On Canonically Fibred Algebraic 3-Folds – Some New Examples

Meng Chen and Aoxiang Cui

To the memory of Eckart Viehweg

Abstract: This note aims to improve known numerical bounds proved earlier by Chen [8] and Chen-Hacon [9] and to present some new examples of smooth minimal 3-folds canonically fibred by surfaces (resp. curves) of geometric genus as large as 19 (resp. 13). As an interesting by-product, we present a new class of general type surfaces which are canonically fibred by curves of genus 13.

Keywords: canonically fibred 3-folds, inequalities of Noether type, Miyaoka-Yau inequality.

1. Introduction

Let V be a nonsingular projective 3-fold of general type. Assume X is a minimal model of V with at worst $\mathbb{Q}$-factorial terminal singularities. When the geometric genus $p_g(X) \geq 2$, the canonical map $\varphi_1 := \Phi_{|K_X|}$ is usually a key tool for birational classification. If φ_1 is generically finite, Hacon [13] gave an example to show that $\deg(\varphi_1)$ is usually not bounded from above. If φ_1 is non-constant and not generically finite (i.e. $0 < \dim \overline{\varphi_1(X)} < 3$), then φ_1 is said to be *of fiber type*. In this situation V (or X) is said to be *canonically of fiber type*. Let F be

---

a birational smooth model of the generic irreducible component in the general fiber of φ_1. Denote by $g(F)$ (resp. $p_g(F)$) the genus (resp. the geometric genus) of F when F is a curve (resp. a surface). Clearly F is of general type and nonsingular by simple addition formula and the Bertini theorem. We say that X is *canonically fibred by curves F (resp. surfaces F)*. It is interesting to see if the birational invariants of F are bounded from above. Unfortunately such kind of boundedness was only proved when X is Gorenstein (see Chen-Hacon [9]).

Restricting our interest to Gorenstein minimal 3-folds X which are canonically of fiber type, Chen-Hacon [9, Theorem 1.1] proved the desired boundedness theorem like $g(F) \leq 487$ (resp. $p_g(F) \leq 434$). However, even if we assume $p_g(X) \gg 0$, the upper bounds (see, for instance, Chen [8, Theorem 0.1]) for $g(F)$ (resp. $p_g(F)$) might be far from optimal. Besides, among all known examples, the biggest value of $g(F)$ (resp. $p_g(F)$) is 5.

The motivation of this paper is to study the following:

Question 1.1. (cf. Chen-Hacon [9, Question 4.2]) (1) For canonically fibred Gorenstein minimal 3-folds X of general type, find optimal upper bounds of the invariants of fibers.

(2) Find new examples of X such that the generic irreducible component in the general fiber of φ_1 has birational invariants as large as possible.

First of all, we aim at improving known upper bounds for $g(F)$ (resp. $p_g(F)$) and shall prove the following theorem.

Theorem 1.2. *Let X be a Gorenstein minimal projective 3-fold of general type. Assume that X is canonically of fiber type. Let F be a smooth model of the generic irreducible component in the general fiber of φ_1. Then*

   (i) $g(F) \leq 91$ *when F is a curve and $p_g(X) \geq 183$;*
   (ii) $p_g(F) \leq 37$ *when F is a surface and $p_g(X) \geq 3890$.*

The second and the more important purpose of this paper is to present several new classes of 3-folds such as:

   (1) $X_{S,19}$ (see Example 5.4), canonically fibred by **surfaces** of general type of geometric genus $p_g(F) = 19$;

(2) $X_{C,\nu,13}$ (see Example 5.5), canonically fibred by **curves** of genus $g(F) = 13$, where $\nu \geq 3$;

(3) $Y_{S,19}$ (see Example 5.6);

(4) $Y_{C,\nu,13}$ (see Example 5.7), $\nu \geq 3$;

(5) $Z_{S,19}$ (see Example 5.8);

(6) $Z_{C,\nu,13}$ (see Example 5.9), $\nu \geq 3$;

(7) $X_{S,16}$ (see Example 5.10);

(8) $X_{C,\nu,11}$ (see Example 5.11), $\nu \geq 3$;

(9) $X_{S,13}$ (see Example 5.12);

(10) $X_{C,\nu,9}$ (see Example 5.13), $\nu \geq 3$.

An earlier result due to Beauville [2] says that an algebraic surface S of general type can be canonically fibred by curves of genus at most 5, as long as $\chi(\mathcal{O}_S) \geq 21$. However, there are no known examples of canonically fibred surfaces with fiber genus > 3. All above examples hint that, unlike the situation of surfaces, there might be plenty of canonically fibred minimal 3-folds.

A reward of above 3-fold structure is that we have actually found, in the last section, a new class of general type surfaces (see Example 6.1) which are canonically fibred by curves of genus 13.

As known to several experts, the main idea to treat this kind of questions is to deduce an effective inequality of Noether type while translating Miyaoka-Yau inequality ([21, 32, 33]) in terms of $p_g(X)$. We will go a little bit further in both sides in order to prove Theorem 1.2. In the authors' opinion, those examples in the last section are more interesting. Naively we feel that $p_g(F) \leq 37$ in Theorem 1.2 (ii) is a nearly optimal upper bound.

Throughout we prefer to use "$\sim$" to denote linear equivalence whereas "$\equiv$" means numerical equivalence.
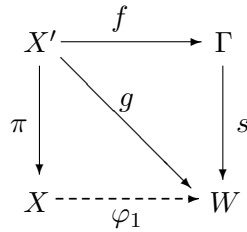
2. **Technical preparation**

2.1. Set up for φ_1.

Let X be a Gorenstein minimal 3-fold of general type with $p_g(X) := h^0(X, \omega_X) \geq 2$. We denote by φ_1 the canonical rational map $\varphi_1 : X \dashrightarrow \mathbb{P}^{p_g(X)-1}$. By Hironaka's theorem on resolution of singularities, we can take successive blow-ups $\pi : X' \to X$ along smooth centers, such that

(i) X' is nonsingular;
(ii) the movable part of $|K_{X'}|$ is base point free;
(iii) for a fixed Weil divisor $K_0 \sim K_X$, the support of $\pi^*(K_0) + E$ is of simple normal crossing, where E is the exceptional divisor of π on X'.

Set $g := \varphi_1 \circ \pi$. Then $g : X' \to W \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Taking the Stein factorization of g, one gets $X' \xrightarrow{f} \Gamma \xrightarrow{s} W$ where s is finite, Γ is normal and f is a fiber space. So we have the following commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{\;\;f\;\;} & \Gamma \\
\;\downarrow{\scriptstyle \pi} & \searrow{\scriptstyle g} & \;\downarrow{\scriptstyle s} \\
X & \dashrightarrow{\;\;\varphi_1\;\;} & W
\end{array}
$$

The fibration $f : X' \to \Gamma$ is referred to as an *induced fibration* of φ_1. Write $K_{X'} = \pi^*(K_X) + E$ and $|K_{X'}| = |M| + Z$ where $|M|$ is the movable part of $|K_{X'}|$, E is exceptional and Z the fixed part. Since $\pi^*(K_X) \geq M$, one may also write $\pi^*(K_X) \sim M + E'$ where E' is an effective divisor. In fact, $E' \leq Z$.

If $\dim \overline{\varphi_1(X)} \geq 2$, a general member S of $|M|$ is a nonsingular projective surface of general type.

