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# Periods of Enriques Manifolds

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*To the memory of Eckart Viehweg*

**Abstract:** Enriques manifolds are complex spaces whose universal coverings are hyperkähler manifolds. We introduce period domains for Enriques manifolds, establish a local Torelli theorem, and apply period maps in various situations, involving punctual Hilbert schemes, moduli spaces of stable sheaves, and Mukai flops.
**Keywords:** Enriques manifolds, period domains, Torelli Theorem, Mukai flop.

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## Introduction

In this paper we continue our study of *Enriques manifolds* started in [23]. By definition, an Enriques manifold is a connected complex space $Y$ that is not simply connected and whose universal covering $X$ is a hyperkähler manifold. The notion of *hyperkähler manifolds* was first investigated by Beauville [2], and denotes simply connected compact Kähler manifolds with $H^{2,0}(X)$ generated by a symplectic form. Such spaces were intensively studied by Huybrechts and others (see to [13] for an introduction). Hyperkähler and Enriques manifolds are the natural generalizations of K3 and Enriques surfaces to higher dimensions. Enriques varieties were introduced independently in [4].

Using punctual Hilbert schemes, moduli spaces of stable sheaves, and generalized Kummer varieties, we constructed several examples of Enriques manifolds [23]. The basic numerical invariant for an Enriques manifold $Y$ called the *index* is the order $d \geq 2$ of its fundamental group, which is necessarily a finite cyclic group. Most constructions yield index $d = 2$ and are related to Enriques surfaces. However, there are also examples with index $d = 3, 4$ coming from bielliptic surfaces.

The goal of this paper is to study *periods* for Enriques manifolds, that is, linear algebra data coming from Hodge theory, which shed some light on deformations and moduli. Throughout, we build on the vast theory of periods for K3 surfaces, Enriques surfaces, and hyperkähler manifolds. The first main result of this paper is a *Local Torelli Theorem* for Enriques manifolds: Roughly speaking, the base of the Kuranishi family for an Enriques manifold is biholomorphic to some open subset of a bounded symmetric domain. It turns out that the bounded symmetric domains in question are of type IV for index $d = 2$. In contrast, for $d \geq 3$ we have domains of type I that are biholomorphic to complex balls.

Our notion of marking for Enriques manifolds depends on two simple observations: First, the fundamental group $\pi_1(Y)$ can be canonically identified with the group of complex roots of unity $\mu_d(\mathbb{C})$, via the trace of the representation on

$H^{2,0}(X)$. Second, complex representations of $\mu_d(\mathbb{C})$ correspond to weight decompositions $V = \bigoplus V_i$, which are indexed by the character group $\mathbb{Z}/d\mathbb{Z}$. Thus our period domains will be of the form

$$\mathcal{D}_L = \{[\sigma] \in \mathbb{P}(L_{\mathbb{C},1}) \mid (\sigma, \sigma) = 0 \quad \text{and} \quad (\sigma, \overline{\sigma}) > 0\},$$

where $L_{\mathbb{C},1}$ is the weight space for the identity character of the complexification of a certain lattice $L$ endowed with an orthogonal representation of $G = \mu_d(C)$, and a marking of an Enriques manifold $Y$ is an isomorphism $\phi : H^2(X, \mathbb{Z}) \to L$, where $H^2(X, \mathbb{Z})$ is the *Beauville–Bogomolov lattice* endowed with the canonical representation of $G = \pi_1(Y)$.

As an application of the Local Torelli Theorem, we shall prove that any small deformation of the known Enriques manifolds

$$\mathrm{Hilb}^n(S)/G \quad \text{and} \quad M_H(\nu)/G \quad \text{and} \quad \mathrm{Km}^n(A)/G,$$

which come from punctual Hilbert schemes, moduli spaces of stable sheaves, and generalized Kummer varieties, is of the same form. Note that the situation for hyperkähler manifolds is rather different.

We also show that birationally equivalent Enriques manifolds have identical periods. Examples of birational maps are given by *Mukai flops* of $\mathrm{Hilb}^n(S)$, where $S$ is a K3 surface arising as a universal covering of an Enriques surface, and the Mukai flop are given with respect to certain $\mathbb{P}^n = \mathrm{Hilb}^n(C)$, where $C \subset S$ are $(-2)$-curves. We give a detailed study of Mukai flops defined on generalized Kummer varieties $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ for certain abelian surfaces $A$ admiting fibrations $\varphi : A \to E$ onto elliptic curves. Here the Mukai flops are defined with the help of relative Hilbert schemes $\mathrm{Hilb}^{n+1}(A/F)$. Along the way, we obtain new examples of nonkähler manifolds with trivial canonical class that are bimeromorphic to hyperkähler manifolds.

This paper is dedicated to the memory of Eckart Viehweg. We both learned a lot from him: about moduli and many other things.

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## 1. Enriques manifolds and Kuranishi family

Recall that a hyperkähler manifold $X$ is a compact complex Kähler manifold that is simply connected, with $H^{2,0}(X) = H^0(X, \Omega^2_X)$ generated by a 2-form that is everywhere nondegenerate. The dimension of such manifolds is even, and usually written as $\dim(X) = 2n$. An *Enriques manifold* is a connected complex manifold $Y$ that is not simply connected, and whose universal covering $X$ is hyperkähler. Such manifolds are necessarily projective. The trace of the representation of $G = \pi_1(Y)$ on $H^{2,0}(X)$ gives a homomorphism $G \to \mathbb{C}^\times$, which induces a canonical bijection $G \to \mu_d(\mathbb{C})$ with the multiplicative group of $d$-th complex roots of unity (see [23], Section 2, and [3], Section 4). Throughout, we identify the groups

$$G = \pi_1(Y) = \mu_d(\mathbb{C}).$$

The integer $d \geq 2$ is called the *index* of the Enriques manifold $Y$.

Recall that the group of characters $\mu_d(\mathbb{C}) \to \mathbb{C}^\times$ is cyclic of order $d$, and contains a canonical generator, the identiy character $\zeta \mapsto \zeta$. Throughout, we use the identification $\mathrm{Hom}(\mu_d(\mathbb{C}), \mathbb{C}^\times) = \mathbb{Z}/d\mathbb{Z}$. A finite-dimensional complex representation of $G$ is nothing but a finite-dimensional complex vector space $V$ endowed with a *weight decomposition* $V = \bigoplus V_i$ indexed by the characters $i \in \mathbb{Z}/d\mathbb{Z}$. Explicitely, the *weight spaces* $V_i \subset V$ is the set of vectors where each group element $\zeta \in G$ acts via multiplication by the complex number $\zeta^i \in \mathbb{C}$. Note that $V_0 \subset V$ is the $G$-invariant subspace, and $V_1 \subset V$ is the subspace where the action of each $\zeta$ is multiplication by itself.

Now let $Y$ be an Enriques manifold of index $d \geq 2$, and $X \to Y$ be its universal covering, such that $X$ is a hyperkähler manifold. The fundamental group $G = \pi_1(Y) = \mu_d(\mathbb{C})$ acts on $H^1(X, \Theta_X)$, such that we have an weight decomposition of cohomology vector spaces

$$H^q(X, \Theta_X) = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} H^q(X, \Theta_X)_i$$

indexed by the characters of $\pi_1(Y) = \mu_d(\mathbb{C})$. Recall that $H^q(X, \Theta_X)_0$ is nothing but the $G$-invariant part.

**Proposition 1.1.** *The group of global vector fields $H^0(Y, \Theta_Y)$ vanishes, and we have $H^1(Y, \Theta_Y) = H^1(X, \Theta_X)_0 \simeq H^{1,1}(X)_1$.*

*Proof.* Choose a nonzero $\sigma_X \in H^{2,0}(X)$. Then $\delta \mapsto (\delta' \mapsto \sigma_X(\delta, \delta'))$ yields an isomorphism $\Theta_X \to \Omega^1_X$. By definition, each group element $\zeta \in G$ acts via multiplication with the complex number $\zeta \in \mathbb{C}$ on $H^{2,0}(X)$. In turn, our isomorphism induces a bijection $H^1(X, \Theta_X)_0 \to H^{1,1}(X)_1$ between weight spaces.

The projection $h : X \to Y$ is finite and étale, such that the canonical map $\Omega^1_Y \otimes_{\mathcal{O}_Y} h_*(\mathcal{O}_X) \to h_*(\Omega^1_X)$ is bijective. Taking duals into $h_*(\mathcal{O}_X)$ and using canoncial identifications, we obtain a bijection $h_*(\Theta_X) \to \Theta_Y \otimes_{\mathcal{O}_Y} h_*(\mathcal{O}_X)$, and the equality $H^q(Y, \Theta_Y) = H^q(X, \Theta_X)_0$ follows. Now $H^0(X, \Theta_X) = 0$ ensures $H^0(Y, \Theta_Y) = 0$. □

Let $\mathfrak{Y} \to B$ be a *Kuranishi family* of $Y = \mathfrak{Y}_0$, that is, a deformation of $Y = \mathfrak{Y}_0$ that is versal and has the property that $\dim H^1(Y, \Theta_Y)$ equals the embedding dimension of $0 \in B$.

