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A Local Version of The Kawamata-Viehweg Vanishing Theorem

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To the memory of Eckart Viehweg

Abstract: We give a criterion for a divisorial sheaf on a log terminal variety to be Cohen-Macaulay. The log canonical case and applications to moduli are also considered.
Keywords: vanishing theorems, Cohen-Macaulay, log canonical.

The, by now classical, Kawamata–Viehweg vanishing theorem [Kaw82, Vie82] says that *global* cohomologies vanish for divisorial sheaves which are $\mathbb{Q}$-linearly equivalent to a divisor of the form (nef and big) + Δ. In this note we prove that *local* cohomologies vanish for divisorial sheaves which are $\mathbb{Q}$-linearly equivalent to a divisor of the form Δ. If X is a cone over a Fano variety, one can set up a perfect correspondence between the global and local versions.

More generally, we study the depth of various sheaves associated to a log canonical pair (X, Δ). The first significant result in this direction, due to [Elk81], says that if $(X, 0)$ is canonical then X has rational singularities. In particular, $\mathcal{O}_X$ is CM. The proof has been simplified repeatedly in [Fuj85], [Kol97, Sec.11] and [KM98, 5.22]. Various generalizations for other divisorial sheaves and to the log canonical case were established in [KM98, 5.25], [Kov00], [Ale08] and [Fuj09b, Secs.4.2–3].

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Here we prove a further generalization, which, I believe, covers all the theorems about depth mentioned above.

We work with varieties over a field of characteristic 0. For the basic definitions and for background material see [KM98].

Definition 1. Let X be normal and D_1, D_2 two $\mathbb{Q}$-divisors. We say that D_1 is *locally* $\mathbb{Q}$*-linearly equivalent* to D_2, denoted by $D_1 \sim_{\mathbb{Q},loc} D_2$, if $D_1 - D_2$ is $\mathbb{Q}$-Cartier. The same definition works if X is not normal, as long as none of the irreducible components of the D_i is contained in $\operatorname{Sing} X$.

Note that this is indeed a local property. That is, if $\{X_i : i \in I\}$ is an open cover of X and $D_1|_{X_i} \sim_{\mathbb{Q},loc} D_2|_{X_i}$ for every i, then $D_1 \sim_{\mathbb{Q},loc} D_2$.

The following can be viewed as a local version of the Kawamata-Viehweg vanishing theorem.

Theorem 2. *Let* (X, Δ) *be dlt,* D *a (not necessarily effective)* $\mathbb{Z}$*-divisor and* $\Delta' \leq \Delta$ *an effective* $\mathbb{Q}$*-divisor on* X *such that* $D \sim_{\mathbb{Q},loc} \Delta'$*. Then* $\mathcal{O}_X(-D)$ *is CM.*

Here dlt is short for divisorial log terminal [KM98, 2.37] and CM for Cohen–Macaulay. A sheaf F is CM iff all its local cohomologies vanish below the maximal dimension; that is, iff $H^i_x(X, F) = 0$ for $i < \operatorname{codim}_X x$ for every point $x \in X$. Similarly, $\operatorname{depth}_x F \geq j$ iff $H^i_x(X, F) = 0$ for $i < j$; see [Gro68, III.3.1] or [Har77, Exrcs.III.3.3–5].

Examples illustrating the necessity of the assumptions are given in (4.5–10).

The proof of Theorem 2 also works in the complex analytic case. (Normally one would expect that proofs of a local statement as above automatically work for analytic spaces as well. However, many of the papers cited above use global techniques, and some basic questions are still unsettled; see, for instance, (4.3).)

Weaker results hold for log canonical and semi log canonical pairs. For basic definitions in the semi log canonical case (abbreviated as *slc*) see [K$^+$92, Sec.12] or [Fuj09b].

Theorem 3. *Let* (X, Δ) *be semi log canonical and* $x \in X$ *a point that is not a log canonical center of* (X, Δ).

(1) *Let D be a $\mathbb{Z}$-divisor such that none of the irreducible components of D is contained in* $\operatorname{Sing} X$. *Let $\Delta' \leq \Delta$ be an effective $\mathbb{Q}$-divisor on X such that $D \sim_{\mathbb{Q},loc} \Delta'$. Then*

$$\operatorname{depth}_x \mathcal{O}_X(-D) \geq \min\{3, \operatorname{codim}_X x\}.$$

(2) *Let $Z \subset X$ be any closed, reduced subscheme that is a union of lc centers of (X, Δ). Then*

$$\operatorname{depth}_x \mathcal{O}_X(-Z) \geq \min\{3, 1 + \operatorname{codim}_Z x\}.$$

In contrast with (2), my proof does not work in the complex analytic case; see (17).

4 (Applications and examples)**.**

(4.1) In (2) we can take $\Delta' = 0$. Then D can be any $\mathbb{Q}$-Cartier divisor, reproving [KM98, 5.25]. The $D = \Delta' \leq \Delta$ case recovers [Fuj09b, 4.13].

(4.2) The $D = 0$ case of (3.1) is a theorem of [Ale08, Fuj09b] which says that if (X, Δ) is lc and $x \in X$ is not a log canonical center then $\operatorname{depth}_x \mathcal{O}_X \geq \min\{3, \operatorname{codim}_X x\}$. This can fail if x is a log canonical center, for instance when $x \in X$ is a cone over an Abelian variety of dimension ≥ 2.

(4.3) If (X, Δ) is slc then $K_X + \Delta$ is $\mathbb{Q}$-Cartier, hence $-K_X \sim_{\mathbb{Q},loc} \Delta$. Then $\mathcal{O}_X\bigl(-(-K_X)\bigr) \cong \omega_X$. Thus if $x \in X$ is not a log canonical center then $\operatorname{depth}_x \omega_X \geq \min\{3, \operatorname{codim}_X x\}$. (Note that while $\mathcal{O}_X$ is CM iff ω_X is CM, it can happen that $\mathcal{O}_X$ is S_3 but ω_X is not; see [Pat10]. Thus (4.3) does not seem to be a formal consequence of (4.2).)

Let $f : (X, \Delta) \to C$ be a semi log canonical morphism to a smooth curve C (cf. [KM98, 7.1]) and X_c the fiber over a closed point. None of the lc centers are contained in X_c, thus if $x \in X_c$ has codimension ≥ 2 then $\operatorname{depth}_x \omega_{X/C} \geq 3$. Therefore, the restriction of $\omega_{X/C}$ to X_c is S_2, hence it is isomorphic to ω_{X_c}. More generally, $\omega_{X/C}$ commutes with arbitrary base change. (When the general fiber is klt, this follows from [Elk81]; for projective morphisms a proof is given in [KK10], but the general case has not been known earlier. As far as I know, the complex analytic case is still unproved.)