If $\dim \overline{\varphi_1(X)} = 1$, denote by F a general fiber of f and set $S = F$. Then S is still a nonsingular projective surface of general type. Under this situation, one has

$$
M = \sum_{i=1}^{a_1} F_i \equiv a_1 F.
$$

where the F_i's are smooth fibers of f and, clearly, $a_1 \geq p_g(X) - 1$.

In both cases, we call S a *generic irreducible element* of $|M|$ or $|K_{X'}|$. Set

$$p = \begin{cases} 1 & \text{if } \dim(\Gamma) = 2, \\ a_1 & \text{if } \dim(\Gamma) = 1. \end{cases}$$

Hence, we always have $M \equiv pS$.

2.2. Volume inequality.

Assume X is canonically of fiber type. Pick a generic irreducible element S of $|K_{X'}|$. Suppose there is a movable linear system $|G|$ on S with a smooth generic irreducible element C. Then, by Kodaira Lemma, there exists a positive rational number β such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. As being recognized in [6, Inequality (2.1)], one has the following inequality:

$$(2.1) \qquad\qquad\qquad K_X^3 \geq p\beta\xi$$

where $\xi := (\pi^*(K_X).C)_{X'}$.

2.3. An inequality bounding ξ from below.

Keep the same notation as above. By [6, Inequality (2.2)], one has

$$(2.2) \qquad\qquad\qquad \xi \geq \frac{2g(C) - 2}{1 + \dfrac{1}{p} + \dfrac{1}{\beta}}.$$

3. The canonical family of curves

Assume $\dim \Gamma = 2$ in this section. We have an induced fibration $f : X' \to \Gamma$ with the general fiber a smooth curve C with $g(C) \geq 2$. Keep the same notation as in 2.1.

Proposition 3.1. *Let X be a Gorenstein minimal projective 3-fold of general type. If $\dim \overline{\varphi_1(X)} = 2$, then*

$$K_X^3 \geq \left\lceil \frac{2g(C) - 2}{2 + \frac{1}{p_g(X) - 2}} \right\rceil (p_g(X) - 2).$$

In particular, $K_X^3 \geq (g(C) - 1)(p_g(X) - 2)$ when $p_g(X) \geq 111$.

Proof. Take $|G| = |S|_S|$ on a general member S. Then $G \equiv \beta C$ with $\beta \geq p_g(X) - 2$. Thus the inequality (2.1) gives

$$K_X^3 \geq (\pi^*(K_X).C)(p_g(X) - 2).$$

By inequality (2.2), one has

$$\pi^*(K_X).C \geq \frac{2g(C) - 2}{2 + \frac{1}{p_g(X) - 2}}.$$

The proof of Chen-Hacon [9, Theorem 1.1(1)] actually implies $g(C) \leq 164$ whenever $p_g(X) \geq 111$. Now under the condition $p_g(X) \geq 111$, one gets $\pi^*(K_X).C > g(C) - 2$. Since $\pi^*(K_X).C$ is an integer, we get $\pi^*(K_X).C \geq g(C) - 1$. □

Combining Miyaoka-Yau inequality (Miyaoka [21] and Yau [32, 33]) and Theorem 3.1, we only need to get an upper bound of $\chi(\omega_X)$. In Chen-Hacon [9, Proposition 2.1], we know that

$$\chi(\omega_X) \leq (1 + \frac{1}{p_g(V_y)})p_g(V),$$

where V is the smooth model of X, and V_y is a generic irreducible component in the general fiber of Albanese morphism of V.

In the following theorem, we will bound $\chi(\omega_X)$ with a careful classification.

Theorem 3.2. *Let X be a Gorenstein minimal projective 3-fold of general type. Assume that X is canonically fibred by curves C. Then $g(C) \leq 91$ whenever $p_g(X) \geq 183$.*

Proof. Assume $q(X) \leq 2$, then $\chi(\omega_X) \leq p_g(X) + 1$ and one can easily verify the statement.

Assume $q(X) \geq 3$. Take a smooth birational model V of X. Denote by V_y a generic irreducible component in the general fiber of the Albanese morphism of V. Note that $p_g(V_y) > 0$ since $p_g(V) = p_g(X) > 0$.

Case (1). As long as V_y is not a surface with $q(V_y) = 0$ and $p_g(V_y) \leq 3$, then Chen-Hacon [9, Proposition 2.1] implies $\chi(\omega_X) \leq \frac{5}{4}p_g(X)$. Provided that $p_g(X) \geq 111$, one has:

$$(g(C) - 1)(p_g(X) - 2) \leq K_X^3 \leq 72 \cdot \frac{5}{4}p_g(X).$$

So, by calculation, one has $g(C) \leq 91$ when $p_g(X) \geq 183$.

Case (2). If V_y is a surface with $q(V_y) = 0$ and $p_g(V_y) = 1$, the canonical map $\Phi_{|K_V|}$ maps V_y to a point, which contradicts to the assumption $\dim \Gamma = 2$. Thus this is an impossible case.

Case (3). If V_y is a surface with $q(V_y) = 0$ and $p_g(V_y) = 2$, the assumption $\dim \Gamma = 2$ implies that the natural restriction $j : H^0(V, K_V) \to H^0(V_y, K_{V_y})$ is surjective. This means $\Phi_{|K_V|}(V_y) = \mathbb{P}^1$. In other words, $\varphi_1|_{V_y} = \Phi_{|K_{V_y}|}$. Taking further birational modification to X' and V, we may assume $X' = V$ and the relative canonical map of Alb_V is a morphism. Thus C, as a general fiber of f, is nothing but a generic irreducible element of the movable part of $|K_{V_y}|$. By the result of Beauville [2], one easily gets, at worst, $g(C) \leq K_{V_y}^2 + 1 \leq 9\chi(\mathcal{O}_{V_y}) + 1 \leq 28$.

Case (4). If V_y is a surface with $q(V_y) = 0$ and $p_g(V_y) = 3$, we will discuss it in details. In fact, $|K_{V_y}|$ can be either composed with a pencil of curves or not. We may still investigate the natural map j. Note that $\dim \mathrm{Im}(j) \geq 2$. First, we consider the case $\dim \mathrm{Im}(j) = 2$. Then $|K_V||_{V_y}$ is a sub-pencil of $|K_{V_y}|$. Clearly the generic irreducible element of the movable part of $|K_V||_{V_y}$ is numerically equivalent to C, a general fiber of f contained in V_y. Denote by $V_{y,0}$ the minimal model of V_y. Noting that $\chi(\mathcal{O}_{V_{y,0}}) \leq 4$, one has

$$2g(C) - 2 \leq 2K_{V_{y,0}}^2 \leq 18\chi(\mathcal{O}_{V_{y,0}}),$$

which yields $g(C) \leq 37$. Next, we consider the case $\dim \mathrm{Im}(j) = 3$. This says j is surjective. So $\Phi_{|K_V|}|_{V_y} = \Phi_{|K_{V_y}|}$. When $|K_{V_y}|$ is composed with a pencil, then the general fiber C of f contained in V_y is nothing but a generic irreducible element in the movable part of $|K_{V,y}|$. By Beauville [2], one has $g(C) \leq 36$. When $\dim \Phi_{|K_V|}(V_y) = 2$, we hope to prove $\dim \varphi_1(V) = 3$ which contradicts to the assumption $\dim \Gamma = 2$. In fact, denote by $a : V \to Y$ the induced fibration after the Stein factorization of the Albanese map of V. Note that $g(Y) \geq q(V) \geq 3$. Replace a by a relative minimal model $\hat{a} : Z \to Y$. Let Z_y be a general fiber of $\hat{a}$. According to Fujita [12], one knows $\hat{a}_* \omega_{Z/Y}$ is semi-positive. The Riemann-Roch on Y gives

$$h^0(Y, \hat{a}_* \omega_Z \otimes \mathcal{O}_Y(-y)) \geq 3(g(Y) - 2) > 0.$$

Take a general point $y_1 \in Y$. Since $|K_Z - Z_{y_1}| \neq \emptyset$, we may take a non-zero section $s_1 \in H^0(Z, \omega_Z)$ such that s_1 vanishes along Z_{y_1}. Write $\mathrm{div}(s_1) =$

