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Free Polygons, Twin Trees, and CAT(1)-Spaces

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Abstract: We give model theoretic constructions for a zoo of examples and counterexamples: first we build generalized n-gons satisfying strong transitivity properties. An ultraproduct of these yields a forest from which we obtain twin trees as well as homogeneous 1-round CAT(1)-spaces.
Keywords: generalized polygons, twin trees, CAT(1)-spaces

1 Introduction

The classification of the Moufang polygons by Tits and Weiss [18] establishes a close connection between Moufang polygons (and higher rank spherical buildings) and classical or algebraic groups. Weak versions of the Moufang condition have been shown to be sufficient for the classification [13, 14, 15]. While in the finite case and in the case of compact connected topological buildings the Moufang condition is in fact equivalent to the existence of a BN-pair, examples (such as the ones constructed below) show that this fails in general. Therefore, it is of interest to explore the extend to which the Moufang condition might be weakened. We here construct examples satisfying rather strong homogeneity conditions. This construction is much more straightforward than the one given in [12] and has different properties. In Section 5 we use these examples to obtain twin trees with

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large automorphism groups. Finally we consider the ultralimit of these spaces in Section 6.

2 Generalized polygons - free constructions

Generalized polygons are exactly the spherical buildings of rank 2. A generalized n-gon Γ is a bipartite graph with valencies at least 3, diameter n and girth $2n$. Without the assumption on the valencies, such a graph is called a *weak* nn-gon. We call $(x_0, \dots x_k)$ a simple path if the x_i are pairwise distinct and x_i is adjacent to x_{i+1} for $i = 0, \dots k-1$. The natural graph theoretic distance function on Γ is denoted by d or sometimes d_n. The set of elements at distance i from some element $x \in \Gamma$ is denoted by $\Gamma_i(x)$. Elements at distance n are called *opposite*.

The following easy lemma is a special case of [8].

2.1 Lemma *If* Γ *is a generalized* n*-gon and* $\alpha \in \mathrm{Aut}(\Gamma)$*, there exists some* $x \in \Gamma$ *with* $d(x, \alpha(x)) \geq n-1$.

Proof. Let $x \in \Gamma$ be such that $k = d(x, \alpha(x))$ is maximal and suppose $k \leq n-2$. For $i = 1, 2, 3$ let $y_i \in \Gamma_1(x)$ and let y_1 be the unique element with $d(y_1, \alpha(x)) = k-1$. Clearly, $\alpha(y_i) \in \Gamma_1(\alpha(x))$. Therefore, there is at most one $i \in \{1, 2, 3\}$ with $d(y_1, \alpha(y_i)) = k-2$. Hence without loss of generality we have $d(y_1, \alpha(y_1)) = d(y_1, \alpha(y_2)) = k$ and therefore $d(y_2, \alpha(y_2)) = k+2$.

□

If $G \leq \mathrm{Aut}(\Gamma)$, we denote by $G^{[i]}_{x_0}$ the subgroup of G fixing all elements of $\Gamma_i(x_0)$ and for elements $x_0, \dots, x_k$, we set $G^{[i]}_{x_0, x_1, \dots, x_k} = G^{[i]}_{x_0} \cap G^{[i]}_{x_1} \cap \dots \cap G^{[1]}_{x_k}$.

For every simple path $(x_0, \dots x_{n+1})$ of length $n+1$ and every i with $0 \leq i \leq n$, we have $G_{x_0, \dots x_{n+1}} \cap G^{[1]}_{x_i, x_{i+1}} = 1$ (see e.g. [20] 4.4.2 (v)).

For $2 \leq k \leq n$, the generalized n-gon Γ is said to be k-Moufang with respect to $G \leq \mathrm{Aut}(\Gamma)$ if for each simple k-path $(x_0, \dots x_k)$ the group $G^{[1]}_{x_1, \dots, x_{k-1}}$ acts transitively on the set of $2n$-cycles through $(x_0, \dots x_k)$. If Γ is 4-Moufang with respect to some group G, then Γ is in fact n-Moufang with respect to the same group G and we say that Γ is a Moufang polygon (see e.g. [20] 6.8.2). If G_{x_0, x_1}

acts transitively on the set of $2n$-cycles through (x_0, x_1) for all paths (x_0, x_1) (sometimes referred to as the 1-Moufang condition), then G acts transitively on the set of ordered $2n$-cycles of Γ, or *strongly transitively* on Γ. This is equivalent to G having a spherical BN-pair of rank 2, which is in general too weak to allow a classification, see examples below and the ones in [17, 12].