(4.4) Assume that (X, Δ) is slc. For any $n \geq 1$, write

$$-nK_X - \lfloor n\Delta \rfloor \sim_{\mathbb{Q}} -n\bigl(K_X + \Delta\bigr) + \bigl(n\Delta - \lfloor n\Delta \rfloor\bigr) \sim_{\mathbb{Q},loc} n\Delta - \lfloor n\Delta \rfloor.$$

Assume now that $\Delta = \sum(1 - \frac{1}{m_i})D_i$ with $m_i \in \mathbb{N} \cup \{\infty\}$. Then $n\Delta - \lfloor n\Delta \rfloor = \sum_i \frac{c_i}{m_i} D_i$ for some $c_i \in \mathbb{N}$ where $0 \le c_i < m_i$ for every i. Thus $c_i \le m_i - 1$ for every i, that is, $n\Delta - \lfloor n\Delta \rfloor \le \Delta$. Thus, if $x \in X$ is not a log canonical center then

$$\operatorname{depth}_x \mathcal{O}_X\big(nK_X + \lfloor n\Delta \rfloor\big) \ge \min\{3, \operatorname{codim}_X x\}.$$

In particular, if $f : (X, \Delta) \to C$ is a proper slc morphism to a smooth curve and X_c is any fiber then the restriction of $\mathcal{O}_X\big(nK_X + \lfloor n\Delta \rfloor\big)$ to any fiber is S_2 and hence the natural map

$$\mathcal{O}_X\big(nK_X + \lfloor n\Delta \rfloor\big)|_{X_c} \to \mathcal{O}_{X_c}\big(nK_{X_c} + \lfloor n\Delta_c \rfloor\big) \quad \text{is an isomorphism.}$$

This implies that the Hilbert function of the fibers

$$\chi\big(X_c, \mathcal{O}_{X_c}\big(nK_{X_c} + \lfloor n\Delta_c \rfloor\big)\big)$$

is deformation invariant. (Note that, because of the rounding down, the Hilbert function is not a polynomial in the usual sense, rather a polynomial whose coefficients are periodic functions of n. The period divides the index of (X, Δ), that is, the smallest $n_0 \in \mathbb{N}$ such that $n_0\Delta$ is a $\mathbb{Z}$-divisor and $n_0\big(K_X + \Delta\big)$ is Cartier.)

(4.5) It is also worthwhile to note that while the assumptions of Theorem 2 depend only on the $\mathbb{Q}$-linear equivalence class of D, being CM is not preserved by $\mathbb{Q}$-linear equivalence in general. For instance, let X be a cone over an Abelian variety A of dimension ≥ 2. Let D_A be a $\mathbb{Z}$-divisor on A such that $mD_A \sim 0$ for some $m > 1$ but $D_A \not\sim 0$. Let D_X be the cone over D_A. Then $D_X \sim_{\mathbb{Q},loc} 0$, $\mathcal{O}_X(D_X)$ is CM but $\mathcal{O}_X$ is not CM.

These assertions follow from the next easy characterization of CM divisorial sheaves on cones:

(4.6) *Claim.* Let $Y \subset \mathbb{P}^n$ be projectively normal, H the hyperplane class on Y and D a Cartier divisor on Y. Let $X \subset \mathbb{A}^{n+1}$ be the cone over Y with vertex v and D_X the cone over D. Then

$$H^i_v\big(X, \mathcal{O}_X(D_X)\big) = \sum_{m \in \mathbb{Z}} H^{i-1}\big(Y, \mathcal{O}_Y(D + mH)\big) \quad \text{for } i \ge 2.$$

In particular, $\mathcal{O}_X(D_X)$ is CM iff

$$H^i\big(Y, \mathcal{O}_Y(D + mH)\big) = 0 \quad \forall\, m \in \mathbb{Z},\ \forall\, 0 < i < \dim Y. \quad \square$$

(4.7) Consider the quadric cone $X := (x_1x_2 = x_3x_4) \subset \mathbb{A}^4$ with vertex $v = (0,0,0,0)$. It is the cone over the quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. It contains two families of planes with typical members $A := (x_1 = x_3 = 0)$ and $B := (x_1 = x_4 = 0)$. By (4.6)

$$\begin{aligned} H^2_v\big(X, \mathcal{O}_X(aA+bB)\big) &= \textstyle\sum_{m\in\mathbb{Z}} H^1\big(Q, \mathcal{O}_Q(a+m, b+m)\big) \\ &= \textstyle\sum_{0\le m\le |b-a|-2} H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)\big) \otimes H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(|b-a|-2-m)\big). \end{aligned}$$

Thus we see that $\mathcal{O}_X(aA+bB)$ is CM only if $|b-a| < 2$.

(4.8) As another application of (2), assume that $(X, \sum a_iD_i)$ is dlt and $1-\frac{1}{n} \le a_i \le 1$ for every i for some $n \in \mathbb{N}$. Then, for every m,

$$\begin{aligned} m\big(K_X + \textstyle\sum_i D_i\big) = & \textstyle\sum_i m(1-a_i)D_i + m\big(K_X + \sum_i a_iD_i\big) \\ \sim_{\mathbb{Q},loc} & \textstyle\sum_i m(1-a_i)D_i. \end{aligned}$$

If $1 \le m \le n-1$ then $0 \le m(1-a_i) \le a_i$, thus by (2) and by Serre duality we conclude that

$$\omega_X^{[-m]}\left(-m\textstyle\sum D_i\right) \quad \text{and} \quad \omega_X^{[m+1]}\left(m\textstyle\sum D_i\right) \quad \text{are CM for } 1 \le m \le n-1. \tag{4.8.1}$$

If, in addition, K_X is $\mathbb{Q}$-Cartier, then $m\sum_i D_i \sim_{\mathbb{Q},loc} \sum_i m(1-a_i)D_i$, hence

$$\mathcal{O}_X\left(-m\textstyle\sum D_i\right) \quad \text{is CM for } 1 \le m \le n-1. \tag{4.8.2}$$

Results like these are quite fragile. As an example, let $X \subset \mathbb{A}^4$ be the quadric cone with the 2 families of planes $|A|$ and $|B|$. Then

$$\big(X, A_1 + \tfrac{1}{2}(B_1+B_2)\big) \quad \text{and} \quad \big(X, \tfrac{9}{10}(A_1+A_2) + \tfrac{6}{10}(B_1+B_2+B_3)\big)$$

are both dlt, giving that

$$\mathcal{O}_X\left(-A_1-B_1-B_2\right) \quad \text{and} \quad \mathcal{O}_X\left(-A_1-A_2-B_1-B_2-B_3\right) \quad \text{are CM.}$$

Note, however, that the sheaves

$$\mathcal{O}_X\left(-B_1-B_2\right), \mathcal{O}_X\left(-B_1-B_2-B_3\right) \quad \text{and} \quad \mathcal{O}_X\left(-A_1-B_1-B_2-B_3\right)$$

are not CM by (4.7).