$Z_{y_1} + G_1$ with $G_1 > 0$. Pick another general fiber Z_{y_2} such that $Z_{y_2} \not\leq Z_{y_1} + G_2$. Then s_1 does not vanish along Z_{y_2}. For the same reason, we may find another non-zero section $s_2 \in H^0(Z, \omega_Z)$ such that s_2 vanishes along Z_{y_2}. This already implies $\Phi_{|K_Z|}(Z_{y_1}) \neq \Phi_{|K_Z|}(Z_{y_2})$. Thus $\dim \overline{\Phi_{|K_Z|}(Z)} > \dim \overline{\Phi_{|K_Z|}(Z_{y_1})} = 2$, a contradiction. We have proved the theorem. □

4. The canonical family of surfaces

In this section we assume $\dim \Gamma = 1$. Essentially we will study the case when $p_g(F)$ is large. Recall that Chen [7, Theorem 1] implies $b = g(\Gamma) \leq 1$ whenever $p_g(F) > 2$. First we will deduce a very delicate inequality of Noether type.

Proposition 4.1. *Let X be a Gorenstein minimal projective 3-fold of general type. Assume X is canonically fibred by surfaces F with $p_g(F) > 2$. Then*

$$K_X^3 \geq \begin{cases} (K_{F_0}^2 + \frac{1}{4(20K_{F_0}^2+1)})p_g(X), & when \quad b = 1; \\[2ex] (K_{F_0}^2 + \frac{1}{4(20K_{F_0}^2+1)})(p_g(X)-1) - \frac{4K_{F_0}^2}{2(20K_{F_0}^2+1)}, & when \ b = 0 \ and \\ & p_g(X) \geq 56. \end{cases}$$

Proof. The property we are discussing here is birationally invariant. For technical reason, we need to choose a suitable minimal model. According to Kawamata [15, Lemma 5.1], any Gorenstein minimal 3-fold is birational to a factorial Gorenstein minimal model with at worst terminal singularities. So we may assume that X is factorial. We keep the same notation as in 2.1. We have an induced fibration $f : X' \to \Gamma$ onto the smooth curve Γ.

Pick a general fiber F of f. Set $N := \pi_*(F)$ and $Z := \pi_*(E')$. Then we have

$$K_X \equiv a_1 N + Z.$$

As one knows (see, for instance, Chen-Chen-Zhang [5, 2.2]), $K_X.N^2$ is a non-negative even integer.

In case $b = 1$, automatically $K_X.N^2 = 0$ since the movable part of $|K_X|$ is base point free.

In case $b = 0$ and $K_X.N^2 > 0$ (which means $K_X.N^2 \geq 2$), one has

$$K_X^3 \geq a_1 K_X^2.N \geq 2a_1^2 \geq 2(p_g(X)-1)^2.$$

Noting that $\chi(\omega_X) \leq \frac{3}{2}p_g(X)$ by Chen-Hacon [9, 2.2(2)], the Miyaoka-Yau inequality implies $p_g(X) \leq 55$. In other words, $K_X.N^2 = 0$ whenever $p_g(X) \geq 56$.

Now we can work under the assumption $K_X.N^2 = 0$. Then one has $\pi^*(K_X)|_F = \sigma^*(K_{F_0})$ by Chen-Chen-Zhang [5, Claim 3.3]. Therefore

$$K_X^2.N = \pi^*(K_X)^2.F = (\pi^*(K_X)|_F)^2 = \sigma^*(K_{F_0})^2 = K_{F_0}^2.$$

Recall that we are studying on the factorial minimal model X. According to Lee [17], $|4K_X|$ is base point free. Take a general member S_4 of $|4K_X|$. Then S_4 is a nonsingular projective surface of general type. We hope to do some delicate calculation on S_4. Clearly $f(\pi^*(S_4)) = \Gamma$, which also means that S has a natural fibration structure since $(N|_{S_4})^2 = 4K_X.N^2 = 0$. Taking the restriction, one has:

$$K_X|_{S_4} \equiv a_1 N|_{S_4} + Z|_{S_4}.$$

The fact $(N|_{S_4})^2 = 0$ also implies that $f|_{\pi^*(S_4)}$ factors through S_4, i.e. there is a fibration $\nu : S_4 \to \Gamma$. Since S_4 and F are both general, one sees that $N|_{S_4} = \pi(F|_{\pi^*(S_4)})$ is irreducible and reduced and is exactly a general fiber of ν.

By abuse of notation, we set $C := N|_{S_4}$ and $G := Z|_{S_4}$. Write $G := G_v + G_h$ where G_v is the vertical part while $G_h := \sum_i m_i G_i$ $(m_i > 0)$ is the horizontal part. Then we have

$$K_X|_{S_4} \equiv a_1 C + G_v + \sum m_i G_i.$$

Note that C is nef, $C^2 = 0$, $C.G_i > 0$ for all i and $p_a(G_i) \geq g(\Gamma)$. One has

$$4K_{F_0}^2 = 4(\pi^*(K_X)|_F)^2 = 4K_X^2.N$$
$$= \sum m_i(C.G_i) \geq \sum m_i.$$

Write $k := 4K_{F_0}^2$. For each i, we have

$$K_{S_4}.G_i + G_i^2 = 2p_a(G_i) - 2 \geq 2b - 2 = 2g(\Gamma) - 2.$$

On S_4, take the divisor $D := kK_{S_4} + 2k(1-b)C + G_v + \sum m_i G_i$. For each i, one has

$$D.G_i \geq k(K_{S_4}.G_i) + 2k(1-b)(C.G_i) + m_i G_i^2$$
$$\geq m_i(K_{S_4}.G_i) + 2m_i(1-b)(C.G_i) + m_i(2b - 2 - K_{S_4}.G_i)$$
$$= 2m_i((1-b)(C.G_i) + b - 1) \geq 0.$$

Thus $D.G_h \geq 0$. Explicitly,

$$D.G_h = k(K_{S_4}.G_h) + 2k(1-b)(C.G_h) + G_v.G_h + G_h^2 \geq 0.$$

Since

$$K_X|_{S_4}.G_h = a_1(C.G_h) + G_v.G_h + G_h^2$$

and $K_{S_4} = (K_X + S_4)|_{S_4} = 5K_X|_{S_4}$, by summing up together, one gets

$$(5k+1)(K_X|_{S_4}.G_h) \geq (a_1 - 2k(1-b))(C.G_h) \geq a_1 - 2k(1-b).$$

Finally, one has

$$K_X^3 = \frac{1}{4}(K_X|_{S_4})^2 \geq \frac{1}{4}a_1(K_X|_{S_4}.N|_{S_4}) + \frac{1}{4}(K_X|_{S_4}.G_h)$$

$$\geq (1 + \frac{1}{k(5k+1)})a_1 K_{F_0}^2 - \frac{k}{2(5k+1)}(1-b)$$

which yields the statements of this proposition. □

Theorem 4.2. *Let X be a Gorenstein minimal projective 3-fold of general type. Assume that X is canonicaly fibred by surfaces. Let $f : X' \to \Gamma$ be an induced fibration and denote by F a general fiber. Then $p_g(F) \leq 37$ when either $b = g(\Gamma) > 0$ or $b = 0$ and $p_g(X) \gg 0$, say $p_g(X) \geq 3890$.*

Proof. If $b > 1$, then $p_g(F) \leq 2$ by Chen [7, Theorem 1].

