Pure and Applied Mathematics Quarterly
Volume 7, Number 3
(*Special Issue: In honor of
Jacques Tits*)
797—858, 2011

Strongly p-embedded Subgroups

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Abstract: For p a prime and G a finite group, a proper subgroup H of G is strongly p-embedded in G if and only if p divides $|H|$ but does not divide $|H \cap H^g|$ for all $g \in G \setminus H$. This article contributes to the various projects to improve the classification of the finite simple groups. One of the main theorems is as follows: Suppose that G is a finite group in which every proper subgroup has composition factors from the known simple groups, p is an odd prime and that H is a strongly p-embedded subgroup of G. Assume that $H \cap K$ is of even order for all non-trivial normal subgroups K of G, $O_{p'}(H) = 1$ and $m_p(C_H(t)) \geq 2$ for every involution t of H. Then there exists $n \geq 2$ such that either $F^*(G) \cong \mathrm{PSU}_3(p^n)$ or $p = 3$ and $F^*(G) \cong 2\mathrm{G}_2(3^{2n-1})$.
Keywords: Group theory, classification, simple groups

1. Introduction

In this paper we study finite groups which possess a strongly p-embedded subgroup for some prime p. Suppose that p is a prime. A subgroup H of the finite group G is said to be *strongly p-embedded in G* if the following two conditions hold.

(i) $H < G$ and p divides $|H|$; and
(ii) if $g \in G \setminus H$, then p does not divide $|H \cap H^g|$.

One of the most important properties of strongly p-embedded subgroups is that $N_G(X) \leq H$ for any non-trivial p-subgroup X of H.

Received January 12, 2009.

Groups with a strongly 2-embedded subgroup have been classified by Bender [2] and Suzuki [17] and their classification forms a pedestal upon which the classification of the finite simple groups stands. The classification of groups with a strongly 2-embedded subgroup states that if G is a finite group with a strongly 2-embedded subgroup, then $O^{2'}(G/O(G))$ is a simple rank 1 Lie type group defined in characteristic 2 or G has quaternion or cyclic Sylow 2-subgroups. Of course rank 1 Lie type groups in characteristic 2 are the building blocks of the groups of Lie type in characteristic 2. When dealing with groups defined in characteristic p for odd p, strongly p-embedded subgroups play an equally influential role.

Assume from here on that p is an odd prime and G is a finite group with $O_p(G) = 1$. If G has cyclic Sylow p-subgroup P, then $N_G(\Omega_1(P))$ is strongly p-embedded in G. There is thus no prospect of listing all such groups. However, if $m_p(G) \geq 2$, then the almost simple groups with a strongly p-embedded subgroup are known as a corollary to the classification of the finite simple groups. They include the rank 1 Lie type groups defined in characteristic p, alternating groups $\mathrm{Alt}(2p)$ for $p \geq 5$, and there are only additional examples if $p \leq 11$. See Proposition 2.5 for a complete list. When we examine the list of simple groups with a strongly p-embedded subgroup H, we see that either H is a p-local subgroup or that $F^*(H) \cong \mathrm{Alt}(p) \times \mathrm{Alt}(p)$ with $p \geq 5$ or $\Omega_8^+(2)$ with $p = 5$. Thus the fact that H is strongly p-embedded severely restricts its structure. One major application of the results of this paper will be to the investigation of groups of local characteristic p orchestrated by Meierfrankenfeld, Stellmacher and Stroth [13]. In the described application, we have a subgroup H of G generated by all the normalizers of non-trivial p-subgroups contained in a fixed Sylow p-subgroup of G. Generally speaking under such circumstances H will be a Lie type group defined in characteristic p. We would like to assert that $H = G$. Assuming that this is not the case, work in progress by Salarian and Stroth will show that H is strongly p-embedded in G. The conclusion of our third theorem, Theorem 1.3, is that under mild restrictions on the structure of H, H cannot in fact be strongly p-embedded. Recall that a $\mathcal{K}$*-group* is a group in which every composition factor is from the list of "known" simple groups. That is, every simple section is either a cyclic group of prime order, an alternating group, a group of Lie type or one of the twenty six sporadic simple groups. A group is $\mathcal{K}$*-proper* if all its proper subgroups are $\mathcal{K}$-groups. Obviously $\mathcal{K}$-proper groups are the focus of attention in the proof of the classification of the finite simple groups.