(4.9) The following example shows that (4.8.2) fails in general if K_X is not $\mathbb{Q}$-Cartier.

Let $Q \subset \mathbb{A}^4$ be the affine quadric $(xy = zt)$. Let $B_1 = (x = z = 0)$ and $B_2 = (y = t = 0)$ be 2 planes in the same family of planes on Y. For some c_i (to be specified later), consider the divisor $c_1B_1 + c_2B_2$. (Note that $K_Y + c_1B_1 + c_2B_2$

is $\mathbb{Q}$-Cartier only if $c_1 + c_2 = 0$, so one of the c_i will have to be negative, but let us not worry about it for now.)

Consider the group action $\tau : (x, y, z, t) \mapsto (\epsilon x, y, \epsilon z, t)$ where ϵ is a primitive nth root of unity. This generates an action of μ_n; let $\pi : Y \to X_n = Y/\mu_n$ be the corresponding quotient. Both of the B_i are τ-invariant; set $B'_i = \pi(B_i)$ (with reduced structure).

Note that the fixed point set of τ is B_1 and π ramifies along B_1 with ramification index n. Thus $K_Y = \pi^*\bigl(K_{X_n} + (1 - \frac{1}{n})B'_1\bigr)$ hence

$$K_Y + c_1 B_1 + c_2 B_2 = \pi^*\Bigl(K_{X_n} + \bigl(1 - \tfrac{1}{n} + \tfrac{c_1}{n}\bigr)B'_1 + c_2 B'_2\Bigr).$$

Now we see that even if $c_1 < 0$, the coefficient of B'_1 could be positive. In particular one computes that

$$K_Y - \bigl(1 - \tfrac{2}{n+1}\bigr)B_1 + \bigl(1 - \tfrac{2}{n+1}\bigr)B_2 = \pi^*\Bigl(K_{X_n} + \bigl(1 - \tfrac{2}{n+1}\bigr)\bigl(B'_1 + B'_2\bigr)\Bigr).$$

Thus $\Bigl(X_n, \bigl(1 - \frac{2}{n+1}\bigr)\bigl(B'_1 + B'_2\bigr)\Bigr)$ is klt but $B'_1 + B'_2$ consists of 2 normal surfaces intersecting at a point, hence it is not S_2. Therefore, $\mathcal{O}_{X_n}\bigl(-B'_1 - B'_2\bigr)$ has only depth 2 at the origin.

(4.10) With D as in (2), $\mathcal{O}_X(-D)$ is CM. What about $\mathcal{O}_X(D)$?

On a proper variety, sheaves of the form $\mathcal{O}_X(D)$ are quite different from ideal sheaves, but being CM is a local condition. If X is affine, then there is always a reduced divisor D' such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(-D')$. Despite this, (2) does not hold for $\mathcal{O}_X(D)$.

To see such an example, let $S := \mathbb{P}^1 \times \mathbb{P}^1$ and $C := \mathbb{P}^1 \times \{(0{:}0)\}$ a line on S. Embed S into $\mathbb{P}^5$ by $|-(K_S + C)|$ and let $X \subset \mathbb{A}^6$ be the affine cone over S and $D \subset X$ the cone over C. Then (X, D) is canonical and so $\mathcal{O}_X(-D)$ is CM.

Let $F_i \subset X$ be cones over lines of the form $\{p_i\} \times \mathbb{P}^1$. Then $D + F_1 + F_2 \sim 0$, hence $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X(-F_1 - F_2)$. Since $F_1 + F_2$ is not S_2, $\mathcal{O}_X(D)$ is not CM.

The proof of Theorem 2 uses the method of two spectral sequences introduced in [KM98, 5.22] in the global case and in [Fuj09b] in the local case.

5 (The method of two spectral sequences)**.** Let $f : Y \to X$ be a proper morphism, $V \subset X$ a closed subscheme and $W := f^{-1}V \subset Y$. For any coherent sheaf F on

Y there is a Leray spectral sequence

$$H^i_V\big(X, R^j f_* F\big) \Rightarrow H^{i+j}_W(Y, F), \tag{5.1}$$

where H^i_V denotes cohomology with supports in V; see [Gro68, Chap.1]. In particular, if $R^j f_* F = 0$ for every $j > 0$ then the spectral sequence degenerates and we get isomorphisms $H^i_V\big(X, f_* F\big) \cong H^i_W(Y, F)$.

Given a map of sheaves $F \to F'$ we get, for each i, a commutative diagram

$$\begin{array}{ccc} H^i_V\big(X, f_* F'\big) & \stackrel{\alpha'_i}{\to} & H^i_W\big(Y, F'\big) \\ \uparrow & & \uparrow \\ H^i_V\big(X, f_* F\big) & \stackrel{\alpha_i}{\to} & H^i_W\big(Y, F\big). \end{array} \tag{5.2}$$

The following simple observation will be a key ingredient in the proof of (2).

Claim 6. With the above notation, assume that for some i,

(1) $f_* F = f_* F'$,
(2) $H^i_W\big(Y, F\big) = 0$ and
(3) α'_i is an injection.

Then $H^i_V\big(X, f_* F\big) = 0$. □

For the cases $i = 1, 2$, somewhat weaker hypotheses suffice. First note that $H^1_V\big(X, f_* F\big) \hookrightarrow H^1_W\big(Y, F\big)$ is injective, thus, if $i = 1$, then (6.2) alone yields $H^1_V\big(X, f_* F\big) = 0$.

The $i = 2$ case is more interesting. If $H^2_W\big(Y, F\big) = 0$ then $\alpha_2 = 0$. Together with (6.1) this implies that $\alpha'_2 = 0$. On the other hand, α'_2 sits in the exact sequence

$$H^0_V\big(X, R^1 f_* F'\big) \to H^2_V\big(X, f_* F'\big) \xrightarrow{\alpha'_2} H^2_W\big(Y, F'\big),$$

hence we get the following, first used in [Ale08].