If $b = 1$, we know that $f_*\omega_{X'} = f_*\omega_{X'/\Gamma}$ and $R^1 f_*\omega_{X'} = R^1 f_*\omega_{X'/\Gamma}$ are both semi-positive (see Fujita [12], Kawamata [14], Kollár [16], Nakayama [22] and Viehweg [26]). Furthermore $R^2 f_*\omega_{X'} = \omega_\Gamma \cong \mathcal{O}_\Gamma$. This implies $\chi(R^1 f_*\omega_{X'}) \geq 0$ and $\chi(R^2 f_*\omega_{X'}) = 0$. Thus

$$\chi(\omega_{X'}) = \chi(f_*\omega_{X'}) - \chi(R^1 f_*\omega_{X'}) + \chi(R^2 f_*\omega_{X'}) \leq \chi(f_*\omega_{X'}) \leq p_g(X).$$

So Proposition 4.1 and Miyaoka-Yau inequality imply

$$(K_{F_0}^2 + \frac{1}{4(20K_{F_0}^2 + 1)})p_g(X) \leq 72p_g(X)$$

which directly gives $K_{F_0}^2 < 72$, whence $p_g(F) \leq 37$ by the Neother inequality $K_{F_0}^2 \geq 2p_g(F) - 4$.

If $b = 0$, we assume $p_g(X) \geq 56$. Clearly

$$\chi(\omega_X) \leq p_g(X) + q(F) - 1.$$

A preliminary estimation in Chen-Hacon [9, 2.2(2)] gives $K_{F_0}^2 \leq 108$ when $p_g(X) \geq 327$. We shall discuss by distinguishing the value of $q(F)$.

Case (1). $q(F) > 0$. By Debarre [11], one has $K_{F_0}^2 \geq 2p_g(F)$. So Proposition 4.1 and Miyaoka-Yau inequality implies

$$K_{F_0}^2 \leq 72 - \frac{1}{4(20K_{F_0}^2 + 1)} + (36K_{F_0}^2 + \frac{2K_{F_0}^2}{20K_{F_0}^2 + 1}) \cdot \frac{1}{p_g(X) - 1}.$$

The calculation shows $K_{F_0}^2 \leq 72$ (whence $p_g(F) \leq 36$) if $p_g(X) \geq 3890$; and $K_{F_0}^2 \leq 71$ (whence $p_g(F) \leq 35$) if $p_g(X) \geq 33616518$.

Case (2). $q(F) = 0$. Similarly, we have

$$K_{F_0}^2 \leq 72 - \frac{1}{4(20K_{F_0}^2 + 1)} + \frac{2K_{F_0}^2}{20K_{F_0}^2 + 1} \cdot \frac{1}{p_g(X) - 1}.$$

At least, when $p_g(X) \geq 865$, one has $K_{F_0}^2 < 72$ or $p_g(F) \leq 37$.

To make the conclusion, when $p_g(X) \geq 3890$ and $b = 0$, one gets $p_g(F) \leq 37$. □

Theorem 3.2 and Theorem 4.2 imply Theorem 1.2.

5. New examples of canonically fibred 3-folds

An equally important task of birational classification is to provide supporting examples. One may refer to Chen [8] and Chen-Hacon [9] for some examples where the largest known value of $g(F)$ (resp. $p_g(F)$) is 5. In fact, it has been an open problem to look for general type 3-folds canonically fibred by curves (resp. surfaces) with invariant of general fiber as large as possible. In this section we would like to present some new examples.

5.1. Standard construction.

Let S be a minimal projective surface of general type with $p_g(S) = 0$. Assume there exists a divisor H on S such that $|K_S + H|$ is composed with a pencil of curves and that $2H$ is linearly equivalent to a smooth divisor R. The existence of such pair (S, H) is secured by Lemma 5.3. Let $\hat{C}$ be a generic irreducible element of the movable part of $|K_S + H|$. Assume $\hat{C}$ is smooth. Set $d := \hat{C}.H$ and $D := \hat{C} \bigcap H$.

Let C_0 be a fixed smooth projective curve of genus 2. Let θ be a 2-torsion divisor on C_0. Set $Y := S \times C_0$. Denote by $p_1 : Y \to S$, $p_2 : Y \to C_0$ the two projections. Take $\delta := p_1^*(H) + p_2^*(\theta)$ and pick a smooth divisor $\Delta \sim p_1^*(2H)$.

Then the pair (δ, Δ) determines a smooth double covering $\pi : X \to Y$ and $K_X = \pi^*(K_Y + \delta)$. Clearly X is smooth, minimal and of general type.

Since $K_Y + \delta = p_1^*(K_S + H) + p_2^*(K_{C_0} + \theta)$, $p_g(Y) = 0$ and $h^0(K_{C_0} + \theta) = 1$, one sees that $|K_X| = \pi^*|K_Y + \delta|$ and that $\Phi_{|K_X|}$ factors through π, p_1 and $\Phi_{|K_S+H|}$. Since $|K_S + H|$ is composed with a pencil of curves $\hat{C}$, X is canonically fibred by surfaces F and F is a double covering over $T := \hat{C} \times C_0$ corresponding to the data $(q_1^*(D) + q_2^*(\theta), q_1^*(2D))$ where q_1 and q_2 are projections. Denote by $\sigma : F \to T$ the double covering. Then $K_F = \sigma^*(K_T + q_1^*(D) + q_2^*(\theta))$. By calculation, one has $p_g(F) = 3g(\hat{C})$ when $d = 0$ and $p_g(F) = 3g(\hat{C}) + d - 1$ whenever $d > 0$.

The 3-folds constructed in the above way will be denoted by $X_{S,p_g(F)}$ in the context.

5.2. Variant—An Infinite Family.

In the construction 5.1, if we replace C_0 by any smooth curve C_ν of genus $\nu \geq 3$, what we obtain is a smooth minimal 3-fold X canonically fibred by curves. In fact, since $h^0(K_{C_0} + \theta) > 1$, $\Phi_{|K_X|}$ factors through π and $p_1 \times p_2$. Thus a generic irreducible component in the fibres of $\Phi_{|K_X|}$ is simply a double covering $\tau : F \to \hat{C}$ branched along the divisor $2D$. Thus Hurwitz formula gives

$$2g(F) - 2 = 2(2g(\hat{C}) - 2) + 2d$$

and hence $g(F) = 2g(\hat{C}) + d - 1$.

Clearly $p_g(X)$ can be arbitrarily large as long as $\nu = g(C_\nu)$ is large. So such kind of 3-folds X form an infinite family. We denote these 3-folds by $X_{C,\nu,g(F)}$.

