K3 Surfaces with Order 11 Automorphisms

Keiji Oguiso and De-Qi Zhang

This paper is dedicated to the memory of Eckart Viehweg.

Abstract: In the present paper we describe the K3 surfaces admitting order 11 automorphisms and apply this to classify log Enriques surfaces of global index 11.

Keywords: K3 surface, automorphism.

1. Introduction

The purpose of this paper is to describe the family of complex K3 surfaces with automorphisms of order 11 and apply this to classify log Enriques surfaces of global canonical index 11 (see [18] for the definition). We note that any automorphism of order 11 of a K3 surface is necessarily non-symplectic, that is, acts on the space of the global two forms non-trivially [9].

Throughout this paper, we consider a pair (X, G) consisting of a complex projective K3 surface X and a finite group G of automorphisms on X which fits in the exact sequence:

$$1 \to G_N \to G \xrightarrow{\rho} \mu_{11n} = \langle \zeta_{11n} \rangle \to 1,$$

where the last map ρ is the natural representaion of G on the space $H^{2,0}(X) = \mathbb{C}\omega_X$ and n is some positive integer. It is known that $n \le 6$ ([9], [5]; see also [7]).

We fix an element $g \in G$ with $\rho(g) = \zeta_{11}$, i.e.,

$$g^* \omega_X = \zeta_{11} \omega_X$$

and set

$$M = H^2(X, \mathbb{Z})^g.$$

For simplicity of description, we also assume that G is maximal in the sense that if (X, G') also satisfies the same conditon as above for some n' and $G \subseteq G'$ then $G = G'$.

In order to state our main Theorem, we first construct three types of examples of such pairs. We denote by U and $U(m)$ the lattices defined respectively by the Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

Denote by A_*, D_*, E_* the negative definite lattices given by the Dynkin diagrams of the indicated types.

Example 1.1. ([5], [7]) Let S_{66} be the K3 surface given by the Weierstrass equation $y^2 = x^3 + (t^{11} - 1)$, and σ_{66} the automorphism of S_{66} given by

$$\sigma_{66}^*(x, y, t) = (\zeta_{66}^{22} x, \zeta_{66}^{33} y, \zeta_{66}^{12} t).$$

Then the pair $(S_{66}, \langle \sigma_{66} \rangle)$ gives an example of (X, G) with $n = 6$ and $G_N = \{1\}$, i.e., $G \simeq \mu_{66}$ and with $M \simeq U$.

Example 1.2. ([5]) Consider the rational, fibered threefold $\varphi : \mathcal{X} \to \mathbb{C}$ defined by

$$y^2 = x^3 + x + (t^{11} - s)$$

and its order 22 automrphism σ given by

$$\sigma^*(x, y, t, s) = (x, -y, \zeta_{11} t, s)$$

where s is the coordinate of $\mathbb{C}$. Then φ is a morphism smooth over $s \neq \pm\sqrt{-4/27}$ and

$$\mathcal{X}_{\sqrt{-4/27}} \ (\simeq \mathcal{X}_{-\sqrt{-4/27}})$$

has a unique singular point of type A_{10}.

The pair $(\mathcal{X}_0, \langle \sigma_{44} \rangle)$, where

$$\sigma_{44}^*(x, y, t) = (\zeta_{44}^{22} x, \zeta_{44}^{11} y, \zeta_{44}^{34} t)$$

gives an example of (X, G) with $n = 4$ and $G_N = \{1\}$, i.e., $G \simeq \mu_{44}$ and with $M = U$. (The minimal resolution of) $(\mathcal{X}_s, \langle \sigma \rangle)$ with $s \neq 0$ gives an example of (X, G) with $n = 2$ (and $G_N = \{1\}$), i.e., $G \simeq \mu_{22}$ and with $M = U$ (resp. $U \oplus A_{10}$) if

$$s \neq 0, \pm\sqrt{-4/27} \ (\text{ resp. if } s = \pm\sqrt{-4/27})$$

(cf. Remark 1.3 and the proof of Claim 2.6 below for the calculation of M and G).

The following remark will help to verify the calculation of G and M in Examples 1.1 and 1.2 above.

Remark 1.3. (1) Let (X, G) be any of the pairs in Examples 1.1 and 1.2 above and let g be the unique order 11 element in G satisfying $g^*\omega_X = \zeta_{11}\omega_X$. The natural G-stable (hence g-stable) Jacobian elliptic fibration $f : X \to \mathbf{P}^1$, with t as the inhomogeneous coordinate of the base space, is the only g-stable elliptic fibration on X (cf. the first paragraph in the proof of Proposition 2.3 below.)

(2) The fixed locus (point wise) X^g is equal to the union of a smooth rational curve in the type I_{11} fiber $X_{t=0}$ and two points on the type II fiber $X_{t=\infty}$ (resp. the union of the smooth fiber $X_{t=0}$ and two points on the type II fiber $X_{t=\infty}$), when X is equal to $\mathcal{X}_{\sqrt{-4/27}}$ (resp. any of other cases in Examples 1.1 and 1.2).

(3) For any $s \neq 0, \pm\sqrt{-4/27}$, four surfaces

$$S_{66}, \ \mathcal{X}_0, \ \mathcal{X}_{\sqrt{-4/27}}, \ \mathcal{X}_s$$

are not isomorphic to one another.