**Proposition 1.2.** *After shrinking $B$ if necessary, the Kuranishi family $\mathfrak{Y} \to B$ of an Enriques manifold $Y$ of index $d \geq 2$ is universal, the base is smooth, and each fiber $\mathfrak{Y}_b$ is an Enriques manifold of index $d$.*

*Proof.* This is a special case of general results due to Fujiki ([10], Lemma 4.14) and Ran ([25], Corollary 2). We recall the arguments, since the explicit construction will be useful later. Let $\mathfrak{X}' \to D'$ be the Kuranishi family of $X = \mathfrak{X}'_0$. After shrinking $D'$ if necessary, we may assume that the family is universal, has smooth base, and all its fibers are hyperkähler manifolds (see [15]). By universality, the fundamental group $G = \pi_1(Y)$ acts on this family, such that the origin $0 \in D'$ is fixed. Since the $G$-fixed locus in $\mathfrak{X}'$ is closed and the projection $\mathfrak{X}' \to D'$ is proper, we may also assume that $G$ acts freely on $\mathfrak{X}'$. It is well-known that there is a regular system of parameters $u_1, \ldots, u_r \in \mathcal{O}^\wedge_{D',0}$ so that the generator of $G$ acts via $u_i \mapsto e^{2\pi\sqrt{-1}n_i/d}u_i$, for certain exponents $n_i$ (see, for example, [26], Lemma 5.4). This implies that the $G$-fixed locus $D \subset D'$ is smooth of dimension $\dim H^1(X, \Theta_X)_0 = \dim H^1(Y, \Theta_Y)$. Consider the induced family $\mathfrak{X} = \mathfrak{X}' \times_{D'} D$. Then $G$ acts fiberwise on $\mathfrak{X}$, and we obtain a family of Enriques manifolds $\mathfrak{X}/G \to D$ of index $d$. We have a commutative diagram

$$\begin{array}{ccc} H^1(Y, \Theta) & \longrightarrow & H^1(X, \Theta) \\ \uparrow & & \uparrow \\ \Theta_D(0) & \longrightarrow & \Theta_{D'}(0) \end{array} \tag{1}$$

where the vertical maps are the Kodaira–Spencer maps. The map on the right is bijective, and the map on the left is the induced map on $G$-invariant subspaces. Consequently, the Kodaira–Spencer map for $\mathfrak{Y} \to D$ is bijective as well. It follows that $\mathfrak{Y}/G \to D$ is versal, and even universal because $H^0(Y, \Theta_Y) = 0$. $\square$

## 2. Period domains and local Torelli

Our next task is to define period domains and period maps for Enriques manifolds, in analogy to the case of Enriques surfaces (for the letter, we refer to [1], Chapter VIII, Section 19). To this end we need a suitable notion of *marking*. Let $Y$ be an Enriques manifold and $X$ be the universal covering, and $H^2(X, \mathbb{Z})$ be the *Beauville–Bogomolov lattice*, which is endowed with the primitive and integral *Beauville–Bogomolov form* (see [13], Section 23) and an orthogonal representation of $G = \pi_1(Y) = \mu_d(\mathbb{C})$. Note that these forms and lattices are also called Beauville–Bogomolov–Fujiki forms and lattices. On the complexification $H^2(X, \mathbb{C})$, we denote by $(\sigma, \sigma')$ the induced bilinear form, such that $(\sigma, \overline{\sigma'})$ is the induced Hermitian form. A little care has to be taken not to confuse bilinear and Hermitian extensions. In the following, we find it practical to say that a nondegenerate lattice or hermitian form has signature of type $(p, *)$ if its signature is $(p, q)$ for some integer $q \geq 0$. Our starting point is the following observation:

**Lemma 2.1.** *The lattice $H^2(X, \mathbb{Z})$ is nondegenerate with signature of type $(3, *)$. The Hermitian form on the weight space $H^2(X, \mathbb{C})_1$ is nondegenerate, and has signature of type $(2, *)$ for $d = 2$, and $(1, *)$ for $d \geq 3$.*

*Proof.* According to [2], Theorem 5, the Beauville–Bogomolov lattice $H^2(X, \mathbb{Z})$ is nondegenerate and has signature $(3, *)$. Since the $G$-action is orthogonal, the eigenspace decomposition on $H^2(X, \mathbb{C})$ is orthogonal, whence the restriction of the Beauville–Bogomolov form to each eigenspace remains nondegenerate.

In the case $d \geq 3$, the weight space $H^2(X, \mathbb{C})_1$ contains $H^{2,0}(X)$, but is orthogonal, with respect to the Hermitian form, to $H^{0,2}(X)$ and the ample class coming from $Y$, whence has signature of type $(1, *)$. In case $d = 2$, we have $1 = -1$ in the character group $\mathbb{Z}/d\mathbb{Z}$, such that the weight space contains also $H^{0,2}(X)$. Consequently, the signature is of type $(2, *)$. $\square$

Now let $L$ be an abstract nondegenerate lattice with signature of type $(3, *)$, endowed with an orthogonal representation of the cyclic group $G = \mu_d(\mathbb{C})$. We further impose the condition that the Hermitian form on the weight space $L_{\mathbb{C},1}$ is nondegenerate, with signature of type $(2,*)$ in case $d=2$, and $(1,*)$ for $d\geq 3$.

An *L-marking* for an Enriques manifold $Y$ of index $d\geq 2$ is an equivariant isometry $\phi: H^2(X,\mathbb{Z})\to L$, where $X$ is the universal covering of $Y$ and $H^2(X,\mathbb{Z})$ is the Beauville–Bogomolov lattice for the hyperkähler manifold $X$, endowed with the canonical action of $G=\pi_1(Y)=\mu_d(\mathbb{C})$. We now define the *period domain* $\mathcal{D}_L$ *for L-marked Enriques manifolds* as

$$\mathcal{D}_L=\{[\sigma]\in\mathbb{P}(L_{\mathbb{C},1})\mid(\sigma,\sigma)=0\quad\text{and}\quad(\sigma,\overline{\sigma})>0\},$$

where $\overline{\sigma}$ denotes complex conjugation inside the complexification $L_{\mathbb{C}}$. Note that for $d=2$, the weight space $L_{\mathbb{C},1}\subset L_{\mathbb{C}}$ is invariant under complex conjugation. On the other hand, for $d\geq 3$, each $\sigma\in L_{\mathbb{C},1}$ satisfies $(\sigma,\sigma)=(\zeta\sigma,\zeta\sigma)=\zeta^2(\sigma,\sigma)$ for all $\zeta\in G$, whence the weight space $L_{\mathbb{C},1}\subset L_{\mathbb{C}}$ is totally isotropic; now the period domain is actually given by

$$\mathcal{D}_L=\{[\sigma]\in\mathbb{P}(L_{\mathbb{C},1})\mid(\sigma,\overline{\sigma})>0\}.$$

Clearly, our period domains inside $\mathbb{P}(L_{\mathbb{C},1})$ are locally closed with respect to the classical topology, whence inherit the structure of a complex manifold.

It turns out that $\mathcal{D}_L$ is a *bounded symmetric domain.* By results of E. Cartan [5], each bounded symmetric domain is the product of irreducible bounded symmetric domains, and the irreducible bounded symmetric domains fall into six classes. The first four are called *Cartan classical domain*, and in Siegel's notation ([27], Chapter XI, §48) are denoted by I, II, III, IV. Recall that the Cartan classical domains of type $\mathrm{I}_{m,n}$ consists of complex matrices $A\in\mathrm{Mat}_{m\times n}(\mathbb{C})$ so that the Hermitian matrix $E_m-A\overline{A}^t$ is positive definite. The Cartan classical domains of type $\mathrm{IV}_n$ is a connected component of the set of all nonzero $z\in\mathbb{C}^{n+2}$ with $z^tHz=0$ and $\bar{z}^tHz>0$, up to nonzero scalar factors, where $H$ is a Hermitian form of signature $(2,n)$. One should bear in mind that the symmetric bounded domain of type $\mathrm{IV}_2$ is not irreducible, rather biholomorphic to $\mathbb{H}\times\mathbb{H}$.

**Proposition 2.2.** *Set* $q=\dim(L_{\mathbb{C},1})$. *For* $d=2$, *the period domain* $\mathcal{D}_L$ *is the disjoint union of two copies of bounded symmetric domains of Type* $\mathrm{IV}_{q-1}$ *of dimension* $q-1$. *For* $d\geq 3$, *the period domains* $\mathcal{D}_L$ *are bounded symmetric domains of type* $\mathrm{I}_{1,q-1}$, *whence biholomorphic to the complex ball of dimension* $q-1$.

