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Hua Structures and Proper Moufang Sets with Abelian Root Groups

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Dedicated to professor Jacques Tits on the occasion of his 80-th birthday.

Abstract: The purpose of this note is to make precise the connection between quadratic Jordan division algebras and proper Moufang sets with abelian root groups. This is done with the aid of Hua structures.
Keywords: Moufang set, quadratic Jordan division algebra

1. Introduction

In this note we discuss the connections between proper Moufang sets with abelian root groups and quadratic Jordan division algebras. It seems that the connections between these two notions were first realized by Loos in [L]. In a recent paper De Medts and Weiss [DW] gave a clear, direct and elegant way to construct a Moufang set from a QJDA. We also refer the reader to [MV, subsection 3.1] for a short discussion on this topic. To describe the connections between Moufang sets and QJDA we introduce structures which we call **Hua structures**. We note that several authors dealt with concepts which are similar, if not equivalent, to our notion of a Hua structure. In fact Faulkner in [F] even calls his structure a Hua structure (but we do not know the precise way it relates to our notion). Further Timmesfeld in [Ti2] and his student S. Weiss [Wei] dealt with structures which are very much related to our structures.

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2. Hua structures

Throughout this note if A is an abelian group, then A^* denote the nonzero elements of A. A Hua structure $(U, \mathcal{H})$ consists of an abelian group U with $|U| \geq 4$, together with a map $\mathcal{H}\colon U^* \to \mathrm{Aut}(U)$ denoted $\mathcal{H}\colon a \mapsto H_a$, such that the following axioms hold: for $x \in U^*$, set $x^{-1} := xH_x^{-1}$, then for all $x, y \in U^*$:

(H1) $H_{x^{-1}} = H_x^{-1}$.
(H2) $H_{-x} = H_x$.
(H3) $H_{xH_y} = H_y H_x H_y$.
(H4) $H_x H_{x^{-1}+y^{-1}} H_y = H_{x+y}$.
(H5) $x^{-1}H_{x,y} = y \cdot 2$, where $H_{x,y} := H_{x+y} - H_x - H_y \in \mathrm{End}(U)$.

In Axioms (H4) and (H5), $x \neq -y$.

Lemma 1. *Let $(U, \mathcal{H})$ be a Hua structure. Pick $e \in U^*$ and define $\mathcal{G}\colon U^* \to \mathrm{Aut}(U)$ by $a \mapsto G_a$, where $G_a = H_e^{-1}H_a$. Then $(U, \mathcal{G})$ is a Hua structure.*

Proof. Note that by (H1), $G_a = H_{e^{-1}}H_a$. For $x \in U^*$, let us denote $\dot{x} := xG_x^{-1}$, so that $\dot{x}$ is the "inverse" of x in $(U, \mathcal{G})$. Notice that

$$\dot{x} = xH_x^{-1}H_e = x^{-1}H_e. \tag{1}$$

(H1): We have

$$G_{\dot{x}} = H_e^{-1}H_{x^{-1}H_e} \overset{\text{(H3)}}{=} H_{x^{-1}}H_e \overset{\text{(H1)}}{=} H_x^{-1}H_e = G_x^{-1}.$$

(H2): $G_{-x} = H_e^{-1}H_{-x} \overset{\text{(H2)}}{=} H_e^{-1}H_x = G_x$.

(H3): We have

$$G_{xG_y} = H_e^{-1}H_{xH_{e^{-1}}H_y} \overset{\text{(H3)}}{=} H_{e^{-1}}H_yH_{e^{-1}}H_xH_{e^{-1}}H_y = G_yG_xG_y.$$

(H4): We have

$$\begin{aligned} G_xG_{\dot{x}+\dot{y}}G_y &\overset{(1)}{=} H_{e^{-1}}H_xH_{e^{-1}}H_{(x^{-1}+y^{-1})H_e}H_{e^{-1}}H_y \\ &\overset{\text{(H3)}}{=} H_{e^{-1}}H_xH_{x^{-1}+y^{-1}}H_y \overset{\text{(H4)}}{=} H_{e^{-1}}H_{x+y} = G_{x+y}. \end{aligned}$$