Example 1.4. Let us consider the following three series of rational Jacobian elliptic surfaces:

(1) $j^{(1)} : J^{(1)} \to \mathbb{P}^1$, defined by the Weierstrass equation

$$y^2 = x^3 + (t - 1)$$

whose singular fibers are $J_1^{(1)}$ of Kodaira's type II and $J_\infty^{(1)}$ of Kodaira's type II^*;

(2) $j^{(2)} : J^{(2)} \to \mathbb{P}^1$, defined by the Weierstrass equation

$$y^2 = x^3 + x + (t - s)$$

with $s \neq \pm\sqrt{-4/27}$, whose singular fibers are $J^{(2)}_\alpha$, $J^{(2)}_\beta$ of Kodaira's type I_1, and $J^{(2)}_\infty$ of Kodaira's type II^*, where $t = \alpha, \beta$ are two distinct non-zero roots of the discriminant $\Delta(t) = 4 + 27(t - s)^2$; and

(3) $j^{(3)} : J^{(3)} \to \mathbb{P}^1$, defined by the Weierstrass equation

$$y^2 = x^3 + x + (t - s)$$

with $s = \sqrt{-4/27}$ whose singular fibers are $J^{(3)}_0$, $J^{(3)}_{2s}$ of Kodaira's type I_1, and $J^{(3)}_\infty$ of Kodaira's type II^*.

Let $p^{(i,e)} : P^{(i,e)} \to \mathbb{P}^1$ be a non-trivial principal homogeneous space of $j^{(i)} : J^{(i)} \to \mathbb{P}^1$ given by an element e of order 11 in $(J^{(i)})_0$. (For the basic results on the principal homogeneous space of rational Jacobian elliptic fibrations, see [3, Chapter V, Section 4].) Then $p^{(i,e)} : P^{(i,e)} \to \mathbb{P}^1$ is a rational elliptic surface with a multiple fiber of multiplicity 11 over 0 (of type I_0 in the cases $i = 1, 2$ and of type I_1 in the case $i = 3$).

Let $Z^{(i,e)}$ be the log Enriques surface of index 11 obtained by the composite of the blow up at the intersection of the components of multiplicities 5 and 6 in $(P^{(i,e)})_\infty$, which is of Kodaira's type II^*, and the blow down of the proper transform of $(P^{(i,e)})_\infty$. Let $X^{(i,e)}$ be the global canonical cover of $Z^{(i,e)}$ and $G^{(i,e)}$ the Galois group of this covering. Then, each of these pairs $(X^{(i,e)}, G^{(i,e)})$ gives an example of (X, G) with $n = 1$ and $G_N = \{1\}$, i.e., $G \simeq \mu_{11}$ and with $M = U(11)$ (see Lemma 2.9 below to verify the calculation of G and M).

Our main result is as follows:

Theorem 1.5 (Main Theorem). *Under the notation above, the following are true.*

(1) *We have $G_N = \{1\}$ so that $G \simeq \mu_{11n}$ and g is unique and of order 11.*

(2) *M is isomorphic to either one of U, $U \oplus A_{10}$ or $U(11)$.*

(3) *In the case where $M \simeq U$ or $U \oplus A_{10}$, (X, G) is isomorphic to either $(S_{66}, \langle\sigma_{66}\rangle)$, $(\mathcal{X}_0, \langle\sigma_{44}\rangle)$, or $(\mathcal{X}_s, \langle\sigma\rangle)$ $(s \neq 0)$ in Examples 1.1 and 1.2.*
   Moreover, $M \simeq U \oplus A_{10}$ if and only if $(X, \langle g \rangle)$ is isomorphic to

$$(\mathcal{X}_{\sqrt{-4/27}}, \langle\sigma^2\rangle) \ (\simeq (\mathcal{X}_{-\sqrt{-4/27}}, \langle\sigma^2\rangle)).$$

(4) *In the case where $M \simeq U(11)$, (X, G) is isomorphic to one of $(X^{(i,e)}, G^{(i,e)})$ in Example 1.4.*

Combining the main Theorem 1.5 with Remark 1.3, we obtain the following, where a log Enriques surface is maximal if, by definition, any birational morphism $Z' \to Z$ from another log Enriques surface Z' must be an isomorphism.

Corollary 1.6. *Maximal log Enriques surfaces of global index* 11 *are isomorphic to either a* $Z^{(i,e)}$ *in Example* 1.4 *or* $\overline{\mathcal{X}}_{\sqrt{-4/27}}/\langle g \rangle$*, where* $\overline{\mathcal{X}}_{\sqrt{-4/27}}$ *is the surface obtained from the surface* $\mathcal{X}_{\sqrt{-4/27}}$ *in Example* 1.2 *with the unique g-fixed curve contracted.*

Remark 1.7. (1) In the main Theorem 1.5 (3) and (4) and Examples 1.2 and 1.4, the pairs (X, G) parametrized by s and $-s$, are isomorphic to each other. In particular, the pair (X, G) with $M \simeq U \oplus A_{10}$ is unique up to isomorphisms.

