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Generalized Ovals in PG$(3n-1,q)$, with q Odd

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Abstract: In 1954 Segre proved that every oval of PG$(2,q)$, with q odd, is a nonsingular conic. The proof relies on the "Lemma of Tangents". A generalized oval of PG$(3n-1,q)$ is a set of q^n+1 $(n-1)$-dimensional subspaces of PG$(3n-1,q)$, every three of them generate PG$(3n-1,q)$; a generalized oval with $n=1$ is an oval. The only known generalized ovals are essentially ovals of PG$(2,q^n)$ interpreted over GF(q). If the oval of PG$(2,q^n)$ is a conic, then we call the corresponding generalized oval classical. Now assume q odd. In the paper we prove several properties of classical generalized ovals. Further we obtain a strong characterization of classical generalized ovals in PG$(3n-1,q)$ and an interesting theorem on generalized ovals in PG$(5,q)$, developing a theory in the spirit of Segre's approach. So for example a "Lemma of Tangents" for generalized ovals is obtained. We hope such an approach will lead to a classification of all generalized ovals in PG$(3n-1,q)$, with q odd.
Keywords: oval, generalized oval, generalized quadrangle, Laguerre plane, quadric

1 Introduction

An *oval* of PG$(2,q)$ is a set of $q+1$ points no three of which are collinear. In 1954 Segre [1954] proved his celebrated theorem, stating that every oval of PG$(2,q)$,

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with q odd, is a nonsingular conic. For q even there are many infinite classes of ovals which are not conics; see e.g. Hirschfeld [1998]. In 1971 Thas [1971] generalized ovals to *pseudo-ovals* or *generalized ovals* of $\mathrm{PG}(3n-1,q)$ with $n \geq 1$. Here the elements are $(n-1)$-dimensional subspaces of $\mathrm{PG}(3n-1,q)$. For $n = 1$ pseudo-ovals are ovals. If the pseudo-oval is embedded in a regular $(n-1)$-spread of $\mathrm{PG}(3n-1,q)$ it is called *regular* or *elementary*. In such a case there exists an oval of $\mathrm{PG}(2,q^n)$ such that its interpretation over $\mathrm{GF}(q)$ is the pseudo-oval. If the oval is a conic of $\mathrm{PG}(2,q^n)$ the corresponding regular pseudo-oval is called a *classical pseudo-oval* or a *pseudo-conic*. In fact all known pseudo-ovals are regular and, for q odd, all known pseudo-ovals are classical.

In 1985 Casse, Thas and Wild [1985] obtained an elegant characterization of classical pseudo-ovals, for q odd, and in Thas, K. Thas and Van Maldeghem [2006] many other characterization theorems of particular pseudo-ovals can be found.

Generalized ovals are equivalent to *translation generalized quadrangles* of order s, with $s \neq 1$; see Thas [1974] and Thas, K. Thas and Van Maldeghem [2006]. Hence many characterizations of generalized ovals are described in terms of generalized quadrangles, and conversely. Also, generalized ovals of $\mathrm{PG}(3n-1,q)$, for q odd, are equivalent to particular Laguerre planes of order $s = q^n$; see Payne and Thas [1976, 1984].

In recent years there has been great interest in generalized ovals of $\mathrm{PG}(5,q)$, with q odd, whose lines are contained in some nonsingular elliptic quadric $Q^-(5,q)$ of $\mathrm{PG}(5,q)$; see Shult [2005], Cossidente, Ebert, Marino and Siciliano [2006] and Cossidente, King and Marino [2006]. It is no surprise that the classical pseudo-oval in $\mathrm{PG}(5,q)$, with q odd, is the only known example.

In this paper we will prove an interesting characterization of pseudo-conics in $\mathrm{PG}(3n-1,q)$ and some strong new results on generalized ovals, for q odd. To that purpose we generalize the approach of Segre, used to prove his famous theorem on ovals. In particular we find an analogue of his well-known "Lemma of Tangents".

2 Ovals, the Lemma of Tangents and Segre's Theorem

A k-*arc* of PG$(2,q)$ is a set of k points of PG$(2,q)$ no three of which are collinear. Then clearly $k \leq q+2$. By Bose [1947], for q odd, $k \leq q+1$. Further, any nonsingular conic of PG$(2,q)$ is a $(q+1)$-arc. It can be shown that each $(q+1)$-arc K of PG$(2,q)$, q even, extends to a $(q+2)$-arc $K \cup \{x\}$ (see, e.g. Hirschfeld [1998], p. 177); the point x, which is uniquely defined by K, is called the *kernel* or *nucleus* of K. The $(q+1)$-arcs of PG$(2,q)$ are called *ovals*.

For any k-arc K with $3 \leq k \leq q+1$, choose three of its points as the vertices of the triangle of reference $u_0u_1u_2$ of the coordinate system. The lines intersecting K in one point are called the *tangent lines* of K. A tangent line of K through one of u_0, u_1, u_2 has respective equation

$$X_1 - dX_2 = 0, X_2 - dX_0 = 0, X_0 - dX_1 = 0,$$

with $d \neq 0$. We call d the *coordinate* of such a line. Suppose the $t = q+2-k$ tangent lines at each of u_0, u_1, u_2 are

$$X_1 - a_iX_2 = 0, X_2 - b_iX_0 = 0, X_0 - c_iX_1 = 0,$$

$i = 1, 2, \cdots, t$. Then Segre [1954] proves the following important lemma.

Lemma 2.1 (Lemma of Tangents). *The coordinates a_i, b_i, c_i of the tangent lines at u_0, u_1, u_2 of a k-arc K through these points satisfy*

$$\Pi_{i=1}^{t} a_ib_ic_i = -1.$$

Proof. Let $K = \{u_0, u_1, u_2, y_1, y_2, \cdots, y_{k-3}\}$, with $y_j = (x_0^j, x_1^j, x_2^j), j = 1, 2, \cdots, k-3$. Then

$$u_0y_j : X_1 - d_0^jX_2 = 0, u_1y_j : X_2 - d_1^jX_0 = 0, u_2y_j : X_0 - d_2^jX_1 = 0,$$

with

$$d_0^j = x_1^j/x_2^j, d_1^j = x_2^j/x_0^j, d_2^j = x_0^j/x_1^j, j = 1, 2, \cdots, k-3.$$

Hence

$$d_0^jd_1^jd_2^j = 1, \text{ with } j = 1, 2, \cdots, k-3.$$

As the product of the nonzero elements of GF(q) is -1, we have

$$\Pi_{i=1}^{t} a_i \Pi_{j=1}^{k-3} d_0^j = -1, \Pi_{i=1}^{t} b_i \Pi_{j=1}^{k-3} d_1^j = -1, \Pi_{i=1}^{t} c_i \Pi_{j=1}^{k-3} d_2^j = -1.$$

Hence

$$\Pi_{i=1}^{t} a_i b_i c_i = -1. \quad \square$$

For an oval K we have $t = 1$, and so the lemma becomes $abc = -1$. Geometrically this means that for q odd the triangles formed by three points of an oval and the tangent lines at these points are in perspective, or, equivalently, that for any three distinct points u_0, u_1, u_2 on an oval there is a (unique) nonsingular conic containing these points u_0, u_1, u_2 and having as tangent lines at u_0, u_1, u_2 the tangent lines of the oval at u_0, u_1, u_2; for q even the condition means that the tangent lines at any three points of the oval are concurrent.

Now we state the celebrated theorem of Segre [1954]. In the original proof Lemma 2.1 is applied to several triangles $u_0u_1u_2$ on the oval K, and this involves some calculations. We present here a proof relying on Lemma 2.1, but without any calculation.

Theorem 2.2 *In* PG$(2, q)$*, q odd, every oval is a nonsingular conic.*

Proof. Let $K = \{u_0, u_1, \cdots, u_q\}$ be an oval of PG$(2, q)$, q odd. The tangent line of K at u_i is denoted by L_i, with $i = 0, 1, \cdots, q$. By Lemma 2.1, for any three points u_i, u_j, u_k of K there is a nonsingular conic C containing u_i, u_j, u_k and tangent to L_i, L_j, L_k. Let C_l be the conic defined by $\{u_i, u_j, u_k\}$, with $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Assume, by way of contradiction, that $C_0 \neq C_1$. Now we consider the cubic curves $C_0 \cup L_0$ and $C_1 \cup L_1$. The common points of these curves are u_0, u_1, u_2, u_3, each counted twice, and $L_0 \cap L_1$. Next, we consider the cubic curve $C_2 \cup L_2$. This curve contains eight common points of $C_0 \cup L_0$ and $C_1 \cup L_1$ (u_0, u_1, u_2, u_3, each counted twice). So $C_2 \cup L_2$ contains $L_0 \cap L_1$; see e.g. Semple and Roth [1949], p. 42. Hence L_2 contains $L_0 \cap L_1$, clearly a contradiction as L_0, L_1, L_2 are tangent to C_3. So $C_0 = C_1$. Similarly, $C_0 = C_1 = C_2 = C_3$. Consequently, C_3 contains u_3; similarly, C_3 contains $u_4, u_5, \cdots, u_q$. So $K \subseteq C_3$ and as $|C_3| = |K| = q + 1$, it follows that $K = C_3$. $\square$

For q even Theorem 2.2 is only valid for $q \in \{2, 4\}$; see e.g. Hirschfeld [1998] Section 8.4.

