degree 0 on B such that*

$$E' = f^*V \otimes \pi^* L.$$

Proof. By the definition, $\{E'_t = E'|_{\{t\}\times B}\}_{t\in C}$ and $\{E'_y = E'|_{C\times\{y\}}\}_{y\in B}$ are families of semi-stable bundles of degree 0. Apply Theorem 2.4 to $f : X \to C$ (resp. $\pi : X \to B$), then $\Delta(E') = 0$ implies that $\{E'_t\}_{t\in C}$ (resp. $\{E'_y\}_{y\in B}$) are isomorphic to each other. By tensoring E (thus E') with $\pi^* L^{-1}$ (where L is a line bundle of degree 0 on B), we can assume that $\mathrm{H}^0(E'_t) \neq 0$ ($\forall\, t \in C$), which has dimension at most 2 since E'_t is semi-stable of degree 0. If $\mathrm{H}^0(E'_t)$ has dimension 2, then $E' = f^*(f_* E')$ and we are done.

If $\mathrm{H}^0(E'_t)$ has dimension 1, we will show a contradiction. In fact, by the definition of E', we have an exact sequence

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{O}(\mu) \to 0 \tag{4.17}$$

where $\mathcal{O}(\mu)$ is a line bundle on $\{t_0\} \times B \cong B$ of degree $\mu < 0$. Then

$$V_1 := f_* E = f_* E'$$

is a line bundle on C. Since $\{E'_t\}_{t\in C}$ are isomorphic to each other and $\mathrm{H}^0(E'_t)$ has dimension 1, we have the exact sequence

$$0 \to f^*V_1 \to E' \to f^*V_2 \otimes \pi^*L_0 \to 0 \tag{4.18}$$

for a line bundle V_2 on C and a degree 0 line bundle L_0 on B. If $L_0 \neq \mathcal{O}_B$, then $R^i f_*(f^*(V_2^{-1} \otimes V_1) \otimes L_0) = V_2^{-1} \otimes V_1 \otimes \mathrm{H}^i(L_0) = 0$ $(i = 0, 1)$, which implies $\mathrm{H}^1(X, f^*(V_2^{-1} \otimes V_1) \otimes L_0) = 0$ and (4.18) is splitting. This is impossible since E'_y is semi-stable of degree 0 and we can show that $\deg(V_1) = \deg(f_*E) \leq -1$ in the following.

To prove that $\deg(f_*E) \leq -1$, we consider the exact sequence

$$0 \to f^*f_*E \to E \to \mathcal{F} \to 0 \tag{4.19}$$

where $\mathcal{F}|_{f^{-1}(C\setminus\{t_0\})}$ is locally free of rank 1 by (4.18). But $\mathcal{F}$ is not locally free (otherwise $c_2(E) = (c_1(E) - c_1(f^*f_*E)) \cdot c_1(f^*f_*E) = 0$) and for any $y \in B$ the restriction of (4.19) to $X_y = \pi^{-1}(y)$

$$0 \to f_*E \to E_y \to \mathcal{F}_y \to 0 \tag{4.20}$$

is exact, which means that $\mathcal{F}$ is B-flat (cf. Lemma 2.1.4 of [5]). Thus, by Lemma 1.27 of [8], there is a $y_0 \in B$ such that $\mathcal{F}_{y_0}$ has torsion $\tau \neq 0$ since $\mathcal{F}$ is not locally free. Then, since E_{y_0} is stable of degree 1,

$$\deg(\mathcal{F}_{y_0}) \geq 1 + \deg(\mathcal{F}_{y_0}/\tau) > 1 + \mu(E_{y_0}) = \frac{3}{2}$$

which implies $\deg(f_*E) \leq -1$ by (4.20).

We have shown that L_0 has to be $\mathcal{O}_B$ and (4.18) has to be

$$0 \to f^*V_1 \to E' \to f^*V_2 \to 0 \tag{4.21}$$

which is determined by a class of $\mathrm{H}^1(X, f^*(V_1 \otimes V_2^{-1}))$. However, noting $R^1 f_*(f^*(V_1 \otimes V_2^{-1})) = V_1 \otimes V_2^{-1} \otimes \mathrm{H}^1(\mathcal{O}_B) = V_1 \otimes V_2^{-1}$ and

$$\mathrm{H}^0(C, V_1 \otimes V_2^{-1}) = 0,$$

by using Leray spectral sequence, we have

$$\mathrm{H}^1(C, V_1 \otimes V_2^{-1}) \cong \mathrm{H}^1(X, f^*(V_1 \otimes V_2^{-1})).$$