Theorem 1.1. *Suppose that G is a finite group, p is an odd prime and that H is a strongly p-embedded subgroup of G such that $H \cap K$ has even order for all non-trivial normal subgroups K of G. Assume that $F^*(H) = O_p(H)$ and $m_p(C_H(t)) \geq 2$ for every involution t in H. If $N_G(T)$ is a $\mathcal{K}$-group for all non-trivial 2-subgroups T of G, then there exists $n \geq 2$ such that either $F^*(G) \cong \mathrm{PSU}_3(p^n)$ or, $p = 3$ and $F^*(G) \cong {}^2\mathrm{G}_2(3^{2n-1})$.*

Theorem 1.2. *Suppose that G is a finite group, p is an odd prime and that H is a strongly p-embedded subgroup of G. Assume that $O_{p'}(H) = 1$ and that $m_p(C_H(t)) \geq 2$ for every involution t of H. If $N_G(T)$ is a $\mathcal{K}$-group for all non-trivial p'-subgroups T of G and H is a $\mathcal{K}$-group, then either $F^*(H) = O_p(H)$ or $F^*(H) = E(H)$ is quasisimple.*

Theorem 1.3. *Suppose that G is a finite group, p is an odd prime and that H is a strongly p-embedded subgroup of G. Assume that $O_{p'}(H) = 1$, $F^*(H) = E(H)$ is quasisimple or $F^*(H) = O_p(H)$ and that $m_p(C_H(t)) \geq 2$ for every involution t of H. If $N_G(T)$ is a $\mathcal{K}$-group for all non-trivial 2-subgroups T of G and H is a $\mathcal{K}$-group, then $F^*(H) = O_p(H)$.*

We choose to compartmentalize our theorems in this way so that we can be precise about the $\mathcal{K}$-group hypothesis we are invoking. For example in the expected application of this work to the project to classify the finite groups of local characteristic p as explained above, H will be a finite quasisimple $\mathcal{K}$-group and this will be known independently with just a $\mathcal{K}$-group assumption on the normalizers of non-trivial p-subgroups of G. Our three theorems combine to give the following theorem which is the main result of the paper.

Theorem 1.4. *Suppose that G is a finite $\mathcal{K}$-proper group, p is an odd prime and that H is a strongly p-embedded subgroup of G such that $H \cap K$ is of even order for all non-trivial normal subgroups K of G. Assume that $O_{p'}(H) = 1$ and that $m_p(C_H(t)) \geq 2$ for every involution t of H. Then there exists $n \geq 2$ such that either $F^*(G) \cong \mathrm{PSU}_3(p^n)$ or, $p = 3$ and $F^*(G) \cong {}^2\mathrm{G}_2(3^{2n-1})$.*

The hypothesis in Theorems 1.1 and 1.4 that $|H \cap K|$ is even for all normal subgroups K of G is there to guarantee that we cannot transfer all the elements of order 2 from H and finish up with a configuration where the strongly p-embedded subgroup has odd order. In particular, we do not want our group G to have $G > O^2(G)$ and $H \cap O^2(G)$ a strongly p-embedded subgroup of $O^2(G)$

of odd order. For example, taking $G \cong \mathrm{PGL}_2(27)$, we see that G has a strongly 3-embedded subgroup of even order, but the normal subgroup of index 2 does not.

Recall that our primary application of our theorem is to the project to classify groups of local characteristic p. With this in mind, we have the following corollary to Theorem 1.4.

Corollary 1.5. *Suppose that p is an odd prime and H is a strongly p-embedded subgroup of G. If $N_G(T)$ is a $\mathcal{K}$-group for all non-trivial 2-subgroups T of G, then $F^*(H)$ is not a simple Lie type group defined in characteristic p of rank at least 3.*

In fact, if G and H are as in Corollary 1.5, then Theorem 1.4 can be used to show that if $F^*(H)$ is a Lie type group in characteristic p of rank 2, then the only possibility is that $F^*(H) \cong \mathrm{PSL}_3(p)$ or $\mathrm{PSp}_4(p)$. In further work by the authors, it is shown that the latter case cannot happen [15]. To eliminate the possibility that $F^*(H) \cong \mathrm{PSL}_3(p)$, it seems that different techniques need to be developed.

We now proceed to describe the contents of the paper and at the same time give an outline of the proof of our theorems. In Section 2, we present the preliminary results that we shall call upon throughout the paper. We begin with our main characterization theorems. These are the results which formally identify the groups in Theorem 1.1. Thus we state Aschbacher's Classical Involution Theorem (Theorem 2.2) which will be used to identify the unitary groups and the theorem which is the culmination of work by Walter, Bombieri, and Thompson (Theorem 2.3) to recognize the Ree groups. We then move on to results about almost simple $\mathcal{K}$-groups. The most prominent and important of these is Proposition 2.5 which describes the structure of the almost simple $\mathcal{K}$-groups of p-rank at least 2 which possess a strongly p-embedded subgroup. This is followed by Proposition 2.7 which catalogues the structure of a strongly p-embedded subgroup in each of the groups listed in Proposition 2.5. Next a series of corollaries to the propositions are presented, perhaps the most useful being Corollary 2.8. We then collect some results related to the Thompson Transfer Lemma and results which limit the structure of 2-groups which admit certain types of automorphisms, see, for example, Lemmas 2.23 and 2.26.