(2) By the main Theorem 1.5, the pairs (X, G) are not finitely many any more and move in a 1-dimensional (non-isotrivial) family, which is one of the main difference from the previous works [7], [12], [16], [5] [6] concerning larger non-symplectic group actions. Indeed, calculating the $J-$invariant and combining with the fact that the pair (X, G) with $\text{ord}(G) = 40$ and its elliptic fiber space structure are both unique [7], we find that the family $\varphi : \mathcal{X} \to \mathbb{C}$ given in Example 1.2 is not isotrivial. Similarly, the uniqueness of the Jacobian elliptic fiber space structure on a rational surface shows that the family given in Example 1.4 is also not isotrivial.

(3) One can also explain the reason why (X, G)'s form a 1-dimensional family from the view point of the period mapping. Since for generic (X, G), the transcendental lattice T_X is of rank 20 and isomorphic to either $U^2 \oplus E_8^2$ or $U \oplus U(11) \oplus E_8^2$; further, the eigenspace with respect to the eigenvalue ζ_{11} of the action g on $T_X \otimes \mathbb{C}$ in which the period $\mathbb{C}\omega_X$ should lie is two dimensional. Conversely a generic one dimensional subspace in this eigen space gives periods of K3 surfaces with order 11 automorphisms g by the surjectivity of the period mapping [1].

(4) In our classification, we make use of the invariant part M of the g-action on $H^2(X, \mathbb{Z})$, instead of the Neron Severi lattice S_X which always contains M and certainly equals M if X is generic in the family. However, for special X, S_X is probably larger than M. So, in our classification, the determination of the Neron Severi lattice [15], which is one of the hardest and most important problems concerning algebraic surfaces, remains unsettled. The reason why we describe the result according to M rather than S_X is that on the one hand, the

Neron Severi lattices are quite unstable under deformations, for instance, in the case of the family of quartic K3 surfaces, and on the other hand, it turns out that the invariant part M is fairly stable under deformation at least in our case.

A group G is called a *K3 group*, if $G \leq \operatorname{Aut}(X)$ for some complex $K3$ surface X.

Proposition 1.8. *No sporadic finite simple group which is different from the Monster group* **M***, contains all finite K3 groups as its subgroups.*

Question 1.9. *Can we embed every finite K3 group into the Monster simple group* **M***?*

Remark 1.10. After this work was done and motivated by Mukai's embedding of *all* finite symplectic $K3$ groups into the sporadic simple Mathieu group M_{23} ($\leq$ **M**) and the observation in Proposition 1.8, the above Question 1.9 crossed our minds. We planned to solve this question and include the current paper as part of the new project [13]. However, this project is unexpectedly complicated and we have not yet completed it. So we decide to publish the current paper as an independent paper.

After the current paper was written in 1999, there have been much progress, especially in positive characteristic, among which is the very significant work of Dolgachev-Keum [4] where the authors successfully extended Mukai's classification of finite symplectic $K3$ groups to positive characteristics. See also [19] for a partial survey.

2. Proof of the main Theorem

We now prove the main Theorem 1.5. We employ the same notation introduced in §1 freely. Let N be the orthogonal lattice of M in $H^2(X, \mathbb{Z})$. Then N is

g−stable and $T_X \subseteq N$ and $M \subseteq S_X$. For a lattice L, we denote by L^* the dual (over) lattice $\mathrm{Hom}(L, \mathbb{Z})$. For a positive integer I, we denote by $\varphi(I)$ the cardinality of the multiplicative group $(\mathbb{Z}/I)^\times$.

Lemma 2.1. *We have $G_N = \{1\}$. In particular, g is unique and is of order 11.*

Proof. Suppose the contrary that $|G_N| \geq 2$. Since $\varphi(11) \,|\, \mathrm{rank}\, T_X$, $\mathrm{rank}\, T_X$ is either 10 or 20. According [17, the list] and its notation,

$$\mathrm{rank}\, T_X = \mathrm{rank}\, T_Y \leq 22 - (c + 1)$$

with $c \geq 8$ (resp. $c \geq 12$) when $|G_N| \geq 2$ (resp. $|G_N| \geq 3$). Thus $|G_N| = 2$ and $\mathrm{rank}\, T_X = 10$. Hence $\mathrm{rank}\, S_X = 12$.

Write $G_N = \langle \iota \rangle \simeq \mathbb{Z}/2$. Then $|G| = 22$. Since $G_N \lhd G$, we have $G = \langle h \rangle \simeq \mathbb{Z}/22$. We may assume that $g = h^2$ and $\iota = h^{11}$. By the topological Lefschetz fixed point formula, we have the diagonalization $\iota^*|(S_X \otimes \mathbb{C}) = \mathrm{diag}[I_4, -I_8]$, relative to some basis. By considering minimal polynomial of ζ_{11} over $\mathbb{Q}$, we have either $g^*|S_X \otimes \mathbb{C} = I_{12}$ (identity matrix), or

$$g^*|S_X \otimes \mathbb{C} = \mathrm{diag}[\zeta_{11}, \zeta_{11}^2, \ldots, \zeta_{11}^{10}, 1, 1].$$

If the second case for g^* occurs, then simultaneously diagonalize g^* and ι^* on $S_X \otimes \mathbb{C}$, we would get a diagonalization of $(g \circ \iota)^*$ whose diagonal entries consist of a few ± 1 and between 6 and 8 entries of $22nd$ primitive roots of the unity, which is impossible because $g \circ \iota$ is of order 22 and the Euler number $\varphi(22) = 10$.

