Pure and Applied Mathematics Quarterly
Volume 7, Number 4
(*Special Issue:
In memory of Eckart Viehweg*)
1517—1527, 2011

Further Examples of Stable Bundles of Rank 2 with 4 Sections

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Dedicated to the memory of Eckart Viehweg

Abstract: In this paper we construct new examples of stable bundles of rank 2 of small degree with 4 sections on a smooth irreducible curve of maximal Clifford index. The corresponding Brill-Noether loci have negative expected dimension of arbitrarily large absolute value.
Keywords: Stable vector bundle, Brill-Noether locus, Clifford index, gonality.

1. Introduction

It has been apparent for some time that the classical Brill-Noether theory for line bundles on a smoooth irreducible curve does not extend readily to bundles of higher rank. Some aspects of this have been clarified recently by the introduction of Clifford indices of higher rank [7]. An example of a stable rank-3 bundle with Clifford index less than the classical Clifford index on a general curve of genus 9 or 11 is given in [8], disproving a conjecture of Mercat [9]. Very recently, it was proved in [4] that there exist curves of any genus ≥ 11 for which the rank-2 Clifford index is strictly smaller than the classical Clifford index. In this paper

Received: Nov. 3, 2010; Revised: Nov. 9, 2010.
1991 *Mathematics Subject Classification.* Primary: 14H60; Secondary: 14F05, 14J28, 32L10
Both authors are members of the research group VBAC (Vector Bundles on Algebraic Curves). The second author would like to thank the Department Mathematik der Universität Erlangen-Nürnberg for its hospitality

we use the methods of [4] to present further examples of this, showing in particular that the difference between the two Clifford indices can be arbitrarily large.

We recall that the classical Clifford index $\gamma_1(C)$ of a smooth projective curve of genus $g \geq 4$ over an algebraically closed field of characteristic 0 is defined as

$$\gamma_1(C) := \min\{d - 2(h^0(L) - 1) \mid L \in \mathrm{Pic}^d(C), d \leq g-1, h^0(L) \geq 2\}.$$

It is a classical fact that $\gamma_1 \leq \left[\frac{g-1}{2}\right]$ with equality for the general curve of genus g. More generally, for any positive integer n the rank-n Clifford index $\gamma_n'(C)$ is defined as follows. For any vector bundle E of rank n and degree d on C define

$$\gamma(E) := \frac{1}{n}(d - 2(h^0(E) - n)).$$

Then

$$\gamma_n' = \gamma_n'(C) := \min\left\{\gamma(E) \;\middle|\; \begin{array}{l} E \text{ semistable of rank } n \text{ with} \\ d \leq n(g-1) \text{ and } h^0(E) \geq 2n \end{array}\right\}.$$

In particular $\gamma_1 = \gamma_1'$ and it is easy to see that $\gamma_n' \leq \gamma_1$ for all n.

The gonality sequence $(d_r)_{r \in \mathbb{N}}$ is defined by

$$d_r := \min_{L \in Pic(C)} \{\deg L \mid h^0(L) \geq r+1\}.$$

In classical terms d_r is the minimum number d for which a g^r_d exists. In the case of a general curve we have for all r,

$$d_r = g + r - \left[\frac{g}{r+1}\right].$$

According to [9], [7] a version of Mercat's conjecture states that

$$\gamma_n' = \gamma_1 \quad \text{for all } n.$$

As mentioned above, counterexamples in rank 3 and rank 2 are now known. For the rest of the paper we concentrate on rank 2.

For $\gamma_1 \leq 4$ it is known that $\gamma_2' = \gamma_1$ (see [7, Proposition 3.8]). In any case, we have according to [7, Theorem 5.2]

$$\gamma_2' \geq \min\left\{\gamma_1, \frac{d_4}{2} - 2\right\}.$$

For the general curve of genus 11 we have $\gamma_1 = 5$ and $d_4 = 13$. So in this case, $\gamma_2' = 5$ or $\frac{9}{2}$. It is shown in [4, Theorem 3.6] that there exist curves C of genus 11 with $\gamma_1 = 5$ and $\gamma_2' = \frac{9}{2}$, but this cannot happen on a general curve of genus 11 [4, Theorems 1.6 and 1.7]. Counterexamples to the conjecture in higher genus were also constructed in [4]. All examples E constructed in [4] have $\gamma(E) = \gamma_1 - \frac{1}{2}$.