We call a generalized n-gon Γ almost-2-Moufang with respect to G if for every finite set $A \subseteq \Gamma_1(x_1)$, and any path (x_0, x_1, x_2) the group G_A acts transitively on the $2n$-cycles containing (x_0, x_1, x_2). Similarly, one can define almost-3-Moufang for paths (x_0, x_1, x_2, x_3) and finite subsets $A \subseteq \Gamma_1(x_1) \cup \Gamma_1(x_2)$.

It was shown in [15] that the 2-Moufang condition implies the Moufang condition for generalized n-gons with $n \leq 6$. We here construct generalized n-gons for all $n \geq 3$ which are almost-2-Moufang, but not Moufang (and in fact not even almost-3-Moufang)

It is well-known that finite or Moufang generalized n-gons exist only for $n = 3, 4, 6, 8$. Background on the Moufang condition for generalized n-gons can be found in [20] and [18].

The following well-known construction shows the existence of many generalized n-gons for any $n \geq 3$.

2.2 Free n-completion: Let Γ_0 be a connected bipartite graph not containing any k-cycles for $k < 2n$. Then we obtain the free n-completion of Γ_0 in stages in the following way: at stage $i \geq 1$ we obtain Γ_i from Γ_{i-1} by adding a new path of length $n-1$ for each pair of elements x, y at distance $n+1$ in Γ_{i-1}. Then $\Gamma = \bigcup \Gamma_i$ is called the free n-completion of Γ_0 and we say that Γ is freely generated over Γ_0. If Γ_0 contains at least two pairs x_1, y_1 and x_2, y_2 of elements with $d(x_1, y_1) = d(x_2, y_2) = n+1$ in Γ_0 and $d(x_1, x_2)$ is prime to n, then Γ is in fact a generalized n-gon (see e.g. [20] 1.3.13).

Obviously, if $\Gamma_0 \cong \Delta_0$ then also their free n-completions are isomorphic, but the converse need not hold: the free n-completions of Γ_0 and Γ_i are obviously the same.

However, there is a necessary criterion for the free completions of connected bipartite graphs to be isomorphic, which can be stated in terms of the rank function δ_n.

2.3 Definition (i) For any finite graph $\Gamma = (V, E)$ with vertex set V and edge set E we define $\delta_n(\Gamma) = (n-1)|V| - (n-2)|E|$.

(ii) We say that the finite graph Γ_0 is n-strong in some graph Γ (and we write $\Gamma_0 \leq_n \Gamma$) if $\Gamma_0 \subseteq \Gamma$ and for each finite graph $A \subseteq \Gamma, \Gamma_0 \subseteq A$ we have $\delta_n(\Gamma_0) \leq \delta_n(A)$.

The function δ_n was denoted y in [12] and was also used for $n = 3$ in [5]. The $\leq$-relation was also considered in [12]. We will use the following easy fact (see [12] 2.4):

2.4 Fact If $A \leq_n B$ and $C \subseteq B$, then $A \cap C \leq_n C$.

We say that a generalized n-gon Γ is generated by a subset $A \subseteq \Gamma$ if no generalized n-gon properly contained in Γ contains A. Γ is said to be finitely generated if it is generated by a finite subset. Similarly, a subpolygon $\Gamma' \subseteq \Gamma$ is generated by $A \subseteq \Gamma$ if Γ' is the smallest subpolygon of Γ containing A.

2.5 Proposition *The following are equivalent for a generalized n-gon Γ generated by a finite connected set Γ_0:*

(i) Γ *is the free n-completion of* Γ_0*;*

(ii) $\Gamma_0 \leq_n \Gamma$.

Proof. To see that (i) implies (ii), just notice that it follows immediately from the definition of δ_n and the definition of free extensions that we have $\delta_n(\Gamma_0) = \delta_n(\Gamma_i)$ for all i. Thus $\Gamma_0 \leq_n \Gamma_i$ for all i. As any finite set B containing Γ_0 will be contained in some stage Γ_i, the claim follows from 2.4.