Our investigation of groups with a strongly p-embedded subgroup starts in earnest in Section 3. We assume that G is a group and that H is a strongly

p-embedded subgroup of G and prove some basic properties about such configurations. In particular, easy, but key facts that are used throughout the work are established such as if K is a subgroup of G which is not contained in H and $H \cap K$ has order divisible by p, then $H \cap K$ is strongly p-embedded in K and, of particular importance, if $m_p(H \cap K) \geq 2$, then $O_{p'}(K) \leq H$ and $F^*(C_G(t)/O_{p'}(C_G(t)))$ is an almost simple group of p-rank at least 2 containing a strongly p-embedded subgroup. This is a simple consequence of the properties of strongly p-embedded subgroups and coprime action. A further consequence of these elementary lemmas is that $F^*(G)$ is a non-abelian simple group. Now the hypothesis in the main theorems that $H \cap K$ has even order immediately implies that $H \cap F^*(G)$ has even order. Thus for our work we may as well suppose that $G = O^2(G)$ and so, in Section 4, where we introduce the main hypothesis Hypothesis 4.1, this is included. In Section 4 we start our exploration of groups G with a strongly p-embedded subgroup H such that $O_{p'}(H) = 1$, $m_p(C_H(t)) \geq 2$ for all involutions $t \in H$, and $G = O^2(G)$. We also impose our main $\mathcal{K}$-group hypothesis, that $N_G(T)$ is a $\mathcal{K}$-group for all 2-groups T of G. We may also suppose that G is not a $\mathcal{K}$-group. Thus H is not strongly 2-embedded in G and so, as H has even order, there is an involution $t \in H$ with $C_G(t) \not\leq H$. Since $m_p(C_H(t)) \geq 2$, we have $O_{p'}(C_G(t)) \leq H$ and $C_H(t)$ is strongly p-embedded in $C_G(t)$. This allows us to use the $\mathcal{K}$-group hypothesis to get hold of the structure of $C_G(t)/O_{p'}(C_G(t))$. One stark consequence of the information that we obtain in this section is that for any involution $t \in H$, $C_G(t) \not\leq H$ and the pertinent structural information about $C_G(t)$ is listed in Lemma 4.4. We close Section 4 with an immediate application of Lemma 4.4 which states that if $E(H) \neq 1$, then $E(H)$ is quasisimple. This is Lemma 4.5.

Sections 5, 6 and 7, study groups satisfying Hypothesis 4.1 which have $Q = F^*(H) = O_p(H)$. Since H is strongly p-embedded in G, we then have $H = N_G(Q)$. Of course we may also assume that G does not have a classical involution. Our objective then in these three sections is to prove that $G \cong {}^2\mathrm{G}_2(3^{2n-1})$ for some $n \geq 2$. To do this we seek to establish the hypothesis of Theorem 2.3. So we need to show that G has Sylow 2-subgroups which have order 8 and an involution t such that $C_G(t)$ contains a normal subgroup isomorphic to $\mathrm{PSL}_2(p^a)$ for some $a \geq 2$. We start this work in Section 5. For t an involution in H, t acts on Q and, of course, either $C_Q(t) \neq 1$ or $C_Q(t) = 1$. In the former case Lemma 5.2 shows that there is a unique component L_t contained in $C_G(t)$ which is normal and has order divisible by p. In the latter case, we can only establish that $C_G(t)$ has a normal 2-component (Lemma 5.3) with the same properties. In both cases, as by hypothesis $C_G(t)$ is a $\mathcal{K}$-group, we may use Proposition 2.5 to describe the possible isomorphism types of $L_t/O(L_t)$. We then make a careful choice of t. We assume that $|C_Q(t)|$ is maximal from among all involutions $t \in H$ with $C_Q(t) \neq 1$ or that $C_Q(t) = 1$. This choice leads to Lemma 5.7 which states that a Sylow 2-subgroup of $O_{p'}(C_G(t))$ is cyclic and that $O_{p'}(C_G(t))$ has a normal 2-complement. In Section 6 our objective is to prove that $L_t/O_{p'}(L_t) \cong \mathrm{PSL}_2(p^a)$ for some $a \geq 2$. The exact conclusion being posted in Theorem 6.1. This section deals with all the other possibilities for $L_t/O_{p'}(L_t)$ the most resilient case arising when H is strongly 3-embedded and $L_t/O(L_t)$ is a covering group of $\mathrm{PSL}_3(4)$. Section 7 is the most technical of the paper. We begin with an involution t in H such that $L_t/O_{p'}(L_t) \cong \mathrm{PSL}_2(p^a)$ and $O_{p'}(C_G(t))$ has cyclic Sylow 2-subgroups. Arguments about the fusion of involutions in G centreing around the Thompson Transfer Lemma and the fact that $G = O^2(G)$ eventually establish the structure of $C_G(t)$ required to invoke Theorem 2.3.