Lemma 5.3. *Let S be any smooth minimal projective surface of general type with $p_g(S) = 0$. Assume $\mu : S \to \mathbb{P}^1$ is a genus 2 fibration. Let H be a general fiber of μ. Then $|K_S + H|$ is composed with a pencil of curves $\hat{C}$ of genus $g(\hat{C})$ and $\hat{C}.H = 2$.*

Proof. By Ramanujam's vanishing theorem [1, P.131, Theorem 8.1], one has $H^1(S, K_S + H) = 0$. Thus

$$h^0(S, K_S + H) = \frac{1}{2}(K_S + H)H + \chi(\mathcal{O}_S) = g(H) = 2$$

which gives that $|K_S + H|$ is naturally composed with a pencil of curves $\hat{C}$ of genus $g(\hat{C})$.

Since $q(S) \leq p_g(S) = 0$, we have the surjective map

$$H^0(S, K_S + H) \to H^0(H, K_H).$$

This means that $|K_S + H|$ is composed with a different pencil from $|H|$. Thus $\mu(\hat{C}) = \mathbb{P}^1$. In other words, $|H|\|_{\hat{C}}$ is movable. The Riemann-Roch and the Clifford theorem simply imply $\hat{C}.H = 2$ since $g(H) = 2$.

Finally, whenever $K_S^2 \geq 2$, Xiao [30, Theorem 6.5] proved that S can not have two different pencils of genus 2 on S. We thus see $g(\hat{C}) \geq 3$. □

We would like to look for those pairs (S, H) satisfying the conditions of Lemma 5.3.

Example 5.4. The 3-fold $X_{\mathbf{S},19}$ which is canonically fibred by surfaces F with $p_g(F) = 19$.

We take a pair (S, H) which was found by Xiao [28, P. 288], where S is a numerical Compedelli surface with $K_S^2 = 2$, $p_g(S) = q(S) = 0$ and $\mathrm{Tor}(S) = (\mathbb{Z}_2)^3$. We need to recall the construction to determine the pencil $|\hat{C}|$ on S.

To start the construction, let $P = \mathbb{P}^1 \times \mathbb{P}^1$. Denote by x, $y \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ the two coordinates of those points in P. Take four curves C_1, C_2, C_3 and C_4 defined by the following equations, respectively:

$$\begin{aligned}
C_1 &: x = y; \\
C_2 &: x = -y; \\
C_3 &: xy = 1; \\
C_4 &: xy = -1.
\end{aligned}$$

These four curves intersect mutually at 12 ordinary double points:

$$(0, 0), \ (\infty, \infty), \ (0, \infty), \ (\infty, 0)$$
$$(\pm 1, \pm 1), \ (\pm \sqrt{-1}, \pm \sqrt{-1}).$$
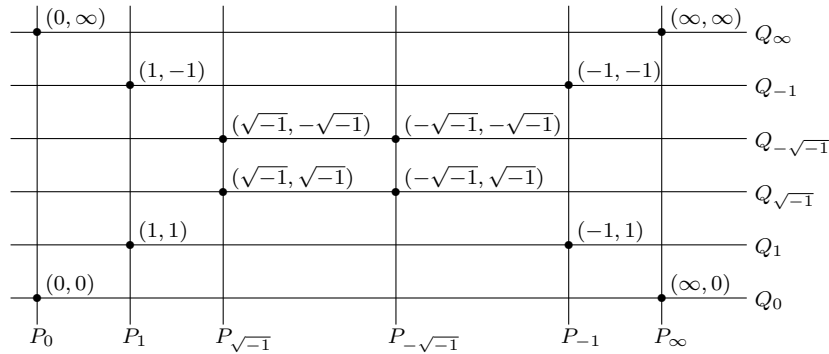
Denote by $\varphi : P \to \mathbb{P}^1$ be the first projection. Let $F_1, \ldots, F_6$ be the fibers of φ over $P_0, P_\infty, P_1, P_{-1}, P_{\sqrt{-1}}, P_{-\sqrt{-1}} \in \mathbb{P}^1$, respectively. Note that each fiber F_i contains exactly two double points. Furthermore we take the following divisors of bi-degree $(4, 2)$:

$$D_1 = C_1 + C_2 + F_1 + F_2,$$
$$D_2 = C_1 + C_3 + F_3 + F_4,$$
$$D_3 = C_2 + C_3 + F_5 + F_6.$$

Being linearly equivalent, passing through the above 12 points and having no common components, D_1, D_2 and D_3 generate a linear system which has the general smooth member D with D again passing through the 12 points. Let

$$R_1 := \sum_{i=1}^{4} C_i + \sum_{i=1}^{6} F_i + D$$

and δ_1 be a divisor of bi-degree $(7, 3)$ on P. Then the data (δ_1, R_1) determines a singular double covering onto P. Noting that R_1 has exactly 12 singularities of multiplicity 4, we take the blowing up $\tau : \tilde{P} \to P$ resolving the 12 points. Then, on $\tilde{P}$, the strict transform $\tilde{R}_1 := \tau_*^{-1}(R_1)$ is smooth and the corresponding double covering $\theta : \tilde{S} \to \tilde{P}$ is smooth. Set $\tilde{f} := \varphi \circ \tau \circ \theta$. Then we have a fibration $\tilde{f} : \tilde{S} \to \mathbb{P}^1$. Clearly, on $\tilde{S}$, the strict transforms of $F_1, \ldots, F_6$ are exactly the only (-1)-curves and are contained in fibers of $\tilde{f}$. These (-1)-curves are contracted to get the relative minimal fibration $f : S \to \mathbb{P}^1$ where S happens to be minimal with $K_S^2 = 2$ and $p_g(S) = q(S) = 0$. Denote by $\sigma : \tilde{S} \to S$ the blow down. Clearly f is a fibration of genus 2. Denote by H a general fiber of f. We would

like to study the linear system $|K_S + H|$.

$$
\begin{array}{ccccc}
S & \xleftarrow{\ \sigma\ } & \tilde{S} & \xrightarrow{\ \theta\ } & \tilde{P} \\
{\scriptstyle f}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \tau} \\
\mathbb{P}^1 & =\!=\!= & \mathbb{P}^1 & \xleftarrow{\ \varphi\ } & P
\end{array}
$$

Denote by $E_i = \theta_*^{-1}(\tilde{F}_i)$ and $\tilde{F}_i := \tau_*^{-1}(F_i)$ for $i = 1, \ldots, 6$. We have known that $E_1, \ldots, E_6$ are all (-1)-curves. On $\tilde{P}$, let $J_1, \ldots, J_{12}$ are 12 exceptional curves after the blowing up τ. Denote $L = \tau^*(1,0)$. Then $\sigma^*(H) \sim \theta^*(L)$.

Take $\tilde{\delta}_1 = \tau^* \delta_1 - 2 \sum_{k=1}^{12} J_k$. We have

$$
K_{\tilde{S}} + \sigma^*(H) = \theta^*(K_{\tilde{P}} + \tilde{\delta}_1 + L)
$$

$$
\sim \theta^*(\tau^*(K_P + \delta_1 + (1,0)) - \sum_{k=1}^{12} J_k)
$$

$$
\sim \theta^*(\tau^*(6,1) - \sum_{k=1}^{12} J_k)
$$

$$
\sim \theta^* \tau^*(0,1) + 2 \sum_{i=1}^{6} E_i.
$$

Thus we see that $\sigma^*(K_S + H) \sim \theta^* \tau^*(0,1) + \sum_{i=1}^{6} E_i$ and that $|K_S + H|$ has exactly 6 base points, but no fixed parts. Clearly a general member $\hat{C} \in |K_S + H|$ is obtained by mapping a general curve $\theta^* \tau^*(0,1)$ onto S and $\hat{C}$ is a smooth curve of genus 6. Lemma 5.3 tells that $|K_S + H|$ is composed with a pencil $\hat{C}$.