*Proof.* By our assumptions on $L$, the weight space $L_{\mathbb{C},1}$ has signature $(2, q-2)$ in case $d = 2$, so the first statement holds. Now suppose $d \geq 3$. Now $L_{\mathbb{C},1}$ has signature $(1, q-1)$, and we may identify $\mathcal{D}_L$ with the set of

$$\left\{(z_0 : \ldots : z_{q-1}) \in \mathbb{P}^{q-1} \mid z_0\overline{z}_0 - \sum_{i=1}^{q-1} z_i\overline{z}_i > 0\right\},$$

which obviously coincides with the complex ball

$$\left\{(z_1, \ldots, z_{q-1}) \in \mathbb{C}^{q-1} \mid \sum_{i=1}^{q-1} z_i\overline{z}_i < 1\right\}.$$

The assertion follows. □

**Remark 2.3.** For $d = 4$, such constructions already appeared in Kondo's study of periods for nonhyperelliptic curves of genus three ([16], §2).

Let $(Y, \phi)$ be an $L$-marked Enriques manifold of index $d \geq 2$, with universal covering $X$. Let $\sigma_X \in H^{2,0}(X)$ be a nonzero form, which is unique up to scalar factors. Considered as a class in $H^2(X, \mathbb{Z})$, we have

$$(\sigma_X, \sigma_X) = 0 \quad \text{and} \quad (\sigma_X, \overline{\sigma}_X) > 0 \quad \text{and} \quad \sigma_X \in H^2(X, \mathbb{C})_1.$$

We thus define the *period point* of our $L$-marked Enriques manifold as the induced point $[\phi(\sigma_X)] \in \mathcal{D}_L$.

Now let $f : \mathfrak{Y} \to B$ be a flat family of Enriques manifolds, say over some simply connected complex space $B$. It follows from Proposition 1.2 that each fiber $\mathfrak{Y}_b$ is an Enriques manifold of index $d$. Moreover, the universal covering $\mathfrak{X} \to \mathfrak{Y}$ is fiber wise the universal covering, and we obtain a flat family $\mathfrak{X} \to B$ of hyperkähler manifolds.

Suppose we have an $L$-marking $\phi : H^2(X, \mathbb{Z}) \to L$, where $X = \mathfrak{X}_0$ is the universal covering of $Y = \mathfrak{Y}_0$. Since the local system $R^2\phi_*\mathbb{Z}_{\mathfrak{X}}$ is constant, our $L$-marking of $Y$ *uniquely* extends to an $L$-marking $\phi : R^2\phi_*\mathbb{Z}_{\mathfrak{X}} \to L_B$ of the flat family of Enriques manifolds. In turn, we obtain a period map

$$p : B \longrightarrow \mathcal{D}_L, \quad b \longmapsto [\phi(\sigma_{\mathfrak{X}_b})].$$

of the marked family. Such period maps are holomorphic, according to general results of Griffiths [11]. It turns out that the *Local Torelli Theorem* holds:

**Theorem 2.4.** *Let $Y$ be an $L$-marked Enriques manifold and $\mathfrak{Y} \to B$ be the Kuranishi family of $Y = \mathfrak{Y}_0$. Then the period map $p : B \to \mathcal{D}_L$ is a local isomorphism at $0 \in B$.*

*Proof.* Since both $B$ and $\mathcal{D}$ are smooth, it suffices to check that the differential of the period map at $0 \in B$ is injective and that $B$ and $\mathcal{D}$ have the same dimension. By the Local Torelli Theorem for hyperkähler manifolds ([2], Theorem 5), the differential of the period map $b \mapsto [\phi(\sigma_{\mathfrak{X}_b})]$ for the Kuranishi family of $X$ is bijective. In light of the commutative diagram (1), the differential of the period map for the Kuranishi family of $Y$ is injective as well.

It remains to compute vector space dimensions. The tangent space at $0 \in B$ is isomorphic to $H^1(Y, \Theta_Y) = H^1(X, \Theta_X)_0 = H^{1,1}(X)_1$. Let us first consider the case $d \geq 3$. Then $\mathcal{D}_L \subset \mathbb{P}(L_{\mathbb{C},1})$ is an open subset, with respect to the classical topology, and the tangent space at the period point is

$$\mathrm{Hom}(\mathbb{C}\phi(\sigma_X), L_{\mathbb{C},1}/\mathbb{C}\phi(\sigma_X)) = \mathrm{Hom}(\mathbb{C}\sigma_X, H^2(X,\mathbb{C})_1/\mathbb{C}\sigma_X).$$

The Hodge decomposition of $H^2(X,\mathbb{C})$ is invariant with respect to automorphisms of $X$, such that

$$H^2(X,\mathbb{C})_1 = H^{0,2}(X)_1 \oplus H^{1,1}(X)_1 \oplus H^{2,0}(X)_1.$$

The first summand vanishes, because $H^{0,2}(X) = H^{0,2}(X)_{-1}$ and $1 \neq -1$ in the character group $\mathbb{Z}/d\mathbb{Z}$. It follows that $H^{1,1}(X)_1$ and $\mathrm{Hom}(\mathbb{C}\sigma_X, H^2(X,\mathbb{C})_1/\mathbb{C}\sigma_X)$ have the same dimensions.

We finally treat the case $d = 2$. Now the period domain $\mathcal{D}_L$ is an open part of a quadratic $(\sigma, \sigma) = 0$ inside $\mathbb{P}(L_{\mathbb{C},1})$, so the tangent space at the period point is given by

$$\mathrm{Hom}(\mathbb{C}\sigma_X, V/\mathbb{C}\sigma_X),$$

where $V \subset H^2(X,\mathbb{C})_1$ is the orthogonal complement of $\sigma_X \in H^2(X,\mathbb{C})$. Clearly, we have

$$H^2(X,\mathbb{C})_1 = H^{1,1}(X)_1 \oplus \mathbb{C}\sigma_X \oplus \mathbb{C}\overline{\sigma}_X.$$

Taking into account that the Beauville–Bogomolov form has $(\sigma_X, \sigma_X) = 0$ and $(\sigma_X, \overline{\sigma}_X) > 0$, the orthogonal complement in question is $V = H^{1,1}(X)_1 \oplus \mathbb{C}\sigma_X$, and the argument concludes as in the preceding paragraph. $\square$

Let $\mathcal{M}_L$ be the set of isomorphism classes of $L$-marked Enriques manifolds. Using the Local Torelli Theorem as in [13], Definition 25.4, we conclude that there is a unique topology and complex structure on $\mathcal{M}_L$ making all the period maps defined on the bases of the Kuranishi family holomorphic. We thus have a *coarse moduli space* $\mathcal{M}_L$ of $L$-marked Enriques manifolds, and the global period map

$$p : \mathcal{M}_L \longrightarrow \mathcal{D}_L$$

is étale. Note, however, that $\mathcal{M}_L$ is not Hausdorff, as we shall see in Section 4. We note in passing that the automorphism group of a marked Enriques manifold is finite, since the same holds for hyperkähler manifolds ([15], Section 9).

## 3. Applications of Local Torelli

Let $S'$ be an Enriques surface. Then $G = \pi_1(S')$ is cyclic of order two, and the universal covering $S$ is a K3 surface. Let $n \geq 1$ be an odd number. According to [23], Proposition 4.1, the induced $G$-action on $X = \mathrm{Hilb}^n(S)$ is free, and $Y = X/G$ is an Enriques manifold of index $d = 2$. Using period maps, we now show that any small deformation of $Y$ is of the same form.

Let $\mathfrak{Y} \to B$ be the Kuranishi family of the Enriques manifold $Y = \mathfrak{Y}_0$, and denote by $\mathfrak{S}' \to B'$ the Kuranishi family of the Enriques surface $S' = \mathfrak{S}'_0$. We may assume that the base spaces $B$ and $B'$ are smooth and contractible. Recall that $\dim(B') = 10$. Let $\mathfrak{S} \to \mathfrak{S}'$ be the universal covering, such that $\mathfrak{S} \to B'$ is a flat family of K3 surfaces. The relative Hilbert scheme, or rather the relative Douady space [24], gives a deformation $\mathrm{Hilb}^n(\mathfrak{S}/B')/G \to B'$ of the Enriques manifold $Y$, which in turn yields a classifying map $h : B' \to B$.

**Proposition 3.1.** *The classifying map $h : B' \to B$ is a local isomorphism at the origin $0 \in B'$. In other words, any small deformation of the Enriques manifold $Y = \mathrm{Hilb}^n(S)/G$ is again of this form.*

*Proof.* Let $L = H^2(X, \mathbb{Z})$ be the Beauville–Bogomolov lattice, endowed with the canonical $G$-action. Recall that Beauville [2] defined an injection of Hodge structure

$$i : H^2(S, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z})$$

compatible with the $G$-action, where the Beauville–Bogomolov form restricts to the cup product. The cokernel is generated by the class of the exceptional divisor of the Hilbert–Chow map $\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$, which is $G$-invariant. It follows that we obtain an identification $H^2(S,\mathbb{C})_1 = H^2(X,\mathbb{C})_1$ of weight spaces.