(H5): We have

$$\begin{aligned} \dot{x}G_{x,y} &= x^{-1}H_e(G_{x+y} - G_x - G_y) \\ = x^{-1}H_e(H_e^{-1}H_{x+y} - H_e^{-1}H_x - H_e^{-1}H_y) &= x^{-1}H_{x,y} = y \cdot 2. \end{aligned}$$ □

Definition 2. Let $(U, \mathcal{H})$ be a Hua structure. We say that $(U, \mathcal{H}) = (U, \mathcal{H}, e)$ is a **Hua structure with an identity** e**,** if $e \in U^*$ satisfies $H_e = \mathrm{id}_U$. Given $e \in U^*$ we let $(U, \mathcal{H}_e, e)$ be the Hua structure such that $\mathcal{H}_e = \mathcal{G}$, where $\mathcal{G}$ is as in Lemma 1. We call $(U, \mathcal{H}_e)$ an **isotope** of $(U, \mathcal{H})$. (The notion of isotope imitates the same notion in the theory of quadratic Jordan algebras, see [Mc2]).

Let us recall that a Moufang set is a doubly transitive permutation group such that the point stabilizer contains a normal subgroup which is regular on the remaining points. These regular normal subgroups are called the **root groups** and they are assumed to be conjugate and to generate the whole group. In [DW, DS1, DST, DS2] the notation $\mathbb{M}(U, \tau)$ is used for a Moufang set (and this notation is of course explained there). The group U in this notation is isomorphic to any one of the root groups of the Moufang set.

Given a Moufang set $\mathbb{M}(U, \tau)$, the **Hua map** h_a, $a \in U^*$ was defined in [DW, Definition 3.2, p. 422] (see also [DS1, Notation 3.2(3)]), and in [DW, Theorem 3.2, p. 423] it is shown that $\mathbb{M}(U, \tau)$ is a Moufang set iff $h_a \in \mathrm{Aut}(U)$, for all $a \in U^*$. Recall that a Moufang set $\mathbb{M}(U, \tau)$ is **proper** if its little projective group is not sharply 2-transitive. The main observation of this note is:

Theorem A. *Let U be an abelian group. Then*

(1) *Assume that $(U, \mathcal{H})$ is a Hua structure and let $\tau: U^* \to U^*$ be defined by $x\tau = -x^{-1}$. Then $\mathbb{M}(U, \tau)$ is a proper Moufang set. Conversely*
(2) *Assume $\mathbb{M}(U, \tau)$ is a proper Moufang set and that $\tau = \mu_e$, for some $e \in U^*$. Then $(U, \mathcal{H})$ is a Hua structure, where $\mathcal{H}: a \mapsto h_a$, for all $a \in U^*$, and where $h_a \in \mathrm{Aut}(U)$ is the Hua map corresponding to a.*

We now recall the definition of a quadratic Jordan division algebra.

3. Quadratic Jordan division algebras

Let k be an arbitrary commutative field, let J be a vector space over k of arbitrary dimension, and let $e \in J^*$ be a distinguished element. For each $x \in J$, let $W_x \in \mathrm{End}_k(J)$, and assume that the map $\mathcal{W} : J \to \mathrm{End}(J)$, defined by $x \mapsto W_x$, is quadratic, i.e.

$$W_{xt} = W_x t^2 \text{ for all } t \in k, \text{ and}$$
$$\text{the map } (x, y) \mapsto W_{x,y} \text{ is } k\text{-bilinear,}$$

(note that we multiply by scalars on the right) where

$$W_{x,y} := W_{x+y} - W_x - W_y$$

for all $x, y \in J$. Let

$$zV_{x,y} := yW_{x,z},$$

for all $x, y, z \in J$. Then the triple $(J, \mathcal{W}, e)$ is a **quadratic Jordan algebra** if the identities

(QJ1) $W_e = \mathrm{id}_J$;
(QJ2) $W_x V_{x,y} = V_{y,x} W_x$;
(QJ3) $W_{yW_x} = W_x W_y W_x$ ["the fundamental identity"]

hold *strictly*, i.e. if they continue to hold in all scalar extensions of J. (It suffices for them to hold in the polynomial extension $J_{k[t]}$ and this is automatically true if the base field k has at least 4 elements.)