Now we take the triple $(S, H, \hat{C})$ and run Construction 5.1. What we get is the 3-fold $X_{S,19}$ which is canonically fibred by surfaces F with $p_g(F) = 19$. This is a new record with regard to [9, Question 4.2].

Example 5.5. The 3-fold family $X_{\mathbf{C},\nu,13}$ which are canonically fibred by curves F of genus $g(F) = 13$.

We take the same triple $(S, H, \hat{C})$ as in Example 5.4 and run Construction 5.2. What we shall get is a 3-fold family $X_{C,\nu,13}$ which are canonically fibred by curves F of genus 13 where $\nu \geq 3$. This is again a new record with regard to [9, Question 4.2].

Example 5.6. The 3-fold $Y_{\mathbf{S},19}$.

We take an alternative pair (S, H) constructed by Weng ([27]) and rephrased by Xiao as [31, Examples 4.4.8]. Let a, $b \in \mathbb{C}$ be two numbers satisfying $a \neq 0, \pm 1$, $b \neq 0$, $ab \neq a + 1$. Take $P = \mathbb{P}^1 \times \mathbb{P}^1$. Denote by x, $y \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ the two coordinates of those points in P. Take four projective curves defined by the following equations:

$$C_1 : xy = 1;$$
$$C_2 : xy = a;$$
$$C_3 : abx^3 + ab(a + 1)x + (a + 1)^2 y = (a + 1)xy^2;$$
$$C_4 : (a + 1)y^2 + (a + 1)bx^2y^2 + (a + 1)bx^3y = abx^2.$$

These curves have bi-degrees $(1, 1)$, $(1, 1)$, $(3, 2)$ and $(3, 2)$, respectively. The divisor $R_p := C_1 + C_2 + C_3 + C_4$ have the following triple points:

$$(0, 0), (0, \infty), (\infty, 0), (\infty, \infty);$$
$$(\alpha_i, \frac{1}{\alpha_i}), (\alpha_i, \frac{a}{\alpha_i}), \quad i = 1, \dots, 4$$

where the α_i's $(i = 1, \dots, 4)$ are the roots of the following equation:

$$bx^4 + (a + 1)bx^2 + (a + 1) = 0.$$

Note that R_p has no more singular points. Take

$$R := R_p + p_1^*(0 + \infty + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).$$

For a divisor δ with $2\delta \sim R$, the pair (δ, R) gives a double covering which induces a minimal relative fibration $f : S \to \mathbb{P}^1$ of genus 2, where S is a minimal surface of general type with $K_S^2 = 2$ and $p_g(S) = q(S) = 0$. Let H be a general fiber of f. Since R has similar singularities as that in Example 5.4, similar calculation shows that $|K_S + H|$ has exactly 6 base points, but no fixed parts. Thus a general member $\hat{C} \in |K_S + H|$ is a smooth curve of genus 6. Lemma 5.3 tells that $|K_S + H|$ is composed with a pencil $\hat{C}$.

Now if we take the triple $(S, H, \hat{C})$ and run Construction 5.1. What we get is the 3-fold $Y_{S,19}$ which is canonically fibred by surfaces F with $p_g(F) = 19$.

Example 5.7. The 3-fold family $Y_{\mathbf{C}, \nu, 13}$.

Again we take the same triple $(S, H, \hat{C})$ as in Example 5.6 and run Construction 5.2. What we obtain is the 3-fold family $Y_{C, \nu, 13}$ which are canonically fibred by curves F of genus 13 with the parameter $\nu \geq 3$.

Example 5.8. The 3-fold $Z_{\mathbf{S},19}$.

We take one more pair (S, H) constructed by Weng and rephrased by Xiao as [31, Examples 4.4.9]. Take $P = \mathbb{P}^1 \times \mathbb{P}^1$. Denote by x, $y \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ the two coordinates of those points in P. Let $a \in \mathbb{C}$ and $a \neq \pm 1, \pm\sqrt{-1}$. For any point $c \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, F_c denotes the fiber over c of the fibration (first projection) $p_1 : P \to \mathbb{P}^1$. Consider the following five curves of bi-degree $(1, 1)$:

$$D_1 : x = y;$$
$$D_2 : xy = 1;$$
$$D_3 : a^2 y = x;$$
$$D_4 : xy = a^2;$$
$$D_5 : a^2 xy = 1.$$

Take two curves of bi-degree $(5, 3)$:

$$B_1 = D_1 + 2D_5 + F_a + F_{-a},$$

$$B_2 = D_3 + 2D_4 + F_1 + F_{-1}.$$

One sees that B_1 and B_2 have no common components and they both pass through the following 12 points:

$$(0, 0), \ (\infty, \infty), \ (1, 1), \ (-1, -1), \ (a, \tfrac{1}{a}), \ (-a, -\tfrac{1}{a});$$
$$(0, \infty), \ (\infty, 0), \ (1, \tfrac{1}{a^2}), (-1, -\tfrac{1}{a^2}), \ (a, a), (-a, -a).$$

Furthermore, B_1 and B_2 each has double points along:

$$(0, \infty), (\infty, 0), (1, \tfrac{1}{a^2}), (-1, -\tfrac{1}{a^2}), (a, a), (-a, -a).$$

By Bertini theorem, the general member B of the linear system generated by B_1 and B_2 is irreducible, B passes through the above 12 points and the above mentioned 6 points are exactly the double points of B. Now take

$$R = B + D_1 + D_2 + D_3 + F_0 + F_\infty + F_1 + F_{-1} + F_a + F_{-a}.$$

Then the 12 points are the only singular points of R and each point is of multiplicity 4. Take a divisor δ such that $R \sim 2\delta$. Then (δ, R) again induces a minimal fibration $f : S \to \mathbb{P}^1$ of genus 2, where S is a minimal surface of general type with $K_S^2 = 2$ and $p_g(S) = q(S) = 0$. Let H be a general fiber of f. Since R has similar singularities as that in Example 5.4, similar calculation shows that $|K_S + H|$ has

exactly 6 base points, but no fixed parts. Thus a general member $\hat{C} \in |K_S + H|$ is a smooth curve of genus 6. Lemma 5.3 tells that $|K_S + H|$ is composed with a pencil $\hat{C}$.

Now if we take the triple $(S, H, \hat{C})$ and run Construction 5.1. What we get is the 3-fold $Z_{S,19}$ which is canonically fibred by surfaces F with $p_g(F) = 19$.

Example 5.9. The 3-fold family $Z_{\mathbf{C},\nu,13}$.

Still we take the same triple $(S, H, \hat{C})$ as in Example 5.8 and run Construction 5.2. What we obtain is a 3-fold family $Z_{C,\nu,13}$ which are canonically fibred by curves F of genus 13.

Example 5.10. The 3-fold $X_{S,16}$.