For the other direction, suppose that $\Gamma_0 \leq \Gamma$. We want to show that Γ is the free completion of Γ_0. Let Γ_i be the i^{th} stage of the free completion of Γ_0. We show by induction on i that $\Gamma_i \subseteq \Gamma$. The claim obviously holds for $i = 0$. Now suppose $\Gamma_i \subseteq \Gamma$ and $x, y \in \Gamma_i$ have distance $n + 1$ in Γ_i. Since Γ is a generalized n-gon, there is a unique path $(x_0 = x, x_1, \ldots x_{n-1} = y)$ in Γ. Clearly, this path cannot lie entirely in Γ_i. We claim that $x_1, \ldots x_{n-2} \notin \Gamma_i$. Suppose $x_j \in \Gamma_i$ with $1 \leq j \leq n-2$ minimal. Then $\Gamma_i \cup \{x_1, \ldots, x_{j-1}\}$ is a finite subgraph of Γ containing Γ_0, but $\delta_n(\Gamma_0) = \delta_n(\Gamma_i) < \delta_n(\Gamma_i \cup \{x_1, \ldots, x_{j-1}\})$ contradicting our assumption.

Since Γ is generated by Γ_0 and $\bigcup \Gamma_i$ is a generalized polygon containing Γ_0, the claim follows.

□

2.6 Corollary *Suppose A_0 and B_0 are finite connected graphs whose free n-completions A and B are isomorphic generalized n-gons. Then $\delta_n(A_0) = \delta_n(B_0)$.*

Proof. Let A_0, B_0 be finite connected graphs and let A, B be their respective n-completions. Suppose that $\delta_n(A_0) > \delta_n(B_0)$, but $A \cong B$ via some isomorphism φ. Then $\varphi(A_0)$ is contained in some finite stage B_i of the free completion and $\varphi^{-1}(B_i)$ contains A_0. But clearly $\delta_n(\varphi^{-1}(B_i)) = \delta_n(B_i) = \delta_n(B_0) < \delta_n(A_0)$, contradicting Proposition 2.5. □

2.7 Lemma *If Γ is a generalized n-gon freely generated over the finite set Γ_0, then every finitely generated sub-n-gon Δ is also freely generated over a finite set.*

Proof. This is clear since there is some stage Γ_i of the free n-completion which contains every element of the finite generating set for Δ. Then Δ is the free completion of $\Delta \cap \Gamma_i$. □

2.8 Corollary *The class of generalized n-gons freely generated over a finite connected subgraph is countably infinite.*

Proof. Each simple path γ of length $k \geq n + 2$ gives rise to a generalized n-gon. Notice that $\delta_n(\gamma) = (n-2) + k$ and so paths of different lengths give rise to non-isomorphic polygons. On the other hand, clearly, there are at most countably many finite connected graphs. □

2.9 Remark *Note that by Tits' fundamental work [16], spherical buildings of rank greater than two all arise from algebraic groups. Thus, there is no way to obtain similar free constructions in this case.*

3 Amalgamation

We will use Fraïssé's Amalgamation technique for a first order language to obtain new generalized polygons. Fraïssé's Theorem states the following (see [6] 7.1.2 for a proof):

3.1 Theorem *Suppose L is a first-order language and $\mathcal{C}$ is a class of finitely generated L-structures which is closed under finitely generated substructures satisfying the following additional properties:*

- *(Joint Embedding Property) for $A, B \in \mathcal{C}$, there is some $C \in \mathcal{C}$ such that both A and B are embeddable in C;*
- *(Amalgamation Property) for $A, B, C \in \mathcal{C}$, and embeddings $e : A \longrightarrow B, f : A \longrightarrow C$, there is some $D \in \mathcal{C}$ and embeddings $g : B \longrightarrow D, h : C \longrightarrow D$ such that $ge = hf$*

Then there is a countable L-structure M, unique up to isomorphism, satisfying:

(i) Every finitely generated substructure of M is isomorphic to an element of $\mathcal{C}$;

(ii) Every element of $\mathcal{C}$ embeds into M.