3 Generalized Ovals

A *generalized oval* or *pseudo-oval* in PG$(3n-1,q)$, $n \geq 1$, is a set of q^n+1 $(n-1)$-dimensional subspaces any three of which generate PG$(3n-1,q)$; see Thas [1971] and Thas, K. Thas and Van Maldeghem [2006]. For $n=1$ a generalized oval is the same as an oval. Let $O=\{\pi_0,\pi_1,\cdots,\pi_{q^n}\}$ be a generalized oval in PG$(3n-1,q)$. Then for each π_i there is exactly one $(2n-1)$-dimensional subspace τ_i such that $\pi_i \subset \tau_i$ and $\tau_i \cap (\pi_0 \cup \cdots \cup \pi_{i-1} \cup \pi_{i+1} \cup \cdots \cup \pi_{q^n}) = \emptyset$; τ_i is called the *tangent space* of O at π_i, $i=0,1,\cdots,q^n$. Generalized ovals play an important role in the theory of finite generalized quadrangles; see Payne and Thas [1984] and Thas, K. Thas and Van Maldeghem [2006]. In fact, the theory of generalized ovals is equivalent to the theory of finite translation generalized quadrangles of order s.

If q is even, then all tangent spaces of a generalized oval contain a common $(n-1)$-dimensional space; see Thas [1971] and Thas, K. Thas and Van Maldeghem [2006]. This common space is called the *kernel* or *nucleus* of the generalized oval. If q is odd, then each point not in an element of the generalized oval O is contained in either 0 or 2 tangent spaces, and each hyperplane not containing a tangent space contains either 0 or 2 elements of O; see e.g. Thas, K. Thas and Van Maldeghem [2006].

Now we define the regular generalized ovals. In the extension PG$(3n-1,q^n)$ of PG$(3n-1,q)$ we consider n planes PG$^{(i)}(2,q^n)=\xi_i$, with $i=1,2,\cdots,n$, which are conjugate with respect to the extension GF(q^n) of GF(q), that is, which form an orbit of the Galois group corresponding to this extension, and which span PG$(3n-1,q^n)$. In ξ_1 we consider an oval $O_1=\{x_0^{(1)},x_1^{(1)},\cdots,x_{q^n}^{(1)}\}$. Further, let $x_i^{(1)},x_i^{(2)},\cdots,x_i^{(n)}$, with $i=0,1,\cdots,q^n$, be conjugate with respect to the extension GF(q^n) of GF(q). The points $x_i^{(1)},x_i^{(2)},\cdots,x_i^{(n)}$ define an $(n-1)$-dimensional space PG$^{(i)}(n-1,q)=\pi_i$ over GF(q), with $i=0,1,\cdots,q^n$. Then $O=\{\pi_0,\pi_1,\cdots,\pi_{q^n}\}$ is a generalized oval of PG$(3n-1,q)$. Here we speak of a *regular* or *elementary* generalized oval. No other generalized ovals are known. If O_1 is a conic, then O is called a *classical* generalized oval or a *pseudo-conic*. By Segre's Theorem, for q odd each regular generalized oval is classical.

Assume that q is odd. Fix a space π_i of the generalized oval $O=\{\pi_0,\pi_1,\cdots,\pi_{q^n}\}$. Let τ_j be the tangent space of O at π_j, with $j=0,1,\cdots,q^n$. Then the elements $\pi_i, \tau_i \cap \tau_j$, with $j \neq i$, are the elements of a $(n-1)$-spread S_i of τ_i, that is, they form a partition of τ_i. Also, the spaces $\langle \pi_i, \pi_j \rangle$ generated by π_i and π_j, with $j \neq i$, together with τ_i intersect a given $\mathrm{PG}(2n-1,q)$ skew to π_i in the elements of a $(n-1)$-spread S_i^* of $\mathrm{PG}(2n-1,q)$. The following characterization theorem is due to Casse, Thas and Wild [1985].

Theorem 3.1 *If at least one of the spreads $S_0, S_1, \cdots, S_{q^n}, S_0^*, S_1^*, \cdots, S_{q^n}^*$ is regular, then they all are and O is classical.*

For characterizations of particular classes of generalized ovals in the even case we refer to Chapter 8 of Thas, K. Thas and Van Maldeghem [2006].

Let q be odd. Then for each n even any classical generalized oval of $\mathrm{PG}(3n-1,q)$ belongs to a nonsingular elliptic quadric $Q^-(3n-1,q)$ and to a nonsingular hyperbolic quadric $Q^+(3n-1,q)$, and for each n odd, any classical generalized oval of $\mathrm{PG}(3n-1,q)$ belongs to a nonsingular parabolic quadric; see Shult and Thas [1995]. In each of these cases the tangent space at any element π of the generalized oval coincides with the tangent space at π of any of the corresponding quadrics. For $n = 2$ any classical generalized oval O is the intersection of some $Q^-(5,q)$ and some $Q^+(5,q)$; it is contained in $(q+1)/2$ nonsingular elliptic quadrics and $(q+1)/2$ nonsingular hyperbolic quadrics, and at each element L of O these quadrics have a common tangent space which coincides with the tangent space of O at L.

The following theorem is due to Shult and Thas [1994].

Theorem 3.2 *If O is a generalized oval of lines contained in a nonsingular hyperbolic quadric $Q^+(5,q)$ of $\mathrm{PG}(5,q)$, with q odd, then O is classical.*

As for q even all tangent spaces of a generalized oval contain a common $(n-1)$-dimensional space, it follows that for $n > 1$ the generalized oval is never contained in a nonsingular quadric.

In recent years there was great interest in generalized ovals consisting of q^2+1 lines of $Q^-(5,q)$, with q odd. Such a generalized oval is equivalent to a set of q^2+1 points of the nonsingular Hermitian variety $H(3,q^2)$ in $\mathrm{PG}(3,q^2)$, such that the plane defined by any three distinct points of this set is nontangent to

$H(3,q^2)$; see Payne and Thas [1984]. This object was studied in different contexts by Shult [2005], Cossidente, Ebert, Marino and Siciliano [2006] and Cossidente, King and Marino [2006].

Let π_1, π_2, π_3 be mutually skew $(n-1)$-dimensional subspaces of PG$(3n-1,q)$, let τ_i be a $(2n-1)$-dimensional space containing π_i but skew to π_j and π_k, with $\{i,j,k\} = \{1,2,3\}$, and let $\tau_i \cap \tau_j = \eta_k$, with $\{i,j,k\} = \{1,2,3\}$. The space generated by η_i and π_i will be denoted by ζ_i, with $i = 1,2,3$. If the $(2n-1)$-dimensional spaces $\zeta_1, \zeta_2, \zeta_3$ have a $(n-1)$-dimensional space in common, then we say that $\{\pi_1, \pi_2, \pi_3\}$ and $\{\tau_1, \tau_2, \tau_3\}$ are in *perspective*; if $\zeta_1, \zeta_2, \zeta_3$ have a nonempty intersection, then we say that $\{\pi_1, \pi_2, \pi_3\}$ and $\{\tau_1, \tau_2, \tau_3\}$ are in *semi-perspective*.

Let O be a regular generalized oval in PG$(3n-1,q)$, with q odd. Then for any three distinct elements π_i, π_j, π_k of O the sets $\{\pi_i, \pi_j, \pi_k\}$ and $\{\tau_i, \tau_j, \tau_k\}$ are in perspective, where τ_l is the tangent space of O at π_l, $l \in \{i,j,k\}$. This follows immediately from the fact that in the odd case, by the Lemma of Tangents, this property holds for every oval.

4 Projective Spaces over Matrix Algebras

Here we rely on Thas [1971] where the m-dimensional projective space $S_m(M_n(\mathrm{GF}(q)))$ over the total matrix algebra $M_n(\mathrm{GF}(q))$ of the $n \times n$-matrices with elements in GF(q) is studied.

Let π be an $(n-1)$-dimensional subspace of PG$(mn+n-1,q)$ and let $p_1(x_0^1, x_1^1, \cdots, x_{mn+n-1}^1), \cdots, p_n(x_0^n, x_1^n, \cdots, x_{mn+n-1}^n)$ be n independent points of π. Further, let

$$\begin{bmatrix} x_0^1 & \cdots x_0^n \\ x_1^1 & \cdots x_1^n \\ \vdots & \vdots \\ x_{mn+n-1}^1 & \cdots x_{mn+n-1}^n \end{bmatrix} = \begin{bmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_m \end{bmatrix} = \mathcal{A},$$

where $\xi_0, \xi_1, \cdots, \xi_m$ are $n \times n$-matrices over GF(q). Clearly,

$$\operatorname{rank} \mathcal{A} = n.$$

We write $\pi = (\xi_0, \xi_1, \cdots, \xi_m)$ or $\pi(\xi_0, \xi_1, \cdots, \xi_m)$; we say that π is a point of the projective space $S_m(M_n(\mathrm{GF}(q)))$, having $\xi_0, \xi_1, \cdots, \xi_m$ as projective coordinates. The coordinates $\xi_0, \xi_1, \cdots, \xi_m$ are determined by π up to a right factor of proportion $\rho \in M_n(\mathrm{GF}(q))$, with $|\rho| \neq 0$.

Let τ be a $(mn-1)$-dimensional subspace of $\mathrm{PG}(mn+n-1, q)$, and let

$$\begin{aligned}
&\beta_1 : a_0^1 X_0 + a_1^1 X_1 + \cdots + a_{mn+n-1}^1 X_{mn+n-1} = 0,\\
&\vdots\\
&\beta_n : a_0^n X_0 + a_1^n X_1 + \cdots + a_{mn+n-1}^n X_{mn+n-1} = 0
\end{aligned}$$

be n independent hyperplanes containing τ. Further, let

$$\begin{bmatrix} a_0^1 & a_1^1 & \cdots & a_{mn+n-1}^1 \\ a_0^2 & a_1^2 & \cdots & a_{mn+n-1}^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_{mn+n-1}^n \end{bmatrix} = [\alpha_0 \, \alpha_1 \cdots \alpha_m] = \mathcal{B},$$

where $\alpha_0, \alpha_1, \cdots, \alpha_m$ are $n \times n$-matrices over $\mathrm{GF}(q)$. Clearly,

$$\mathrm{rank}\ \mathcal{B} = n.$$

We write $\tau = (\alpha_0, \alpha_1, \cdots, \alpha_m)$ or $\tau(\alpha_0, \alpha_1, \cdots, \alpha_m)$; we say that τ is a hyperplane of the projective space $S_m(M_n(\mathrm{GF}(q)))$, having $\alpha_0, \alpha_1, \cdots, \alpha_m$ as projective coordinates. We also say that

$$\alpha_0 \chi_0 + \alpha_1 \chi_1 + \cdots + \alpha_m \chi_m = 0$$

is an equation of the hyperplane. The coordinates $\alpha_0, \alpha_1, \cdots, \alpha_m$ are determined by τ up to a left factor of proportion $\rho' \in M_n(\mathrm{GF}(q))$, with $|\rho'| \neq 0$.