Claim 7. With the above notation, assume that

(1) $f_* F = f_* F'$,
(2) $H^2_W\big(Y, F\big) = 0$ and
(3) $H^0_V\big(X, R^1 f_* F'\big) = 0$.

Then $H^2_V\big(X, f_* F\big) = 0$. □

8 (A special case)**.** As a warm up we prove, following the methods of [KM98, 5.22] and [Fuj09b, 4.2.1.App], that if (X, Δ) is klt then X is CM and has rational singularities.

We need to prove that $H^i_x(X, F) = 0$ for $i < \operatorname{codim}_X x$ for every point $x \in X$. We can localize at x; thus from now on assume that x is a closed point. Let $f : Y \to X$ be a resolution such that $W := f^{-1}(x)$ is a divisor. Choose $F := \mathcal{O}_Y$ and $F' := \mathcal{O}_Y(B)$ where B is an effective, f-exceptional divisor to be specified later. Then $f_*\mathcal{O}_Y(B) = f_*\mathcal{O}_Y = \mathcal{O}_X$ hence (6.1) holds and we have a commutative diagram

$$\begin{array}{ccc} H^i_x\big(X, \mathcal{O}_X\big) & \stackrel{\alpha'_i}{\to} & H^i_W\big(Y, \mathcal{O}_Y(B)\big) \\ || & & \uparrow \\ H^i_x\big(X, \mathcal{O}_X\big) & \to & H^i_W\big(Y, \mathcal{O}_Y\big). \end{array} \tag{8.1}$$

By (20), $H^i_W\big(Y, \mathcal{O}_Y\big) = 0$ for $i < \dim X$, hence (6.2) also holds.

In order to prove (6.3), we finally use that (X, Δ) is klt. By definition, this means than we can choose $f : Y \to X$ such that

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + B - A \tag{8.2}$$

where B is an effective, f-exceptional, $\mathbb{Z}$-divisor, A is a simple normal crossing divisor and $\lfloor A \rfloor = 0$. Then $B \sim_{\mathbb{Q}} K_Y + A - f^*(K_X + \Delta)$, hence we conclude from (13) that $R^i f_*\mathcal{O}_Y(B) = 0$ for $i > 0$. Therefore the spectral sequence for $\mathcal{O}_Y(B)$ degenerates and $H^i_x\big(X, \mathcal{O}_X\big) \cong H^i_W\big(Y, \mathcal{O}_Y(B)\big)$ for every i.

Thus, for $i < \dim X$, the commutative diagram (8.1) becomes

$$\begin{array}{ccc} H^i_x\big(X, \mathcal{O}_X\big) & \cong & H^i_W\big(Y, \mathcal{O}_Y(B)\big) \\ || & & \uparrow \\ H^i_x\big(X, \mathcal{O}_X\big) & \to & 0. \end{array} \tag{8.3}$$

This implies that $H^i_x\big(X, \mathcal{O}_X\big) = 0$ for $i < \dim X$, hence X is CM.

Next we prove that $R^j f_*\mathcal{O}_Y = 0$ for $j > 0$. By induction on the dimension and localization, we may assume that $\operatorname{Supp} R^j f_*\mathcal{O}_Y \subset \{x\}$ for $j > 0$. Then

$$H^i_x\big(X, R^j f_*\mathcal{O}_Y\big) = 0 \quad \text{unless } i = 0 \text{ or } (i, j) = (n, 0).$$

Since $H^i_W\big(Y, \mathcal{O}_Y\big) = 0$ for $i < \dim X$, we conclude that $R^j f_*\mathcal{O}_Y = 0$ for $0 < j < n - 1$ and we have an exact sequence

$$0 \to R^{n-1} f_*\mathcal{O}_Y \to H^n_x\big(X, \mathcal{O}_X\big) \stackrel{\alpha_n}{\to} H^n_W\big(Y, \mathcal{O}_Y\big). \tag{8.4}$$

Note that α_n also sits in the diagram

$$\begin{array}{ccc} H^n_x\big(X,\mathcal{O}_X\big) & \cong & H^n_W\big(Y,\mathcal{O}_Y(B)\big) \\ || & & \uparrow \\ H^n_x\big(X,\mathcal{O}_X\big) & \xrightarrow{\alpha_n} & H^n_W\big(Y,\mathcal{O}_Y\big) \end{array} \tag{8.5}$$

which implies that α_n is injective. Thus $R^{n-1}f_*\mathcal{O}_Y = 0$ as required. □

The proof of the general case is quite similar.

9 (Proof of Theorem 2)**.**

We may assume that X is affine and $x \in X$ is a closed point. Write $\Delta = \Delta' + \Delta''$. As in [KM98, 2.43], there are effective $\mathbb{Q}$-divisors $\Delta'_1 \sim_{\mathbb{Q},loc} \Delta'$ and $\Delta''_1 \sim_{\mathbb{Q},loc} \Delta''$ such that

$$\big(X,(1-\epsilon)\Delta + \epsilon(\Delta'_1 + \Delta''_1)\big) \quad \text{is klt.}$$

Furthermore, $D \sim_{\mathbb{Q},loc} \Delta' \sim_{\mathbb{Q},loc} (1-\epsilon)\Delta' + \epsilon\Delta'_1$. Thus we may assume that (X,Δ) is in fact klt.

We need to prove that for every $x \in X$,

$$H^i_x\big(X,\mathcal{O}_X(-D)\big) = 0 \quad \text{for } i < \operatorname{codim}_X x.$$

We can localize at x, hence we may assume that x is a closed point.

Choose a log resolution $f : Y \to X$ of (X,Δ) such that $W := f^{-1}(x)$ is a divisor and write

$$f^*\big(D - \Delta'\big) = f^{-1}_*D - f^{-1}_*\Delta' - F, \tag{9.1}$$

where F is f-exceptional. Set $D_Y := f^{-1}_*D - \lfloor F \rfloor$ and note that

$$D_Y = f^{-1}_*\Delta' + \{F\} + f^*\big(D - \Delta'\big).$$

Thus, from (12), we obtain that if B_Y is effective and f-exceptional, then $f_*\mathcal{O}_Y\big(B_Y - D_Y\big) = \mathcal{O}_X(-D)$. Thus (6.1) holds and we have a commutative diagram

$$\begin{array}{ccc} H^i_x\big(X,\mathcal{O}_X(-D)\big) & \xrightarrow{\alpha'_i} & H^i_W\big(Y,\mathcal{O}_Y(B_Y - D_Y)\big) \\ || & & \uparrow \\ H^i_x\big(X,\mathcal{O}_X(-D)\big) & \to & H^i_W\big(Y,\mathcal{O}_Y(-D_Y)\big) \end{array} \tag{9.2}$$

By (9.1), $D_Y \sim_{\mathbb{Q}} \big(f\text{-nef}\big) + \big(f^{-1}_*\Delta' + \{F\}\big)$, thus, by (20), $H^i_W\big(Y,\mathcal{O}_Y(-D_Y)\big) = 0$ for $i < \dim X$ and so (6.2) is satisfied.