(iii) If $A \in \mathcal{C}$, then $\mathrm{Aut}(M)$ acts transitively on the set of substructures of M isomorphic to A.

The model M is also called the *Fraïssé limit* of the class $\mathcal{C}$. Note that Fraïssé limits are existentially closed (that is, if there is some existential sentence which is true in some extension of the structure then it is already true in that structure).

We now fix a first order language $L = \{f_k : k \in \mathbb{N}\}$ containing binary functions f_k. Any (partial) generalized n-gon becomes an L-structure if we interpret these functions as follows: $f_k(x, y) = x_k$ if $(x = x_0, \ldots, x_k, \ldots y)$ is the unique shortest path from x to y. If there is no unique such path, then we let $f_k(x, y) = x$. Notice that edges of the graph can be defined in this language: in a generalized n-gon Γ the pair (x, y) is an edge if and only if $f_1(x, y) = y$. Thus, the axioms of a generalized n-gon are expressible in this language.

Clearly, the language L has the following property: If Γ is a generalized n-gon and $A \subseteq \Gamma$, then the L-substructure $\langle A \rangle$ of Γ generated by A is the same as the (possibly weak) sub-n-gon generated by A in Γ.

3.2 Definition Let $\mathcal{C}_n$ be the class of all finitely generated L-substructures of all free n-completions of finite connected bipartite graphs not containing any k-cycles for $k < 2n$.

Then $\mathcal{C}_n$ is countable by Corollary 2.8 and closed under finitely generated L-substructures.

3.3 Remark $\mathcal{C}_n$ contains in particular the following structures:

(i) the empty structure (making the Joint Embedding Property a special case of the Amalgamation Property) ;

(ii) all paths of length at most n, and more generally any 'hat-rack', i.e. any path $(x_0, \ldots x_k), k \leq n$ together with finite subsets of $\Gamma_1(x_i), i = 1, \ldots, k-1$;

(iii) a $2n$-cycle, and more generally all finite weak n-gons containing at most 2 thick elements (at distance n);

(iv) any finite generalized n-gon (these exist for $n = 3, 4, 6, 8$ only);

(v) arbitrarily large finite discrete sets if n is even and a discrete set of order two if n is odd.

In order to show that we may apply Fraïssé's Theorem to our class $\mathcal{C}_n$, it suffices to show that the Amalgamation Property holds for this class (as the empty structure is included in $\mathcal{C}_n$, which makes the Joint Embedding Property a special case of the Amalgamation Property.

3.4 Lemma *$\mathcal{C}_n$ has the Amalgamation Property.*

Proof. Let $A, B, C \in \mathcal{C}_n$, and suppose that A is a substructure in both B and C. Then let D be the free n-completion of the free amalgam of B and C over A. Obviously, D is in $\mathcal{C}_n$.

□

3.5 Theorem *For all $n \geq 3$ there is a countable generalized n-gon Γ_n whose automorphism group acts transitively on all finitely generated L_n-substructures of given isomorphism type.*

Proof. This follows at once from Fraïssé's Theorem if we show that the Fraïssé limit Γ_n of $\mathcal{C}_n$ is indeed a generalized n-gon. We first show that Γ_n has diameter n. Let $x, y \in \Gamma_n$ and let A be the L_n-substructure generated by x, y. Then either A is a path from x to y (this happens if $d(x, y) < n$), or $A = \{x, y\}$. Let $\gamma = (x_0, \ldots x_n)$ be a path of length n, then $\gamma \in \mathcal{C}_n$ and hence γ can be embedded into $\tilde{\Gamma}$. The images z_0, z_n of x_0, x_n will have distance n in Γ_n, and hence $\{z_0, z_n\}$ is a finitely generated L_n-substructure of Γ_n isomorphic to A. By homogeneity there is an automorphism of Γ_n taking $\{z_0, z_n\}$ to $\{x, y\}$, taking a path from z_0 to z_n to a path from x to y and proving $d(x, y) = n$ in Γ_n.

Clearly, Γ_n does not contain any k-cycles for $k < 2n$ as such a cycle would generate a substructure isomorphic to an element of $\mathcal{C}_n$, which is impossible.