If $\pi(\xi_0, \xi_1, \cdots, \xi_m)$ is a point of $S_m(M_n(\mathrm{GF}(q)))$ and if $\tau(\alpha_0, \alpha_1, \cdots, \alpha_m)$ is a hyperplane of $S_m(M_n(\mathrm{GF}(q)))$, then we say that π is contained in or is incident with τ if the corresponding $(n-1)$-dimensional space is contained in the corresponding $(mn-1)$-dimensional projective space. Algebraically this is equivalent to

$$\alpha_0 \xi_0 + \alpha_1 \xi_1 + \cdots + \alpha_m \xi_m = 0.$$

For more details on $S_m(M_n(\mathrm{GF}(q)))$ we refer to Thas [1971].

5 A Property of Quadrics

Now we prove a property of quadrics in PG$(5, q)$, with q odd. Then this result will be generalized to PG$(3n-1, q)$, with $n \geq 1$ and q odd.

Theorem 5.1 *Let Q be a nonsingular quadric in* PG$(5, q)$*, with q odd, and let L_1, L_2, L_3 be distinct lines on Q with generate* PG$(5, q)$*. The tangent space of Q at L_i is denoted by τ_i, with $i = 1, 2, 3$. Further,assume that $L_i \cap \tau_j = \emptyset$, for all $i \neq j$. If $\{L_1, L_2, L_3\}$ and $\{\tau_1, \tau_2, \tau_3\}$ are in semi-perspective, then they are in perspective.*

Proof. Consider a nonsingular quadric Q in PG$(5, q)$, q odd, with equation

$$X_0^2 + aX_1^2 + X_2X_3 + X_4X_5 = 0, \text{with } a \neq 0.$$

Coordinates are chosen in such a way that

$$\begin{aligned}
L_1 &= \langle (0,0,1,0,0,0), (0,0,0,0,1,0) \rangle, \\
L_2 &= \langle (0,0,0,1,0,0), (0,0,0,0,0,1) \rangle, \\
L_3 &= \langle (1,0,a_2,a_3,a_4,a_5), (0,1,b_2,b_3,b_4,b_5) \rangle,
\end{aligned}$$

with

$$\begin{aligned}
&1 + a_2a_3 + a_4a_5 = 0, && (1)\\
&a + b_2b_3 + b_4b_5 = 0, && (2)\\
&a_2b_3 + a_3b_2 + a_4b_5 + a_5b_4 = 0, && (3)\\
&a_3b_5 - a_5b_3 \neq 0, && (4)\\
&a_2b_4 - a_4b_2 \neq 0. && (5)
\end{aligned}$$

Let θ be the orthogonal polarity defined by Q. Then

$$\begin{aligned}
\tau_1 &= L_1^\theta : X_3 = X_5 = 0, \\
\tau_2 &= L_2^\theta : X_2 = X_4 = 0, \\
\tau_3 &= L_3^\theta : 2X_0 + a_3X_2 + a_2X_3 + a_5X_4 + a_4X_5 = 0, \\
&\qquad 2aX_1 + b_3X_2 + b_2X_3 + b_5X_4 + b_4X_5 = 0.
\end{aligned}$$

Further,

$$\begin{aligned}
M_2 &= \tau_1 \cap \tau_3 : X_3 = X_5 = 2X_0 + a_3X_2 + a_5X_4 = 2aX_1 + b_3X_2 + b_5X_4 = 0, \\
M_1 &= \tau_2 \cap \tau_3 : X_2 = X_4 = 2X_0 + a_2X_3 + a_4X_5 = 2aX_1 + b_2X_3 + b_4X_5 = 0, \\
M_3 &= \tau_1 \cap \tau_2 : X_2 = X_3 = X_4 = X_5 = 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\zeta_1 = \langle L_1, M_1\rangle :\ & 2X_0 + a_2X_3 + a_4X_5 = 2aX_1 + b_2X_3 + b_4X_5 = 0,\\
\zeta_2 = \langle L_2, M_2\rangle :\ & 2X_0 + a_3X_2 + a_5X_4 = 2aX_1 + b_3X_2 + b_5X_4 = 0,\\
\zeta_3 = \langle L_3, M_3\rangle = & \{(u, v, ra_2 + r'b_2, ra_3 + r'b_3, ra_4 + r'b_4, ra_5 + r'b_5) :\\
& \qquad (u, v, r, r') \neq (0,0,0,0)\}.
\end{aligned}$$

The points of $\zeta_1 \cap \zeta_2 \cap \zeta_3$ are obtained by solving the following system of linear equations:

$$\begin{cases}
2u + a_2(ra_3 + r'b_3) + a_4(ra_5 + r'b_5) = 0, & (6)\\
2av + b_2(ra_3 + r'b_3) + b_4(ra_5 + r'b_5) = 0, & (7)\\
2u + a_3(ra_2 + r'b_2) + a_5(ra_4 + r'b_4) = 0, & (8)\\
2av + b_3(ra_2 + r'b_2) + b_5(ra_4 + r'b_4) = 0. & (9)
\end{cases}$$

Adding (6) and (8), and taking account of (1) and (3), we obtain $r = 2u$; similarly $r' = 2v$. So the system of equations becomes

$$\begin{cases}
r = 2u,\\
r' = 2v,\\
r(b_2a_3 + b_4a_5) = 0,\\
r'(b_2a_3 + b_4a_5) = 0.
\end{cases}$$

Consequently, $\zeta_1 \cap \zeta_2 \cap \zeta_3 = \emptyset$ if $b_2a_3 + b_4a_5 \neq 0$ and $\zeta_1 \cap \zeta_2 \cap \zeta_3$ is a line if $b_2a_3 + b_4a_5 = 0$. $\square$

For general n we have the following theorem.

Theorem 5.2 *Let Q be a nonsingular quadric in $\mathrm{PG}(3n-1, q)$, q odd, and let π_1, π_2, π_3 be distinct $(n-1)$-dimensional spaces on Q which generate $\mathrm{PG}(3n-1, q)$. The tangent space of Q at π_i is denoted by τ_i, with $i = 1, 2, 3$. Further,assume that $\pi_i \cap \tau_j = \emptyset$, for all $i \neq j$. Let $\eta_k = \tau_i \cap \tau_j$ with $\{i, j, k\} = \{1, 2, 3\}$ and let $\zeta_i = \langle \eta_i, \pi_i\rangle$, with $i = 1, 2, 3$. Then the dimension of $\zeta_1 \cap \zeta_2 \cap \zeta_3$ has the same parity as $n - 1$. In particular, if n is odd, then $\{\pi_1, \pi_2, \pi_3\}$ and $\{\tau_1, \tau_2, \tau_3\}$ are always in semi-perspective.*

Sketch of the proof. The proof is quite similar to the proof of Theorem 5.1, so we just give a sketch of it.

Consider a nonsingular quadric Q in PG$(3n-1,q)$, q odd, with equation

$$X_0^2 + X_1^2 + \cdots + aX_{n-1}^2 + X_nX_{n+1} + X_{n+2}X_{n+3} + \cdots + X_{3n-2}X_{3n-1} = 0, a \neq 0.$$

Let e_i be the point with 1 in the $(i+1)$-th position and 0 elsewhere. Coordinates are chosen in such a way that

$$\begin{aligned}
\pi_1 &= \langle e_n, e_{n+2}, \cdots, e_{3n-2}\rangle,\\
\pi_2 &= \langle e_{n+1}, e_{n+3}, \cdots, e_{3n-1}\rangle,\\
\pi_3 &= \langle y_1, y_2, \cdots, y_n\rangle, \text{where}
\end{aligned}$$

$$y_i = (0, \cdots, 0, 1, 0, \cdots, 0, a_n^{(i)}, a_{n+1}^{(i)}, \cdots, a_{3n-1}^{(i)}),$$
with $i = 1, 2, \cdots, n$ and the 1 in position i.

Also,

$$1 + a_n^{(i)}a_{n+1}^{(i)} + a_{n+2}^{(i)}a_{n+3}^{(i)} + \cdots + a_{3n-2}^{(i)}a_{3n-1}^{(i)} = 0, \text{ for } i = 1, 2, \cdots, n-1,$$

$$a + a_n^{(n)}a_{n+1}^{(n)} + a_{n+2}^{(n)}a_{n+3}^{(n)} + \cdots + a_{3n-2}^{(n)}a_{3n-1}^{(n)} = 0,$$

and

$$a_n^{(i)}a_{n+1}^{(j)} + a_n^{(j)}a_{n+1}^{(i)} + \cdots + a_{3n-2}^{(i)}a_{3n-1}^{(j)} + a_{3n-2}^{(j)}a_{3n-1}^{(i)} = 0, \text{ for all } i \neq j$$
with $i, j \in \{1, 2, \cdots, n\}$.