If the first case for g^* occurs, then we get the following diagonalizations, relative to two possibly different bases (up to re-ordering):

$$(g \circ \iota)^*|(S_X \otimes \mathbb{C}) = \mathrm{diag}[I_4, -I_8], \quad (g \circ \iota)^*|(T_X \otimes \mathbb{C}) = \mathrm{diag}[\zeta_{11}, \zeta_{11}^2, \ldots, \zeta_{11}^{10}].$$

Thus $\chi_{topol}(X^{g \circ \iota}) = -3$ by the topological Lefschetz fixed point formula. In particular, $X^{g \circ \iota}$ contains a curve. On the other hand, since $(g \circ \iota)^{11} = \iota$, we have $X^{g \circ \iota} \subseteq X^\iota$, so that $X^{g \circ \iota}$ consists of finitely many points, a contradiction.

$\square$

Lemma 2.2. *M is isomorphic to either U, $U(11)$ or $U \oplus A_{10}$.*

Proof. Since M is a primitive sublattice of the unimodular lattice $H^2(X, \mathbb{Z})$, we have a natural isomorphism $M^*/M \simeq N^*/N$. Noting that $N^{g^*} = \{0\}$, we can

apply the same argument as in [7, Lemmas (1.1), (3.2)] and [12, Lemmas (1.2), (1.3)] for the pair (M, N) (instead of (S_X, T_X) there) to get $\varphi(11) = 10 \,|\, \operatorname{rank} N$ and

$$M^*/M \simeq N^*/N \simeq (\mathbb{Z}/11)^{\oplus s}$$

for some integer s with $0 \leq s \leq \operatorname{rank} N/10$. Since $\operatorname{rank} N \leq 21$, we have

$$(\operatorname{rank}(M), \det(M)) = (22 - \operatorname{rank}(N), -11^s) =$$

$$(2, -1), (2, -11), (2, -11^2), (12, -1), (12, -11).$$

By the classification of indefinite unimodular even lattices, the case $(\operatorname{rank}(M), \det(M)) = (12, -1)$ is impossible and in the case $(\operatorname{rank}(M), \det(M)) = (2, -1)$ we have $M = U$.

By [14], a p-elementary $(p > 2)$ even hyperbolic lattice of rank > 2, is determined uniquely by its rank and discriminant. So, $M = U \oplus A_{10}$ when $(\operatorname{rank}(M), \det(M)) = (12, -11)$.

Suppose that rank $M = 2$. Write $M = (a_{ij})$, where $a_{11} = 2a, a_{22} = 2c, a_{12} = a_{21} = b$ for integers a, b, c. Then $\det M \equiv 0, -1 \pmod{4}$ and hence the case $(\operatorname{rank}(M), \det(M)) = (2, -11)$ is impossible. We consider the case where $(\operatorname{rank}(M), \det(M)) = (2, -11^2)$. Note that M^* is generated by a $\mathbb{Z}$-basis

$$(\varepsilon_1 \ \varepsilon_2) = (e_1 \ e_2)M^{-1} = (e_1 \ e_2)(-1/11^2)(b_{ij})$$

where

$$b_{11} = 2c, b_{22} = 2a, b_{12} = b_{21} = -b.$$

Here e_i's form the basis of M with (a_{ij}) as the intersection matrix. Since $M^*/M = (\mathbb{Z}/11)^{\oplus s}$, each b_{ij} (and hence each a_{ij}) is divisible by 11. So $M = M_1(11)$ with an indefinite even unimodular lattice M_1. Thus $M = U(11)$ under a suitable basis.

$$\square$$

Proposition 2.3. *Assume that $M \simeq U$. Then (X, G) is isomorphic to either $(S_{66}, \langle \sigma_{66} \rangle)$, $(\mathcal{X}_0, \langle \sigma_{44} \rangle)$, or $(\mathcal{X}_s, \langle \sigma \rangle)$ $(s \neq 0, \pm\sqrt{-4/27})$ in Examples* 1.1 *and* 1.2.

Proof. If $n \geq 3$, the result follows from [7, Main Theorem]. Let us consider the case $n \leq 2$. Since $M \simeq U$, X admits a g-stable Jacobian fibration $f : X \to \mathbb{P}^1$ by [10]. Let E and C be a general fiber of f and the unique g-stable section of

f. Here the uniqueness of the $g-$stable section follows from the fact that if C' is also a $g-$stable section then $[C'] = a[C] + b[E]$ and

$$((aC + bE).E) = 1, \quad (aC + bE)^2 = -2.$$

We see then these equalities imply $a = 1$ and $b = 0$.

Let $\bar{g}$ be the automorphism of the base space $\mathbb{P}^1$ induced by g. Since there are no elliptic curves admitting Lie automorphism of order 11, $\bar{g}$ is also of order 11. We may then adjust an inhomogeneous coordinate t of $\mathbb{P}^1$ so that $(\mathbb{P}^1)^{\bar{g}} = \{0, \infty\}$. We note that X_0 and X_∞ are both irreducible, because the irreducible component R of X_0 meeting C is $g-$stable so that rank $M \geq 3$ unless $R = X_0$.