In this paper we use the methods of [4] to generalize these examples. Our main result is the following theorem.

Theorem 1.1. *Suppose $d = g - s$ with an integer $s \geq -1$ and*

$$g \geq \max\{4s + 14, 12\}.$$

Suppose further that the quadratic form

$$3m^2 + dmn + (g-1)n^2$$

cannot take the value -1 for any integers $m, n \in \mathbb{Z}$. Then there exists a curve C of genus g having $\gamma_1(C) = \left[\frac{g-1}{2}\right]$ and a stable bundle E of rank 2 on C with $\gamma(E) = \frac{g-s}{2} - 2$ and hence

$$\gamma_1 - \gamma_2' \geq \left[\frac{g-1}{2}\right] - \frac{g-s}{2} + 2 > 0.$$

In particular the difference $\gamma_1 - \gamma_2'$ can be arbitrarily large.

This statement can also be written in terms of the Brill-Noether loci $B(2, d, 4)$ which are defined as follows. Let $M(2, d)$ denote the moduli space of stable bundles of rank 2 and degree d on C. Then

$$B(2, d, 4) := \{E \in M(2, d) \mid h^0(E) \geq 4\}.$$

Theorem 1.1 says that under the given hypotheses $B(2, g - s, 4)$ is non-empty. It may be noted that the expected dimension of $B(2, g - s, 4)$ is $-4s - 11 < 0$.

The key point in proving this theorem is the construction of the curves C, which all lie on K3-surfaces and are therefore not general, although they do have maximal Clifford index.

Theorem 1.2. *Suppose $d = g - s$ with an integer $s \geq -1$ and*

$$g \geq \max\{4s + 14, 12\}.$$

Then there exists a smooth K3-surface S of type $(2, 3)$ in $\mathbb{P}^4$ containing a smooth curve C of genus g and degree d with

$$Pic(S) = H\mathbb{Z} \oplus C\mathbb{Z},$$

where H is the polarization, such that S contains no divisor D with $D^2 = 0$. Moreover, if S does not contain a (-2)-curve, then C is of maximal Clifford index $\left[\frac{g-1}{2}\right]$.

The proof of Theorem 1.2, which uses the methods of [3] and [4], is given in Section 2. This is followed in Section 3 by the proof of Theorem 1.1.

2. Proof of Theorem 1.2

Lemma 2.1. *Let $d = g - s$ with $g \geq 4s + 14$ and $s \geq -1$. Then $d^2 - 6(2g-2)$ is not a perfect square.*

Proof. If $d^2 - 6(2g-2) = g^2 - (2s+12)g + s^2 + 12 = m^2$ for some non-negative integer m, then the discriminant

$$(s+6)^2 - (s^2 + 12 - m^2) = 12s + 24 + m^2$$

is a perfect square of the form $(m+b)^2$ with $b \geq 2$. Solving the equation $g^2 - (2s+12)g + (s^2 + 12 - m^2) = 0$ for g, we get

$$g = s + 6 \pm (m+b). \tag{2.1}$$

Now, since $b \geq 2$, we have $(m+b-2)^2 \geq m^2$ and hence

$$4(m+b) - 4 = (m+b)^2 - (m+b-2)^2 \leq 12s + 24,$$

which gives $m + b \leq 3s + 7$. So (2.1) implies $g \leq 4s + 13$, which contradicts the hypothesis. □

Proposition 2.2. *Let $g \geq 4s + 14$ with $s \geq -1$. Then there exists a smooth K3-surface S of type $(2,3)$ in $\mathbb{P}^4$ containing a smooth curve C of genus g and degree $d = g - s$ with*

$$Pic(S) = H\mathbb{Z} \oplus C\mathbb{Z},$$

where H is the polarization, such that S contains no divisor D with $D^2 = 0$.