Further,

$$|a_j^{(i)}| \neq 0, \text{ with } i = 1, 2, \cdots, n \text{ and } j = n+1, n+3, \cdots, 3n-1,$$

and

$$|a_j^{(i)}| \neq 0, \text{ with } i = 1, 2, \cdots, n \text{ and } j = n, n+2, \cdots, 3n-2.$$

The points

$$z = (u_0, u_1, \cdots, u_{n-1}, \sum_{i=1}^{n} r_i a_n^{(i)}, \sum_{i=1}^{n} r_i a_{n+1}^{(i)}, \cdots, \sum_{i=1}^{n} r_i a_{3n-1}^{(i)})$$

of ζ_3 in $\zeta_1 \cap \zeta_2$ are determined by the following system of linear equations:

$r_{i+1} = 2u_i$,
with $i = 0, 1, \cdots, n-1$,
$\sum_{i=1}^{n} r_i b_{ij} = 0$, with $j = 1, 2, \cdots, n$,
where $b_{ij} = a_n^{(i)}a_{n+1}^{(j)} + a_{n+2}^{(i)}a_{n+3}^{(j)} + \cdots + a_{3n-2}^{(i)}a_{3n-1}^{(j)}$,
with $i \neq j$, and $b_{ii} = 0$.

The system consisting of the last n equations has a skew-symmetric matrix $\mathcal{M}$. Hence rank $\mathcal{M}$ is even. This proves the theorem. □

Remark 5.3 . *Since we are mainly interested in generalized ovals we assume in Theorem 5.2 that q is odd. In the even case the statement is quite different, so for example if q is even and $n = 2$ we always have $\zeta_1 \cap \zeta_2 \cap \zeta_3 = \emptyset$.*

Theorem 5.4 *Let O be a generalized oval of $\mathrm{PG}(3n-1, q)$, with q odd, contained in a nonsingular quadric Q of $\mathrm{PG}(3n-1, q)$. If $\pi \in O$, then the tangent spaces at π of O and Q coincide.*

Proof. Let $\pi \in O$, let τ be the tangent space of Q at π, and let η be a $(n-1)$-dimensional subspace of τ skew to π. Further, let $\eta \cap Q = Q'$. If $y \in \eta - Q'$, then $\langle \pi, y \rangle$ has no point in common with $Q - \pi$, so has no point in common with any element of $O - \{\pi\}$. So $\langle y, \pi \rangle$ belongs to the tangent space τ' of O at π. The spaces $\langle \pi, y \rangle$ generate τ, so $\tau \subseteq \tau'$. As dim τ = dim τ', we have $\tau = \tau'$. □

Corollary 5.5 *Let O be a generalized oval of $\mathrm{PG}(3n-1, q)$, with q odd, contained in a nonsingular quadric Q of $\mathrm{PG}(3n-1, q)$. Then any three distinct elements π_1, π_2, π_3 of O satisfy the requirements in the statement of Theorem 5.2.*

6 A Characterization of Pseudo-conics

Assume that the triples $\{\pi_0, \pi_1, \pi_2\}$ and $\{\tau_0, \tau_1, \tau_2\}$ are in perspective in $\mathrm{PG}(3n-1, q)$, with q odd; here π_i is $(n-1)$-dimensional, τ_i is $(2n-1)$-dimensional and $\pi_i \subset \tau_i$, with $i = 0, 1, 2$. Let $\tau_i \cap \tau_j = \eta_k$, with $\{i, j, k\} = \{0, 1, 2\}$, $\langle \eta_k, \pi_k \rangle = \zeta_k$ with $k = 0, 1, 2$, and $\zeta_0 \cap \zeta_1 \cap \zeta_2 = \zeta$; the space ζ is $(n-1)$-dimensional.

Coordinates can be chosen in such a way that $\pi_0(\varepsilon, 0, 0), \pi_1(0, \varepsilon, 0)$, $\pi_2(0, 0, \varepsilon), \tau_0(0, \varepsilon, \varepsilon), \tau_1(\varepsilon, 0, \varepsilon), \tau_2(\varepsilon, \varepsilon, 0), \zeta(\varepsilon, \varepsilon, \varepsilon)$, with ε the identity matrix of order n, and 0 the zero matrix of order n.

The spaces $\pi_i, \tau_i, \eta_i, \zeta_i, \zeta$, with $i = 0, 1, 2$, all belong to the Segre variety

$$\mathcal{S}_{n-1;2} = \{(x_0y_0, x_0y_1, \cdots, x_0y_{n-1}, x_1y_0, x_1y_1, \cdots, x_1y_{n-1},$$
$$x_2y_0, x_2y_1, \cdots, x_2y_{n-1}) : (x_0, x_1, x_2) \in \mathrm{GF}(q)^3 - \{(0,0,0)\},$$
$$(y_0, y_1, \cdots, y_{n-1}) \in \mathrm{GF}(q)^n - \{(0,0,\cdots,0)\}\}.$$

For more on Segre varieties see Sections 25.5 and 25.6 of Hirschfeld and Thas [1991].

Let π_3 be an $(n-1)$-dimensional space of PG$(3n-1,q)$, let τ_3 be a $(2n-1)$-dimensional space of PG$(3n-1,q)$, let $\pi_3 \subset \tau_3$, and assume that $\langle \pi_i, \pi_j, \pi_3 \rangle =$ PG$(3n-1,q)$, $\pi_3 \cap \tau_i = \emptyset$ and $\tau_3 \cap \pi_i = \emptyset$, with $i \neq j$ and $i,j \in \{0,1,2\}$. Further, let $\pi_3(\xi_0,\xi_1,\xi_2)$ and $\tau_3(\alpha_0,\alpha_1,\alpha_2)$. By Section 4 we have

$$\alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 = 0.$$

Now we put

$$\gamma_0 = \xi_1 + \xi_2,$$
$$\gamma_1 = \xi_2 + \xi_0,$$
$$\gamma_2 = \xi_0 + \xi_1.$$

Further, let us assume $\gamma_0 = \varepsilon$ (as $\pi_3 \not\subset \tau_0$ we have $\gamma_0 \neq 0$, and $\gamma_0, \gamma_1, \gamma_2$ are determined up to a right factor).

Lemma 6.1 *If for all $i \neq j$ and $i,j \in \{0,1,2\}$ the triples $\{\pi_i, \pi_j, \pi_3\}$ and $\{\tau_i, \tau_j, \tau_3\}$ are in perspective, then*

$$\gamma_1\gamma_2 = \gamma_2\gamma_1, \text{ and}$$
$$(\varepsilon - \gamma_1 - \gamma_2)^2 = 4\gamma_1\gamma_2.$$

Proof. Coordinates of $\tau_3 \cap \tau_0 = \eta_0' : (\alpha_0^{-1}(\alpha_2 - \alpha_1), \varepsilon, -\varepsilon)$.
Coordinates of $\tau_3 \cap \tau_1 = \eta_1' : (\varepsilon, \alpha_1^{-1}(\alpha_2 - \alpha_0), -\varepsilon)$.
Coordinates of $\tau_0 \cap \tau_1 = \eta_2 : (\varepsilon, \varepsilon, -\varepsilon)$.
Now we intersect $\langle \pi_1, \eta_0' \rangle$ and $\langle \pi_0, \eta_1' \rangle$ and obtain

$$(\alpha_0^{-1}(\alpha_2 - \alpha_1), \alpha_1^{-1}(\alpha_2 - \alpha_0), -\varepsilon).$$

Then we express that this element is contained in $\langle \pi_3, \eta_2 \rangle$:

$$\begin{cases} \alpha_0^{-1}(\alpha_2 - \alpha_1) = \xi_0\rho + \rho', \\ \alpha_1^{-1}(\alpha_2 - \alpha_0) = \xi_1\rho + \rho', \\ -\varepsilon = \xi_2\rho - \rho'. \end{cases}$$

It follows that

$$\begin{cases} \alpha_0(\xi_2\rho + \varepsilon) = \alpha_2 - \alpha_1 - \alpha_0\xi_0\rho, \\ \alpha_1(\xi_2\rho + \varepsilon) = \alpha_2 - \alpha_0 - \alpha_1\xi_1\rho. \end{cases}$$

So

$$\begin{cases} \alpha_0(\xi_2 + \xi_0)\rho = \alpha_2 - \alpha_1 - \alpha_0, \\ \alpha_1(\xi_2 + \xi_1)\rho = \alpha_2 - \alpha_1 - \alpha_0. \end{cases}$$

Consequently,

$$\alpha_0(\xi_2 + \xi_0) = \alpha_1(\xi_2 + \xi_1).$$

Similarly,

$$\begin{cases} \alpha_1(\xi_0 + \xi_1) = \alpha_2(\xi_0 + \xi_2), \\ \alpha_2(\xi_1 + \xi_2) = \alpha_0(\xi_1 + \xi_0). \end{cases}$$

Also,

$$\alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 = 0.$$

We may put $\alpha_0 = \varepsilon$. Consequently,

$$\begin{cases} \gamma_1 = \alpha_1, \\ \alpha_1\gamma_2 = \alpha_2\gamma_1, \\ \gamma_2 = \alpha_2, \\ (\gamma_1 + \gamma_2 - \varepsilon) + \alpha_1(\varepsilon + \gamma_2 - \gamma_1) + \alpha_2(\varepsilon + \gamma_1 - \gamma_2) = 0 \end{cases}$$

It follows that

$$\begin{aligned} &\gamma_1\gamma_2 = \gamma_2\gamma_1, \text{ and} \\ &(\varepsilon - \gamma_1 - \gamma_2)^2 = 4\gamma_1\gamma_2. \qquad \square \end{aligned}$$

In Lemma 6.2 and Lemma 6.3 we assume that q is odd and that $\{\pi_i, \pi_j, \pi_k\}$ and $\{\tau_i, \tau_j, \tau_k\}$ are in perspective for all $i \neq j \neq k \neq i$ and $i, j, k \in \{0, 1, 2, 3\}$.