In Section 8, we start to study the situation when $O_{p'}(H) = 1$ and $E = E(H) \neq 1$. By Lemma 4.5 we already know that E is a quasisimple group. We further assume that E is a $\mathcal{K}$-group. In Lemma 8.1 we show that $O_p(H) \leq E(H)$. We remark that our proof of this result requires the $\mathcal{K}$-group hypothesis on H as well as on $N_G(T)$ for non-trivial p'-subgroups T of G. A very useful though easy consequence of Lemma 4.4 in conjunction with Propositions 2.5 and 2.7 is that if $C_H(t)/O_{p'}(C_G(t))$ is not soluble, then $F^*(C_H(t)/O_{p'}(C_G(t))) \cong \mathrm{Alt}(p) \times \mathrm{Alt}(p)$ with $p \geq 5$ or to $\Omega_8^+(2)$ and $p = 5$ (see Lemma 8.3). Now for most candidates for E, we can select an involution $t \in E$ such that $C_E(t)$ is non-soluble and, since we can show that p divides $|C_E(t)|$ (Lemma 8.7), we finish up only having to consider small Lie rank simple groups defined over small fields in any significant detail. In these cases, when there is more than one class of involutions in E, it is usually possible to choose a further involution s such that $m_p(C_H(s)) < 2$. Thus the meat in this section is contained in dealing with the possibility that E might be a rank 1 Lie type group or that $E \cong \mathrm{PSL}_3(2^a)$ for some $a \geq 1$. The arguments dispatching these possibilities are presented in Lemmas 8.8 and 8.9. Finally we present proofs of Theorems 1.1, 1.2 and 1.4 as well as Corollary 1.5 in Section 9.

Our group theoretical notation is mostly standard and can be found in [9]. Particularly we use the following notation and conventions. Suppose that X is a finite group. We use $O(X)$ to represent $O_{2'}(X)$, the largest normal subgroup of X of odd order. As usual $Z^*(X)$ is the subgroup of X which contains $O(X)$ and satisfies $Z^*(X)/O(X) = Z(X/O(X))$. For $Y \leq X$ and $x \in X$, x^Y denotes the set of Y-conjugates of x. If $a \in x^Y$, we write $a \sim_Y x$ to indicate that a and x are conjugate by an element of Y. If p is a prime, $P \in \mathrm{Syl}_p(G)$ and $x \in Z(P)^{\#}$, then x is called a p-central element. Our notation for the simple groups is also standard or self-explanatory. The dihedral group of order n is denoted by $\mathrm{Dih}(n)$, the semi-dihedral group of order n is denoted by $\mathrm{SDih}(n)$ and the generalized quaternion group of order 2^a is denoted by Q_{2^a} and is usually referred to as a quaternion group of order 2^a. The extraspecial 2-groups P of order 2^{1+2n} are denoted 2_+^{1+2n} if P has an elementary abelian subgroup of order 2^{n+1} and are otherwise denoted by 2_-^{1+2n}. If A and B are groups, then $A * B$ represents the central product of A and B, $A : B$ denotes a semidirect product of A and B with undefined action, and $A{}^{\cdot}B$ denotes a non-split extension of A by B.

Acknowledgement. The first author thanks the Institut für Mathematik, Universität Halle–Wittenberg and the second author for their hospitality during visits to Halle to carry out research on this project. Both authors also thank Ulrich Meierfrankenfeld for various improvements and for pointing out a gap in an earlier version of the paper.

2. Preliminaries

In this section we gather together an eclectic collection of preliminary results which we will invoke at various places throughout the paper. We begin with the principal recognition theorems that we shall apply.

Definition 2.1. *Suppose that G is a group, t is an involution in G and $C = C_G(t)$. Then t is a* classical involution in G *provided there exists a subnormal subgroup R of C which satisfies the following conditions.*

(i) *R has non-abelian Sylow 2-subgroups and t is the unique involution in R;*
(ii) *$r^G \cap C \subset N_G(R)$ for all 2-elements $r \in R$; and*
(iii) *$[R^g, R] \leq O_{2'}(C)$ for all $g \in C \setminus N_G(R)$.*

Notice that if G is a group, t is an involution in G and R is a normal subgroup of $C_G(t)$ containing t which has quaternion Sylow 2-subgroups, then t is a classical involution. The following result is of fundamental importance in our work.