Now set $L = H^2(X,\mathbb{Z})$, and define $L' \subset L$ as the image of $i$. In this way we obtain a marking of the Enriques manifold $Y$ and the Enriques surface $S'$. These marking extends uniquely to markings of the families $\mathfrak{Y} \to \mathfrak{U}$ and $\mathfrak{S}' \to \mathfrak{U}'$. Now recall that the period domain for $L$-marked Enriques manifolds is

$$\mathcal{D} = \{[\sigma] \in \mathbb{P}(L_{\mathbb{C},1}) \mid (\sigma,\sigma) = 0 \quad \text{and} \quad (\sigma,\overline{\sigma}) > 0\}.$$

This coincides with the period domain for $L'$-marked Enriques surfaces as described in [1], Chapter VIII, Section 19, because we have an equality of weight spaces $L'_{\mathbb{C},1} = L_{\mathbb{C},1}$. Now consider the diagram

$$\begin{tikzcd} B' \arrow[rr, "h"] \arrow[dr, "p'"'] & & B \arrow[dl, "p"] \\ & \mathcal{D} & \end{tikzcd}$$

where $p', p$ are the period maps for the marked families of Enriques surfaces and Enriques manifolds, respectively. By the Local Torelli Theorems, both period maps are local isomorphisms at the origins. It thus remains to check that the diagram is commutative. But this follows from the very definition of the period map and the fact that the map $i$ of local systems sends the lines $H^{2,0}(\mathfrak{S}_b)$ to $H^{2,0}(\mathfrak{X}_b)$. □

**Remark 3.2.** According to [9], every small deformation of a punctual Hilbert scheme of an Enriques surface is of the same form, and our Enriques manifolds $Y = \mathrm{Hilb}^n(S)/G$ show the same behavior. The situation for the hyperkähler manifold $X = \mathrm{Hilb}^n(S)$ is different: Its Kuranishi family has a 21-dimensional base, whereas the Kuranishi family for the K3 surface has only dimension 20. As explained in [2], Theorem 6, a very general small deformation of $X$ is not isomorphic to a punctual Hilbert scheme.

Keeping the previous assumptions, we now additionally assume that our Enriques surface $S'$ is general in the sense that the $K3$ surface $S$ has the minimal possible Picard number $\rho(S) = 10$. Let $\nu = (r, l, \chi - r) \in H^{\mathrm{ev}}(S,\mathbb{Z})$ be a primitive *Mukai vector*, with $l \in \mathrm{Pic}(S)$ and $\nu^2 \geq 0$ and $\chi$ odd, and $H \in \mathrm{NS}(S)$ a very general polarization. Then the moduli space of $X = M_H(v)$ of $H$-stable sheaves on $S$ with Mukai vector $\nu(\mathcal{F}) = \nu$ is a hyperkähler manifold of dimension $\nu^2 + 2$.

According to [23], Theorem 5.3, the canonical $G$-action leaves this moduli space invariant and acts freely on it, such that $Y = X/G$ is an Enriques manifold of index $d = 2$. We shall see that any small deformation of $Y$ is of the same form.

Let $\mathfrak{Y} \to B$ be the Kuranishi family of the Enriques manifold $Y = \mathfrak{Y}_0$, and $\mathfrak{S}' \to B'$ be the Kuranishi family of the Enriques surface $S' = \mathfrak{S}_0$. Then we have a relative moduli space $\mathfrak{M}_H(\nu)$ for the induced family $\mathfrak{S} \to B'$ of K3 surfaces, where $H$ now denotes a very general relative polarization. The fiber wise $G$-action on $\mathfrak{M}_H(\nu)$ is free, according to [23], Proposition 5.2, and we obtain a flat family $\mathfrak{M}_H(\nu)/G \to B'$ of Enriques manifolds. Let $h : B' \to B$ be the classifying map.

**Proposition 3.3.** *The classifying map $h : B' \to B$ is a local isomorphism at the origin $0 \in B'$. In other words, any small deformation of the Enriques manifold $Y = M_H(\nu)/G$ is again of this form.*

*Proof.* Mukai (see [19] and [18]) defined a homomorphism

$$\theta : \nu^\perp \longrightarrow H^2(M_H(\nu), \mathbb{C})$$

where $\nu^\perp \subset H^{\mathrm{ev}}(S, \mathbb{C})$ denotes the orthogonal complement with respect to the *Mukai pairing*, and O'Grady [21] showed in full generality that it is bijective, orthogonal, and respects the integral structure as well as the Hodge structure. The function $\theta(\alpha)$ can be defined on the full Mukai lattice as a Künneth component of

$$\frac{1}{\sigma} \operatorname{pr}_{2*}(\operatorname{ch}(\mathcal{Q})(1 + \operatorname{pr}_1^*[S]) \operatorname{pr}_1^*(\alpha))$$

where $\operatorname{pr}_1 : S \times M_H(\nu) \to S$ and $\operatorname{pr}_2 : S \times M_H(\nu) \to M_H(\nu)$ are the projections and $\mathcal{Q}$ is a *quasitautological bundle*, that is, a coherent sheaf on $S \times M_H(\nu)$ whose restrictions to $S \times \{[\mathcal{F}]\}$ are isomorphic to $\mathcal{F}^{\oplus\sigma}$ for some $\sigma \geq 1$, and satisfying the obvious universality property. On the orthogonal complement $\nu^\perp$, the expression $\theta(\alpha)$ does not depend on the choice of the quasitautological bundle. From this one infers that $\theta : \nu^\perp \to H^2(M_H(\nu), \mathbb{C})$ is natural, in particular, equivariant with respect to the canonical action of $G$. Since the Mukai vector $\nu \in H^{\mathrm{ev}}(S, \mathbb{Z})$ is $G$-fixed, we have $H^2(S, \mathbb{C})_1 \subset \nu^\perp$, and this yields an identification $H^2(S, \mathbb{C})_1 = H^2(M_H(\nu), \mathbb{C})_1$. Now the argument concludes as in the previous proof. $\square$

**Remark 3.4.** Let $\mathfrak{S} \to B$ be the flat family of K3 surfaces induced from the Kuranishi family $\mathfrak{S}' \to B$ of the Enriques surface $S'$. According to [22], every neighborhood of the origin $0 \in B$ contains points $b$ so that the fiber $\mathfrak{S}_b$ has Picard number $\rho > 10$. Hence there are Enriques manifolds of the form $M_H(\nu)$ arising from Enriques surfaces $S'$ that are more special than the general ones considered in [23], Section 5.

Now suppose that $S$ is a *bielliptic surface*, such that $\omega_S \in \mathrm{Pic}(S)$ has order $d \in \{2,3,4,6\}$ and that the corresponding canonical covering $A$ is an abelian surface. Then $A \to S$ is an étale Galois covering, with Galois group $G = \mu_d(\mathbb{C})$. Let $n \geq 2$ be an integer with $d \mid n+1$, and consider the generalized Kummer variety $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ comprising those zero cycles mapping to the origin $0 \in A$ under the summation map. According to the results of [23], Section 6, with a suitable choice of the origin $0 \in A$ and with $d \neq 6$ and with one exception for $d = 3$, the generalized Kummer variety $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ is invariant under the canonical $G$-action on $\mathrm{Hilb}^{n+1}(A)$ and the induced $G$-action on the hyperkähler manifold $X = \mathrm{Km}^n(A)$ is free, such that $Y = X/G$ is an Enriques manifold. Using similiar arguments as for Proposition 3.1, one shows:

**Proposition 3.5.** *Any small deformation of the Enriques manifold* $Y = \mathrm{Km}^n(A)/G$ *is of the same form.*

**Remark 3.6.** One may show that the period domain of marked $Y = \mathrm{Km}^n(A)/G$ is a bounded symmetric domain of type $I_{1,2}$ for $d = 2$, whence biholomorphic to $\mathbb{H} \times \mathbb{H}$, and of type $I_{1,1}$ for $d = 3, 4, 6$, whence biholomorphic to $\mathbb{H}$. In both cases, it coincides with the period domain of the originial biellliptic surface $S$, and one may use periods of elliptic curves to describe the period map explicitely.

The results of this section trigger several questions:

**Question 3.7.** Are the Enriques manifolds of the form $\mathrm{Hilb}^n(S)/G$ and $M_H(\nu)/G$ with index $d = 2$ and same dimension $2n = \nu^2 + 2$ deformation equivalent? This is actually true for the universal covering hyperkähler manifolds by the work of Yoshioka ([28], Theorem 8.1, under some technical assumptions; see also the discussion after [17], Theorem 2.3.), but the techniques of deforming through elliptic surfcaes do not seem to carry over to an equivariant setting.