Since $g^* \omega_X = \zeta_{11} \omega_X$, an easy local coordinate calculation shows that neither of X_0, X_∞ is of Kodaira's type I_1. Moreover, noting that g permutes the other singular fibers, we have

$$24 = \chi_{topol}(X) = \chi_{topol}(X_0) + \chi_{topol}(X_\infty) + 11m$$

for some positive integer m. Thus after suitable change of inhomogeneous coordinate t if necessary, (X_0, X_∞) is of type (I_0, II) and the set of the other singular fibers is either

(1) $\{X_{\zeta_{11}^i} | 0 \leq i \leq 10\}$, all of Kodaira's type II,

(2) $\{X_{\zeta_{11}^i}, X_{\zeta_{11}^j \alpha} | 0 \leq i, j \leq 10\}$ $(\alpha \notin \mu_{11})$, all of Kodaira's type I_1, or

(3) $\{X_{\zeta_{11}^i} | 0 \leq i \leq 10\}$, all of Kodaira's type I_2.

□

Claim 2.4. *The case* (3) *can not happen.*

Proof. Assuming the contrary that Case(3) occurs, we denote by R the irreducible component of X_1 meeting C. Since

$$S := \sum_{0 \leq i \leq 10} g^i(R)$$

is $g-$stable, we have $[S] = a[C] + b[E]$. Now $(S.E) = 0$ implies that $a = 0$ and hence $S = b[E]$. This leads to

$$-22 = (S)^2 = (bE)^2 = 0$$

which is a contradiction.

$\square$

Claim 2.5. *The case* (1) *can not happen under the assumption that* $n \leq 2$.

Proof. Assuming the contrary that Case (1) happens, we will determine the Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t)$$

of $f : X \to \mathbb{P}^1$. Since the singular fibers of f are all of type II, the J−function

$$J(t) := 4a(t)^3/(4a(t)^3 + 27b(t)^2) = 0$$

as a rational function. Thus, $a(t) = 0$ and the equation is $y^2 = x^3 + b(t)$.

Let us consider the discriminant divisor

$$\Delta(t) = 27b(t)^2.$$

Since the singular fibers of f over $t \neq \infty$ are $X_{\zeta_{11}^i}$ and these are all of type II, we have $\Delta(t) = c(t^{11} - 1)^2$ for some nonzero constant c. Then $b(t) = c'(t^{11} - 1)$ for some nonzero constant c'. Changing x, y by suitable multiples, we finally find that f is given by the equation

$$y^2 = x^3 + (t^{11} - 1)$$

which is isomorphic to S_{66} in Example 1.1. In particular, $G \simeq \mu_{66}$ by [7]. Thus $n = 6$, a contradiction. The argument above is similar to [5, (5.1)]; we keep this argument for readers' convenience, and thank the referee for pointing this out. $\square$

Claim 2.6. *Assume that* $f : X \to \mathbb{P}^1$ *satisfies the condition of the case* (2) *and* $M \simeq U$ *and* $n \leq 2$. *Then* $f : X \to \mathbb{P}^1$ *is isomorphic to a Jacobian elliptic fibration given by a Weierstrass equation*

$$y^2 = x^3 + x + (t^{11} - s)$$

for some $s \neq 0, \pm\sqrt{-4/27}$, *and under this isomorphism, we have* $G \simeq \langle \sigma \rangle$, *where*

$$\sigma^*(x, y, t) = (x, -y, \zeta_{11}t).$$

In particular, $n = 2$.

Proof. Again we will determine the Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t)$$

of $f : X \to \mathbb{P}^1$, where $a(t), b(t)$ are polynomials in t. First note that $\deg a(t) \le 8$ and $\deg b(t) \le 12$ by the canonical bundle formula. Since f has singular fibers

$$\{X_{\zeta_{11}^i}, X_{\zeta_{11}^j \alpha} \,|\, 0 \le i, j \le 10\}$$

of type I_1, the discriminant divisor $\Delta(t)$ is equal to

$$\delta(t^{11} - \alpha^{11})(t^{11} - 1)$$

for some non-zero constant δ. Since the $J-$function

$$J(t) = 4a(t)^3 / \Delta(t)$$

is $\bar{g}-$invariant, $a(t)$ (and hence $b(t)$) are also $\bar{g}-$semi invariant. Thus

$$a(t) = At^m, \ b(t) = t^n(B_1 + B_2 t^{11})$$

where A, B_i are constants, $m \le 8, n \le 12$, and $n \le 1$ when $B_2 \ne 0$. Comparing coefficients of the equality

$$\Delta(t) = 4a(t)^3 + 27b(t)^2$$

we see that

$$a(t) = A, \ b(t) = B_1 + B_2 t^{11}.$$

Noting that $A \ne 0$ because of the existence of singular fibers of type I_1. We have also $B_2 \ne 0$, otherwise, X is birational to a product of a fibre and the parameter space $\mathbb{P}^1$ and hence is not a $K3$ surface, absurd! We can, by a suitable coordinate change, normalize the Weierstrass equation of X as

$$X = \mathcal{X}_s : \ y^2 = x^3 + x + (t^{11} - s).$$

Here s is a constant, and $s \ne 0$ for otherwise $n = 4$ by [7].