We call the endomorphisms W_x the **structure maps** of $(J, \mathcal{W}, e)$, and we call $e \in J$ an **identity element**. An element $x \in J$ is called **invertible** if there exists $y \in J$ such that

$$yW_x = x \quad \text{and} \quad eW_y W_x = e.$$

In this case y is called the **inverse** of x and is denoted $y = x^{-1}$. By [Mc2, 6.1.2], an element $x \in J$ is invertible if and only if W_x is invertible; we then have $W_x^{-1} = W_{x^{-1}}$. In particular,

$$(x^{-1})^{-1} = x \quad \text{and} \quad x^{-1} = xW_x^{-1}.$$

If all elements in J^* are invertible, then $(J, \mathcal{W}, e)$ is called a quadratic Jordan **division** algebra.

Remark 3. The notion of a quadratic Jordan algebra (QJA) was introduced by McCrimmon in [Mc1]. We mention that our notation for QJA differ slightly from those in [Mc1] and [Mc2], e.g. we use W_x in place of the more common notation U_x, to avoid confusion with our notation for the root groups.

We also mention that when looking at results taken from [DW], the reader should be cautioned that in [DW] the notation U_x is used (and not W_x) and the maps in [DW] are applied on the left of the variable while we apply them on the right.

As we observe in Proposition 9 below (its proof comes from [DW, section 4]) if $(J, \mathcal{W}, e)$ is a quadratic Jordan division algebra, then it is also a Hua structure (with $U = (J, +)$ and $\mathcal{H} = \mathcal{W}$). The question arises as to whether the converse holds.

Questions. Let $(U, \mathcal{H}, e)$ be a Hua structure with an identity. As indicated in Proposition 10(1) below, U is a vector space over $\mathrm{GF}(p)$ (and we write $\mathrm{char}(U) = p$) or over $\mathbb{Q}$ (and we write $\mathrm{char}(U) = 0$).

(1) Assume $\mathrm{char}(U) \notin \{2, 3\}$. Does $(U, \mathcal{H}, e)$ satisfy the hypotheses of Theorem 11 (below), so that $(U, \mathcal{H}, e)$ is a quadratic Jordan division algebra?
(2) What happens when $\mathrm{char}(U) \in \{2, 3\}$? Is the map $(x, y) \mapsto H_{x,y}$ biadditive and is $(U, \mathcal{H}, e)$ a quadratic Jordan division algebra (possibly **without** the requirement that the identities (QJ1)–(QJ3) hold "strictly")?

We now proceed with the proof of Theorem A. Proposition 5 below proves part (1) of Theorem A and Proposition 7 proves part (2).

Lemma 4. *Let $(U, \mathcal{H})$ be a Hua structure. Then*

(1) $(x^{-1})^{-1} = x$, *for all $x \in U^*$, in particular, the map $x \to x^{-1}$ is a permutation of U^*.*
(2) $(-x)^{-1} = -(x^{-1})$, *for all $x \in U^*$.*
(3) *There exists $a, b \in U^*$ with $H_a \neq H_b$.*
(4) *Given* (H1) – (H3), *axiom* (H4) *implies that*

$$(x^{-1} - a)^{-1} H_{a - xH_a} = (x - aH_x)H_a.$$

for all $a, x \in U^*$, such that $x \neq a^{-1}$.

(5) *Given* (H1) – (H3), *axiom* (H5) *is equivalent to*

$$a^{-1}H_{a-xH_a} = a + aH_xH_a - xH_a \cdot 2,$$

for all $a, x \in U^*$, *such that* $x \neq a^{-1}$.

Proof. (1): By definition $(x^{-1})^{-1} = x^{-1}H_{x^{-1}}^{-1} \overset{(H1)}{=} x^{-1}H_x = x$.

(2): We have $(-x)^{-1} = (-x)H_{-x}^{-1} \overset{(H2)}{=} -(xH_x^{-1}) = -(x^{-1})$.

(3): Suppose that $h := H_x = H_y$, for all $x, y \in U^*$, then by (H5),

$$y \cdot 2 = x^{-1}H_{x,y} = -x^{-1}h = -x^{-1}H_x = -x,$$

for all $x, y \in U^*$ such that $x \neq -y$. It follows that $|U| \leq 3$, a contradiction.