More strongly, one may ask whether each Enriques manifold of the form $M_H(\nu)/G$ is birational to an Enriques manifold of the form $\mathrm{Hilb}^{n+1}(S)/G$. See Huybrechts work [14] for results on hyperkähler manifolds.

**Question 3.8.** What can be said about the image of the global period map $p : \mathcal{M}_L \to \mathcal{D}_L$? This is particularly interesting for marked Enriques manifolds of the form $\mathrm{Hilb}^n(S)/G$ and $M_H(\nu)/G$ of index $d = 2$ coming from Enriques surface. In contrast to K3 surfaces, the image of the period map for Enriques surfaces is not surjective, since it misses the classes $[\sigma] \in \mathbb{P}(L_{\mathbb{C},1})$ orthogonal to some of the $l \in L_{\mathbb{C},1}$ with $l^2 = -2$.

On the other hand, the global period map for marked Enriques manifolds of the form $\mathrm{Km}^n(A)/G$ is surjective, as is the case for bielliptic surfaces.

## 4. Birational Enriques manifolds

We shall next study birational Enriques manifolds and show that they have identical periods. Let $Y$ be an Enriques manifold of index $d \geq 2$, and $X$ be its universal covering. Set $L = H^2(X, \mathbb{Z})$, and let $\phi : L \to H^2(X, \mathbb{Z})$ be the identity map, regarded as an $L$-marking of $Y$.

**Theorem 4.1.** *Let $Y'$ be another Enriques manifold that is birational to $Y$, with universal covering $X'$. Then $Y'$ also has index $d$, and there is an $L$-marking $\phi' : L \to H^2(X', \mathbb{Z})$ so that $(Y, \phi)$ and $(Y', \phi')$ have the same period point in the period domain $\mathcal{D}_L$.*

*Proof.* The fundamental group is a birational invariant for smooth compact complex manifolds. Let $\varphi : Y \dashrightarrow Y'$ be a birational map, $\pi : X \to Y$ and $\pi' : X' \to Y'$ be the universal covering maps of $Y$ and $Y'$ respectively. Then $\varphi \circ \pi$ is a rational map from $X$ to $Y'$. Let $\nu : Z \to X$ be a Hironaka's resolution of indeterminacy of $\varphi \circ \pi$. Similarly, we choose a Hironaka's resolution of indeterminacy $\nu' : Z' \to X'$ of the rational map $\varphi^{-1} \circ \pi'$. Since $Z$ is smooth and birational to $X$, it follows that $Z$ is simply connected. The same is true for $Z'$. Thus, the morphism $\varphi \circ \pi \circ \nu : Z \to Y'$ can be lifted to a morphism to $X'$, say $\varrho : Z \to X'$. Similarly, we have a morphism $\varrho' : Z' \to X$ which is a lift of $\varphi^{-1} \circ \pi' \circ \nu'$. By (1), $\varrho$ and $\varrho'$ are both birational. Thus

$$f := \varrho \circ \nu^{-1} : X \dashrightarrow X'$$

is a birational map. Note that $f$ is isomorphic in codimension 1. This is because $K_X$ and $K_{X'}$ are both trivial. Thus, we can naturally pull back the 2-form on $X'$ to $X$. Thus, we can choose a generator $\sigma_X$ of $H^2(X, \Omega_X^2)$ and a generator $\sigma_{X'}$ of $H^2(X', \Omega_{X'}^2)$ such that $\sigma_X = f^*\sigma_{X'}$. Moreover, by [13], Proposition 25.14 (see also Pages 213–214), $f$ naturally induces a Hodge isometry with respect to the

Beauville–Bogomolov form:

$$f^* : H^2(X', \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \ .$$

On the other hand, by the construction, we have $\varphi \circ \pi = \pi' \circ f$. Thus, $f \circ g = g \circ f$. Here $g$ is a generator of $\pi_1(Y) = \mu_d(\mathbb{C}) = \pi_1(Y')$. Hence $f^*$ induces a bijection of weight spaces:

$$H^2(X', \mathbb{C})_1 \simeq H^2(X, \mathbb{C})_1.$$

Hence, by $f^*$, the period of $Y$ and the period of $Y'$ become the same point. □

To give examples of birationally equivalent Enriques manifolds, we have to recall certain birational transformations for hyperkähler manifolds. Let $X$ be a hyperkähler manifold of dimension $\dim(X) = 2n$, with $n \geq 2$. Suppose there is a closed subspace $P \subset X$ with $P \simeq \mathbb{P}^n$. As described in [14], the *Mukai flop* $\check{X}$ of $X$ with respect to $P \subset X$ is defined via a commutative diagram

(2)
$$\begin{tikzcd} & \tilde{X} \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & \check{X} \arrow[dl] \\ & \bar{X} & \end{tikzcd}$$

Here $\tilde{X} \to X$ is the blowing-up with center $P \subset X$. Its exceptional divisor is isomorphic to the incidence scheme $E \subset P \times \check{P}$, where $\check{P}$ denotes the dual projective space, and $\tilde{X} \to X$ contracts $E$ along the first projection of $P \times \check{P}$. The morphism $\tilde{X} \to \check{X}$ is defined as the contraction of $E$ along the second projection, which is also the blowing-up of $\check{P} \subset \check{X}$. Moreover, the morphisms $X \to \bar{X}$ and $\check{X} \to \bar{X}$ are the contractions of $P$ and $\check{P}$, respectively. Note that the Mukai flop is simply connected and $H^{2,0}(\check{X})$ is generated by a symplectic form, such that $\check{X}$ is hyperkähler if and only if it is Kähler. Given several disjoint copies $P_1, \ldots, P_d \subset X$, we may also perform a Mukai flop $\check{X}$ with respect to $P_1 \cup \ldots \cup P_d \subset X$ simultaneously.

Now let $X$ be the universal covering of an Enriques manifold $Y$ of index $d \geq 2$, and suppose there is a copy $Q \subset Y$ of $\mathbb{P}^n$. Since $X \to Y$ is a local isomorphism on a small neighborhood of $P_i \subset X$ with respect to the classical topology, we obtain a Mukai flop $\check{Y}$ of $Y$ with respect to $Q \subset Y$, whose universal covering is the Mukai flop $\check{X}$ of $X$ with respect to $P_1 \cup \ldots \cup P_d \subset X$. Note, however, that it is in general not so easy to determine whether or not the Mukai flops are Kähler, such that $\check{Y}$ is indeed an Enriques manifold.

Here is an example: Let $S'$ be an Enriques surface, with universal covering $S$, and $n \geq 0$ be an odd number. Then the induced action of $G = \pi_1(S')$ on the punctual Hilbert scheme $X = \mathrm{Hilb}^n(S)$ is free, such that $Y = X/G$ is an Enriques manifold of index $d = 2$. Suppose furthermore that $S'$ is nodal, that is, there is a curve $C' \subset S'$ of arithmetic genus zero. Then $C' \simeq \mathbb{P}^1$ and $C'^2 = -2$, that is, $C' \subset S'$ is a $(-2)$-curve. The preimage $C_1 \cup C_2 \subset S$ is a union of two disjoint $(-2)$-curves. In turn, we obtain two disjoint copies

$$P_i = \mathrm{Hilb}^n(C_i) \subset \mathrm{Hilb}^n(S) = X, \quad i = 1, 2$$

of $\mathbb{P}^n = \mathrm{Sym}^n(\mathbb{P}^1) = \mathrm{Hilb}^n(\mathbb{P}^1)$ inside the hyperkähler manifold, which are interchanged by the $G$-action. Set $Q = (P_1 \cup P_2)/G \subset Y$.

**Proposition 4.2.** *Assumptions as above. Then the Mukai flop $\check{Y}$ of $Y$ with respect to $Q \subset Y$ is projective, whence an Enriques manifold birational to $Y$.*

*Proof.* This is a variation of an argument of Debarre [6], where maps to Grassmannians are exploited. To carry it out, we have to verify that there is some $\mathcal{L}' \in \mathrm{Pic}(S')$ with the following properties: $\mathcal{L}' \cdot C' = 1$ and both $\mathcal{L}'$ and its preimage $\mathcal{L} \in \mathrm{Pic}(S)$ are very ample. In other words, $C'$ and $C_1, C_2$ become *lines* under suitable embeddings into projective spaces. To see this, consider the contraction $S' \to \bar{S}'$ of the $(-2)$-curve $C'$. Then the proper normal surface $\bar{S}'$ is projective. Let $D_1$ be the pullback of some ample line bundle. Since the intersection form on $\mathrm{NS}(S')$ is unimodular, there is a divisor $D_2$ with $D_2 \cdot C' = 1$. Consider the invertible sheaf $\mathcal{L}' = \mathcal{O}_{S'}(nD_1 + D_2)$. Then $\mathcal{L}'(-C' - K_{S'})$ is relatively ample over $\bar{S}'$, whence ample for $n \gg 0$, such that $H^1(S', \mathcal{L}'(-C')) = 0$ by Kodaira Vanishing. Arguing similarly on $S$, we easily infer that $\mathcal{L}'$ has the desired properties for $n \gg 0$. Increasing $n$ if necessary, we furthermore achieve that $C' \subset S'$ and $C_1, C_2 \subset S$ are the only lines with respect to $\mathcal{L}'$ and $\mathcal{L}$, respectively. Using maps to Grassmannians as in [6], Section 3.2, we see that the Mukai flop $\check{X}$ is indeed projective, and this then holds for $\check{Y}$ as well. $\square$

**Remark 4.3.** As discussed in [15], Section 2.4, the existence of such Mukai flops implies that the coarse moduli space of $L$-marked Enriques manifolds $\mathcal{M}_L$ is not Hausdorff: any two birationally equivalent Enriques manifolds give rise to *nonseparated points* of the moduli space.