Finally we need α'_i to be an isomorphism. By the klt assumption,

$$K_Y + f_*^{-1}\Delta \sim_{\mathbb{Q}} f^*(K_X+\Delta) + B - A \tag{9.3}$$

where A, B are effective, f-exceptional, B is a $\mathbb{Z}$-divisor, and $\lfloor A \rfloor = 0$. Write $B - A + \{F\} =: B_Y - A_Y$ where A_Y, B_Y are effective, f-exceptional, B_Y is a $\mathbb{Z}$-divisor, and $\lfloor A_Y \rfloor = 0$. Note that

$$\begin{aligned} B_Y - D_Y &\sim_{\mathbb{Q}} B - A + \{F\} + A_Y - f_*^{-1}\Delta' - \{F\} - f^*\big(D - \Delta'\big) \\ &\sim_{\mathbb{Q}} K_Y + f_*^{-1}\Delta + A_Y - f_*^{-1}\Delta' - f^*\big(K_X + \Delta + D - \Delta'\big) \\ &\sim_{\mathbb{Q}} K_Y + f_*^{-1}\Delta'' + A_Y - f^*\big(K_X + \Delta + D - \Delta'\big). \end{aligned} \tag{9.4}$$

Thus $R^i f_* \mathcal{O}_Y\big(B_Y - D_Y\big) = 0$ for $i > 0$ by (13) and so α'_i is an isomorphism.

These imply that $H^i_x\big(X, \mathcal{O}_X(-D)\big) = 0$ for $i < \dim X$ hence $\mathcal{O}_X(-D)$ is CM. □

10 (Proof of Theorem 3)**.**

We start with (3.1) and first consider the case when X is normal, that is, when (X, Δ) is lc. There are a few places where we have to modify the previous proof (9).

We may assume that $\lfloor \Delta \rfloor = 0$. (If $\Delta = \sum d_i D_i$ then we can replace Δ by $\sum (d_i/2)(D_{1i} + D_{2i})$ where D_{1i}, D_{2i} are general members of the linear system $|D_i|$, cf. [KM98, 2.33].) Let $f : Y \to X$ be a log resolution of (X, Δ) and write

$$K_Y + f_*^{-1}\Delta \sim_{\mathbb{Q}} f^*(K_X+\Delta) + B - A - E, \tag{9.5}$$

where A, B, E are effective, f-exceptional, B, E are $\mathbb{Z}$-divisors, E is reduced and $\lfloor A \rfloor = 0$.

Pick D_Y as in (9.1). By (12), if B_Y is effective and f-exceptional, then $f_*\mathcal{O}_Y\big(B_Y - D_Y\big) = \mathcal{O}_X(-D)$ and $H^i_W\big(Y, \mathcal{O}_Y(-D_Y)\big) = 0$ for $i < \dim X$ by (13). It remains to check (7.3), that is, the vanishing of $H^0_x\big(X, R^1 f_* \mathcal{O}_Y\big(B_Y - D_Y\big)\big)$.

To this end choose B_Y, A_Y, E_Y such that

$$B_Y - A_Y - E_Y = B - A - E + \{F\}.$$

where A_Y, B_Y, E_Y are effective, f-exceptional, B_Y, E_Y are $\mathbb{Z}$-divisors, $E_Y \le E$ is reduced, $\lfloor A_Y \rfloor = 0$ and F as in (9.1). With this choice, as in (9.4),

$$B_Y - D_Y \sim_{\mathbb{Q}} K_Y + f_*^{-1}\Delta'' + A_Y + E_Y - f^*\big(K_X + \Delta + D - \Delta'\big).$$

By assumption x is not a log canonical center, hence by (16) x is not an associated prime of $R^i f_*\mathcal{O}_Y(B_Y - D_Y)$. Thus $H^0_x\big(X, R^1 f_*\mathcal{O}_Y(B_Y - D_Y)\big) = 0$ and hence $H^2_x\big(X, \mathcal{O}_X(-D)\big) = 0$.

The above proof should work without changes if X is not normal, that is, when (X, Δ) is slc, but (16) is not stated for semi-resolutions in the references. We go around this problem as follows.

By (18) there is a double cover $\pi : (\tilde{X}, \tilde{\Delta}) \to (X, \Delta)$, étale in codimension 1 such that every irreducible component of $(\tilde{X}, \tilde{\Delta})$ is smooth in codimension 1. Set $\tilde{D} := \pi^{-1}(D)$. Then $\mathcal{O}_X(-D)$ is a direct summand of $\pi_*\mathcal{O}_{\tilde{X}}(-\tilde{D})$, hence it is enough to prove the depth bounds for $\mathcal{O}_{\tilde{X}}(-\tilde{D})$.

As in [Kol08, 20] we can construct a semi-resolution $\tilde{f} : (\tilde{Y}, \tilde{\Delta}_Y) \to (\tilde{X}, \tilde{\Delta})$ such that $(\tilde{Y}, \tilde{\Delta}_Y)$ is an embedded simple normal crossing pair, as required in (16). The rest of the proof works as before.

Next consider (3.2). The only interesting case is when $\operatorname{codim}_Z x \geq 2$. Since x is not a lc center, this implies that $\operatorname{codim}_{Z_i} x \geq 2$ for every irreducible component $Z_i \subset Z$. (It is, however, possible that x has codimension 1 in some lc center that is contained in Z.) By localizing, we may assume that x is a closed point.