Theorem 2.2 (Aschbacher)**.** *Suppose that G is a group and $F^*(G)$ is simple. If G has a classical involution then G is a $\mathcal{K}$-group.*

Proof. This is the main theorem in [1]. □

Theorem 2.3 (Walter, Bombieri, Thompson)**.** *Suppose that G is a perfect group, $S \in \mathrm{Syl}_2(G)$ and $t \in G$ is an involution. If S is abelian of order 8 and $C_G(t)/O(C_G(t))$ contains a normal subgroup isomorphic to $\mathrm{PSL}_2(p^a)$, $p^a > 3$, then $G \cong \mathrm{J}_1$ or ${}^2\mathrm{G}_2(3^a)$ with $a \geq 3$, a odd.*

Proof. From the main result in [19] we have that $C_{F^*(G)}(t) = \langle t \rangle \times \mathrm{PSL}_2(p^a)$. The assertion then follows from [18] and [3]. □

We now move on to properties of $\mathcal{K}$-groups with strongly p-embedded subgroups.

Definition 2.4. *Suppose that p is a prime, X is a group and $P \in \mathrm{Syl}_p(X)$. Then the* 1-generated p-core *of X is defined as follows:*

$$\Gamma_{P,1}(X) = \langle N_X(R) \mid 1 \neq R \leq P \rangle.$$

We say that a proper subgroup M of X is *strongly p-embedded* if and only if p divides $|M|$ and $\Gamma_{P,1}(X) \leq M$ for some Sylow p-subgroup P of M. That this definition is equivalent to the one given in the introduction is an easy application of Sylow's Theorem (see [9, 17.10] for example).

Proposition 2.5. *Suppose that p is a prime, X is a $\mathcal{K}$-group and $K = F^*(X)$ is simple. Let $P \in \mathrm{Syl}_p(X)$ and $Q = P \cap K$. If $m_p(P) \geq 2$ and $\Gamma_{P,1}(X) < X$, then $\Gamma_{Q,1}(K) < K$ and p and K are as follows.*

(i) *p is arbitrary, $a \geq 1$ and $K \cong \mathrm{PSL}_2(p^{a+1})$, or $\mathrm{PSU}_3(p^a)$, ${}^2\mathrm{B}_2(2^{2a+1})$ ($p = 2$) or ${}^2\mathrm{G}_2(3^{2a+1})$ ($p = 3$) and X/K is a p'-group.*
(ii) *$p > 3$, $K \cong \mathrm{Alt}(2p)$ and $|X/K| \leq 2$.*
(iii) *$p = 3$, $K \cong \mathrm{PSL}_2(8)$ and $X \cong \mathrm{PSL}_2(8) : 3$.*
(iv) *$p = 3$, $K \cong \mathrm{PSL}_3(4)$ and X/K is a 2-group.*
(v) *$p = 3$ and $X = K \cong \mathrm{M}_{11}$.*
(vi) *$p = 5$ and $K \cong {}^2\mathrm{B}_2(32)$ and $X \cong {}^2\mathrm{B}_2(32) : 5$.*
(vii) *$p = 5$, $K \cong {}^2\mathrm{F}_4(2)'$ and $|X/K| \leq 2$.*

(viii) $p = 5$, $K \cong \mathrm{McL}$ *and* $|X/K| \leq 2$.
(ix) $p = 5$, $K \cong \mathrm{Fi}_{22}$ *and* $|X/K| \leq 2$.
(x) $p = 11$ *and* $X = K \cong \mathrm{J}_4$.

Proof. See [10, 7.6.1] or [8, 24.1]. □

Notation 2.6. *We let $\mathcal{E}$ be the set of pairs (K, p) in the conclusion of Proposition 2.5 with p odd.*

Proposition 2.7. *Suppose that X, K, P and Q are as in Proposition 2.5. Let $H = \Gamma_{Q,1}(K)$. Then the following hold.*