(4): We have

$$\begin{aligned}
(x^{-1} - a)^{-1}H_{a-xH_a} &= (x - aH_x)H_a && \iff \\
(x^{-1} - a)^{-1}H_{a-xH_a} &= (x^{-1} - a)H_xH_a && \iff \\
(x^{-1} - a)^{-1}H_aH_a^{-1}H_{a-xH_a}H_a^{-1}H_{x^{-1}} &= x^{-1} - a && \iff \\
(x^{-1} - a)^{-1}H_aH_{aH_a^{-1}-x}H_{x^{-1}} &= x^{-1} - a \quad \text{(by (H3))} && \iff \\
(x^{-1} - a)^{-1}H_aH_{a^{-1}-x}H_{x^{-1}} &= x^{-1} - a.
\end{aligned}$$

Using part (2), the fact that $H_{x^{-1}} = H_{-x^{-1}}$ and (H4) we get

$$H_aH_{a^{-1}-x}H_{x^{-1}} = H_aH_{a^{-1}+(-x)}H_{(-x)^{-1}} = H_{a+(-x)^{-1}} = H_{a-x^{-1}}.$$

Now (4) follows from the definition of $(x^{-1} - a)^{-1}$.

(5): Let $y \in U^*$ and set $x := -yH_a^{-1}$, then

$$\begin{aligned}
a^{-1}H_{a,y} = y \cdot 2 \iff a^{-1}H_{a,-xH_a} &= -xH_a \cdot 2 \iff \\
a^{-1}(H_{a-xH_a} - H_a - H_aH_xH_a) &= -xH_a \cdot 2 \iff \\
a^{-1}H_{a-xH_a} - a - aH_xH_a &= -xH_a \cdot 2 \iff \\
a^{-1}H_{a-xH_a} &= a + aH_xH_a - xH_a \cdot 2,
\end{aligned}$$

so (5) holds. □

The following proposition comes from [DW, Theorem 4.1, p. 427].

Proposition 5. *Let $(U, \mathcal{H})$ be a Hua structure and let $\tau\colon U^* \to U^*$ be defined by $x\tau = -x^{-1}$. Then $\mathbb{M}(U, \tau)$ is a proper Moufang set, and moreover $h_a = H_a$, for all $a \in U^*$.*

Proof. First, by Lemma 4(1 & 2), $\tau^{-1} = \tau$. By the definition of the Hua maps h_a we have

$$h_a = \tau\alpha_a\tau^{-1}\alpha^{-1}_{a\tau^{-1}}\tau\alpha^{-1}_{(-(a\tau^{-1}))\tau} = \tau\alpha_a\tau\alpha_{a^{-1}}\tau\alpha_a.$$

Hence,

$$(a^{-1})h_a = a^{-1}\tau\alpha_a\tau\alpha_{a^{-1}}\tau\alpha_a = (-a)\alpha_a\tau\alpha_{a^{-1}}\tau\alpha_a = a = (a^{-1})H_a.$$

For $x \neq a^{-1}$ we have

$$xh_a = x\tau\alpha_a\tau\alpha_{a^{-1}}\tau\alpha_a = a - \big(a^{-1} - (a - x^{-1})^{-1}\big)^{-1}$$

Thus the assertion that $H_a = h_a$ is equivalent to

$$(a - xH_a)^{-1} = a^{-1} + (x^{-1} - a)^{-1}, \quad \text{for all } x \in U^* \setminus \{a^{-1}\}. \tag{2}$$

Hence we may assume that $x \neq a^{-1}$.

Applying H_{a-xH_a} to both sides of equation (2) we get that it is equivalent to

$$a - xH_a = (a^{-1} + (x^{-1} - a)^{-1})H_{a-xH_a}. \tag{3}$$

Using (H5) and Lemma 4(5) we see that equation (3) is equivalent to

$$a - xH_a = (x^{-1} - a)^{-1}H_{a-xH_a} + a + aH_xH_a - xH_a \cdot 2,$$

or

$$(x^{-1} - a)^{-1}H_{a-xH_a} = (x - aH_x)H_a.$$

But this last equality holds by Lemma 4(4). Hence $h_a = H_a \in \mathrm{Aut}(U)$, for all $a \in U^*$, so by [DW, Theorem 3.2], $\mathbb{M}(U, \tau)$ is a Moufang set.