If X is normal, let $f : Y \to X$ be a log resolution. If X is not normal, as before, by first passing to a double cover (18) we may assume that there is a semi log resolution $f_1 : Y_1 \to X$ to which (16) applies. We may also assume that every irreducible component of Z is the image of some divisor $E_j \subset Y$ with discrepancy -1. We have to be more careful if Z contains one of the codimension 1 components of $\operatorname{Sing} X$. In this case, we have a divisor $E_j \subset \operatorname{Sing} Y_1$ mapping to Z. We blow up E_j to get $f : Y \to X$ and replace E_j with both of the irreducible components of its preimage on Y. Set $E = \sum_j E_j$. By (11),

$$\mathcal{O}_X(-Z) = f_*\big(\mathcal{O}_Y(-E)\big) = f_*\big(\mathcal{O}_Y(B - E)\big)$$

for any effective f-exceptional divisor B whose support does not contain any of the E_j. As usual, write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta) + B - E - E' - A.$$

By (19), $H^i_W\big(Y, \mathcal{O}_Y(-E)\big) = 0$ is dual to $\big(R^{n-i} f_*\omega_Y(E)\big)_x$. We have assumed that $\dim f(E_j) \geq 2$ for every j and $\dim f(E_j \cap E_k) \geq 1$ for every $j \neq k$ since x is not a lc center. Thus, by (15) and (19), $H^i_W\big(Y, \mathcal{O}_Y(-E)\big) = 0$ for $i \le 2$ and so (7.2) holds.

From $B - E \sim_{\mathbb{Q}} K_Y + E' + A - f^*(K_X + \Delta)$ and (16) we conclude that every associated prime of $R^i f_* \mathcal{O}_Y(B - E)$ is an lc center of (X, Δ). Since x is not an lc center, this implies that $H^0_x\big(X, R^1 f_* \mathcal{O}_Y(B - E)\big) = 0$, giving (7.3). Therefore $H^2_x\big(X, \mathcal{O}_X(-Z)\big) = 0$ and so $\mathcal{O}_X(-Z)$ has depth ≥ 3 at x. □

Conditions for (6.1).

11. The assumption (6.1) is easy to satisfy in many cases. Let $f : Y \to X$ be a proper, birational morphism to a normal scheme X. For any closed subscheme $Z_Y \subset Y$, $f_* \mathcal{O}_Y(-Z_Y) = \mathcal{O}_X\big(-f(Z_Y)\big)$ where $f(Z_Y)$ is the scheme theoretic image. If Z_Y is reduced, then $Z_X := f(Z_Y) \subset X$ is also reduced. Thus if B is f-exceptional and $f(B)$ does not contain any of the irreducible components of Z_X, then $\mathcal{O}_X(-Z_X) = f_* \mathcal{O}_Y(-Z_Y) = f_* \mathcal{O}_Y(B - Z_Y)$.

The last equality holds even if X is not normal.

Another easy case is the following (cf. [Fuj85]).

Lemma 12. *Let $f : Y \to X$ be a proper, birational morphism. Let D be a $\mathbb{Z}$-divisor on X and assume that $D \sim_{\mathbb{Q},f} D_h + D_v$ where D_v is f-exceptional, $\lfloor D_v \rfloor = 0$ and D_h is effective without exceptional components. Let B be an effective, f-exceptional divisor. Assume that*

(1) *either X and Y are normal,*
(2) *or X and Y are S_2, f is an isomorphism outside a codimension 2 subscheme of X and Y is normal at the generic point of every exceptional divisor.*

Then

$$\mathcal{O}_X\big(-f_* D\big) = f_* \mathcal{O}_Y(-D) = f_* \mathcal{O}_Y(B - D).$$

Proof. Note that in general $f_* \mathcal{O}_Y(-D) \subset \mathcal{O}_X\big(-f_* D\big)$ and there is an effective, f-exceptional divisor B' such that $\mathcal{O}_X\big(-f_* D\big) \subset f_* \mathcal{O}_Y(B' - D)$. Thus it is enough to prove that $f_* \mathcal{O}_Y(-D) = f_* \mathcal{O}_Y(B - D)$.

For some B_1, fix a section $s \in H^0\big(X, f_* \mathcal{O}_Y(B_1 - D)\big)$ and then take $0 \le B \le B_1$ as small as possible such that $s \in H^0\big(X, f_* \mathcal{O}_Y(B - D)\big)$ still holds. We need to prove that $B = 0$.

By assumption, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M on X such that $D + f^*M = D_h + D_v$. Choose $n \in \mathbb{N}$ such that $nD_h, nD_v, f^*(nM)$ are all $\mathbb{Z}$-divisors. Then

$$\begin{aligned}\big(f^*s\big)^n &\in H^0\big(Y, \mathcal{O}_Y(nB - nD)\big)\\ &= H^0\big(Y, \mathcal{O}_Y(nB - nD_h - nD_v + f^*(nM)\big)\\ &\subset H^0\big(Y, \mathcal{O}_Y(nB - nD_v + f^*(nM)\big).\end{aligned}$$

As noted in (11), adding effective exceptional divisors to a pull-back never creates new sections. By assumption, every irreducible component of B appears in $nB - nD_v$ with positive coefficient. Thus $\big(f^*s\big)^n$ vanishes along every irreducible component of B and so does f^*s, contradicting the minimality of B. Thus $B = 0$ and so s is a section of $f_*\mathcal{O}_Y(-D)$. □

Conditions for (6.2).

The following is the relative version of the Kawamata–Viehweg vanishing theorem [Kaw82, Vie82].

Theorem 13. *Let $f : Y \to X$ be a projective, birational morphism, Y smooth and Δ an effective simple normal crossing divisor on Y such that $\lfloor \Delta \rfloor = 0$. Let M be a line bundle on Y and assume that $M \sim_{\mathbb{Q},f} K_Y + (f\text{-nef}) + \Delta$. Then $R^i f_*M = 0$ for $i > 0$.* □

If f is not birational, Y is not smooth or if $\lfloor \Delta \rfloor \neq 0$, then vanishing fails in general. There are, however, some easy consequences that can be read off from induction on the number of irreducible components and by writing down the obvious exact sequences

$$0 \to M \to M(H) \to M(H)|_H \to 0 \quad \text{and} \quad 0 \to M(-D) \to M \to M|_D \to 0$$

where H is a smooth sufficiently ample divisor on Y and $D \subset \lfloor \Delta \rfloor$ is a smooth divisor.

Let W be a smooth variety and $\sum_{i\in I} E_i$ a snc divisor on W. Write $I = I_V \cup I_D$ as a disjoint union. Set $Y := \sum_{i\in I_V} E_i$ as a subscheme and $D_Y := \sum_{i\in I_D} E_i|_Y$ as a divisor on Y. We call (Y, D_Y) an *embedded snc pair.* Anything isomorphic to such a pair is called an *embeddable snc pair.* A pair is called an *snc pair* if it is locally an embeddable snc pair.