Proof. The conditions of [6, Theorem 6.1,2.] are fulfilled to give the existence of S and C. Let

$$D \sim mH + nC \quad \text{with} \quad m, n \in \mathbb{Z}.$$

We want to show that the equation $D^2 = 0$ does not have an integer solution. Now

$$D^2 = 6m^2 + 2dmn + (2g-2)n^2.$$

For an integer solution we must have that the discriminant $d^2 - 6(2g-2)$ is a perfect square and this contradicts Lemma 2.1. □

Lemma 2.3. *Under the hypotheses of Proposition 2.2, the curve C is an ample divisor on S.*

Proof. We show that $C \cdot D > 0$ for any effective divisor on S which we may assume to be irreducible. So let $D \sim mH + nC$ be an irreducible curve on S. So

$$C \cdot D = m(g-s) + n(2g-2).$$

Note first that, since H is a hyperplane, we have

$$D \cdot H = 6m + (g-s)n > 0. \tag{2.2}$$

If $m, n \geq 0$, then one of them has to be positive and then clearly $C \cdot D > 0$. The case $m, n \leq 0$ contradicts (2.2).

Suppose $m > 0$ and $n < 0$. Then, using (2.2) we have

$$C \cdot D = m(g-s) + n(2g-2) > -n\left(\frac{(g-s)^2}{6} - (2g-2)\right).$$

So $C \cdot D > 0$ for $g > s + 6 + 2\sqrt{3s+6}$, which holds, since $g \geq 4s + 14$.

Finally, suppose $m < 0$ and $n > 0$. Then, since we assumed D irreducible,

$$nC \cdot D = -mD \cdot H + D^2 \geq -mD \cdot H - 2 \geq -m - 2.$$

If $m \leq -3$, then $nC \cdot D > 0$. If $m = -1$, we have

$$C \cdot D = -(g-s) + n(2g-2) \geq g + s - 2 > 0.$$

The same argument works for $m = -2,\ n \geq 2$. Finally, if $m = -2$ and $n = 1$, we still get $C \cdot D > 0$ unless $D \cdot H = 1$ and $D^2 = -2$. Solving these equations gives $s = 1, g = 14$, contradicting the hypotheses. □

Theorem 2.4. *Let the situation be as above with $d = g - s$, $s \geq -1$ and*

$$g \geq \max\{4s + 14, 12\}.$$

If S does not contain a (-2)-curve, then C is of maximal Clifford index $\left[\frac{g-1}{2}\right]$.

Note that a stronger form of this has been proved for $s = -2$ and g odd in [4, Theorem 3.6] and for $s = -1$ and g even in [4, Theorem 3.7]. The proof follows closely that of [3, Theorem 3.3], but, since some of the estimates are delicate and our hypotheses differ, we give full details.

Proof. Since C is ample by Lemma 2.3, it follows from [1, Proposition 3.3] that $\gamma_1(C)$ is computed by a pencil. If $\gamma_1(C) < \left[\frac{g-1}{2}\right]$, it then follows from [2] that there is an effective divisor D on S such that $D|_C$ computes $\gamma_1(C)$ and satisfying

$$h^0(S,D) \geq 2, \quad h^0(S,C-D) \geq 2 \quad \text{and} \quad \deg(D|_C) \leq g-1.$$

We consider the exact cohomology sequence

$$0 \to H^0(S,D-C) \to H^0(S,D) \to H^0(C,D|_C) \to H^1(S,D-C).$$

Since $C-D$ is effective, and not equivalent to zero, we get

$$H^0(S,D-C) = 0.$$

By assumption S does not contain (-2)-curves, so $|C-D|$ has no fixed components. According to Proposition 2.2 the equation $(C-D)^2 = 0$ has no solutions, therefore $(C-D)^2 > 0$ and the general element of $|C-D|$ is smooth and irreducible. It follows that

$$H^1(S,D-C) = H^1(S,C-D)^* = 0$$

and

$$\gamma_1(C) = \gamma(D|_C) = D \cdot C - 2\dim|D| = D \cdot C - D^2 - 2$$

by Riemann-Roch. We shall prove that

$$D \cdot C - D^2 - 2 \geq \left[\frac{g-1}{2}\right],$$

a contradiction.