Lemma 6.2 *Assume γ_1 has n distinct eigenvalues which form an orbit of the Galois group of* $\mathrm{GF}(q^n)$ *with respect to* $\mathrm{GF}(q)$*. Then the spaces $\pi_0, \pi_1, \pi_2, \pi_3$ are elements of a common pseudo-conic O of* $\mathrm{PG}(3n-1, q)$*, such that the tangent spaces of O at $\pi_0, \pi_1, \pi_2, \pi_3$ respectively are $\tau_0, \tau_1, \tau_2, \tau_3$.*

Proof. Coordinates are chosen in such a way that $\pi_0(0, \varepsilon, \varepsilon), \pi_1(\varepsilon, 0, \varepsilon)$,
$\pi_2(\varepsilon, \varepsilon, 0)$ and $\zeta_1 \cap \zeta_2 \cap \zeta_3 = \zeta(\varepsilon, \varepsilon, \varepsilon)$. Then $\pi_3(\varepsilon, \gamma_1, \gamma_2)$. Further,
$\tau_0(\varepsilon, 0, 0), \tau_1(0, \varepsilon, 0), \tau_2(0, 0, \varepsilon)$ and $\tau_3(\gamma_1 + \gamma_2 - \varepsilon, \varepsilon - \gamma_1 + \gamma_2, \varepsilon + \gamma_1 - \gamma_2)$.

Now we determine the intersections of π_3 with the Segre variety $\mathcal{S}_{n-1;2}$ defined earlier in this section. Let

$$\gamma_1 = [a_{ij}]_{0\le i,j\le n-1}, \gamma_2 = [b_{ij}]_{0\le i,j\le n-1}.$$

Then $\mathcal{S}_{n-1;2} \cap \pi_3$ is determined by

$$\begin{cases} x_0 y_0 = r_0, \\ x_0 y_1 = r_1, \\ \vdots \\ x_0 y_{n-1} = r_{n-1}, \\ x_1 y_0 = r_0 a_{00} + \cdots + r_{n-1} a_{0,n-1}, \\ x_1 y_1 = r_0 a_{10} + \cdots + r_{n-1} a_{1,n-1}, \\ \vdots \\ x_1 y_{n-1} = r_0 a_{n-1,0} + \cdots + r_{n-1} a_{n-1,n-1}, \\ \vdots \\ x_2 y_0 = r_0 b_{00} + \cdots + r_{n-1} b_{0,n-1}, \\ \vdots \\ x_2 y_{n-1} = r_0 b_{n-1,0} + \cdots + r_{n-1} b_{n-1,n-1}. \end{cases}$$

Put $x_0 = 1$. Then we first solve the following equation:

$$\begin{bmatrix} a_{00}-x_1 & a_{01} & \cdots & a_{0,n-1} \\ a_{10} & a_{11}-x_1 & \cdots & a_{1,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1}-x_1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} = 0. \quad (10)$$

This yields n solutions for $(x_1, r_0, r_1, \cdots, r_{n-1})$, up to a factor of proportion for $(r_0, r_1, \cdots, r_{n-1})$. From Lemma 6.1 it follows that the solution spaces for $(r_0, r_1, \cdots, r_{n-1})$ of (10) are also solution spaces for $(r_0, r_1, \cdots, r_{n-1})$ of

$$\begin{bmatrix} b_{00}-x_2 & b_{01} & \cdots & b_{0,n-1} \\ b_{10} & b_{11}-x_2 & \cdots & b_{1,n-1} \\ \vdots & \vdots & & \vdots \\ b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-1}-x_2 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} = 0. \quad (11)$$

Coordinates of n linearly independent hyperplanes containing τ_3 are determined by the n rows of the matrix

$$[\gamma_1 + \gamma_2 - \varepsilon \; \varepsilon - \gamma_1 + \gamma_2 \; \varepsilon + \gamma_1 - \gamma_2].$$

The intersection of these hyperplanes with $\mathcal{S}_{n-1;2}$ are determined by

$$[\gamma_1 + \gamma_2 - \varepsilon \; \varepsilon - \gamma_1 + \gamma_2 \; \varepsilon + \gamma_1 - \gamma_2] \begin{bmatrix} x_0 y_0 \\ x_0 y_1 \\ \vdots \\ x_2 y_{n-1} \end{bmatrix} = 0. \quad (12)$$

Assume that $(x'_1, r_0, r_1, \cdots, r_{n-1})$ and $(x'_2, r_0, r_1, \cdots, r_{n-1})$ are solutions (with $(r_0, r_1, \cdots, r_{n-1}) \neq (0, 0, \cdots, 0)$) of (10) and (11).
Equation (12) is equivalent to

$$[x_0(\gamma_1 + \gamma_2 - \varepsilon) + x_1(\varepsilon - \gamma_1 + \gamma_2) + x_2(\varepsilon + \gamma_1 - \gamma_2)] \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = 0.$$

Now we consider equation

$$[x_0(\gamma_1 + \gamma_2 - \varepsilon) + x_1(\varepsilon - \gamma_1 + \gamma_2) + x_2(\varepsilon + \gamma_1 - \gamma_2)] \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} = 0.$$

This equation becomes

$$(x_0(x'_1 + x'_2 - 1) + x_1(1 - x'_1 + x'_2) + x_2(1 + x'_1 - x'_2)) \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} = 0,$$

hence

$$x_0(x'_1 + x'_2 - 1) + x_1(1 - x'_1 + x'_2) + x_2(1 + x'_1 - x'_2) = 0. \quad (13)$$

With the n solutions of (10) and the corresponding solutions of (11) there correspond n planes $\beta_1, \beta_2, \cdots, \beta_n$ of the extension of $\mathcal{S}_{n-1;2}$ to GF(q^n), which are conjugate with respect to that extension and generate PG$(3n-1, q^n)$. From (13) follows that the extension of τ_3 to GF(q^n) intersects each of the planes $\beta_1, \beta_2, \cdots, \beta_n$ in a line. The extension of π_i to GF(q^n) is also denoted by π_i, and the extension of τ_i to GF(q^n) is also denoted by τ_i, with $i = 0, 1, 2, 3$. Let $\pi_i \cap \beta_j = x_{ij}$ and $\beta_j \cap \tau_i = L_{ij}$, $i = 0, 1, 2, 3$ and $j = 1, 2, \cdots, n$. Then $\{x_{ij}, x_{kj}, x_{lj}\}$ and $\{L_{ij}, L_{kj}, L_{lj}\}$, with $j \in \{1, 2, \cdots, n\}$ and i, l, k distinct elements of $\{0, 1, 2, 3\}$, are in perspective. From the proof of Theorem 2.2 follows that there is a nonsingular conic C_j in β_j, which contains $x_{0j}, x_{1j}, x_{2j}, x_{3j}$ and for which L_{ij} is the tangent line at x_{ij}, with $i = 0, 1, 2, 3$. Hence $\pi_0, \pi_1, \pi_2, \pi_3$ are elements of a pseudo-conic O of PG$(3n-1, q)$ such that the tangent spaces of O at $\pi_0, \pi_1, \pi_2, \pi_3$ respectively are $\tau_0, \tau_1, \tau_2, \tau_3$. □

Lemma 6.3 *Assume that $s_1, s_2, \cdots, s_h$ are h distinct simple eigenvalues of γ_1 which form an orbit of the Galois group of* GF(q^h) *with respect to* GF(q)*. Then there are h distinct planes $\beta_1, \beta_2, \cdots, \beta_h$ over* GF(q^h) *which generate a* $(3h-1)$*-dimensional subspace of* PG$(3n-1, q)$ *and with each plane β_i containing a nonsingular conic C_i such that $C_i \cap \pi_j$ is a point p_{ij} and $C_i \cap \tau_j$ is the tangent line of C_i at p_{ij}, with $i = 1, 2, \cdots, h$.*

Proof. This is similar to the proof of Lemma 6.2. □

Theorem 6.4 *Assume that $O = \{\pi_0, \pi_1, \cdots, \pi_{q^n}\}$ is a pseudo-oval of* PG$(3n-1, q)$ *, q odd, and let τ_i be the tangent space of O at π_i, with $i = 0, 1, \cdots, q^n$. If for any three distinct i, j, k with $i, j, k \in \{0, 1, \cdots, q^n\}$ the triples $\{\pi_i, \pi_j, \pi_k\}$ and $\{\tau_i, \tau_j, \tau_k\}$ are in perspective, then O is a pseudo-conic. The converse also holds.*

Proof. Consider $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ and choose coordinates as in the proof of Lemma 6.2. Hence $\pi_0 = (0, \varepsilon, \varepsilon), \pi_1 = (\varepsilon, 0, \varepsilon), \pi_2 = (\varepsilon, \varepsilon, 0), \tau_0 = (\varepsilon, 0, 0), \tau_1 = (0, \varepsilon, 0), \tau_2 = (0, 0, \varepsilon), \pi_3 = (\varepsilon, \gamma_1, \gamma_2), \pi_4 = (\varepsilon, \delta_1, \delta_2)$.
Then

$$\begin{aligned}
&\gamma_1\gamma_2 = \gamma_2\gamma_1,\\
&(\varepsilon - \gamma_1 - \gamma_2)^2 = 4\gamma_1\gamma_2,\\
&\delta_1\delta_2 = \delta_2\delta_1,\\
&(\varepsilon - \delta_1 - \delta_2)^2 = 4\delta_1\delta_2.
\end{aligned}$$

Now we express that $\{\pi_0, \pi_3, \pi_4\}$ and $\{\tau_0, \tau_3, \tau_4\}$ are in perspective. By the proof of Lemma 6.2 we have

$$\tau_3(\gamma_1+\gamma_2-\varepsilon, \varepsilon+\gamma_2-\gamma_1, \varepsilon+\gamma_1-\gamma_2),$$
$$\tau_4(\delta_1+\delta_2-\varepsilon, \varepsilon+\delta_2-\delta_1, \varepsilon+\delta_1-\delta_2).$$

Also,

$$\tau_0 \cap \tau_3 = (0, \varepsilon+\gamma_1-\gamma_2, \gamma_1-\gamma_2-\varepsilon).$$