Hence there exists an extension $0 \to V_1 \to V \to V_2 \to 0$ on C such that $E' \cong f^*V$, which contradicts the assumption

$$\dim(\mathrm{H}^0(\{t\} \times B, E'_t)) = 1.$$

□

Proposition 4.5. *When E is semi-stable of degree 0 on the generic fiber of $f : X \to C$, we have $\Delta(E) \geq 8$. If C is not hyper-elliptic and $\phi : B \to M$ passes through a $(1,1)$-stable bundle, assuming that E defines an essential elliptic curve, then $\Delta(E) \geq 12$.*

Proof. If E is semi-stable on each fiber $X_t = f^{-1}(t)$, then E induces a non-trivial morphism $\varphi_E : C \to \mathbb{P}^1$. By (4.16), $\Delta(E) \geq 8$.

If there is a $t_0 \in C$ such that E_{t_0} is not semi-stable, then we have

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{O}(\mu) \to 0$$

where $\mathcal{O}(\mu)$ is a line bundle of degree μ on B. If $\Delta(E') \neq 0$, then $\Delta(E') > 0$ by Theorem 2.4. On the other hand, $c_1(E')^2 = 0$ since E' has degree 0 on the generic fiber of $X \to C$ and $\mathrm{Pic}(C \times B) = \mathrm{Pic}(C) \times \mathrm{Pic}(B)$. Thus $\Delta(E') = 4 \cdot c_2(E') \geq 4$, and by Lemma 2.6

$$\Delta(E) = \Delta(E') - 4\mu \geq 8.$$

If $\Delta(E') = 0$, by Lemma 4.4, we can assume that $E' = f^*V$, then the sequence (4.17) induces a nontrivial morphism $\varphi : B \to \mathbb{P}(V_{t_0}^{\vee})$ such that $\mathcal{O}(-\mu) = \varphi^*\mathcal{O}_{\mathbb{P}(V_{t_0}^{\vee})}(1)$. Thus $\Delta(E) = -4\mu \geq 8$.

Now we assume that C is not hyper-elliptic and $\phi : B \to M$ passes through a $(1,1)$-stable bundle. If E is semi-stable on each fiber X_t, then $\Delta(E) = 4 \cdot \deg(\varphi_E) \geq 12$ by (4.16) since C is not hyper-elliptic.

If there is $t_0 \in C$ such that E_{t_0} is not semi-stable, we claim $\Delta(E') > 0$ since $\phi : B \to M$ passes through a $(1,1)$-stable bundle. Otherwise, $E' = f^*V$ where V is a $(1,0)$-stable by Lemma 3.2, then sequence (4.17) implies that $\phi : B \to M$ factors through a Hecke curve, which implies that $\phi : B \to M$ is not an essential elliptic curve. If E' is semi-stable on each fiber X_t, then E' defines a nontrivial morphism $\varphi_{E'} : C \to \mathbb{P}^1$ such that $\varphi^*\mathcal{O}_{\mathbb{P}^1}(1) = \Theta(E') = (\det f_! E')^{-1} = c_2(E')$. Thus $\Delta(E') = 4 \cdot \deg(\varphi_{E'}) \geq 12$ and $\Delta(E) = \Delta(E') - 4\mu \geq 16$.

If there is $t_0' \in C$ such that $E'_{t_0'}$ is not semi-stable, then we have

$$0 \to \mathcal{F} \to E' \to {}_{X_{t_0'}}\mathcal{O}(\mu') \to 0 \tag{4.22}$$

where $\mathcal{F}_y = \mathcal{F}|_{C \times \{y\}}$ is stable of degre -1 for generic $y \in B$ since E'_y is stable of degree 0. If $\Delta(\mathcal{F}) \neq 0$, it is clear that $\Delta(\mathcal{F}) = 4 \cdot c_2(\mathcal{F}) \geq 4$ and $\Delta(E) =$

$\Delta(\mathcal{F})-4\mu'-4\mu \geq 12$. If $\Delta(\mathcal{F})=0$, by Theorem 2.4, there is a stable vector bundle V' on C such that $\mathcal{F}_y \cong V'$ for all $y \in B$. Then we can choose $\mathcal{F}=f^*V'$ such that the sequence (4.22) induces a nontrivial morphism $\varphi : B \to \mathbb{P}(V'^{\vee}_{t'_0})$ with $\mathcal{O}(-\mu') = \varphi^*\mathcal{O}_{\mathbb{P}(V'^{\vee}_{t'_0})}(1)$. Thus $\Delta(E') = -4\mu' \geq 8$ and $\Delta(E) = \Delta(E') - 4\mu \geq 12$.