Finally, since the substructure generated by a single vertex in Γ_n is just this vertex, the automorphism group is transitive on vertices. Now the existence of vertices of valency at least 3 immediately implies that all vertices have valency at least 3.

□

3.6 Corollary *In particular, $G = \mathrm{Aut}(\Gamma_n)$ has the following transitivity properties:*

(i) G acts transitively on ordered $2n$-cycles, so G has a BN-pair;

(ii) for any $x \in \Gamma_n$, G_x acts highly transitively on $\Gamma_1(x)$, i.e., G_x is k-transitively for any $k \in \mathbb{N}$ on $\Gamma_1(x)$;

(iii) G acts transitively on finite weak n-gons of the same cardinality. In particular, Γ_n is almost-2-Moufang.

(iv) Let $\gamma = (x_0, x_1, \ldots x_{2n} = x_0)$ be a $2n$-cycle. Then the pointwise stabilizer of γ acts highly transitively on $\Gamma_1(x_1) \setminus \{x_0, x_2\}$.

(v) For any finite set A of vertices of Γ, the elements $\mathrm{Fix}(G_A)$ fixed by G_A are exactly the substructure of Γ_n generated by A.

(vi) Γ_n is not almost-3-Moufang.

Proof. Properties (i)-(iv) all follow from the fact that the corresponding structures are isomorphic to elements of $\mathcal{C}_n$ and their isomorphism types are uniquely determined by their cardinality.

One inclusion of the claim in (v) is obvious. For the other inclusion, it suffices to notice that if $a \notin \langle A \rangle$, then there is $b \notin \langle A \rangle$ with $\langle A, a \rangle \cong \langle A, b \rangle$. Thus there is an automorphism of Γ_n fixing A pointwise and taking a to b, so $a \notin \mathrm{Fix}(G_A)$.

For (vi), let $\gamma = (x_0, \ldots x_{2n-1}, x_0)$ be a $2n$-cycle and $a \in \Gamma_1(x_1) \setminus \{x_0, x_2\}, b \in \Gamma_1(x_2) \setminus \{x_1, x_3\}$. Then the free n-completion A of $\gamma \cup \{a, b\}$ and also a completion B of $\gamma \cup \{a, b\}$ which is not free in a finite number of stages are both contained in $\mathcal{C}_n$. Since G acts transitively on hat-racks of the form $(x_0, x_1, x_2, x_3) \cup \{a, b\}$, we find embeddings φ_1, φ_2 of A and B into Γ which agree on $(x_0, x_1, x_2, x_3) \cup \{a, b\}$. The generalized polygons generated by the respective images of $\gamma \cup \{a, b\}$ in Γ are not isomorphic, so there is no $g \in G_{a,b}$ taking $\phi_1(\gamma)$ to $\phi_2(\gamma)$. □

3.7 Remark We have shown that for a path (x_0, x_1, x_2, x_3) and $a \in \Gamma_1(x_1) \setminus \{x_0, x_2\}$,
$b \in \Gamma_1(x_2) \setminus \{x_1, x_3\}$ the group $G_{a,b}$ is not transitive on $2n$-cycles containing $(x_0, \ldots x_3)$. In particular, Γ is not Moufang.

The proof of (vi) also shows that G is not transitive on $2n + 2$-cycles of Γ_n in contrast to the examples constructed in [12].

4 Groups with highly transitive torus

4.1 Definition Let Γ be a generalized n-gon and $G \leq \mathrm{Aut}(\Gamma)$. Then G has a highly transitive torus if for every $2n$-cycle $\gamma = (x_0, x_1, \ldots x_{2n} = x_0)$ the pointwise stabilizer G_γ acts highly transitively on $\Gamma_1(x_1) \setminus \{x_0, x_2\}$.

Note that a highly transitive torus implies the existence of a BN-pair. Also, if G has a highly transitive torus, then G_x acts highly transitively on $\Gamma_1(x)$. As stated in Corollary 3.6, for each of the generalized n-gons Γ_n constructed in Section 3, $\mathrm{Aut}(\Gamma_n)$ has a highly transitive torus.