Of course, it is another matter whether or not a Mukai flop $Y \dashrightarrow \hat{Y}$ yields Enriques manifolds that are nonisomorphic. Examples of bimeromorphic yet nonisomorphic hyperkähler manifolds were constructed by Debarre [6]. Namikawa [20] even found hyperkähler manifolds having the same periods that are not bimeromorphic. Both constructions, however, use nonprojectivity in an essential way, and apparently do not apply to Enriques manifolds.

## 5. Rational maps to Grassmannians

Recall that the geometry of the punctual Hilbert scheme $\mathrm{Hilb}^{n+1}(E) = \mathrm{Sym}^{n+1}(E)$ of an elliptic curve $E$ is very simple: The canonical map

$$\mathrm{Hilb}^{n+1}(E) \longrightarrow \mathrm{Pic}^{n+1}(E), \quad [Z] \longmapsto \mathscr{O}_E(Z)$$

is a fibration whose fiber $\mathrm{Hilb}^{n+1}_{\mathscr{N}}(E)$ over an invertible sheaf $\mathscr{N} \in \mathrm{Pic}^{n+1}(E)$ is isomorphic to the projectivization of $H^0(E, \mathscr{N})$. Whence $\mathrm{Hilb}^{n+1}(E) \to \mathrm{Pic}^{n+1}(E)$ is a $\mathbb{P}^n$-bundle. Moreover, we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Hilb}^{n+1}(E) \arrow[dl, "+"'] \arrow[dr] & \\
E & & \mathrm{Pic}^{n+1}(E) \arrow[ll]
\end{tikzcd}$$

where the map on the left is the composition of Hilbert–Chow addition map, and the horizontal map makes this diagram commutative. The latter is bijective (and not just an isogeny), because both diagonal arrows have connected fibers. Note that it sends the invertible sheaf associated to the divisor $(n+1)0 \subset E$ to the origin $0 \in E$.

Beauville [3] and Debarre [6] exploited that maps to Grassmannians yield interesting contractions of punctual Hilbert schemes for K3 surfaces. We now continue this line of thought in the following situation: Let $A$ be an abelian surface, and consider the Hilbert scheme of points $\mathrm{Hilb}^{n+1}(A)$ for some fixed integer $n \geq 1$. Given an ample $\mathscr{L} \in \mathrm{Pic}(A)$, we obtain for each closed subscheme $Z \subset A$ of length $n+1$ a restriction map

$$H^0(A, \mathscr{L}) \longrightarrow H^0(Z, \mathscr{L}_Z),$$

where the vector space $H^0(Z, \mathcal{L}_Z)$ is $(n+1)$-dimensional. By the Kodaira Vanishing Theorem, $H^0(A, \mathcal{L})$ has dimension $c_1^2(\mathcal{L})/2$ and $H^1(A, \mathcal{L}) = 0$. Whence the restriction map is surjective if and only if $H^1(A, \mathcal{I}_Z \otimes \mathcal{L}) = 0$, where $\mathcal{I}_Z \subset \mathcal{O}_A$ is the ideal of $Z \subset A$. The set of such $[Z] \in \mathrm{Hilb}^{n+1}(A)$ is open an non-empty, and we get a rational map

$$r = r_{\mathcal{L}} : \mathrm{Hilb}^{n+1}(A) \dashrightarrow \mathrm{Grass}(V, n+1), \quad [Z] \longmapsto (H^0(A, \mathcal{L}) \to H^0(Z, \mathcal{L}_Z))$$

into the Grassmannian of $(n+1)$-dimensional quotients of $V = H^0(A, \mathcal{L})$. We say that this rational map *is defined* at $[Z] \in \mathrm{Hilb}^{n+1}(A)$ if $H^1(A, \mathcal{I}_Z \otimes \mathcal{L}) = 0$. Now the basic observation is:

**Lemma 5.1.** *Suppose $E \subset A$ is an elliptic curve and $\mathcal{N} \in \mathrm{Pic}^{n+1}(E)$ satisfies $\mathcal{N} \neq \mathcal{L}_E$ and $\mathcal{L} \cdot E = n+1$ and $H^1(A, \mathcal{L}(-E)) = 0$. Then the rational map $r : \mathrm{Hilb}^{n+1}(A) \dashrightarrow \mathrm{Grass}(V, n+1)$ is defined in a neighborhood of $\mathrm{Hilb}^{n+1}_{\mathcal{N}}(E)$, and maps $\mathrm{Hilb}^{n+1}_{\mathcal{N}}(E)$ to a point.*

*Proof.* The exact sequence

$$H^0(A, \mathcal{L}) \longrightarrow H^0(E, \mathcal{L}_E) \longrightarrow H^1(A, \mathcal{L}(-E))$$

shows that the restriction map $H^0(A, \mathcal{L}) \to H^0(E, \mathcal{L}_E)$ is surjective. To check that the rational map is defined in a neighborhood of $\mathrm{Hilb}^{n+1}_{\mathcal{N}}(E)$, it suffices to verify that for any closed subscheme $Z \subset E$ of length $n+1$ with $\mathcal{N} \simeq \mathcal{O}_E(Z)$, the restriction map $H^0(E, \mathcal{L}_E) \to H^0(Z, \mathcal{L}_Z)$ is surjective. Indeed, the outer terms in the long exact sequence

$$H^0(E, \mathcal{L}_E(-Z)) \longrightarrow H^0(E, \mathcal{L}_E) \longrightarrow H^0(Z, \mathcal{L}_Z) \longrightarrow H^1(E, \mathcal{L}_E(-Z))$$

vanish, because the invertible sheaf $\mathcal{L}_E(-Z)$ has degree zero but is nontrivial. Thus the rational map is defined at the point $[Z]$. Furthermore, the restriction map $H^0(A, \mathcal{L}) \to H^0(Z, \mathcal{L}_Z)$ factors over the bijection $H^0(E, \mathcal{L}_E) \longrightarrow H^0(Z, \mathcal{L}_Z)$, which means that the image $r([Z])$ does not depend on the point $[Z] \in \mathrm{Hilb}^{n+1}_{\mathcal{N}}(E)$. □

For the rest of this section, we assume that our abelian surface $A$ is endowed with a homomorphism $\varphi : A \to F$ onto an elliptic curve $F$, such that its fibers are elliptic curves. Fix an integer $n \geq 1$, and consider the inclusion of the relative into the absolute Hilbert scheme $\mathrm{Hilb}^{n+1}(A/F) \subset \mathrm{Hilb}^{n+1}(A)$. The relative Hilbert scheme comes along with the structure map $\mathrm{Hilb}^{n+1}(A/F) \to F$, which factors over the canonical map

$$\mathrm{Hilb}^{n+1}(A/F) \longrightarrow \mathrm{Pic}^{n+1}(A/F),$$

and the latter is a $\mathbb{P}^n$-bundle. Throughout, we denote by $\mathrm{Pic}^{n+1}(A/F) \subset \mathrm{Pic}(A)$ the subset of invertible sheaves $\mathcal{L}$ that have degree $n+1$ on the fibers of $\varphi : A \to F$, and by $\mathrm{Hilb}^{n+1}_{\mathcal{L}}(A/F)$ the family of zero cycles $Z$ on fibers $E = \varphi^{-1}(f)$ with $\mathcal{O}_E(Z) \simeq \mathcal{L}_E$.