Conversely, by the standard algorithm to finding out the singular fibers [8], we see that this elliptic surface $\mathcal{X}_s$ has 22 singular fibers of type I_1 and a singular fiber of type II if and only if $s \ne \pm\sqrt{-4/27}$. Moreover, $\mathcal{X}_s$ admits an automorphism g_s of order 11 given by

$$g_s^*(x, y, t) = (x, y, \zeta^{11} t).$$

Since g and g_s make the fibration f and the section C stable and satisfy

$$g^* \omega_X = g_s^* \omega_X = \zeta_{11} \omega_X$$

we have $g = g_s$. Now the condition that $n \leq 2$ implies that $n = 2$ and $G \simeq \mu_{22}$, by the maximality of G and by considering

$$G_s = \langle g_s, \iota_s \rangle \simeq \mu_{22}$$

where

$$\iota_s^*(x, y, t) = (x, -y, t)$$

acts on f as the involution around C.

Write $G = \langle g, \iota \rangle$ with an involution ι. Since $\iota \circ g = g \circ \iota$, we see that C and f are both ι−stable, and 0 and ∞ are two ι-fixed points. In other words, ι does not switch 0 and ∞, because the fibres X_0 and X_∞ are of different types: I_0, II. If ι acts on the base space $\mathbb{P}^1$ as an involution, G permutes the 22 singular fibers of type I_1 as well as the 22 roots of the discriminant divisor

$$\Delta(t) = 4 + 27(t^{11} - s)^2$$

whence $s = 0$, a contradiction. Thus, ι is the involution of f around C, i.e., $\iota = \iota_s$. This means $G = G_s$ and we are done. $\qquad\square$

This completes the proof of Proposition 2.3.

Next we consider the case where $M \simeq U \oplus A_{10}$. In this case, $M = S_X$ and $\operatorname{rank} T_X = \varphi(11) = 10$. So (X, G) is equivariantly isomorphic to the pair $(\mathcal{X}_{\sqrt{-4/27}}, \langle \sigma \rangle)$ in Example 1.2, by [12, Theorem 2] and by making use of the maximality of G as in the previous paragraph. This also proves the main Theorem 1.5 in the case of $M = U$ or $U \oplus A_{10}$.

Finally we consider the case where $M \simeq U(11)$. As before, since $U(11)$ represents zero, X admits a g−stable elliptic fibration $f : X \to \mathbb{P}^1$ and the induced action $\bar{g}$ on the base space is of order 11. We adjust an inhomogeneous coordinate t of the base so that $(\mathbb{P}^1)^{\bar{g}} = \{0, \infty\}$. We need further coordinate change later, but we always keep this condition.

Lemma 2.7. *After a suitable coordinate change, f satisfies either one of the following three cases.*

(1) *X_0 is smooth and $g|X_0$ is a translation of order 11; the remaining singular fibers are $X_{\zeta_{11}^k}$ $(0 \leq k \leq 10)$ and these are all of type II.*

(2) X_0 *is smooth and* $g|X_0$ *is a translation of order* 11; *the remaining singular fibers are* $X_{\zeta_{11}^k}$ *and* $X_{\zeta_{11}^k}\alpha$ $(0 \le k \le 10$ *and* $\alpha \notin \mu_{11})$ *and these are all of type* I_1.

(3) X_0 *is of Type* I_{11} *and* $g|X_0$ *is a translation of order* 11 *(which permutes the fiber components cyclically); the remaining singular fibers are* $X_{\zeta_{11}^k}$ *and these are all of type* I_1.

Moreover, in all three cases, X_∞ *is of type* II *with* $X^g = (X_\infty)^g = \{P_1, P_2\}$, *where* P_1 *is the singular point of* X_∞. *The action of* g *around* P_i *is of type* $1/11(5, 7)$ *if* $i = 1$ *and* $1/11(2, 10)$ *if* $i = 2$.

Proof. The proof is almost identical to the situation where $M \supseteq U$, except that f does not admit $g-$stable sections and we use the assumption that $M \simeq U(11)$ and the fact that X^g is smooth. The type of the action is determined by an elementary local coordinate calculation of the normalization of X_∞ and the fact that $g^*\omega_X = \zeta_{11}\omega_X$. Actually, we have one more possible case in which X_0 is smooth with $g|X_0 = \mathrm{id}$ and X_∞ is of type II with $(X_\infty)^g = \{P_1, P_2\}$. But then the relatively minimal model of $X/\langle g \rangle \to \mathbb{P}^1/\langle \overline{g} \rangle$ is a rational elliptic surface with no multiple fibers and hence has a section C. Now the pullback on X of C is a g-stable section, which contradicts $M \simeq U(11)$. $\qquad \square$

Note that the fibration f on X induces an elliptic fibration $f' : X/\langle g \rangle \to \mathbb{P}^1/\langle \overline{g} \rangle$ on the quotient surface, a log Enriques surface of index 11. Let $S \to X/\langle g \rangle$ be the minimal resolution. Then the proper transform D_0 of $X_\infty/\langle g \rangle$ is a (-1)-curve on S. This is because that the total transform D of $X_\infty/\langle g \rangle$ is a non-relatively minimal fibre of an elliptic fibration on the smooth surface S; to be precise, every irreducible component D_i $(\ne D_0)$ of this fibre D is a curve with self-intersection ≤ -2, and at least one curve say D_1 has $D_1^2 \le -3$ since P_1 (and also P_2) is not a rational double point.