Finally, $H_a = h_a = \tau\mu_a$, for all $a \in U^*$ and so by [DW, Theorem 3.1(ii), p. 420], the Hua subgroup H of $\mathbb{M}(U, \tau)$ is generated by $\{H_a^{-1}H_b \mid a, b \in U^*\}$. By Lemma 4(3), H is non-trivial, so $\mathbb{M}(U, \tau)$ is proper. □

Remark 6. Given a Hua structure $(U, \mathcal{H})$ denote by $\mathbb{M}(U, \mathcal{H}) = \mathbb{M}(U, \tau)$ the Moufang set obtained in Proposition 5. Let $(U, \mathcal{H})$ be a Hua structure and let $(U, \mathcal{H}_e)$ be an isotope of $(U, \mathcal{H})$. Notice that $\mathbb{M}(U, \mathcal{H}_e) = \mathbb{M}(U, \mu_e)$. Indeed, $\mathbb{M}(U, \mathcal{H}_e) = \mathbb{M}(U, \tau')$, where $\tau'\colon x \mapsto -x^{-1}H_e$ (see equation (1)). It follows that $\tau' = \tau H_e = \tau h_e = \tau(\tau\mu_e) = \mu_e$. In particular, $\mathbb{M}(U, \mathcal{H}) = \mathbb{M}(U, \mathcal{H}_e)$.

Next we indicate how a proper Moufang set with abelian root groups leads to a Hua structure.

Proposition 7. *Let* $\mathbb{M}(U,\tau)$ *be a proper Moufang set with abelian root groups, and assume that* $\tau = \mu_e$*, for some* $e \in U^*$*. Let* $\mathcal{H}\colon U^* \to \mathrm{Aut}(U)$ *be the map* $a \mapsto h_a$*. Then* $(U, \mathcal{H}, e)$ *is a Hua structure with an identity and* $x^{-1} = -x\tau$*, for all* $x \in U^*$*.*

Proof. By [S], $\mathbb{M}(U,\tau)$ is special, so the μ-maps of $\mathbb{M}(U,\tau)$ are involutions. We have $x^{-1} = xh_x^{-1} = x\mu_x\tau = -x\tau$. Next by [DS1, Proposition 5.2(1)], (H2) holds, and by [DS1, Proposition 5.2(1 & 3)],

$$h_{x^{-1}} = h_{-x\tau} = h_{x\tau} = h_x^{-1},$$

so (H1) holds. By [DS1, Proposition 5.2(4)], (H3) holds. Axion (H5) is [DS1, Lemma 5.7(2)] and (H4) is proved along the proof of [DS1, Proposition 5.8(2)]. □

Remark 8. If in Proposition 7 we choose $e' \in U^*$ in place of e, and we let g_a be the Hua map corresponding to the choice $\tau' = \mu_{e'}$, then $g_a = \mu_{e'}\mu_a = \mu_{e'}\mu_e h_a = h_{e'}^{-1}h_a$. We thus see that if we let $\mathcal{H}'\colon U^* \to \mathrm{Aut}(U)$ be defined by $a \mapsto g_a$, then $(U, \mathcal{H}', e')$ is an isotop of $(U, \mathcal{H}, e)$.

Here is how to get a Hua structure from a QJDA.

Proposition 9. *Let* $(J, \mathcal{W}, e)$ *be a quadratic Jordan division algebra with structure maps* $a \mapsto W_a$*. Assume that* $|J| \geq 4$ *and let* U *be the additive group of* J*. Let* $\mathcal{H}\colon U^* \to \mathrm{Aut}(U)$ *be defined by* $\mathcal{H}\colon a \mapsto H_a := W_a$*. Then* $(U, \mathcal{H}, e)$ *is a Hua structure with an identity.*

Proof. Axiom (H1) follows from [Mc2, 6.1.2, p. 212], (H2) holds since $x \mapsto W_x$ is quadratic, and (H3) is (QJ3). Axiom (H5) is [DW, equation (18), p. 427].