Let

$$\alpha_0\chi_0 + \alpha_1\chi_1 + \alpha_2\chi_2 = 0$$

be the 3-dimensional space ψ_4 joining $\tau_0 \cap \tau_3$ to π_4. Then we may put

$$\begin{aligned}
\alpha_1 &= \varepsilon - \gamma_1 + \gamma_2 \\
\alpha_2 &= \varepsilon + \gamma_1 - \gamma_2 \\
\alpha_0 &= (\gamma_1 - \gamma_2 - \varepsilon)\delta_1 + (\gamma_2 - \gamma_1 - \varepsilon)\delta_2.
\end{aligned}$$

Similarly, if

$$\beta_0\chi_0 + \beta_1\chi_1 + \beta_2\chi_2 = 0$$

is the 3-dimensional space ψ_3 joining $\tau_0 \cap \tau_4$ to π_3,then

$$\begin{aligned}
\beta_1 &= \varepsilon - \delta_1 + \delta_2 \\
\beta_2 &= \varepsilon + \delta_1 - \delta_2 \\
\beta_0 &= (\delta_1 - \delta_2 - \varepsilon)\gamma_1 + (\delta_2 - \delta_1 - \varepsilon)\gamma_2.
\end{aligned}$$

The space joining $\psi_3 \cap \psi_4$ and $\pi_0(0, \varepsilon, \varepsilon)$ is represented by

$$\alpha\chi_0 + \chi_1 - \chi_2 = 0$$

for some α. As $\alpha_1 - \beta_1 = \beta_2 - \alpha_2$, we have

$$\alpha = (\alpha_1 - \beta_1)^{-1}(\alpha_0 - \beta_0),$$

hence

$$\begin{aligned}
\alpha = (\gamma_2 - \gamma_1 + \delta_1 - \delta_2)^{-1}(&\gamma_1\delta_1 + \gamma_2\delta_2 - \delta_1\gamma_1 - \delta_2\gamma_2 + \delta_2\gamma_1 \\
&+\delta_1\gamma_2 - \gamma_2\delta_1 - \gamma_1\delta_2 - \delta_1 - \delta_2 + \gamma_1 + \gamma_2).
\end{aligned}$$

The space joining $\tau_3 \cap \tau_4$ and $\pi_0(0, \varepsilon, \varepsilon)$ is represented by

$$\beta\chi_0 + \chi_1 - \chi_2 = 0$$

for some β. Substracting corresponding coordinates of τ_3 and τ_4, we see that

$$\beta = (\gamma_2 - \gamma_1 + \delta_1 - \delta_2)^{-1}(\gamma_1 + \gamma_2 - \delta_1 - \delta_2).$$

As $\{\pi_0, \pi_3, \pi_4\}$ and $\{\tau_0, \tau_3, \tau_4\}$ are in perspective, we have $\alpha = \beta$ and so

$$\gamma_1\delta_1+\gamma_2\delta_2-\delta_1\gamma_1-\delta_2\gamma_2+\delta_2\gamma_1+\delta_1\gamma_2-\gamma_2\delta_1-\gamma_1\delta_2-\delta_1-\delta_2+\gamma_1+\gamma_2 = \gamma_1+\gamma_2-\delta_1-\delta_2.$$

It follows that

$$(\gamma_1 - \gamma_2)(\delta_1 - \delta_2) = (\delta_1 - \delta_2)(\gamma_1 - \gamma_2).$$

Let $\mathcal{R} = (r_0, r_1, \cdots, r_{n-1})$ be an eigenvector of γ_1 and γ_2 with corresponding eigenvalues x_1 and x_2. Clearly $\mathcal{R}$ is an eigenvector of $\gamma_1 - \gamma_2$ with eigenvector $x_1 - x_2$. Hence

$$(\gamma_1 - \gamma_2)(\delta_1 - \delta_2)\mathcal{R} = (\delta_1 - \delta_2)(\gamma_1 - \gamma_2)\mathcal{R} = (x_1 - x_2)(\delta_1 - \delta_2)\mathcal{R},$$

and so $(\delta_1 - \delta_2)\mathcal{R} = \mathcal{R}'$ is an eigenvector of $\gamma_1 - \gamma_2$ with eigenvalue $x_1 - x_2$. In the eigenspace of $\gamma_1 - \gamma_2$ which corresponds with $x_1 - x_2$, there is at least one eigenvector $\mathcal{R}_0 \neq 0$, with coordinates in GF(q^n), for which $(\delta_1 - \delta_2)\mathcal{R}_0 = l\mathcal{R}_0$ for some l in GF(q^n). By Lemma 6.1 we have $(\delta_1 - \delta_2)^2 = 2(\delta_1 + \delta_2) - \varepsilon$, and so $\mathcal{R}_0$ is an eigenvector of $\delta_1 + \delta_2$. It follows that $\mathcal{R}_0$ is an eigenvector of δ_1 and of δ_2. Also, $\mathcal{R}_0$ is an eigenvector of γ_1 and γ_2. It is clear that the eigenvector $\mathcal{R}_0$ of γ_1 corresponds to the eigenvalue x_1. Hence, by the proof of Lemma 6.2, for each eigenvalue of γ_1 there is a plane β of PG$(3n-1, q^n)$ which intersects each of $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ in a point and each of $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4$ in a line. Now we consider all eigenvectors $\mathcal{R}_0$ with eigenvalue x_1 of γ_1, eigenvalue x_2 of γ_2, which are also eigenvectors of δ_1. Then a similar reasoning yields that for $\pi_5 \in O - \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ with corresponding tangent space τ_5, there is a plane of PG$(3n-1, q^n)$ which intersects each of $\pi_0, \pi_1, \cdots, \pi_5$ in a point and each of $\tau_0, \tau_1, \cdots, \tau_5$ in a line. Proceeding like this we obtain for each eigenvalue of γ_1 a plane β of PG$(3n-1, q^n)$ which intersects each element of O in a point and each tangent space of O in a line.

Let β be such a plane corresponding to the eigenvalue x_1 of γ_1. Put $\beta \cap \pi_i = \{y_i\}$, with $i = 0, 1, \cdots, q^n$. Assume y_i is defined over GF(q^{h_i}) with $h_i \leq n$ minimal, $i = 0, 1, \cdots, q^n$. If G is the Galois group defined by the extension GF(q^n) of GF(q), then let us consider the planes β^θ with $\theta \in G$. If not all h_i are equal, then there are distinct planes β^θ and $\beta^{\theta'}$, with $\theta, \theta' \in G$, which intersect. This contradicts the fact that β^θ and $\beta^{\theta'}$ are generators of the Segre-variety $\mathcal{S}_{n-1;2}$ defined by π_0, π_1, π_2; here the extension of $\mathcal{S}_{n-1;2}$ to $\mathrm{GF}(q^n)$ is also denoted by $\mathcal{S}_{n-1;2}$. Hence $h_0 = h_1 = \cdots = h_{q^n} = h$. Consequently $y_0, y_1, \cdots, y_{q^n}$ belong to $\mathrm{PG}(2, q^h)$. As $\{y_0, y_1, \cdots y_{q^n}\} = C$ is a conic, we necessarily have $h = n$. Now it easily follows that each point of β is defined over $\mathrm{GF}(q^n)$, but over no smaller field.

If for some $i \in \{0, 1, \cdots, q^n\}$ the n points y_i^θ, with $\theta \in G$, are linearly independent, then, as the n planes β^θ are generators of $\mathcal{S}_{n-1;2}$, the n points y_j^θ are linearly independent for all $j = 0, 1, \cdots, q^n$.

Assume, by way of contradiction, that the planes β and β', with $\beta \neq \beta'$, correspond to the same eigenvalue x_1 of γ_1. If $\beta' = \beta^\theta$, with $\theta \in G$, then $y_i \mapsto y_i^\theta$, with $i = 0, 1, \cdots, q^n$, is induced by a linear projectivity of β onto β^θ. This yields a contradiction. Hence γ_1 has n distinct eigenvalues which form an orbit of the Galois group G. It follows that the points y_3^θ, $\theta \in G$, of π_3 are linearly independent. Hence the planes β^θ, $\theta \in G$, generate the space $\mathrm{PG}(3n-1, q^n)$.

We conclude that O is a pseudo-conic of $\mathrm{PG}(3n-1, q)$. □

We will now give the formulation of Theorem 6.4 in terms of generalized quadrangles and Laguerre planes. Generalized quadrangles were introduced by Tits [1959] in his celebrated paper on triality, and in Payne and Thas [1984] it is shown that generalized quadrangles of odd order s, $s \neq 1$, with an antiregular point are equivalent to Laguerre planes of odd order s.

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a generalized quadrangle of order s, with $s \neq 1$. Then $\mathcal{S}$ is called a *translation generalized quadrangle* with *center* or *base point* x if $\mathcal{S}$ admits an abelian group of automorphisms fixing every line incident with x and acting regularly on the points of $\mathcal{S}$ not collinear with x. In Payne and Thas [1984] it is shown that the theory of translation generalized quadrangles is equivalent to the theory of generalized ovals; see also Thas, K. Thas and Van Maldeghem [2006]

Theorem 6.5 *Let $\mathcal{S} = (P, B, \mathrm{I})$ be a translation generalized quadrangle of order s, s odd and $s \neq 1$, with base point x. Further let L_1, L_2, L_3 be distinct lines incident with x, and let $y_i \mathrm{I} L_i$, $x \neq y_i$, with $i = 1, 2, 3$. Then $\mathcal{S}$ is isomorphic to* *the generalized quadrangle* $Q(4, s)$ *arising from a nonsingular quadric of* $\mathrm{PG}(4, s)$ *if and only if there are* s *points* $z_1, z_2, \cdots, z_s$ *not collinear with* x *such that the line* M_{ij} *incident with* z_i *and concurrent with* L_j *contains a point collinear with* y_k *and* y_l*, with* $\{j, k, l\} = \{1, 2, 3\}$ *and* $i = 1, 2, \cdots, s$ *(one of the points* $z_1, z_2, \cdots, z_s$ *is collinear with* y_1, y_2, y_3*).*

Next, let $\mathcal{L} = (P, B_1 \cup B_2, \mathrm{I})$ be a Laguerre plane of odd order s; here P is the point set, B_1 the line set, B_2 the circle set, and I the (symmetric) incidence relation. For more details on Laguerre planes we refer to Thas, K. Thas and Van Maldeghem [2006]. We say that $\mathcal{L}$ is a *translation Laguerre plane* if $\mathcal{L}$ admits and abelian automorphism group fixing each element of B_1 and acting regularly on B_2.