We let $c : S \to T$ be the contraction of this (-1) curve and $\overline{f} : T \to \mathbb{P}^1$ the induced relatively minimal rational elliptic fibration. We immediately get the following lemma from the construction.

Lemma 2.8. *According to the cases* $(1), (2), (3)$ *in Lemma* 2.7, *the singular fibers of* $\overline{f}$ *are:*

(1) T_0 *of type* $_{11}I_0$, T_∞ *of type* II^*, *and* T_1 *of type* II.

(2) T_0 of type $_{11}I_0$, T_∞ of type II^*, and T_1 and $T_{\alpha^{11}}$ of type I_1.

(3) T_0 of type $_{11}I_1$, T_∞ of type II^*, and T_1 of type I_1.

Note that we can recover (X, f) in Lemma 2.7 easily from $(T, \overline{f})$ in Lemma 2.8. Indeed, let $\overline{f} : T \to \mathbb{P}^1$ be a relatively minimal rational elliptic surface with one of the properties (1), (2) and (3) in Lemma 2.8. Blow up the point of the intersection of the components of multiplicities 5 and 6 in T_∞ and then contract the two connected components of the proper transform of T_∞. We now get a rational elliptic surface $f' : S \to \mathbb{P}^1$ with two singular points of types $1/11(5, 7)$ and $1/11(2, 10)$ and with $11K_S$ linearly equivalent to 0. Let $X \to S$ be the global canonical $\mathbb{Z}/11$-cover of S. Then (X, f), where f is induced from $\overline{f}$, fits corresponding cases in Lemma 2.7.

Moreover, if we let F be the Galois group $\mathrm{Gal}(X/S)$, then we have:

Lemma 2.9. (X, F) *satisfies the condition in the second paragraph of the Introduction. Further, $M \simeq U(11)$ and F is maximal. In particular, in the situation of Lemma 2.7, one has $(X, F) = (X, \langle g \rangle)$.*

Proof. The first assertion is clear. We use the notation, like g, M in the Introduction for F.

If $M \not\simeq U(11)$, then M is either U or $U \oplus A_{10}$ by Lemma 2.2. However, then X^g contains a curve by the main Theorem 1.5 (proved already when $M \supseteq U$) and Remark 1.3, which contradicts the fact that the canonical covering map is étale in codimension one. Thus $M \simeq U(11)$.

Next, we show that F is maximal. Assume that $F \subset H$ and H also satisfies the condition in the Introduction. By Lemma 2.1, $H_N = \{1\}$. By [7], it is enough to eliminate the case where $H = \langle h \rangle \simeq \mathbb{Z}/22$.

Assume the contrary that this case happens. We may assume that the order 11 element $g := h^2$ is as in the Introduction. Since rank $M = 2$, $F_N = \{1\}$ and rank T_X is either 10 or 20, we have either

$$h^*|T_X \otimes \mathbb{C} = \mathrm{diag}[-\zeta_{11}^j | 1 \leq j \leq 10]$$

and $h^*|S_X \otimes \mathbb{C}$ equals one of:

$$\mathrm{diag}[1, \pm 1, \zeta_{11}^j | 1 \leq j \leq 10], \quad \mathrm{diag}[1, \pm 1, -\zeta_{11}^j | 1 \leq j \leq 10]$$

in the case where rank $T_X = 10$, or

$$h^*|T_X \otimes \mathbb{C} = \mathrm{diag}[-\zeta_{11}^j | 1 \leq j \leq 10]^{\oplus 2}, \quad h^*|S_X \otimes \mathbb{C} = \mathrm{diag}[1, \pm 1]$$

in the case where rank $T_X = 20$.

Since $h(X^g) = X^g$ and $X^h \subseteq X^g$, we have $X^h = \{P_1, P_2\}$, noting that the actions of g around two points P_i are different. Thus the topological Lefschetz formula shows that the only possible case is:

$$h^*|T_X \otimes \mathbb{C} = \mathrm{diag}[-\zeta_{11}^j | 1 \leq j \leq 10], \quad h^*|S_X \otimes \mathbb{C} = \mathrm{diag}[1, -1, \zeta_{11}^j | 1 \leq j \leq 10].$$

Let $\iota := h^{11}$. Then,

$$\iota^*|T_X \otimes \mathbb{C} = -I_{10}, \quad \iota^*|S_X \otimes \mathbb{C} = \mathrm{diag}[I_{11}, -1].$$

In particular, $\chi_{\mathrm{top}}(X^\iota) = 2$. This, together with the fact that $\iota^* \omega_X = -\omega_X$, implies that X^ι consists of smooth curves and at least one of them is a smooth rational curve, say C. Write the (disjoint) irreducible decomposition of X^ι as

$$X^\iota = C \cup E_1 \cup ... \cup E_m.$$

Since $g \circ \iota = \iota \circ g$, the g acts on the set $\{C, E_1, ..., E_m\}$.