**Proposition 5.2.** *For each* $\mathcal{L} \in \mathrm{Pic}^{n+1}(A/F)$*, the rational map* $r : \mathrm{Hilb}^{n+1}(A) \dashrightarrow \mathrm{Grass}(V, n+1)$ *is not defined on the closed subset* $\mathrm{Hilb}^{n+1}_{\mathcal{L}}(A/F)$*.*

*Proof.* Let $E = \varphi^{-1}(f)$ be a fiber and $Z \subset E$ be a divisor of length $n+1$ with $\mathcal{O}_E(Z) \simeq \mathcal{L}_E$. The restriction map $H^0(A, \mathcal{L}) \longrightarrow H^0(Z, \mathcal{L}_Z)$ factors over the map on the left of the exact sequence

$$H^0(E, \mathcal{L}_E) \longrightarrow H^0(Z, \mathcal{L}_Z) \longrightarrow H^1(E, \mathcal{L}(-Z)) \longrightarrow 0.$$

Using that $H^1(E, \mathcal{L}(-Z)) \neq 0$, we conclude that the restriction map is not surjective. $\square$

**Proposition 5.3.** *For all ample* $\mathcal{L}' \in \mathrm{Pic}^{n+1}(A/F)$ *and each* $\mathcal{N} \in \mathrm{Pic}(F)$ *of degree* $\deg(\mathcal{N}) \geq n^2 + 1$*, the rational map* $r : \mathrm{Hilb}^{n+1}(A) \dashrightarrow \mathrm{Grass}(V, n+1)$ *given by* $\mathcal{L} = \mathcal{L}' \otimes \varphi^*(\mathcal{N})$ *is defined on the complement of* $\mathrm{Hilb}^{n+1}_{\mathcal{L}}(A/F)$*.*

*Proof.* Let $Z \subset A$ be a subscheme of length $n+1$ with $[Z] \notin \mathrm{Hilb}^{n+1}_{\mathcal{L}}(A/F)$. First, suppose that $Z \subset E = \varphi^{-1}(f)$ is contained in a fiber, and $\mathcal{O}_E(Z) \not\simeq \mathcal{L}_E$. Since $\mathcal{L} = \mathcal{L}' \otimes \varphi^*(\mathcal{N}(-f))$ is ample, we have $H^1(A, \mathcal{L}(-E)) = 0$, whence Lemma 5.1 tells us that the rational map is defined at $[Z]$.

Now assume that $Z$ is not contained in any fiber of $\varphi : A \to F$. We have do distinguish two cases: To start with, suppose that $Z$ is reducible, and decompose $Z = Z_1 + \ldots + Z_r$, $2 \leq r \leq n$ into parts whose support is contained in pairwise different fibers $E_i = \varphi^{-1}(f_i)$. The sheaf $\mathcal{L}(-n(E_1 + \ldots + E_r)) = \mathcal{L}' \otimes \varphi^*(\mathcal{N}(-n(f_1 + \ldots + f_r)))$ is ample, whence the term on the right in the exact sequence

$$H^0(A, \mathcal{L}) \longrightarrow \bigoplus_{i=1}^{r} H^0(nE_i, \mathcal{L}_{nE_i}) \longrightarrow H^1(A, \mathcal{L}(-n(E_1 + \ldots + E_r)))$$

vanishes. Thus it suffices to check that for each zero cycle $W \subset A$ of length $\leq n$ whose support is contained in a fiber $E = \varphi^{-1}(f)$, the restriction map $H^0(nE, \mathcal{L}) \to H^0(W, \mathcal{L}_W)$ is surjective. To this end, set $W_i = W \cap iE$, $1 \leq i \leq n$. Then $W_1 \subset W_2 \subset \ldots \subset W_n$ is a sequence of zero cycles with $W_1 \subset E$ and $W_n = W$. Clearly, the restriction map $H^0(E, \mathcal{L}_E) \to H^0(W_1, \mathcal{L}_{W_1})$ is surjective, because $\deg(W_1) < \deg(\mathcal{L}_E)$. Let $\mathcal{I} \subset \mathcal{O}_{W_{i+1}}$ be the ideal of $W_i \subset W_{i+1}$. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_E(-E) \arrow[r] \arrow[d] & \mathcal{O}_{(i+1)E} \arrow[r] \arrow[d] & \mathcal{O}_{iE} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{W_{i+1}} \arrow[r] & \mathcal{O}_{W_i} \arrow[r] & 0
\end{tikzcd}$$

shows that $\mathcal{I}$ is isomorphic to $\mathcal{O}_D$ for some divisor $D \subset E$ of length $\leq n$. Tensoring with $\mathcal{L}$, using $\mathcal{O}_E(-E) \simeq \mathcal{O}_E$ and taking cohomology, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^0(E, \mathcal{L}_E) \arrow[r] \arrow[d] & H^0((i+1)E, \mathcal{L}_{(i+1)E}) \arrow[r] \arrow[d] & H^0(iE, \mathcal{L}_{iE}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(D, \mathcal{L}_D) \arrow[r] & H^0(W_{i+1}, \mathcal{L}_{W_{i+1}}) \arrow[r] & H^0(W_i, \mathcal{L}_{W_i}) \arrow[r] & 0.
\end{tikzcd}$$

The vertical map on the left is surjective because $\deg(D) < \deg(\mathcal{L}_E)$, and the vertical map on the right is surjective by induction. We conclude that $H^0(nE, \mathcal{L}) \to H^0(W, \mathcal{L}_W)$ is surjective.

It remains to treat the case that $Z \subset A$ is irreducible, say with support contained in $E = \varphi^{-1}(f)$, but not contained in $E$ as a subscheme. We then argue as in the preceding paragraph, taking into account that $Z_1 = Z \cap E$ has length $\leq n$. □

**Proposition 5.4.** *For all ample $\mathcal{L}' \in \mathrm{Pic}^{n+1}(A/F)$ and each $\mathcal{N} \in \mathrm{Pic}(F)$ of degree $\deg(\mathcal{N}) \geq n^2 + 2$, the rational map $r : \mathrm{Hilb}^{n+1}(A) \dashrightarrow \mathrm{Grass}(V, n+1)$ given by $\mathcal{L} = \mathcal{L}' \otimes \varphi^*(\mathcal{N})$ is injective on the complement of $\mathrm{Hilb}^{n+1}(A/F)$.*

*Proof.* Let $Z, Z' \subset A$ be two different zero cycles of length $n+1$, none of them contained in fibers of $\varphi : A \to F$. We have to find some section $s \in H^0(A, \mathcal{L})$ vanishing on one but not on both cycles, for then the kernels of the two surjections $H^0(A, \mathcal{L}) \to H^0(Z, \mathcal{L}_Z)$ and $H^0(A, \mathcal{L}) \to H^0(Z', \mathcal{L}_{Z'})$ are different.

Suppose that we can find a zero cycle $Z \subset W \subset Z \cup Z'$ with $\mathrm{length}(W) = n+2$ so that the intersection $W \cap \varphi^{-1}(f)$ with every fiber has length $\leq n$. Arguing as in the preceding proof, one can infer that the restriction map $H^0(A, \mathcal{L}) \to H^0(W, \mathcal{L}_W)$ is surjective, and we are done.

It remains to treat the case that there is no such zero cycle in neither $Z \subset Z \cup Z'$ nor $Z' \subset Z \cup Z'$. Then it easily follows that some fiber $E = \varphi^{-1}(f)$ intersects $Z$ in length $n$, and this fiber intersects $Z'$ in length $n$ as well, and $Z, Z'$ are supported on $E$. We are done if $W_1 = (Z \cup Z') \cap E$ has length $\geq n+2$, because a nonzero section of $\mathcal{L}_E$ cannot vanish on a subscheme of length greater than $\deg(\mathcal{L}_E) = n+1$. It remains to treat the case that $W_1$ has length $n+1$. We then argue on the infinitesimal neighborhood $2E$ as in the preceding proof. Details are left to the reader. $\square$

## 6. Mukai flops of generalized Kummer varieties

Let $X$ be a hyperkähler manifold of dimension $\dim(X) = 2n$, this time with $n \geq 2$. Recall that for each closed subspace $P \subset X$ with $P \simeq \mathbb{P}^n$, the diagram

$$
\begin{tikzcd}
& \tilde{X} \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & \check{X} \arrow[dl] \\
& \bar{X} &
\end{tikzcd}
\tag{3}
$$

defines the Mukai flop $\check{X}$ of $X$ with respect to $P \subset X$, and that the Mukai flop $\check{X}$ is hyperkähler provided that it is Kähler. The following two results, important in the sequel, are well-known:

**Lemma 6.1.** *Assume that the Mukai flop $\check{X}$ is Kähler. Then $\check{X}$ is projective if and only if $X$ is projective.*

*Proof.* If $X$ is projective, then $\check{X}$ is Moishezon. Being Kähler, it must be projective. The reverse implication holds by symmetry. $\square$

Now let $P, P' \subset X$ be two copies of $\mathbb{P}^n$. We say that $P, P' \subset X$ are *numerically equivalent* if $\deg(\mathcal{L}_P) = \deg(\mathcal{L}_{P'})$ for all $\mathcal{L} \in \operatorname{Pic}(X)$.