Theorem 6.6 *Let* $\mathcal{L} = (P, B_1 \cup B_2, \mathrm{I})$ *be a translation Laguerre plane of odd order* s*, let* L_1, L_2, L_3 *be distinct lines in* B_1 *and let* $y_i \mathrm{I} L_i$*, with* $i = 1, 2, 3$*. Then* $\mathcal{L}$ *is isomorphic to the classical Laguerre plane arising from a quadratic cone of* $\mathrm{PG}(3, s)$ *if and only if there are* s *circles* $C_1, C_2, \cdots, C_s$ *in* B_2 *such that* C_i *is tangent at the common point* m_{ij} *of* C_i *and* L_j *to the circle containing* m_{ij}*,* y_k *and* y_l*, with* $\{j, k, l\} = \{1, 2, 3\}$ *and* $i = 1, 2, \cdots, s$ *(one of these circles is the circle containing* y_1, y_2, y_3*).*

7 A Second Property on Quadrics

In this section we generalize a well-known property on conics.

Theorem 7.1 *Let* π_0, π_1, π_2 *be mutually skew* $(n-1)$*-dimensional subspaces of* $\mathrm{PG}(3n-1, q)$*, and let* τ_i *be a* $(2n-1)$*-dimensional space containing* π_i *but skew to* π_j *and* π_k*, with* $\{i, j, k\} = \{0, 1, 2\}$*. Coordinates are chosen in such a way that* $\pi_0(\varepsilon, 0, 0), \pi_1(0, \varepsilon, 0), \pi_2(0, 0, \varepsilon)$*. Then* $\tau_0(0, \varepsilon, \alpha), \tau_1(\beta, 0, \varepsilon), \tau_2(\varepsilon, \gamma, 0)$ *with* $det(\alpha\beta\gamma) \neq 0$*. Then there is a quadric containing* π_0, π_1, π_2 *and having* τ_i *as tangent space at* π_i*, with* $i = 0, 1, 2$*, if and only if the matrix equation*

$$Z\theta = Z^T, \text{ with } \theta = \alpha\beta\gamma,$$

has a nonsingular solution for the $n \times n$*-matrix* Z*. Moreover, for* q *even with* n *even and for* q *odd, the quadric is nonsingular if and only if* $\theta + \varepsilon$ *is nonsingular, that is, if and only if* $\tau_0 \cap \tau_1 \cap \tau_2 = \emptyset$*.*

Proof. Let Q be any quadric containing π_0, π_1, π_2. Then $Q : \sum_{\substack{i,j=0\\ i\langle j}}^{3n-1} a_{ij}X_iX_j = 0$, where i, j do not both belong to either $\{0, 1, \cdots, n-1\}$, or $\{n, n+1, \cdots, 2n-1\}$, or $\{2n, 2n+1, \cdots, 3n-1\}$.
For $i > j$ we define a_{ij} to be a_{ji}.

Let

$$U = \begin{bmatrix} a_{0n} & a_{0,n+1} & \cdots a_{0,2n-1} \\ a_{1n} & a_{1,n+1} & \cdots a_{1,2n-1} \\ \vdots & \vdots & \vdots \\ a_{n-1,n} & a_{n-1,n+1} & \cdots a_{n-1,2n-1} \end{bmatrix},$$

$$V = \begin{bmatrix} a_{n,2n} & a_{n,2n+1} & \cdots a_{n,3n-1} \\ a_{n+1,2n} & a_{n+1,2n+1} & \cdots a_{n+1,3n-1} \\ \vdots & \vdots & \vdots \\ a_{2n-1,2n} & a_{2n-1,2n+1} & \cdots a_{2n-1,3n-1} \end{bmatrix},$$

$$W = \begin{bmatrix} a_{2n,0} & a_{2n,1} & \cdots a_{2n,n-1} \\ a_{2n+1,0} & a_{2n+1,1} & \cdots a_{2n+1,n-1} \\ \vdots & \vdots & \vdots \\ a_{3n-1,0} & a_{3n-1,1} & \cdots a_{3n-1,n-1} \end{bmatrix}.$$

Then

$$\tau_0 : U \begin{bmatrix} X_n \\ \vdots \\ X_{2n-1} \end{bmatrix} + W^T \begin{bmatrix} X_{2n} \\ \vdots \\ X_{3n-1} \end{bmatrix} = 0,$$

$$\tau_1 : U^T \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix} + V \begin{bmatrix} X_{2n} \\ \vdots \\ X_{3n-1} \end{bmatrix} = 0,$$

$$\tau_2 : W \begin{bmatrix} X_0 \\ \vdots \\ X_{n-1} \end{bmatrix} + V^T \begin{bmatrix} X_n \\ \vdots \\ X_{2n-1} \end{bmatrix} = 0.$$

Now we express that $\tau_0(0, \varepsilon, \alpha), \tau_1(\beta, 0, \varepsilon), \tau_2(\varepsilon, \gamma, 0)$, that is,

$$\begin{aligned}\alpha &= U^{-1}W^T\\ \beta &= V^{-1}U^T\\ \gamma &= W^{-1}V^T.\end{aligned}$$

This yields a system of $3n^2$ linear homogeneous equations with $3n^2$ unknowns.

If such a quadric Q exists, then

$$\alpha\beta\gamma = U^{-1}W^TV^{-1}U^TW^{-1}V^T,$$

so

$$Z\theta = Z^T, \quad (14)$$

with

$$\theta = \alpha\beta\gamma, Z = V(W^T)^{-1}U \text{ and } Z \text{ nonsingular.}$$

Conversely, assume that (14), with $\theta = \alpha\beta\gamma$, has a nonsingular solution Z. Then we put

$$\begin{aligned}V &= Z\alpha,\\ U &= (\gamma^T)^{-1}Z,\\ W &= \alpha^T Z^T \gamma^{-1}.\end{aligned}$$

Now assume that either q is even with n even or that q is odd. Then the quadric is nonsingular if and only if the matrix

$$\begin{bmatrix} 0 & U & W^T\\ U^T & 0 & V\\ W & V^T & 0\end{bmatrix}$$

is nonsingular, that is, if and only if $Z^T + Z$ is nonsingular, that is, if and only if $\theta + \varepsilon$ is nonsingular. It is easy to check that this condition is equivalent to $\tau_0 \cap \tau_1 \cap \tau_2 = \emptyset$. □

Corollary 7.2 (i) *We adopt the notations of Theorem 7.1. There is a quadric containing π_0, π_1, π_2 and having τ_i as tangent space at π_i, with $i = 0, 1, 2$, if and only if the following three conditions hold*

(a) *the matrix θ is similar to its inverse,*

(b) *the rank of $(\theta - \varepsilon)^{2j+1}$ has the same parity for every nonnegative j,*

(c) *for q odd, the rank of $(\theta+\varepsilon)^{2j}$ has the same parity for every nonnegative j;*

(ii) *For $n = 2$ such a quadric exists if and only if det$(\theta) = 1$ with either Tr$(\theta) \neq 2$ or $\theta = \varepsilon$, that is, with either det$(\theta - \varepsilon) \neq 0$ or $\theta = \varepsilon$.*

Proof. (i) For the proof of (i) we refer to Theorem 3.8 of Ballantine [1978/79].

(ii) For $n = 2$ the equation $Z\theta = Z^T$ has nonzero solution for Z if and only if either $\det(\theta) = 1$ or $\det(\theta - \varepsilon) = 0$. If $\theta = \varepsilon$, then the quadric exists. Assume now $\theta \neq \varepsilon$. If $\det(\theta - \varepsilon) = 0$, then one checks that $\det(Z) = 0$ for each solution of $Z\theta = Z^T$. If $\det(\theta) = 1$, then $Z\theta = Z^T$ has a solution with $\det(Z) \neq 0$ if and only if $\mathrm{Tr}(\theta) \neq 2$. □

Remark 7.3 *The condition $\theta \in \{\varepsilon, -\varepsilon\}$ is equivalent for $\{\pi_0, \pi_1, \pi_2\}$ and $\{\tau_0, \tau_1, \tau_2\}$ to be in perspective ($\theta = -\varepsilon$ is equivalent with $\tau_0 \cap \tau_1 \cap \tau_2$ being $(n-1)$-dimensional); det$(\theta-\varepsilon)$det$(\theta+\varepsilon) = 0$ is equivalent for $\{\pi_0, \pi_1, \pi_2\}$ and $\{\tau_0, \tau_1, \tau_2\}$ to be in semi-perspective.*

8 A "Lemma of Tangents" for Generalized Ovals in PG$(5, q)$

In this section we will prove a "Lemma of Tangents" for generalized ovals O of PG$(5, q)$, with q odd.