We will need the following easy fact (see e.g. [4] Ex. 2.1.6):

4.2 Fact Every nontrivial normal subgroup of a highly transitive group is again highly transititive.

4.3 Lemma *If G has a highly transitive torus in its action on a generalized n-gon Γ, then for any $x \in \Gamma$, the group G_x acts faithfully on $\Gamma_1(x)$.*

Proof. Suppose $G_x^{[1]} \neq 1$. Let i be maximal such that for some path $(x_1 \ldots x_i)$ the group $G_{x_1,\ldots,x_i}^{[1]}$ is nontrivial. Then we must have $i < n$ and we can extend $(x_1 \ldots x_i)$ to a $2n$-cycle $\gamma = (x_0, x_1, \ldots, x_{2n} = x_0)$. Let $\gamma' = (x_j \ldots x_0, x_1 \ldots x_{i+1}, x_{i+2}, \ldots x_k)$ be a maximal path in γ such that $H = G_{x_1,\ldots,x_i}^{[1]} \cap G_{\gamma'} \neq 1$. We claim that $\gamma' = \gamma$. For if $H \neq 1$, then $H \trianglelefteq G_{\gamma'}$ acts highly transitively on $\Gamma_1(x_k) \setminus \{x_{k-1}\}$ and hence $H_{x_{k+1}} \neq 1$. Thus, $\gamma' = \gamma$ and therefore $i = 1$. Since $G_{x_1}^{[1]} \cap G_\gamma$ is normal in G_γ, it acts highly transitively on $\Gamma_1(x_2) \setminus \{x_1\}$. Let $u \in G_{x_0}^{[1]} \cap G_{x_2} \setminus G_{x_3}$. (Such an element exists.) Then for any $v \in (G_{x_1}^{[1]} \cap G_\gamma) \setminus G_{x_3^u}$ we have $[u, v] \in G_{x_0,x_1}^{[1]} \setminus \{1\}$, a contradiction. $\square$

4.4 Proposition *Let Γ be a generalized n-gon, and suppose that G acts strongly transitively on Γ. Suppose that $\mathrm{Fix}(G_{x_0,x_1,x_2}) = \{x_0, x_1, x_2\}$ and $\mathrm{Fix}(G_{x_0,\ldots,x_4}) = \{x_0, \ldots, x_4\}$ for any path $(x_0, x_1, x_2, x_3, x_4)$ of length 4 (where we allow $x_1 = x_3$). Then $N_{x_0} \neq 1$ for any non-trivial normal subgroup N of G.*

Proof. Since G is primitive on vertices of a fixed type, N is transitive on each type of vertices. If $N_{x_0} = 1$, then $G = N \rtimes G_{x_0}$ and N acts regularly on the vertices. Now let $y_1, y_2 \in \Gamma_2(x_0)$ and for $i = 1, 2$ let $n_i \in N$ with $x_0^{n_i} = x_i$. Then G_{x_0,y_i} equals the centralizer of n_i in G_{x_0}. By assumption the only vertices of this type fixed by G_{x_0,y_i} are x_0 and y_i showing n_i to be an involution for $i = 1, 2$. Similarly we see that n_1n_2 is an involution. Thus, the elements of N corresponding to $\Gamma_2(x_0)$ form an elementary abelian 2-group. This is invariant under the action of G_{x_0} contradicting the maximality of G_{x_0}.

4.5 Theorem *If G has a highly transitive torus in its action on a generalized n-gon Γ, then any nontrivial normal subgroup N also has a highly transitive torus and hence a BN-pair.*

Proof. Let $\gamma = (x_0, x_1, \ldots x_{2n} = x_0)$ be a $2n$-cycle and let N be a normal subgroup of G. By Lemma 4.2 it suffices to show that N_γ is non-trivial.

Since G has a highly transitive torus, it is easy to see that the conditions of Proposition 4.4 are satisfied and so N_{x_0} is non-trivial and does not act trivially on $\Gamma_1(x_0)$ by Lemma 4.3. Since $N_{x_0} \trianglelefteq G_{x_0}$, N_{x_0} is highly transitive on $\Gamma_1(x_0)$. Thus, N_{x_0,x_1} is nontrivial and acts nontrivially and hence highly transitively on $\Gamma_1(x_1)$ by Lemmas 4.3 and 4.2. Inductively, we thus show that N_γ is nontrivial and acts nontrivially and hence highly transitively on $\Gamma_1(x_1)$, proving the claim.