Corollary 14. *Let* $(Y, \sum_i D_i)$ *be an snc pair and* $f : Y \to X$ *a projective morphism. Let* M *be a line bundle on* Y *and assume that*

$$M \sim_{\mathbb{Q},f} K_Y + (f\text{-nef}) + \textstyle\sum_i a_i D_i \quad \text{where } 0 \le a_i \le 1.$$

Then $R^i f_* M = 0$ *for* $i > \max\{\dim f^{-1}(x) : x \in X\}$. □

Corollary 15. *Notation and assumptions as in (14). Set* $\Delta = \sum_i a_i D_i$. *Assume that* f *is birational and set* $n = \dim Y$. *Then*

(1) $R^n f_* M = 0$.
(2) $R^{n-1} f_* M = 0$ *unless there is a divisor* $B \subset \operatorname{Sing} Y \cup \lfloor \Delta \rfloor$ *such that* $\dim f(B) = 0$.
(3) $R^{n-2} f_* M = 0$ *unless there are divisors* $B_1, B_2 \subset \operatorname{Sing} Y \cup \lfloor \Delta \rfloor$ *such that either* $\dim f(B_1) \le 1$ *or* $\dim f(B_1 \cap B_2) = 0$. □

In the log canonical setting we also used the $i = 1$ case of the following result of [Amb03, 3.2], [Fuj09b, 2.39], [Fuj09a, 6.3]. In contrast with (14) and (15), its proof is quite difficult and subtle.

Theorem 16. *Let* $(Y, \sum_i D_i)$ *be an embeddable snc pair and* $f : Y \to X$ *a projective morphism. Let* M *be a line bundle on* Y *and assume that*

$$M \sim_{\mathbb{Q},f} K_Y + (f\text{-semi-ample}) + \textstyle\sum_i a_i D_i \quad \text{where } 0 \le a_i \le 1.$$

Then every associated prime of $R^i f_* M$ *is the* f*-image of an irreducible component of some intersection* $B_1 \cap \cdots \cap B_r$ *for some divisors* $B_j \subset \operatorname{Sing} Y \cup \{D_i : a_i = 1\}$. □

Remark 17 (The analytic case)**.** The complex analytic version of (13) is proved in [Tak85, Nak87] but the complex analytic version of (16) is not known.

If X is a complex analytic space, we can choose $f : Y \to X$ to be projective. By pushing forward the sequence

$$0 \to M\big(-\textstyle\sum D_i\big) \to M \to M|_{\sum D_i} \to 0$$

shows that $R^i f_* M \cong R^i f_*\big(M|_{\sum D_i}\big)$. Thus we need to prove the analog of (16) for the projective morphism $\sum D_i \to X$. This is in fact how the proofs of [Amb03, Fuj09b] work. However, their proofs rely on a global compactification of $\sum D_i \to X$.

18 (A natural double cover)**.** Let X^0 be a scheme whose singularities are double normal crossing points only. Let $\pi^0 : \bar{X}^0 \to X^0$ denote its normalization with conductors $D^0 \subset X^0$, $\bar{D}^0 \subset \bar{X}^0$ and Galois involution $\tau : \bar{D}^0 \to \bar{D}^0$.

Take two copies $\bar{X}^0_1 \amalg \bar{X}^0_2$ and on $\bar{D}^0_1 \amalg \bar{D}^0_2$ consider the involution

$$\sigma(p,q) = \bigl(\tau(q), \tau(p)\bigr).$$

Note that $\bigl(\bar{D}^0_1 \amalg \bar{D}^0_2\bigr)/\sigma \cong \bar{D}^0$ but the isomorphism is non-canonical.

We obtain $\tilde{X}^0$ either as the universal push-out of

$$\bigl(\bar{D}^0_1 \amalg \bar{D}^0_2\bigr)/\sigma \leftarrow \bigl(\bar{D}^0_1 \amalg \bar{D}^0_2\bigr) \hookrightarrow \bigl(\bar{X}^0_1 \amalg \bar{X}^0_2\bigr)$$

or as the spectrum of the preimage of the σ-invariant part

$$\bigl(\mathcal{O}_{\bar{D}^0_1} + \mathcal{O}_{\bar{D}^0_2}\bigr)^\sigma \subset \mathcal{O}_{\bar{D}^0_1} + \mathcal{O}_{\bar{D}^0_2} \quad \text{in} \quad \mathcal{O}_{\bar{X}^0_1} + \mathcal{O}_{\bar{X}^0_2}.$$

Then $\pi^0 : \tilde{X}^0 \to X^0$ is an étale double cover and the irreducible components of $\tilde{X}^0$ are smooth. The normalization of $\tilde{X}^0$ is a disjoint union of two copies of the normalization of X^0.

Let now X be slc and $j : X^0 \hookrightarrow X$ an open subset with double nc points only and such that $X \setminus X^0$ has codimenson ≥ 2. Let $\pi^0 : \tilde{X}^0 \to X^0$ be as above. Then $j_*\pi^0_*\mathcal{O}_{\tilde{X}^0}$ is a coherent sheaf of algebras on X. Set

$$\tilde{X} := \operatorname{Spec}_X j_*\pi^0_*\mathcal{O}_{\tilde{X}^0}$$

with projection $\pi : \tilde{X} \to X$.

By construction, $\tilde{X}$ is S_2, π is étale in codimension 1 and the normalization of $\tilde{X}$ is a disjoint union of two copies of the normalization of X. Furthermore, the irreducible components of $\tilde{X}$ are smooth in codimension 1.

Conditions for (6.3).

These are reduced to the previous vanishing theorems using the following duality.

Proposition 19. *Let $f : Y \to X$ be a proper morphism, Y CM, M a vector bundle on Y and $x \in X$ a closed point. Set $n = \dim Y$ and let $W \subset Y$ be a subscheme such that* $\operatorname{Supp} W = \operatorname{Supp} f^{-1}(x)$*. Then there is a natural bilinear pairing*

$$H^i_W\bigl(Y, \omega_Y \otimes M^{-1}\bigr) \times \bigl(R^{n-i} f_* M\bigr)_x \to k(x)$$

which has no left or right kernel, where the subscript denotes the stalk at x.

In particular, if either $H^i_W\big(Y, \omega_Y \otimes M^{-1}\big)$ or $\big(R^{n-i} f_* M\big)_x$ is a finite dimensional $k(x)$-vector space then so is the other and they are dual to each other.

Proof. Let $mW \subset Y$ be the subscheme defined by the ideal sheaf $\mathcal{O}_Y(-W)^m$. By [Gro68, II.6]

$$H^i_W\big(Y, \omega_Y \otimes M^{-1}\big) = \varinjlim \operatorname{Ext}^i_Y\big(\mathcal{O}_{mW}, \omega_Y \otimes M^{-1}\big). \tag{19.1}$$

On the other hand, by the theorem on formal functions,

$$\big(R^{n-i} f_* M\big)^{\wedge} = \varprojlim H^{n-i}\big(mW, M|_{mW}\big) \tag{19.2}$$

where $\wedge$ denotes completion at $x \in X$.