(i) *If 2.5 (i) holds, then $H = N_K(Q)$ is a Borel subgroup of K. In particular, $H = JQ$ where J is cyclic and operates irreducibly on each of $Q/\Phi(Q)$ and $Z(Q)$.*
(ii) *If 2.5(ii) holds, then H is isomorphic to the subgroup of $\mathrm{Sym}(p) \wr \mathrm{Sym}(2)$ consisting of even permutations. In particular, if $p > 3$, then $O_p(H) = 1$.*
(iii) *If 2.5(iii) holds, then Q is cyclic of order 9, $H = N_K(Q)$ is dihedral of order 18 and $N_X(P) \cong 3^{1+2}_{-}.2$. In particular, $N_K(Q)$ does not act irreducibly on $P/\Phi(P)$.*
(iv) *If 2.5(iv) holds, then Q is elementary abelian of order 9, $H = N_K(Q)$, $C_K(Q) = Z(Q)$, $H/Q \cong \mathrm{Q}_8$ and a complement to Q in H acts irreducibly on Q.*
(v) *If 2.5(v) holds, then Q is elementary abelian of order 9, $H = N_K(Q)$, $C_H(Q) = Q$, $H/Q \cong \mathrm{SDih}(16)$ and a complement to H in K acts irreducibly on Q.*
(vi) *If 2.5(vi) holds, then Q is cyclic of order 25, $H = N_K(Q)$ is a Frobenius group of order 100, and $N_X(P) \cong 5^{1+2}_{-}.4$. In particular, a complement to Q in $N_K(Q)$ does not act irreducibly on $P/\Phi(P)$.*
(vii) *If 2.5(vii) holds, Q is elementary abelian of order 25, $H = N_K(Q)$, $Q = C_H(Q)$, H/Q is isomorphic to a central product of a cyclic group of order 4 with $\mathrm{SL}_2(3)$ and a complement to Q in H acts irreducibly on Q.*
(viii) *If 2.5(viii) holds, then Q is extraspecial of order 5^3, $H = N_K(Q)$, $C_K(Q) = Z(Q)$, H/Q isomorphic to a non-abelian extension of a cyclic group of order 3 by a cyclic group of order 8 and a complement to Q in H acts irreducibly on $Q/\Phi(Q)$.*
(ix) *If 2.5(ix) holds, then Q has order 25 and $H \cong \mathrm{Aut}(\Omega_8^+(2))$. In particular $O_5(H) = 1$.*

(x) *If 2.5(x) holds, then Q is extraspecial of order 11^3, $H = N_K(Q)$, $C_K(Q) = Z(Q)$, H/Q is isomorphic to the direct product of a cyclic group of order 5 and $\mathrm{GL}_2(3)$. In particular, a complement to Q in H acts irreducibly on $Q/\Phi(Q)$.*

In all cases $N_X(Q)$ is maximal subgroup of G. So $N_X(Q)$ is the only strongly p-embedded subgroup of X containing P.

Proof. This is mostly [10, Theorem 7.6.2], the statement regarding the action of complements to Q in H are easily deduced and are well-known. □

Corollary 2.8. *Assume that X, K, P, Q and H are as in Proposition 2.7. If $O_p(H) \neq 1$, then $N_H(P)$ acts irreducibly on $P/\Phi(P)$, unless either*

(i) $p = 3$ *and* $X \cong \mathrm{PSL}_2(8) : 3$; *or*
(ii) $p = 5$ *and* $X \cong {}^2\mathrm{B}_2(32) : 5$.

Proof. The assertion follows directly from Proposition 2.7. □

Corollary 2.9. *Assume that X, K, P, Q and H are as in Proposition 2.7 with p odd. Then H contains an involution inverting $P/\Phi(P)$ unless cases (i) or (ii) of Corollary 2.8 holds or $K \cong \mathrm{PSL}_2(p^a)$ with $p^a \equiv 3 \pmod 4$.*

Proof. As long as $N_H(P)/C_H(P)P$ has a central involution the result follows with Corollary 2.8. By Proposition 2.7, if $N_H(P)/C_H(P)$ does not have a central involution, we must be in case (i) of Proposition 2.5. Furthermore the Borel subgroup must be of odd order. Therefore $K \cong \mathrm{PSL}_2(p^a)$ with $p^a \equiv 3 \pmod 4$. □

Lemma 2.10. *Let $F^*(X) = K \cong \mathrm{PSL}_3(4)$ and $t \in X$ be an involution which induces an outer automorphism on L. Then $C_K(t) \cong 3^2.\mathrm{Q}_8 \cong \mathrm{PSU}_3(2)$, $\mathrm{PSL}_3(2)$ or $\mathrm{Alt}(5)$.*

Proof. By [10, 2.5.12] K just posseses involutory outer automorphisms which are field-, graph,- or graph-field automorphisms. A field automorphism centralizes $\mathrm{PSL}_3(2)$, a graph automorphism centralizes $\mathrm{PSL}_2(4) \cong \mathrm{Alt}(5)$ and a graph-field automorphism centralizes $\mathrm{PSU}_3(2)$. □

Corollary 2.11. *Assume that X, K, P, Q are as in Proposition 2.5. Set $H = \Gamma_{P,1}(X)$. If $O_{p'}(H) \neq 1$, then $K \cong \mathrm{PSL}_3(4)$ or Fi_{22}, $|X/K| \geq 2$ and $O_{p'}(H) = Z(H)$ has order 2.*

Proof. This is [8, 24.2].