□

The random graph can be constructed in an entirely similar way as the Fraisse limit of finite graphs. As its automorphism group turns out to be simple (see [19] or, more generally, [7]) we conjecture that also $G = \mathrm{Aut}(\Gamma_n)$ is simple. Theorem 4.5 seems a first step towards showing this simplicity.

5 Twin trees and multiple trees

Recall the definition of a twin tree [11]:

5.1 Definition Let T_+, T_- be trees without endpoints. A twinning on (T_+, T_-) is given by a codistance function cod : $T_+ \times T_- \cup T_- \times T_+ \longrightarrow \mathbb{N}$ satisfying the following condition for all $x_+ \in T_+, y_- \in T_-$:

If $\mathrm{cod}(x_+, y_-) = m$, then $\mathrm{cod}(x_+, y'_-) \in \{m+1, m-1\}$ for every neighbour y'_- of $y_- \in T_-$. If $m > 0$, there is a unique y'_- with $\mathrm{cod}(x_+, y'_-) = m+1$, similarly with $+$ and $-$ interchanged.

We call x_+ and y_- opposite if $\mathrm{cod}(x_+, y_-) = 0$. Similarly, edges e_+, e_- are called opposite if the vertices of e_+ are opposite the vertices of e_-. Recall also that a group G acting transitively on the ordered doubly infinite paths through T_+ has an affine BN-pair of type $\tilde{A}_1$. If G acts codistance-preserving and transitively on pairs of opposite edges in (T_+, T_-), then G has a twin BN-pair of type $\tilde{A}_1$.

For each $n \in \mathbb{N}, n \geq 3$, we now fix a generalized almost 2-Moufang n-gon Γ_n with a highly transitive torus as constructed in Section 3 and let $G_n = \mathrm{Aut}(\Gamma_n)$. We now consider the 3-sorted structures consisting of the graphs Γ_n with distance function d, its automorphism group G_n, and the natural numbers $\mathbb{N}$, where the distance function takes its values. Note that each Γ_n comes with a natural opposition relation opp and codistance function cod defined by $\mathrm{cod}(x, y) = n - d(x, y)$ for $x, y \in \Gamma_n$. For $x, y \in \Gamma_n$ the codistance $\mathrm{cod}(x, y)$ is the smallest $k \leq n$ such that there is some z opposite x with $k = d(z, y)$.

Let μ be a nonprincipal ultrafilter on $\mathbb{N}$ and consider the ultraproduct

$$(\Gamma, G, \mathbb{N}^*) = \Pi_\mu(\Gamma_n, G_n, \mathbb{N})$$

again in the language of metric graphs with a distance function d taking values in $\mathbb{N}$. This ultraproduct is obtained from the cartesian product $\Pi_{n\in\mathbb{N}}(\Gamma_n, G_n, \mathbb{N})$ by taking its elements to be equivalence classes modulo the equivalence relation $\sim_\mu$ where

$$(a_i)_{i\in\mathbb{N}} \sim_\mu (b_i)_{i\in\mathbb{N}} \Leftrightarrow \{i \in \mathbb{N}\colon a_i = b_i\} \in \mu.$$

We denote the elements of the ultraproduct by $(a_n)_\mu$. Then in the ultraproduct the sorts carry the structure of a graph with distance function d, a group of automorphisms of Γ and a discretely ordered (non-archimedean) abelian semi-group, respectively. (See [6] for details.)

It is easy to see that Γ is a forest, i.e., a graph with the property that each connected component is a tree, and the distance function takes values in $\mathbb{N}^* = \Pi_\mu \mathbb{N}$. Let $\omega = (n)_\mu \in \mathbb{N}^*$, so $x, y \in \Gamma$ are opposite if and only if $d(x, y) = \omega$. Then Γ is a generalized $\omega_{\mathbb{N}^*}$-gon in the sense of C. Bennett [2].