We take a pair (S, H) found by Oort and Peters [23], where S is a numerical Godeaux surface with $K_S^2 = 1$, $p_g(S) = q(S) = 0$. Consider four curves in $\mathbb{P}^2$ defined by the following equations:

$$C_1: \quad Y^2 + (X - 1)(2X - 3 - 2Y) = 0;$$
$$C_2: \quad Y^2 + (X - 1)(2X - 3 + 2Y) = 0;$$
$$C_3: \quad Y^2 + X(X - 1)(X - 3) = 0;$$
$$C_4: \quad (2 - X)C_3 + (X^2 - 3X + 3)^2 = 0.$$

where C_1, C_2 are quadratic curves and C_3, C_4 are cubic curves. These four plane curves intersect at seven points:

$$P_4 = (\frac{3}{2}, 0), Q_1 = (1, 0), Q_2 = (x_+, x_+), Q_3 = (x_-, x_-),$$
$$Q_4 = (x_+, -x_+), Q_5 = (x_-, -x_-), P_2 = (0, 1).$$

where

$$x_+ = \frac{3 + \sqrt{-3}}{2}, \quad x_- = \frac{3 - \sqrt{-3}}{2}.$$

	P_4	Q_1	Q_2	Q_3	Q_4	Q_5	P_2
C_1	1	1	1	1			
C_2	1	1			1	1	
C_3		1	1	1	1	1	1
C_4	2		1	1	1	1	1

Let $D := C_1 + C_2 + C_3 + C_4$ which is a curve of degree 10. Since P_4 is an ordinary double point of C_4 and both C_1, C_2 pass through P_4 with distinct tangent directions, P_4 is indeed an ordinary quadruple point of D. Triple points $Q_1, \ldots, Q_5$ are of type $(3 \to 3)$(i.e. after single blowing ups, the triple points become ordinary triple points) and P_2 is an ordinary double point of D.

Let $\delta := \mathcal{O}_{\mathbb{P}^2}(5)$ be a divisor on $\mathbb{P}^2$. Covering data (δ, D) determines a singular double covering over $\mathbb{P}^2$. D has five singular points of type $(3 \to 3)$, one singular point of multiplicity 2 and one point of multiplicity 4. Take successive blow-ups $\tau : \tilde{P} \to \mathbb{P}^2$ to resolve these seven singularities.

Denote by E_{P_2}, $E_{P_4} \in \tilde{P}$ the complete transform of the exceptional curves of P_2 and P_4, respectively. The resolution for Q_i needs two-step blow-ups, denote by E_{Q_i}, E'_{Q_i} the two corresponding complete transforms of the exceptional curves.

Divisor $\tilde{D} := \tau_*^{-1}(D) + \sum E_{Q_i} - \sum E'_{Q_i}$ on $\tilde{P}$ is smooth, hence the double covering $\theta : \tilde{S} \to \tilde{P}$ is smooth. Denote by $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ the rational map determined by the pencil of lines passing through P_4 and let $\tilde{f} := \varphi \circ \tau \circ \theta$. By Hurwitz formula we know that $\tilde{f}$ is a fibration of genus 2. On $\tilde{S}$, all the (-1)-curves are introduced by the desingularity of $Q_1, \ldots, Q_5$. Contracting these five (-1)-curves gives a relative minimal fibration $f : S \to \mathbb{P}^1$ where S is in fact a minimal general type surface with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$. Clearly f is also a fibration of genus 2. Denote by H a general fiber of f. Again we can determine the movable part of the linear system $|K_S + H|$.

$$
\begin{array}{ccccc}
S & \xleftarrow{\ \sigma\ } & \tilde{S} & \xrightarrow{\ \theta\ } & \tilde{P} \\
{\scriptstyle f}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \tau} \\
\mathbb{P}^1 & = & \mathbb{P}^1 & \xleftarrow{\ \varphi\ } & \mathbb{P}^2
\end{array}
$$

Take

$$
\tilde{\delta} = \tau^* \delta - E_{P_2} - 2E_{P_4} - \sum E_{Q_i} - 2\sum E'_{Q_i}.
$$

We have

$$
K_{\tilde{S}} + \sigma^*(H) = \theta^*(K_{\tilde{P}} + \tilde{\delta}) + \theta^*(\tau^*(\mathcal{O}_{\mathbb{P}^2}(1) - E_{P_4}))
$$
$$
\sim \theta^*(\tau^*(\mathcal{O}_{\mathbb{P}^2}(3)) - 2E_{P_4} - \sum E'_{Q_i}).
$$

Denote by E_i the five (-1)-curves on $\tilde{S}$, we have

$$\sigma^*(K_S + H) = K_{\tilde{S}} + \sigma^*(H) - \sum E_i$$
$$= \theta^*(\tau^*(\mathcal{O}_{\mathbb{P}^2}(3)) - 2E_{P_4} - \sum E_{Q_i}) + \sum E_i.$$

Now we want to find curves of degree 3 in $\mathbb{P}^2$ satisfying the following conditions:

(1) they pass through P_4 with multiplicity 2 and separated tangent directions;

(2) they pass through $Q_1, \ldots, Q_5$ with different tangent directions from that of $C_1, \ldots, C_4$ at these five points.

A direct computation shows that these curves form a linear system. In fact, a general curve of the following form:

$$u(2(X - \frac{3}{2})^2 Y - 3(X - \frac{3}{2})Y + Y^3) + v(4(X - \frac{3}{2})^3 + 2(X - \frac{3}{2})^2 + Y^2) = 0$$

can be an appropriate candidate. So we have seen that $|K_S + H|$ has exactly 6 base points, but no fixed parts. Thus a general member $\hat{C} \in |K_S + H|$ is a smooth curve of genus 5. Lemma 5.3 implies that $|K_S + H|$ is composed with a pencil $\hat{C}$.

Now if we take the triple $(S, H, \hat{C})$ and run Construction 5.1. What we get is the 3-fold $X_{S,16}$ which is canonically fibred by general type surfaces F with $p_g(F) = 16$.

Example 5.11. The 3-fold family $X_{C,\nu,11}$.

We take the same triple $(S, H, \hat{C})$ as in Example 5.10 and run Construction 5.2. What we obtain is the 3-fold family $X_{C,\nu,11}$ which are canonically fibred by curves F of genus 11 with the parameter $\nu \geq 3$.

Example 5.12. The 3-fold $X_{\mathbf{S},13}$.

Let S be a minimal surface of general type with $K_S^2 = 1$ and $p_g(S) = 0$. Take $H = K_S$. Then, among all known examples in Catanese-Pignatelli [4], Lee [18, 19] and Reid [24], one knows that $|2K_S|$ is composed with a pencil of curves $\hat{C}$ of genus 4, $|2K_S|$ has no fixed part and a generic irreducible element $\hat{C}$ of $|2K_S|$ is smooth. Thus $g(\hat{C}) = 4$. Also one gets $d = \hat{C}.H = 2K_S^2 = 2$. Take the triple $(S, H, \hat{C})$ and fill in Construction 5.1. Then we get the 3-fold $X_{S,13}$ which is canonically fibred by general type surfaces F with $p_g(F) = 13$.

Example 5.13. The 3-fold family $X_{\mathbf{C},\nu,9}$.

Take the same triple $(S, H, \hat{C})$ as in Example 5.12 and run Construction 5.2, one gets the 3-fold family $X_{C,\nu,9}$ which are canonically fibred by curves F of genus $g(F) = 2g(\hat{C}) + d - 1 = 9$.

Finally it is very interesting to know the answer to the following question:

Question 5.14. Are there smooth (Gorenstein) minimal projective 3-folds of general type which are canonically fibred by surfaces (resp. curves) F with $p_g(F) > 19$ (resp. $g(F) > 13$)?