Let $D \sim mH + nC$ with $m,n \in \mathbb{Z}$. Since D is effective and S contains no (-2)-curves, we have $D^2 > 0$ and $D \cdot H > 2$. Since $C-D$ is also effective, we have $(C-D)\cdot H > 2$, i.e. $D \cdot H < d-2$. These inequalities and $\deg(D|_C) \leq g-1$ translate to the following inequalities

$$3m^2 + mnd + n^2(g-1) > 0, \tag{2.3}$$

$$2 < 6m + nd < d-2, \tag{2.4}$$

$$md + (2n-1)(g-1) \leq 0. \tag{2.5}$$

Consider the function

$$f(m,n) := D \cdot C - D^2 - 2 = -6m^2 + (1-2n)dm + (n-n^2)(2g-2) - 2,$$

and denote by

$$a := \frac{1}{6}(d + \sqrt{d^2 - 12(g-1)}) \quad \text{and} \quad b := \frac{1}{6}(d - \sqrt{d^2 - 12(g-1)})$$

the solutions of the equation $6x^2 - 2dx + 2g - 2 = 0$. Note that $d^2 > 12(g-1)$. So a and b are positive real numbers.

Suppose first that $n < 0$. From (2.3) we have either $m < -bn$ or $m > -an$. If $m < -bn$, then (2.4) implies that $2 < n(d - 6b) < 0$, because $n < 0$ and $d - 6b = \sqrt{d^2 - 12(g-1)} > 0$, which gives a contradiction.

If $n < 0$ and $m > -an$, from (2.5) we get

$$-an < m \leq \frac{(g-1)(1-2n)}{d} < \frac{(1-2n)d}{12},$$

since $d^2 > 12(g-1)$. For a fixed n, $f(m,n)$ is increasing as a function of m for $m \leq \frac{(1-2n)d}{12}$ and therefore

$$\begin{aligned} f(m,n) &> f(-an,n) \\ &= \frac{d^2 - 12(g-1) + d\sqrt{d^2 - 12(g-1)}}{6} \cdot (-n) - 2 \\ &\geq \frac{d^2 - 12(g-1) + d\sqrt{d^2 - 12(g-1)}}{6} - 2 \\ &\geq \frac{g-1}{2}, \end{aligned}$$

which gives a contradiction. Here the last inequality reduces to

$$d\sqrt{d^2 - 12(g-1)} \geq 15g - 3 - d^2$$

which certainly holds if $d^2 \geq 15g - 3$. This is true under our hypotheses on g if $s \geq 1$. The inequality can be checked directly in the cases $s = 0$ and $s = -1$.

Now suppose $n > 0$. From (2.3) we get that either $m < -an$ or $m > -bn$. If $m < -an$, we get from (2.4), $2 < n(-6a + d) < 0$, a contradiction.

When $m > -bn$, first suppose $n = 1$. Then (2.5) gives

$$-b < m \leq -\frac{g-1}{d}. \tag{2.6}$$

We claim that

$$1 < b < \frac{4}{3}. \tag{2.7}$$

In terms of s we have

$$\begin{aligned}6b &= g - s - \sqrt{(g-s)^2 - 12(g-1)}\\ &= g - s - \sqrt{(g-(s+6))^2 - 12s - 24}\\ &> g - s - (g-(s+6)) = 6,\end{aligned}$$

since $s \geq -1$. This gives $1 < b$. For the second inequality note that $b = \frac{4}{3}$ gives $s = \frac{g-13}{4}$ and b is a strictly increasing function of s in the interval $\left[-1, \frac{g-13}{4}\right]$. Since certainly $s < \frac{g-13}{4}$, we obtain $b < \frac{4}{3}$.

So there are no solutions of (2.6) unless $d \geq g-1$, i.e. $s = 1, 0$ or -1. For these values of s we must have $m = -1$ and

$$f(m,n) = f(-1,1) = d - 8.$$

So $f(-1,1) \geq \left[\frac{g-1}{2}\right]$ if and only if $g \geq 2s + 14$.