**Proposition 6.2.** *Assume that $P, P' \subset X$ are two disjoint copies of $\mathbb{P}^n$ that are numerically equivalent. Then the Mukai flop $\check{X}$ of $X$ with respect to $P \subset X$ is not projective. If $X$ is projective, $\check{X}$ is not even Kähler.*

*Proof.* Seeking a contradiction, we assume that $\check{X}$ is projective. Then $X$ is projective as well, by Proposition 6.1. Choose an ample $\check{\mathcal{L}} \in \operatorname{Pic}(\check{X})$, and let $\mathcal{L} \in \operatorname{Pic}(X)$ be its strict transform. Then

$$\deg(\mathcal{L}_P) = \deg(\mathcal{L}_{P'}) = \deg(\check{\mathcal{L}}_{P'}) > 0,$$

where we regard $P' \subset X \smallsetminus P = \check{X} \smallsetminus \check{P}$ also as a closed subspace of the Mukai flop. It follows that $\mathcal{L}$ is relatively ample for the contraction $h : X \to \bar{X}$, and obviously $\check{\mathcal{L}}$ is relatively ample for $\check{h} : \check{X} \to \bar{X}$. Using that $P \subset X$ and $\check{P} \subset \check{X}$ have codimension $n \geq 2$, the equality $X \smallsetminus P = \check{X} \smallsetminus \check{P}$ induces an identification

$$\bigoplus_{i\geq 0} h_*(\mathcal{L}^{\otimes i}) = \bigoplus_{i\geq 0} \check{h}_*(\check{\mathcal{L}}^{\otimes i})$$

of $\mathcal{O}_{\bar{X}}$-algebras. Taking the relative homogeneous spectra, we see that the rational map $X \dashrightarrow \check{X}$ extends to a $\bar{X}$-isomorphism $X \to \check{X}$. In turn, the pullbacks of $\mathcal{L}$ and $\check{\mathcal{L}}$ to $\tilde{X}$ become isomorphic, which gives a contradiction.

Finally, suppose that $X$ is projective. If $\check{X}$ is Kähler, then it is also projective. But Proposition 6.2 tells us that it is not projective. ☐

Now let $A$ be again an 2-dimensional complex torus endowed with a homomorphism $\varphi : A \to F$ onto some elliptic curve. The punctual Hilbert scheme $\operatorname{Hilb}^{n+1}(A)$ contains the generalized Kummer variety $\operatorname{Km}^n(A)$, which is defined as the cartesian diagram

$$\begin{array}{ccc} \operatorname{Km}^n(A) & \longrightarrow & \operatorname{Hilb}^{n+1}(A) \\ \downarrow & & \downarrow + \\ 0 & \longrightarrow & A, \end{array}$$

and the relative Hilbert scheme $\operatorname{Hilb}^{n+1}(A/F)$.

**Proposition 6.3.** *The intersection $\operatorname{Km}^n(A) \cap \operatorname{Hilb}^{n+1}(A/F)$ inside $\operatorname{Hilb}^{n+1}(A)$ is the disjoint union of $(n+1)^2$ copies of $\mathbb{P}^n$.*

*Proof.* Let $Z \subset E = \varphi^{-1}(f)$ be a zero cycle of length $n+1$ contained in a fiber, and write $Z = \sum_{i=1}^{n+1} z_i$ with $z_i \in E$. Applying $\varphi : A \to F$ to the sum of $Z$ in $A$, we obtain $(n+1)f \in F$. Thus $[Z] \in \mathrm{Km}^n(A/F)$ implies that $f \in F[n+1]$, hence there are only $(n+1)^2$ possibilities for $f$.

Now suppose that $f \in F[n+1]$, and set $E = \varphi^{-1}(f)$. Consider the map

$$h_f : \mathrm{Hilb}^{n+1}(E) \longrightarrow \ker(\varphi)$$

that sends a zero cycle $Z \subset E$ into its sum in $A$. Clearly, $\mathrm{Km}^n(A) \cap \mathrm{Hilb}^{n+1}(A/F)$ is the disjoint union of the preimages $h_f^{-1}(0)$, whence it is the disjoint union of $(n+1)^2$ copies of $\mathbb{P}^n$. $\square$

**Proposition 6.4.** *The* $(n+1)^2$ *components of* $\mathrm{Km}^n(A) \cap \mathrm{Hilb}^{n+1}(A/F) \subset \mathrm{Km}^n(A)$ *are pairwise numerically equivalent.*

*Proof.* According to [2], Section 7, the restriction map

$$H^2(\mathrm{Hilb}^{n+1}(A), \mathbb{Q}) \to H^2(\mathrm{Km}^n(A), \mathbb{Q})$$

is surjective. It follows that each invertible sheaf on $\mathrm{Km}^n(A)$ has a multiple that is the restriction of an invertible sheaf on $\mathrm{Hilb}^{n+1}(A)$. The latter have constant degree on the fibers of the flat family $\mathrm{Hilb}^{n+1}(A/F) \to \mathrm{Pic}^{n+1}(A/F)$, and our $P_i \subset \mathrm{Km}^n(A)$ are fibers of this family. $\square$

Let us write $P_i \subset \mathrm{Km}^n(A)$, $1 \leq i \leq (n+1)^2$ for these copies of $\mathbb{P}^n$, and $P_I = \cup_{i \in I} P_i$ for the disjoint union for a given subset $I \subset \{1, \ldots, (n+1)^2\}$.

**Theorem 6.5.** *Let* $I \subsetneqq \{1, \ldots, (n+1)^2\}$ *be a proper subset. Then the Mukai flop* $\hat{X}_I$ *of* $X = \mathrm{Km}^n(A)$ *along* $P_I \subset X$ *is not Kähler.*

*Proof.* This follows from Proposition 6.4 and Proposition 6.2. $\square$

**Remark 6.6.** Yoshioka constructed first examples of nonkähler manifolds with trivial canonical class bimeromorphic to hyperkähler manifolds, using Mukai flops of moduli spaces of stable sheaves on abelian surfaces ([28], Section 4.4). Note also that Guan [12] has used primary Kodaira surface, which are not in class $\mathcal{C}$, to construct compact symplectic manifolds that are not in class $\mathcal{C}$.

On the other hand, the results of the previous section give a projectivity statement:

**Theorem 6.7.** *Let $\hat{X}$ be the simultaneous Mukai flop of $X$ with respect to the full intersection $\mathrm{Km}^n(A) \cap \mathrm{Hilb}^{n+1}(A/F) = P_1 \cup \ldots \cup P_{(n+1)^2}$. If the homomorphism $\varphi : A \to F$ admits a section, then the Mukai flop $\hat{X}$ is projective.*

*Proof.* Write $A = E \times F$ with $E = \ker(\varphi)$. Choose some divisor $D_1 \subset E$ of degree $n+1$ not linearly equivalent to $(n+1)0$, and some $D_2 \subset E$ of degree $\geq n^2+2$. Consider the invertible sheaf $\mathcal{L} = \mathcal{O}_A(D_1 \times F + E \times D_2)$ and the rational map

$$r : \mathrm{Hilb}^{n+1}(A) \dashrightarrow \mathrm{Grass}(V, n+1), \quad Z \longmapsto (H^0(A, \mathcal{L}) \to H^0(Z, \mathcal{L}_Z))$$

studied in the previous section. According to Proposition 5.3, this rational map is defined on some neighborhood of $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$. Furthermore, it sends $P_1, \ldots, P_{(n+1)^2}$ to points, and is injective on the complement, by Lemma 5.1 and Proposition 5.4. Hence the Stein factorization of $r : \mathrm{Km}^n(A) \to \mathrm{Grass}(V, n)$ factors over the contraction $X \to \bar{X}$, such that $\bar{X}$ is projective. Using that $\tilde{X}$ is projective, one easily infers that the resolution $\check{X} \to \bar{X}$ is relatively projective. The upshot is that $\check{X}$ is projective. □

Now let $A = E \times F$ be a product of two elliptic curves, on which $G = \mu_2(\mathbb{C})$ acts freely via the involution

$$(e, f) \longmapsto (e + 1/2, -f + z)$$

as explained in [23]. Suppose $n \geq 2$ is an odd integer, and $z \in F$ is a point with $(n+1)z = 0$ but $(n+1)/2 \cdot z \neq 0$. According to loc. cit., Theorem 6.4, the $G$-action on $\mathrm{Hilb}^{n+1}(A)$ leaves the subset $\mathrm{Km}^n(A)$ invariant and acts freely there, such that $Y = \mathrm{Km}^n(A)/G$ is an Enriques manifold of index $d = 2$. The group $G$ permutes the $(n+1)^2$ copies $P_i$ of $\mathbb{P}^n$ inside $X = \mathrm{Hilb}^{n+1}(A)$ considered above, whence the Mukai flop $\bar{X}$ with respect to any $G$-invariant union $P_I$ induces a Mukai flop $\bar{Y}$ of our Enriques manifold $Y$. The upshot is:

**Theorem 6.8.** *There are Enriques manifolds $Y$ admitting Mukai flops that are nonkähler (whence not Enriques manifolds in the strict sense, rather "nonkähler Enriques manifolds"), and other Mukai flops $\hat{Y}$ that are Kähler (so again true Enriques manifolds).*

**Question 6.9.** Is such $\hat{Y}$ isomorphic to $Y$ as an abstract complex space?

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