□

Lemma 2.12. *Suppose that p is an odd prime and $(K,p) \in \mathcal{E}$. If two divides the order of the Schur multiplier of K, then K is isomorphic to $\mathrm{PSL}_2(p^a)$ for some a, $\mathrm{Alt}(2p)$, $\mathrm{PSL}_3(4)$, or Fi_{22}.*

Proof. For this we just consult [10, Theorem 6.1.4]. □

Corollary 2.13. *Assume that p is an odd prime, $(K,p) \in \mathcal{E}$ and $H = \Gamma_{Q,1}(K)$ where $Q \in \mathrm{Syl}_p(K)$. If $\widehat{K}$ is a non-trivial perfect central extension of K with $Z(K)$ a 2-group and if $\widehat{H}$ has quaternion or cyclic Sylow 2-subgroups, then $\widehat{K} \cong \mathrm{SL}_2(p^a)$ for some a.*

Proof. Suppose that $K \not\cong \mathrm{PSL}_2(p^a)$. Then, by Lemma 2.12, $K \cong \mathrm{Alt}(2p)$, $\mathrm{PSL}_3(4)$ or Fi_{22}. In these cases, Proposition 2.7 shows that H does not have cyclic or dihedral Sylow 2-subgroups and so these groups do not arise. It is well-known that the Sylow 2-subgroups of $\mathrm{SL}_2(p^a)$ are quaternion. □

Lemma 2.14. *Suppose that p is an odd prime and X is group with $F^*(X) = K$ where $(K,p) \in \mathcal{E}$. Let $P \in \mathrm{Syl}_p(X)$ and $Q = P \cap K$. Set $H = \Gamma_{Q,1}(K)$. If $|K : H|$ is odd, then $K \cong \mathrm{M}_{11}$.*

Proof. This follows by inspection of the groups in $\mathcal{E}$ and the subgroups corresponding to H. □

Lemma 2.15. *Suppose that p is an odd prime and X is group with $F^*(X) = K$ where $(K,p) \in \mathcal{E}$. Let $P \in \mathrm{Syl}_p(X)$ and $Q = P \cap K$. Set $H = \Gamma_{Q,1}(K)$. If $|H|$ is odd, then $K \cong \mathrm{PSL}_2(p^f)$ for some odd f and $p \equiv 3 \pmod 4$.*

Proof. This follows by inspection of the groups in $\mathcal{E}$ and the subgroups corresponding to H described in Proposition 2.7. □

Lemma 2.16. *Suppose that K is a simple $\mathcal{K}$-group and that p is an odd prime. Then $m_p(\mathrm{Out}(K)) \leq 2$ and, if $P \leq \mathrm{Out}(K)$ has p-rank 2, then $P \not\leq O^p(\mathrm{Out}(K))$.*

Proof. Since the alternating groups and sporadic simple groups have outer automorphism groups which are 2-groups (perhaps trivial), we may suppose that E is a Lie type group. Then the result can be deduced from [10, Theorem 2.5.12]. $\square$

Lemma 2.17. *Assume that G is a group with $Z^*(G) = O_{2'}(G)$. If $t \in G$ is an involution, then $t^G \cap C_G(t) \neq \{t\}$.*

Proof. Let $R \in \mathrm{Syl}_2(C_G(t))$ and assume that $t^G \cap C_G(t) \neq \{t\}$. Then $t^G \cap R = \{t\}$ and so $t \in Z(N_G(R))$. Therefore, $R \in \mathrm{Syl}_2(G)$ and Glauberman's Z^*-Theorem [6] implies that $t \in Z^*(G) = O_{2'}(G)$ which is impossible. Hence the lemma holds. $\square$

The next result is the famous Thompson Transfer Lemma.

Lemma 2.18. *Let G be a group, $S \in \mathrm{Syl}_2(G)$, $T \trianglelefteq S$ with $S = TA$, $A \cap T = 1$, A cyclic and non-trivial. If G has no subgroup of index two and u is the involution in A, then there is some $g \in G$ with $u^g \in T$ and $C_S(u^g) \in \mathrm{Syl}_2(C_G(u^g))$. In particular $|C_S(u)| \leq |C_S(u^g)|$.*

Proof. This is [9, (15.16)]. $\square$

Lemma 2.19. *Suppose that G is a group and $S \in \mathrm{Syl}_2(G)$. Then at least one of the following hold.*

(i) *$G \neq O^2(G)$;*
(ii) *$\Omega_1(Z(S)) \leq \Phi(S)$; or*
(iii) *$N_G(S)$ acts non-trivially on $\Omega_1(Z(S))$.*

Proof. Suppose that (i) and (ii) do not hold. Then there is a maximal subgroup M of S and an involution $t \in \Omega_1(Z(S))$ such that $t \notin M$. By the Thompson Transfer Lemma 2.18, there is $x \in G$ such $t^x \in M$ and $C_S(t^x) \in \mathrm{Syl}_2(C_G(t^x))$. Since t is 2-central, it follows that $t^x \in \Omega_1(Z(S))$. Burnside's Lemma [9, 16.2] now implies that t and t^x are conjugate by an element of $N_G(S)$. Thus (iii) holds. $\square$