Now suppose $m > -bn$ and $n \geq 2$. Then (2.5) gives

$$f(m,n) \geq \min\left\{ f\left(-\frac{(g-1)(2n-1)}{d}, n\right), f(-bn,n)\right\}.$$

We have

$$f\left(-\frac{(g-1)(2n-1)}{d}, n\right) = \frac{g-1}{2}\left((2n-1)^2\left(1 - \frac{12(g-1)}{d^2}\right) + 1\right) - 2.$$

It is easy to see that $f\left(-\frac{(g-1)(2n-1)}{d}, n\right) \geq \frac{g-1}{2}$ for $n \geq 2$. Moreover,

$$f(-bn,n) = -bdn + n(2g-2) - 2 = n(2g-2-bd) - 2.$$

Note that

$$2g - 2 - bd = \frac{\sqrt{d^2 - 12(g-1)}}{6}(d - \sqrt{d^2-12(g-1)}) > 0.$$

So $f(-bn,n)$ is a strictly increasing function of n. Hence it suffices to show that $f(-2b,2) \geq \frac{g-1}{2}$ or equivalently

$$7(g-1) - 4bd - 4 \geq 0.$$

According to (2.7) we have $b < \frac{4}{3}$. So, since $d \leq g+1$, we have

$$\begin{aligned}7(g-1) - 4bd - 4 &\geq 7(g-1) - \frac{16}{3}d - 4\\ &\geq 7g - 7 - \frac{16}{3}g - \frac{16}{3} - 4 = \frac{1}{3}(5g - 49) > 0.\end{aligned}$$

This completes the argument for $m > -bn,\ n > 0$.

Finally, suppose $n = 0$. Then

$$f(m,0) = -6m^2 + dm - 2.$$

As a function of m this takes its maximum value at $\frac{d}{12}$. By (2.5), $m \leq \frac{g-1}{d} \leq \frac{d}{12}$. So $f(m,0)$ takes its minimal value in the allowable range at $m = 1$. Since $f(1,0) = d - 8$, we require $d - 8 \geq \left[\frac{g-1}{2}\right]$ or equivalently

$$g \geq 2s + 14,$$

which is valid by hypothesis. □

This completes the proof of Theorem 1.2.

Remark 2.5. For $s = 0$ or -1 the assumptions of the theorem are best possible, since in these cases $\gamma(H|_C) = \gamma((C-H)|_C) = d - 8$ would otherwise be less than $\left[\frac{g-1}{2}\right]$. For $s \geq 1$ the conditions can be relaxed. For example, if $s \geq 1$ and $g = 4s + 12$, the only places where the argument can fail are in the proofs of Lemma 2.1 and formula (2.7). In the first case, one can show directly that $d^2 - 6(2g-2)$ is not a perfect square; in the second, one can show that $b < \frac{3}{2}$, which is sufficient.

Remark 2.6. The condition that S does not contain a (-2)-curve certainly holds if $3m^2 + dmn + (g-1)n^2 = -1$ has no solutions. We do not know precisely when this is true, but it certainly holds if both $g - 1$ and $g - s$ are divisible by 3. So the conclusion of Theorem 2.4 holds for $s \equiv 1 \mod 3$, if $g \geq 4s + 14$ and $g \equiv 1 \mod 3$. The conclusion also holds, for example, for $g = 16$ and $s = 1$ (see Remark 2.5).

3. Proof of Theorem 1.1

Lemma 3.1. *Let C and H be as in Proposition 2.2 with $d = g - s, s \geq -1$ and suppose that S has no (-2)-curves. Then $H|_C$ is a generated line bundle on C with $h^0(\mathcal{O}_C(H|_C)) = 5$ and*

$$S^2H^0(\mathcal{O}_C(H|_C)) \to H^0(\mathcal{O}_C(H^2|_C))$$

is not injective.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_S(H - C) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H|_C) \to 0.$$

The divisor $H - C$ is not effective, since $(H - C) \cdot H = 6 - d < 0$. So we have

$$0 \to H^0(\mathcal{O}_S(H)) \to H^0(\mathcal{O}_C(H|_C)) \to H^1(\mathcal{O}_S(H - C)) \to 0.$$