First assume that $g(C) \neq C$. Then $g^i(C)$ would be mutually disjoint 11 rational curves with

$$\mathbb{Q}\langle g^i(C) \rangle \subseteq S_X^{\iota^*} \otimes \mathbb{Q}$$

where both sides of the inclusion are of rank 11, whence they are equal. However, $S_X^{\iota^*}$ then contains no ample classes, a contradiction. Thus $g(C) = C$ and $P_1, P_2 \in C$. But, this can not happen, because the action of g around P_i are of types $1/11(5, 7)$ if $i = 1$ and $1/11(2, 10)$ if $i = 2$, and there are no $a \in \{5, 7\}, b \in \{2, 10\}$ with $a + b \equiv 0 \,(\mathrm{mod}\, 11)$. Therefore, F is maximal and Lemma 2.9 is proved. $\square$

Now the only remaining task is to describe rational elliptic surfaces with the property (1), (2), or (3) in Lemma 2.8. However, each of these is obtained as a principal homogeneous space of a Jacobian rational elliptic surface $j : J \to \mathbb{P}^1$ whose singular fiber type is equal to one of the three types in Example 1.4. Now a similar (and easier) calculation shows that the Weierstrass equation of $j : J \to \mathbb{P}^1$ is the same as one of those in Example 1.4. This completes the proof of the main Theorem 1.5.

2.10. **Proof of Proposition 1.8**

Let I be either 54 or a prime number ≤ 19, then the order-I cyclic group μ_I acts purely non-symplectically on some $K3$ surface and hence is a $K3$ group (cf. [7, Main Theorem 3]). Among the 26 sporadic simple groups in [2], only the Monster $\mathbf{M}$ contains all such μ_I as subgroups (neither the baby Monster B nor the Mathieu group M_{23} contains μ_{54} as its subgroup). This proves the proposition.

References

[1] W. Barth, C. Peters and A. Van de Ven, Compact complex surfaces, Springer-Verlag, 1984.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray*, Oxford University Press, Eynsham, 1985.

[3] F. R. Cossec and I. V. Dolgachev, Enriques surfaces I, Progress in Mathematics, Vol. **76**, Birkhauser 1989.

[4] I. V. Dolgachev and J. Keum, Finite groups of symplectic automorphisms of $K3$ surfaces in positive characteristic, Ann. of Math. (2) **169** (2009), no. 1, 269–313.

[5] S. Kondo, Automorphisms of algebraic $K3$ surfaces which act trivially on Picard groups, J. Math. Soc. Japan, **44** (1992), 76-98.

[6] S. Kondo, The maximum order of finite groups of automorphisms of $K3$ surfaces, Amer. J. Math. **121** (1999), no. 6, 1245–1252.

[7] N. Machida and K. Oguiso, On $K3$ surfaces admitting finite non-symplectic group actions, J. Math. Sci. Univ. Tokyo, **5** (1998), 273-297.

[8] A. Neron, Modéles minimaux des variétés abéliennes sur les corp locaux et globaux, Publ. Math. I.H.E.S. **21** (1964).

[9] V. V. Nikulin, Finite automorphism groups of Kahler $K3$ surfaces, Trans. Moscow Math. Soc. **38** (1980), 71-135.

[10] K. Oguiso and J. Sakurai, Calabi-Yau threefolds of quotient type, Asian J. Math. **5** (2001), no. 1, 43–77.

[11] K. Oguiso and D. -Q. Zhang, On the most algebraic $K3$ surfaces and the most extremal log Enriques surfaces, Amer. J. Math. **118** (1996), 1277-1297.

[12] K. Oguiso and D. -Q. Zhang, On Vorontsov's theorem on $K3$ surfaces with non-symplectic group actions, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1571–1580.

[13] K. Oguiso and D. -Q. Zhang, Finite automorphism groups of $K3$ surfaces, in preparation.

[14] A. N. Rudakov and I. R. Shafarevich, Surfaces of type K3 over fields of finite characteristics, Current Problems in Math. **18** (1981), 115-207.

[15] T. Shioda, On the Mordell-Weil lattices, Comment. Math. Univ. Sancti Pauli, **39** (1990), 211-240.

[16] G. Xiao, Non-symplectic involutions of a $K3$ surface, preprint 1996.

[17] G. Xiao, Galois covers between $K3$ surfaces, Ann. Inst. Fourier, Grenoble, **46** (1996), 73-88.

[18] D. -Q. Zhang, Logarithmic Enriques surfaces, I, II; J. Math. Kyoto Univ. **31** (1991), 419-466; **33** (1993), 357-397.

[19] D. -Q. Zhang, Automorphisms of K3 surfaces, Proceedings of the International Conference on Complex Geometry and Related Fields, pp. 379–392, AMS/IP Studies in Advanced Mathematics, 39, Amer. Math. Soc., Providence, RI, 2007.

Keiji Oguiso

Department of Mathematics, Osaka University, Toyonaka,

Osaka 560-0043, Japan; and

Korea Institute for Advanced Study,

Hoegiro 87, Seoul, 130-722, Korea

E-mail: oguiso@math.sci.osaka-u.ac.jp

De-Qi Zhang

Department of Mathematics, National University of Singapore,

10 Lower Kent Ridge Road, Singapore 119076

E-mail: matzdq@nus.edu.sg