6. A new class of canonically fibred surfaces of general type

Let M be a minimal projective surface of general type and assume that $|K_M|$ is composed with a pencil of curves of genus g. The existence of such surfaces with $p_g(M) \geq 3$ was known by Pompilij as early as 1984. In fact, there have been studies by Beauville [2], Catanese [3], Debarre [10], Sun [25], Miyanishi-Yang [20], Xiao [29] and others. Especially a minimal surface M with $p_g = 2$ is automatically canonically fibred by curves. However, among all known canonically fibred surfaces, very few examples with $g > 3$ are known in literature.

Inspired by our construction in the last section. We are able to present here at least 3 new examples with $g = 13$. Of course, our construction below has the potential to illustrate other examples with $g > 3$.

Example 6.1. Take a pair (S, H) satisfying Lemma 5.3. Let $\zeta : \hat{S} \longrightarrow S$ be the double cover corresponding to the datum $(\delta, \Delta) = (H, H_1 + H_2)$ with $H_1 \sim H$, $H_2 \sim H$. Since Δ is smooth, we see that $\hat{S}$ is a minimal surface of general type with $K_{\hat{S}}^2 = 2(K_S + H)^2$ and $p_g(\hat{S}) = h^0(S, K_S + H) = 2$. Thus $|K_{\hat{S}}|$ is automatically composed with a pencil of curves of genus g.

(6.1.1) The surface $\hat{S}_1$ with $g = 13$. If we take the same pair (S, H) as in Example 5.4, what we get is a new surface $\hat{S}_1$ with $K_{\hat{S}_1}^2 = 12$ and, corresponding to the structure of Example 5.5, $|K_{\hat{S}_1}|$ is composed with a pencil of curves of genus $g = 13$, since a general member $\hat{C} \in |K_S + H|$ is smooth as proved in Example 5.4.

(6.1.2) The surface $\hat{S}_2$ with $g = 13$. Similarly if we take the same pair (S, H) as in Example 5.6, what we get is another new surface $\hat{S}_2$ with $K^2 = 12$ and $g = 13$. This example is corresponding to Example 5.7.

(6.1.3) The surface $\hat{S}_3$ **with** $g = 13$. If we take the same pair (S, H) as in Example 5.8, what we get is again a new surface $\hat{S}_3$ with $K^2 = 12$ and $g = 13$. This example is corresponding to Example 5.9.

More concrete examples may be found by choosing a suitable pair (S, H).

Finally we would like to ask the following:

Question 6.2. Can one find canonically fibred general type surfaces with $g > 13$?

REFERENCES

[1] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **4**. Springer-Verlag, Berlin, 1984. x+304 pp.

[2] A. Beauville, *L'application canonique pour les surfaces de type général*, (French) Invent. Math. **55** (1979), no. 2, 121–140.

[3] F. Catanese, *Singular bidouble covers and the construction of interesting algebraic surfaces.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97-120, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999.

[4] F. Catanese, R. Pignatelli, *On simply connected Godeaux surfaces*, Complex analysis and algebraic geometry, 117–153, de Gruyter, Berlin, 2000

[5] J. A. Chen, M. Chen, D. Q. Zhang, *The 5-canonical system on 3-folds of general type*, J. Reine Angew. Math. **603** (2007), 165–181.

[6] J. A. Chen, M. Chen, *Explicit birational geometry for threefolds of general type, II*, Journal of Differential Geometry **86** (2010), 237–271.

[7] M. Chen, *Complex varieties of general type whose canonical systems are composed with pencils*, J. Math. Soc. Japan **51** (1999), no. 2, 331–335.

[8] M. Chen, *Weak boundedness theorems for canonically fibered Gorenstein minimal 3-folds*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1291–1298.

[9] M. Chen, C. D. Hacon, *On the geography of Gorenstein minimal 3-folds of general type*, Asian J. Math. **10** (2006), no. 4, 757–763.

[10] O. Debarre, *Inégalitiés numériques pour les surfaces de type général*, Bull. Soc. Math. France **110** (1982), 319–346.

[11] O. Debarre, *Addendum, inégalitiés numériques pour les surfaces de type général*, Bull. Soc. Math. France **111** (1983), 301–302.

[12] T. Fujita, *On Kaehler fiber spaces over curves*, J. Math. Soc. Japan **30** (1978), no. 4, 779–794.

[13] C. D. Hacon, *On the degree of the canonical maps of 3-folds*, Proc. Japan Acad. Ser. A Math. Sci. **80** (2004), no. 8, 166–167.

[14] Y. Kawamata, *Kodaira dimension of algebraic fiber space over curves*, Invent. Math. **66** (1982), 57–71.

[15] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) **127** (1988), no. 1, 93–163.

[16] J. Kollár, *Higher direct images of dualizing sheaves*, I, Ann. of Math. **123** (1986), 11–42; II, ibid. **124** (1986), 171–202.

[17] S. Lee, *Quartic-canonical systems on canonical threefolds of index 1*, Comm. Algebra **28** (2000), no. 12, 5517–5530.

[18] Y. Lee, *Biconanical pencil of a determinantal Barlow surface*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 893–905.

[19] Y. Lee, *Special members in the bicanonical pencil of Godeaux surfaces*, Osaka J. Math. **42** (2005), no. 1, 163–171

[20] M. Miyanishi, J. G. Yang, *Surfaces of general type whose canonical map is composed of a pencil of genus 3 with small invariants*, J. math. Kyoto Univ. **38** (1998), 123–149.

[21] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai, 1985, 449–476, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[22] N. Nakayama, *Hodge filtrations and the higher direct images of canonical sheaves*, Invent. Math. **85** (1986), 237–251.

[23] F. Oort, C. Peters, *A Campedelli surface with torsion group $\mathbb{Z}/2\mathbb{Z}$*, Indagationes Math. **43**, (1981), 399–407.

[24] M. Reid, *Surfaces with $p_g = 0$, $K^2 = 1$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **25** (1978), no. 1, 75–92.

[25] X. T. Sun, *Surfaces of general type with canonical pencil*. (Chinese) Acta Math. Sinica **33** (1990), no. 6, 769-773.

[26] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension of certain fibre spaces*, Adv. Studies in Pure Math. **1** (1983), 329–353.

[27] L. Weng, *Surfaces of general type of $\chi(\mathcal{O}) = 1$ admitting a genus 2 fibration*, Ph.D. thesis at the University of Science and Technology of China, 1987.

[28] G. Xiao, *Finiteness of the bicanonical mapping of surfaces of general type*, Bull. Soc. Math. France **113** (1985), no. 1, 23–51.

[29] G. Xiao, *L'irrégularité des surfaces de type général dnt le système canonique est composé d'un pinceau*, Compositio Math. **56** (1985), 251–257.

[30] G. Xiao, *Surfaces fibrées en courbes de genre deux*, (French) Lecture Notes in Mathematics, **1137**. Springer-Verlag, Berlin, 1985. x+103 pp.

[31] G. Xiao, *The fibrations of algebraic surfaces* (in Chinese), Modern Mathematics Series, Shanghai Scientific & Technical Publishers, 1991

[32] S. T. Yau, *Calabis conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA **74** (1977), 1798–1799.

[33] S. T. Yau, *On the Ricci curvature of a complex Kähler manifold and the complex Monge-Ampere equations*, Commun. Pure and Appl. Math. **31** (1978), 339–411

Meng Chen
School of Mathematical Sciences & LMNS,
Fudan University,
Shanghai, 200433,
China
E-mail: mchen@fudan.edu.cn

Aoxiang Cui
Department of Mathematics,
Fudan University,
Shanghai, 200433,
China
E-mail: 081018001@fudan.edu.cn