Now

$$(C - H)^2 = 2g - 2 - 2d + 6 = 2s + 4 \geq 2$$

and

$$H^2(\mathcal{O}_S(C - H)) = H^0(\mathcal{O}_S(H - C))^* = 0.$$

So by Riemann-Roch $h^0(\mathcal{O}_S(C - H)) \geq 3$. Since S has no (-2)-curves, it follows that the linear system $|C - H|$ has no fixed components and hence its general element is smooth and irreducible (see [10]). Hence $h^1(\mathcal{O}_S(H - C)) = 0$ and therefore $h^0(\mathcal{O}_C(H|_C)) = h^0(\mathcal{O}_S(H)) = 5$. The last assertion follows from the fact that S is contained in a quadric. □

Remark 3.2. Lemma 3.1 implies that $H|_C$ belongs to W^4_{g-s}. So $g - s \geq d_4$. Since the generic value of d_4 is $g + 4 - \left[\frac{g}{5}\right]$, it follows that C has non-generic d_4 if $g < 5s + 20$.

Lemma 3.3. *Let C be a smooth irreducible curve and M a generated line bundle on C of degree $d < 2d_1$ with $h^0(M) = 5$ and such that $S^2H^0(M) \to H^0(M^2)$ is not injective. Then $B(2, d, 4) \neq \emptyset$.*

The proof is identical with that of [5, Theorem 3.2 (ii)]. □

Theorem 3.4. *Let C be as in Theorem 2.4. Then*

(i) $B(2, g - s, 4) \neq \emptyset$;
(ii) $\gamma'_2(C) \leq \frac{g-s}{2} - 2 < \gamma_1(C)$.

Proof. This follows from Theorem 2.4 and Lemmas 3.1 and 3.3. □

This completes the proof of Theorem 1.1, where the last assertion follows from Remark 2.6.

Corollary 3.5. *$\gamma'_{2n}(C) < \gamma_1(C)$ for every positive integer n.*

Proof. This follows from Theorem 3.4 and [7, Lemma 2.2]. □

Remark 3.6. Under the conditions of Theorem 1.1, for any stable bundle E of rank 2 and degree $g-s$ on C with $h^0(E)=4$, it follows from [5, Proposition 6.1] that the coherent system $(E, H^0(E))$ is α-stable for all $\alpha > 0$. So the corresponding moduli spaces of coherent systems are non-empty.

References

[1] C. Ciliberto, G. Pareschi: *Pencils of minimal degree on curves on a K3-surface.* J. reine angew. Math. 460 (1995), 15-36.

[2] R. Donagi, D. Morrison: *Linear systems on K3-sections.* J. Diff. Geom. 29 (1989), 49-64.

[3] G. Farkas: *Brill-Noether loci and the gonality stratification of* $\mathcal{M}_g$. J. reine angew. Math. 539 (2001), 185-200.

[4] G. Farkas, A. Ortega: *The minimal resolution conjecture and rank two Brill-Noether theory.* Pure and Appl. Math. Quarterly 7, no. 4 (2011), 1517-1527.

[5] I. Grzegorczyk, V. Mercat, P. E. Newstead: *Stable bundles of rank 2 with 4 sections.* arXiv: 1006.1258, to appear in Int. J. Math.

[6] A. Knutsen: *Smooth curves on projective K3-surfaces.* Math. Scandinavia 90 (2002), 215-231.

[7] H. Lange and P. E. Newstead: *Clifford indices for vector bundles on curves.* in: A. Schmitt (Ed.) Affine Flag Manifolds and Principal Bundles. Trends in Mathematics, 165-202. Birkhäuser (2010).

[8] H. Lange, V. Mercat and P. E. Newstead: *On an example of Mukai.* arXiv:1003.4007.

[9] V. Mercat: *Clifford's theorem and higher rank vector bundles.* Int. J. Math. 13 (2002), 785-796.

[10] B. Saint-Donat: *Projective models of K3-surfaces.* Amer. J. Math. 96 (1974), 602-639.

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