Next we prove (H4). By [DW, Equation (19), p. 427],

$$V_{x,z} = H_{x^{-1},z}H_x, \tag{4}$$

and by [DW, Equation (20), p. 427]

$$V_{z,x} = H_{x,zH_x}H_x^{-1}. \tag{5}$$

Hence

$$
\begin{aligned}
H_x H_{x^{-1}+y^{-1}} H_y &= H_{x+y} && \iff \\
H_{x^{-1}+y^{-1}} H_y &= H_{x^{-1}} H_{x+y} && \iff \\
(H_{x^{-1},y^{-1}} - H_{x^{-1}} - H_{y^{-1}}) H_y &= H_{x^{-1}} (H_{x,y} - H_x - H_y) && \iff \\
H_{x^{-1},y^{-1}} H_y &= H_{x^{-1}} H_{x,y} && \iff \\
V_{y,x^{-1}} &= H_{x^{-1}} H_{x,y} H_{x^{-1}} H_x \quad \text{(by equation (4))} && \iff \\
V_{y,x^{-1}} &= H_{xH_{x^{-1}}, yH_{x^{-1}}} H_x && \iff \\
V_{y,x^{-1}} &= H_{x^{-1}, yH_{x^{-1}}} H_x && \iff \\
V_{y,x^{-1}} &= V_{y,x^{-1}} \quad \text{(by equation (5))}.
\end{aligned}
$$

□

Let us now deduce several properties of Hua structures that follow from Proposition 5.

Proposition 10. *Let $(U, \mathcal{H})$ be a Hua structure. Then*

(1) *U is a vector space over $\mathrm{GF}(p)$, p a prime, or over $\mathbb{Q}$; we let $\mathbb{F} = \mathrm{GF}(p)$ in the former case and $\mathbb{F} = \mathbb{Q}$ in the latter case.*
(2) *If we let $K := \langle H_a \mid a \in U^* \rangle$ be the subgroup of $\mathrm{Aut}(U)$ generated by $\{H_a \mid a \in U^*\}$, then either $\mathbb{F} = \mathrm{GF}(2)$, or K acts irreducibly on U.*
(3) *For all $t \in \mathbb{F}^*$ and $a \in U^*$ we have $H_{a \cdot t} = H_a \cdot t^2$, where $a \cdot t$ is the scalar multiplication.*
(4) *$x^{-1} H_a = x$ iff $a \in \{x, -x\}$.*
(5) *For all $a, b \in U^*$ we have $H_a = H_b$ iff $b \in \{a, -a\}$.*

Proof. Let $\tau\colon x \to -x^{-1}$, for all $x \in U^*$. Then, by Proposition 5, $\mathbb{M}(U, \tau)$ is a special Moufang set. Hence Part (1) is proved in [Ti1, Theorem 5.2(a), p. 55] (see also [DS1, Proposition 4.6(5)]). For (2) notice that $H_a = h_a = \tau\mu_a$, for all $a \in U^*$, where μ_a is the μ-map of $\mathbb{M}(U, \tau)$ corresponding to a. Hence $H_a^{-1} H_b = \mu_a \mu_b$ (because μ_a is an involution). Since the Hua subgroup of $\mathbb{M}(U, \tau)$ is generated by $\{\mu_a \mu_b \mid a, b \in U^*\}$ (see [DW, Theorem 3.1(ii), p. 420]) K contains H. Hence, part (2) follows from the main result in [SW].

Part (3) is [DS1, Proposition 4.6(6)]. For (4) note that $x^{-1}H_a = (-x\tau)\tau\mu_a = -x\mu_a$, and hence $x^{-1}H_a = x$ iff $x\mu_a = -x$, so part (4) is [DST, Lemma 7.6]. Part (5) is a consequence of (4). □

For the sake of completeness we repeat here the main result of [DS1, section 5] in the current language.

Theorem 11 (Theorem 5.11 in [DS1]). *Let $(U, \mathcal{H}, e)$ be an Hua structure with an identity and for $a, b \in U^*$, let $H_{a,b} := H_{a+b} - H_a - H_b$. Assume that U is not a group of exponent 2 or 3, and that*

(i) $H_{-a,b} = -H_{a,b}$, *for all* $a, b \in U^*$;
(ii) $aH_{a,b+c} = aH_{a,b} + aH_{a,c}$, *for all* $a, b, c \in U^*$.

Let $J = U$ viewed as a vector space over $\mathbb{F}$ (see Proposition 10(1)) and let $W_a := H_a$, for all $a \in U^*$ and $\mathcal{W}\colon J^* \to \mathrm{End}_{\mathbb{F}}(J)$ be the map $a \mapsto W_a$. Then $(J, \mathcal{W}, e)$ is a quadratic Jordan division algebra.

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