5.2 Theorem *(i) If T is a connected component of Γ, then $G_{\{T\}}$ acts faithfully on T and has a BN-pair of type $\tilde{A}_1$. Here, $G_{\{A\}}$ denotes the setwise stabilizer of a set A.*

(ii) Connected components T_1, T_2 of Γ containing opposite vertices form a twin tree with the twinning given by $\mathrm{cod}(x, y) = \omega - d(x, y)$ *for $x \in T_1, y \in T_2$.*

(iii) If T_1, T_2 are connected components of Γ containing opposite vertices, then $G_{\{T_1\}} \cap G_{\{T_2\}}$ has a twin BN-pair of type $\tilde{A}_1$.

Proof. Part (i) and (iii) follow from Lemma 4.3 and the fact that each G_n acts transitively on pairs of opposite vertices in Γ_n. It is left to see that the condistance function $\mathrm{cod} : T_1 \times T_2 \cup T_2 \times T_1 \longrightarrow \mathbb{N}^*$ has the required properties. Clearly, if $x \in T_1, y \in T_2$ are vertices with $\mathrm{cod}(x, y) = m \in \mathbb{N}^*$, then by the corresponding properties of each Γ_n it is easy to see that if $m > 0$, then y has a unique neighbour $z \in T_2$ with $\mathrm{cod}(x, z) = m + 1$ and $\mathrm{cod}(x, z') = m - 1$ for all other neighbours z' of y. If $m = 0$, then for all neighbours z of y the codistance $\mathrm{cod}(z, x) = 1$. Since T_1, T_2 contains a pair of opposite vertices, whose codistance is 0, it follows inductively that for all $x \in T_1, y \in T_2$ we have in fact $\mathrm{cod}(x, y) \in \mathbb{N}$.

5.3 Remark $\bigcap G_{\{T\}} = 1$ *where the intersection is taken over all connected components T of Γ and if T_1, T_2 are connected components of Γ containing opposite vertices, then $G_{T_1 \cup T_2} = 1$.*

Proof. The first part follows easily from Lemma 2.1. The second part follows from the fact that $G_{x,y}^{[2]} = \{1\}$ for any group G acting on a generalized n-gon with opposite elements x, y (see e.g. [20] 4.4.2 (v)). □

If the even numbers have measure zero with respect to μ (i.e., $\{2n\colon n \in \mathbb{N}\} \notin \mu$), then a set of pairwise opposite elements will have at most two elements, otherwise we can find a countably infinite such set by Remark 3.3. By pruning the trees adequately, it might also be possible to construct multiple trees in the sense of Ronan [10].

6 Homogeneous CAT(1)-spaces

For each $n \in \mathbb{N}, n \geq 3$, let Γ_n and $G_n = \mathrm{Aut}(\Gamma_n)$ be as in Section 5 and let Γ be the ultraproduct with respect to some nonprincipal ultrafilter μ. We now construct the ultralimit $\tilde{\Gamma}$ of the (bounded metric) spaces Γ_n from Γ by scaling the $\mathbb{N}^*$-metric d on Γ by the factor π/ω and identifying points $x, y \in \Gamma$ such that $d(x,y)\pi/\omega < 1/n$ for all $n \in \mathbb{N}$. (Note that we could just as well have rescaled the metric on each Γ_n by the factor π/n.) In particular, each connected component of Γ is 'shrunk' to a point. By construction, $\tilde{\Gamma}$ contains antipodal points and any two antipodal points are contained in an isometrically embedded 1-sphere, but $\tilde{\Gamma}$ does not contain any manifold points. Similar examples without group actions were constructed by Nagano [9]. By Lemma 5.3 the group G acts faithfully on $\tilde{\Gamma}$, and if T is a connected component of Γ, then $G_{\{T\}} \leq G_x$ for some (any) point $x \in T$. Also, by construction G acts transitively on the set of pointed 1-spheres contained in $\tilde{\Gamma}$.

Balser and Lytchak [1] prove that if X is an n-dimensional CAT(1) space that has at least one pair of antipodes and such that each pair of antipodes is contained in some S^n, then X is a spherical building provided that X contains a relatively compact open subset. If a group acts transitively on the pointed n-spheres of a compact spherical building X, the work of Burns and Spatzier [3] shows that such a group is a non-compact real Lie group. Thus a similar statement is far from true without further topological assumptions.

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