We show below that for every m, the groups on the right hand sides of (19.1–2) are dual to each other. This gives the required bilinear pairing which has no left or right kernel.

(Duality using a compactification.) Let $\bar{Y} \supset Y$ be a CM compactification such that M extends to a vector bundle $\bar{M}$ on $\bar{Y}$. Since W is disjoint from $\bar{Y} \setminus Y$,

$$\operatorname{Ext}^i_Y\big(\mathcal{O}_{mW}, \omega_Y \otimes M^{-1}\big) = \operatorname{Ext}^i_{\bar{Y}}\big(\mathcal{O}_{mW}, \omega_{\bar{Y}} \otimes \bar{M}^{-1}\big) = \operatorname{Ext}^i_{\bar{Y}}\big(\mathcal{O}_{mW} \otimes \bar{M}, \omega_{\bar{Y}}\big)$$

and, by Serre duality, the latter is dual to

$$H^{n-i}\big(\bar{Y}, \mathcal{O}_{mW} \otimes \bar{M}\big) = H^{n-i}\big(mW, \bar{M}|_{mW}\big) = H^{n-i}\big(mW, M|_{mW}\big).$$

(It is not known that such $\bar{Y}$ exists, but we could have used any compactification and Grothendieck duality. For the complex analytic case see [RRV71].)

(Duality without compactification.) This proof works if W is an effective Cartier divisor. In most applications, X is given and Y is a suitable resolution, hence this assumption is easy to achieve.

We use the local-to-global spectral sequence for Ext

$$H^i\big(Y, \mathcal{E}xt^j_Y(N, N')\big) \Rightarrow \operatorname{Ext}^{i+j}_Y(N, N').$$

Since $\mathcal{O}_{mW}$ has projective dimension 1 as an $\mathcal{O}_Y$-sheaf, $\mathcal{E}xt^i_Y\big(\mathcal{O}_{mW}, \omega_Y\big) = 0$ for $i \neq 1$ and $\mathcal{E}xt^1_Y\big(\mathcal{O}_{mW}, \omega_Y\big) = \omega_{mW}$. Thus for any locally free $\mathcal{O}_{mW}$-sheaf N, the local-to-global spectral sequence for $\operatorname{Ext}^*_Y(N,\omega_Y)$ degenerates and

$$\begin{aligned}\operatorname{Ext}^i_Y(N,\omega_Y) &= H^{i-1}\big(Y,\mathcal{E}xt^1_Y(N,\omega_Y)\big)\\ &= H^{i-1}\big(Y,N^{-1}\otimes\omega_{mW}\big) \;=\; H^{i-1}\big(mW,N^{-1}\otimes\omega_{mW}\big).\end{aligned}$$

Setting $N=\mathcal{O}_{mW}\otimes M$ gives the isomorphisms

$$\operatorname{Ext}^i_Y\big(\mathcal{O}_{mW},\omega_Y\otimes M^{-1}\big)=H^{i-1}\big(mW,M^{-1}\otimes\omega_{mW}\big).$$

Since mW is a proper CM scheme over a field, Serre duality gives that the latter group is dual to

$$H^{n-1-(i-1)}\big(mW,M|_{mW}\big)=H^{n-i}\big(mW,M|_{mW}\big). \quad \square$$

Combining this with (13) we obtain the following.

Corollary 20. *Let $f:Y\to X$ be a proper morphism, Y smooth. Let $x\in X$ be a closed point and assume that $W:=\operatorname{Supp} f^{-1}(x)$ is a Cartier divisor. Let L be a line bundle on Y such that $L\sim_{\mathbb{Q},f}$ (f-nef) $+\,\Delta$ for some simple normal crossing divisor Δ such that $\lfloor\Delta\rfloor=0$. Then $H^i_W\big(Y,L^{-1}\big)=0$ for $i<\dim X$.* $\square$

There are obvious dual versions of (14) and of (15).

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References

[Ale08] Valery Alexeev, *Limits of stable pairs*, Pure Appl. Math. Q. **4** (2008), no. 3, part 2, 767–783. MR 2435844 (2009j:14020)

[Amb03] F Ambro, *Quasi-log varieties*, Proc. Steklov Inst. Math. **240** (2003), 214–233.

[Elk81] R. Elkik, *Rationalité des singularités canoniques*, Inv. Math. **64** (1981), 1–6.

[Fuj85] T. Fujita, *A reative versions of Kawamata-Viehweg's vanishing theorem*, preprint, Univ. of Tokyo, 1985.

[Fuj09a] Osamu Fujino, *Fundamental theorems for the log minimal model program*, http://www.citebase.org/abstract?id=oai:arXiv.org:0909.4445, 2009.

[Fuj09b] ———, *Introduction to the log minimal model program for log canonical pairs, version 6.01*, http://www.citebase.org/abstract?id=oai:arXiv.org:0907.1506, 2009.

[Gro68] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), North-Holland Publishing Co., Amsterdam, 1968, Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du

Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2. MR 0476737 (57 #16294)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[K+92] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Soc. Math. France, Astérisque vol. 211, 1992.

[Kaw82] Yujiro Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), no. 1, 43–46. MR 675204 (84i:14022)

[KK10] János Kollár and Sándor J. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813. MR 2629988

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)

[Kol97] János Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR 1492525 (99m:14033)

[Kol08] ———, *Semi log resolution*, http://www.citebase.org/abstract?id=oai:arXiv:0812.3592, 2008.

[Kov00] Sándor J. Kovács, *A characterization of rational singularities*, Duke Math. J. **102** (2000), no. 2, 187–191. MR 1749436 (2002b:14005)

[Nak87] Noboru Nakayama, *The lower semicontinuity of the plurigenera of complex varieties*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 551–590. MR 946250 (89h:14028)

[Pat10] Zsolt Patakfalvi, *Base change for the relative canonical sheaf in families of normal varieties*, ArXiv e-prints (2010).

[RRV71] J. P. Ramis, G. Ruget, and J. L. Verdier, *Dualité relative en géométrie analytique complexe*, Invent. Math. **13** (1971), 261–283. MR 0308439 (46 #7553)

[Tak85] Kenshō Takegoshi, *Relative vanishing theorems in analytic spaces*, Duke Math. J. **52** (1985), no. 1, 273–279. MR 791302 (86i:32049)

[Vie82] Eckart Viehweg, *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1–8. MR 667459 (83m:14011)

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