Lemma 8.1 (Lemma of Tangents). *Let $O = \{L_0, L_1, \cdots, L_{q^2}\}$ be a generalized oval in* PG$(5, q)$*, q odd, and let τ_i be the tangent space of O at $L_i, i = 0, 1, \cdots, q^2$. If $L_0(\varepsilon, 0, 0), L_1(0, \varepsilon, 0), L_2(0, 0, \varepsilon), \tau_0(0, \varepsilon, -\alpha)$, $\tau_1(-\beta, 0, \varepsilon)$, $\tau_2(\varepsilon, -\gamma, 0)$, then*

$$\mathit{det}(\alpha\beta\gamma) = 1.$$

Proof. Let $L_i(\xi_0^i, \xi_1^i, \xi_2^i)$, with $i = 3, 4, \cdots, q^2$. Remark that $\det(\xi_0^i \xi_1^i \xi_2^i) \neq 0$. For $i \in \{3, 4, \cdots, q^2\}$ the space $\langle L_0, L_i\rangle$ has equation

$$\chi_1 = \xi_1^i (\xi_2^i)^{-1} \chi_2,$$

the space $\langle L_1, L_i\rangle$ has equation

$$\chi_2 = \xi_2^i(\xi_0^i)^{-1}\chi_0,$$

and the space $\langle L_2, L_i\rangle$ has equation

$$\chi_0 = \xi_0^i(\xi_1^i)^{-1}\chi_1.$$

If

$$\xi_1^i(\xi_2^i)^{-1} = \delta_0^i, \xi_2^i(\xi_0^i)^{-1} = \delta_1^i, \xi_0^i(\xi_1^i)^{-1} = \delta_2^i,$$

then for $i = 3, 4, \cdots, q^2$ we have

$$\delta_0^i\delta_1^i\delta_2^i = \varepsilon.$$

Then

$$\begin{aligned}\langle L_0, L_i\rangle \cap \langle L_1, L_2\rangle &= L_i^0(0, \delta_0^i, \varepsilon),\\ \langle L_1, L_i\rangle \cap \langle L_2, L_0\rangle &= L_i^1(\varepsilon, 0, \delta_1^i),\\ \langle L_2, L_i\rangle \cap \langle L_0, L_1\rangle &= L_i^2(\delta_2^i, \varepsilon, 0),\end{aligned}$$

with $i = 3, 4, \cdots, q^2$.

If $\tau_i \cap \langle L_j, L_k\rangle = T_i'$, with $\{i, j, k\} = \{0, 1, 2\}$, then

$$\begin{aligned}&\{T_0', L_1, L_2, L_3^0, L_4^0, \cdots, L_{q^2}^0\} \text{ is a linespread } S_0 \text{ of } \langle L_1, L_2\rangle,\\ &\{T_1', L_2, L_0, L_3^1, L_4^1, \cdots, L_{q^2}^1\} \text{ is a linespread } S_1 \text{ of } \langle L_2, L_0\rangle, \text{ and}\\ &\{T_2', L_0, L_1, L_3^2, L_4^2, \cdots, L_{q^2}^2\} \text{ is a linespread } S_2 \text{ of } \langle L_0, L_1\rangle.\end{aligned}$$

Replacing each space of the form

$$\chi_1 = \rho\chi_2$$

by its "coordinate" ρ, and defining the coordinate of $\chi_2 = 0$ to be ∞, we obtain the set

$$\{\alpha, 0, \infty, \delta_0^3, \delta_0^4, \cdots, \delta_0^{q^2}\}.$$

Similarly, we obtain

$$\begin{aligned}&\{\beta, 0, \infty, \delta_1^3, \delta_1^4, \cdots, \delta_1^{q^2}\}, \text{and}\\ &\{\gamma, 0, \infty, \delta_2^3, \delta_2^4, \cdots, \delta_2^{q^2}\}.\end{aligned}$$

Clearly we have

$$\det(\alpha\delta_0^3\cdots\delta_0^{q^2}\beta\delta_1^3\cdots\delta_1^{q^2}\gamma\delta_2^3\cdots\delta_2^{q^2}) = \det(\alpha\beta\gamma). \quad (15)$$

Now we calculate

$$\det(\gamma\delta_2^3\delta_2^4\cdots\delta_2^{q^2}).$$

In the space $\langle L_0, L_1\rangle$, that is, $\chi_2 = 0$, we consider the plane $\langle L_i^2, e_3\rangle = \varphi_i^2$, with $e_3(0,0,0,1,0,0)$. The plane φ_i^2 has equations

$$X_4 = X_5 = 0, \text{ and}$$

$$\begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ 0 & 0 & 0 & 1 \\ \multicolumn{2}{c}{(\delta_2^i)^T} & & \varepsilon \end{vmatrix} = 0.$$

This last equation is equivalent to

$$-d_2^i X_0 + c_2^i X_1 + \det(\delta_2^i)X_2 = 0,$$

where

$$\delta_2^i = \begin{bmatrix} a_2^i & c_2^i \\ b_2^i & d_2^i \end{bmatrix}.$$

Similarly, the plane $\langle T_2', e_3\rangle = \varphi^2$ has equations

$$X_4 = X_5 = 0 \text{ and } -d_2X_0 + c_2X_1 + \det(\gamma)X_2 = 0,$$

where

$$\gamma = \begin{bmatrix} a_2 & c_2 \\ b_2 & d_2 \end{bmatrix}.$$

In this way there arise the q^2-1 planes $l_0X_0+l_1X_1+X_2 = 0$, with $(l_0, l_1) \neq (0,0)$, of $\langle L_0, L_1\rangle$ through e_3.

First assume that $d_2 \neq 0$. Then

$$\Pi^{q^2}_{\substack{i=3\\ d_2^i\neq 0}}(d_2^i)d_2 = -\Pi^{q^2}_{\substack{i=3\\ d_2^i\neq 0}}\det(\delta_2^i)\det(\gamma).$$

There are $q-1$ indices i for which $d_2^i = 0$. Now we intersect the lines L_2^i and T_2' with the plane $X_2 = 0$. We obtain the points $(c_2^i, d_2^i, 0, 1, 0, 0)$ and $(c_2, d_2, 0, 1, 0, 0)$. Hence

$$\Pi^{q^2}_{\substack{i=3\\ d_2^i\neq 0}}(d_2^i)d_2 = -1.$$

Consequently

$$\Pi^{q^2}_{\substack{i=3\\ d_2^i\neq 0}} \det(\delta_2^i)\det(\gamma)=1.$$

Now we have

$$\Pi^{q^2}_{\substack{i=3\\ d_2^i=0}} c_2^i = -\Pi^{q^2}_{\substack{i=3\\ d_2^i=0}} \det(\delta_2^i).$$

As

$$\Pi^{q^2}_{\substack{i=3\\ d_2^i=0}} c_2^i = -1,$$

we have

$$\Pi^{q^2}_{\substack{i=3\\ d_2^i=0}} \det(\delta_2^i)=1.$$

Hence

$$\Pi^{q^2}_{i=3}\det(\delta_2^i)\det(\gamma)=1.$$

Next, we assume that $d_2 = 0$. Then $c_2 \neq 0$. We have

$$\Pi^{q^2}_{\substack{i=3\\ c_2^i\neq 0}} (c_2^i)c_2 = -\Pi^{q^2}_{\substack{i=3\\ c_2^i\neq 0}} \det(\delta_2^i)\det(\gamma).$$

There are $q-1$ indices i for which $c_2^i = 0$. As

$$\Pi^{q^2}_{\substack{i=3\\ c_2^i\neq 0}} (c_2^i)c_2 = -1,$$

we have

$$\Pi^{q^2}_{\substack{i=3\\ c_2^i\neq 0}} \det(\delta_2^i)\det(\gamma)=1.$$

Now

$$\Pi^{q^2}_{\substack{i=3\\ c_2^i=0}} d_2^i = -\Pi^{q^2}_{\substack{i=3\\ c_2^i=0}} \det(\delta_2^i),$$

and as

$$\Pi^{q^2}_{\substack{i=3\\ c_2^i=0}} d_2^i = -1,$$

we have

$$\Pi^{q^2}_{\substack{i=3\\ c_2^i=0}} \det(\delta_2^i)=1.$$

Hence

$$\Pi^{q^2}_{i=3}\det(\delta_2^i)\det(\gamma)=1.$$

Similarly

$$\Pi_{i=3}^{q^2}\det(\delta_1^i)\det(\beta) = \Pi_{i=3}^{q^2}\det(\delta_0^i)\det(\alpha) = 1.$$

By (15) it then follows that

$$\det(\alpha\,\beta\,\gamma) = 1.$$

□

Remark 8.2 *Lemma 8.1 can be extended to all $n \geq 2$, but calculations become messy. Instead of considering the planes $\langle L_i^2, e_3\rangle$ one has to consider the hyperplanes of $\langle \pi_0, \pi_1\rangle$ generated by the spaces π_i^2 and $n-1$ points e_i belonging to $\pi_0 \cup \pi_1$ (here notations π_0, π_1, π_i^2 are used instead of L_0, L_1, L_i^2).*

Theorem 8.3 *Let $O = \{L_0, L_1, \cdots, L_{q^2}\}$ be a generalized oval in $\mathrm{PG}(5, q)$, with q odd, and let τ_i be the tangent space of O at L_i, with $i = 0, 1, \cdots, q^2$. Then for any three distinct i, j, k in $\{0, 1, \cdots, q^2\}$ there is either a nonsingular quadric containing L_i, L_j, L_k and having τ_i, τ_j, τ_k as tangent spaces at respectively L_i, L_j, L_k or $\{L_i, L_j, L_k\}$ and $\{\tau_i, \tau_j, \tau_k\}$ are in semi-perspective but not in perspective.*

Proof. With the notations of Lemma 8.1 we have $\det(\alpha\beta\gamma) = 1$. Assume there is no nonsingular quadric containing L_0, L_1, L_2 and having τ_0, τ_1, τ_2 as tangent spaces at respectively L_0, L_1, L_2. Put $\theta = -\alpha\beta\gamma$. By Corollary 7.2(ii) we have $\mathrm{Tr}(\theta) = 2$ with $\theta \neq \varepsilon$ (if $\theta = \varepsilon$, then $\{L_0, L_1, L_2\}$ and $\{\tau_0, \tau_1, \tau_2\}$ are in perspective, and hence there is such a quadric). Consequently $\det(\theta - \varepsilon) = 0$ with $\theta \neq \varepsilon$. By Remark 7.3 it follows that $\{L_0, L_1, L_2\}$ and $\{\tau_0, \tau_1, \tau_2\}$ are in semi-perspective but not in perspective. □

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