□

We have seen in Example 3.6 the existence of essential elliptic curves of degree $6(r,d)$ (which is 6 in our case). Then we have shown

Theorem 4.6. *Let $M = \mathcal{SU}_C(2,\mathcal{L})$ be the moduli space of rank two stable bundles on C with a fixed determinant of degree 1. Then, when C is generic, any essential elliptic curve $\phi : B \to M$ has degree*

$$\deg\phi^*(-K_M) \geq 6$$

and $\deg\phi^*(-K_M) = 6$ *if and only if $\phi : B \to M$ factors through*

$$\phi : B \xrightarrow{\psi} q^{-1}(\xi) = \mathbb{P}(\mathrm{H}^1(V_2^{\vee} \otimes V_1)) \xrightarrow{\Phi_\xi} M$$

for some $\xi = (V_1, V_2)$ such that $\psi^*\mathcal{O}_{\mathbb{P}(\mathrm{H}^1(V_2^{\vee}\otimes V_1))}(1)$ has degree 3.

Proof. By Proposition 4.2, Proposition 4.3 and Proposition 4.5, we have $\Delta(E) \geq 6$. The possible case $\Delta(E) = 6$ occurs only in Proposition 4.2 when $c_2(F_2) = 0$. This implies that E must satisfy

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \to E \to f^*V_2 \to 0$$

which defines $\psi : B \to \mathbb{P}(\mathrm{H}^1(V_2^{\vee} \otimes V_1))$ such that $\psi^*\mathcal{O}_{\mathbb{P}(\mathrm{H}^1(V_2^{\vee}\otimes V_1))}(1)$ has degree $\mu_1 - \mu_2$. Then $\Delta(E) = 6$ and (4.1) imply $\mu_1 - \mu_2 = 3$. □

Theorem 4.7. *When $g > 4$ and C is generic, any essential elliptic curve $\phi : B \to M = \mathcal{SU}_C(2,\mathcal{L})$ that passes through the generic point must have* $\deg\phi^*(-K_M) \geq 12$.

For $r > 2$, let $M = \mathcal{SU}_C(r,\mathcal{L})$ where $\mathcal{L}$ is a line bundle of degree d. What is the minimal degree of essential elliptic curves on M ? I expect the following conjecture to be true.

Conjecture 4.8. *Let $\phi : B \to M = \mathcal{SU}_C(r,\mathcal{L})^s$ be an essential elliptic curve defined by a vector bundle E on $C \times M$. Then, when C is a generic curve, we* *have*

$$\deg\phi^*(-K_M) \geq 6(r,d).$$

When $(r,d) \neq r$*, then* $\deg\phi^*(-K_M) = 6(r,d)$ *if and only if it is an elliptic curve of split type in Example 3.6. If* $\phi : B \to M$ *passes through the generic point and* $g > 4$*, then* $\deg\phi^*(-K_M) \geq 6r$*.*

Remark 4.9. One of referees asked to strengthen Theorem 4.7 to characterize the elliptic curves of Hecke type, so that the statement of Theorem 4.7 is comparable with Theorem 4.6. However, when $r = 2$, there is no essential elliptic curve of Hecke type. In fact, when $r = 2$, I have no easy example of essential elliptic curve through the generic point of M. But it is still meaningful to ask the characterization of essential elliptic curves of degree 12 that pass through the generic point (if any). Now we assume that E defines an essential elliptic curve through generic point with $\Delta(E) = 12$. Then, firstly, it can only happen in the situation of Proposition 4.5, namely, $E_t = E|_{\{t\}\times B}$ are semi-stable of degree 0 for generic points $t \in C$. Secondly, if C admits no cover $C \to \mathbb{P}^1$ of degree 2 and degree 3, then there are t_0', $t_0 \in C$ such that either (i) there exists a stable bundle V' of degree -1 on C suited in the following exact sequences

$$0 \to f^*V' \to E' \to {}_{X_{t_0'}}\mathcal{O}(-2) \to 0,$$

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{O}(-1) \to 0$$

or (ii) there exists a bundle $\mathcal{F}$ on X with $c_2(\mathcal{F}) = 1$ suited in the following exact sequences

$$0 \to \mathcal{F} \to E' \to {}_{X_{t_0'}}\mathcal{O}(-1) \to 0,$$

$$0 \to E' \to E \to {}_{X_{t_0}}\mathcal{O}(-1) \to 0$$

where $\mathcal{F}_y = \mathcal{F}|_{C\times\{y\}}$ is stable of degree -1 for generic $y \in B$.

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