Lemma 2.20. *Suppose that G is a group with $G = O^2(G)$, $t \in G$ is an involution and $S \in \mathrm{Syl}_2(C_G(t))$. Assume that $S = \langle y \rangle \times S_0$, $t \in \langle y \rangle$ and $Z(S_0)$ is elementary abelian. Then $\langle t \rangle = \langle y \rangle$.*

Proof. Assume that y has order greater than 2. Then $Z(S) = \langle y \rangle \times Z(S_0)$ and so t is the unique involution in $\Phi(Z(S))$. Therefore $\langle t \rangle$ is a characteristic subgroup of

S. Hence $t \in Z(N_G(S))$ and so $S \in \mathrm{Syl}_2(G)$. As $S = \langle y \rangle \times S_0$ and $G = O^2(G)$, we may now apply the Thompson Transfer Lemma 2.18 to see that there exists $g \in G$ such that $t^g \in S_0$ and $C_S(t^g) \in \mathrm{Syl}_2(C_G(t^g))$. It follows that $t^g \in Z(S)$. Hence t and t^g are conjugate in $N_G(S)$ by Burnside's Lemma. But we have already noted that $\langle t \rangle$ is a characteristic subgroup of S and so we have $\langle t \rangle = \langle t^g \rangle$ which is a contradiction. Thus $\langle t \rangle = \langle y \rangle$ as claimed. □

Lemma 2.21. *Suppose that G is a group, r is an odd prime and $G/O_2(G) \cong \mathrm{Dih}(2r)$. Let $R \in \mathrm{Syl}_r(G)$ and assume that $C_{O_2(G)}(R) = 1$. Then $N_G(R) \cong \mathrm{Dih}(2r)$ and, for a an involution in $N_G(R)$, all the involutions of $G \setminus O_2(G)$ are conjugate to a and $|C_{O_2(G)}(a)|^2 = |O_2(G)|$.*

Proof. Set $Q = O_2(G)$. Since $N_Q(R) = C_Q(R) = 1$, the Frattini argument shows that $N_G(R) \cong \mathrm{Dih}(2r)$. Let $a, b \in N_G(R)$ be involutions with $a \neq b$. Assume that $c \in bQ$ is an involution. Then $\langle a, c \rangle$ is a dihedral group of order divisible by r. Thus $\langle a, c \rangle$ contains a conjugate, R^x of R. Since $\langle a, c \rangle \cap Q \le C_Q(R^x) = 1$, we have that $\langle a, c \rangle \cong \mathrm{Dih}(2r)$. Thus a and c are G-conjugate. It follows that every involution in $G \setminus Q$ is conjugate to a. Assume that a normalizes R^x for some $x \in Q$. Then $xax^{-1} \in N_G(R)$ and $xax^{-1}a \in Q \cap N_G(R) = 1$. Thus $x \in C_Q(a)$. It follows that a normalizes exactly $|C_Q(a)|$ conjugates of R. Let $x \in bQ$ be an involution. Then $\langle a, x \rangle$ contains a conjugate of R and if $y \in bQ$ is an involution with $\langle a, x \rangle = \langle a, y \rangle$, then $x = y$. Thus a normalizes at least $|Q : C_Q(b)|$ conjugates of R. It follows that $|Q| \le |C_Q(b)||C_Q(a)| = |C_Q(b)|^2$. On the other hand, as $C_Q(b) \cap C_Q(a) \le C_Q(R) = 1$, we have $|C_Q(b)|^2 \le |Q|$ and this completes the proof of the lemma. □

We recall that if $A = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ and $G = \mathrm{Aut}(A)$. Then $G \cong \mathrm{GL}_2(\mathbb{Z}_{2^k})$ and that

$$O_2(G) = \left\{ \begin{pmatrix} a\,c \\ d\,b \end{pmatrix} \mid a, b \in \mathbb{Z}_{2^k}^*, c, d \in 2\mathbb{Z}_{2^k} \right\},$$

where $\mathbb{Z}_{2^k}^*$ denotes the groups of units of $\mathbb{Z}_{2^k}$. In particular, we have that $|O_2(G)| = 2^{4(k-1)}$ and $G/O_2(G) \cong \mathrm{SL}_2(2)$.

Lemma 2.22. *Suppose that $k \ge 2$, $A = \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$ and $G = \mathrm{Aut}(A)$. If $H \le G \cong \mathrm{GL}_2(\mathbb{Z}_{2^k})$ and $H \cong \mathrm{Alt}(4)$, then*

$$O_2(H) = \left\langle \begin{pmatrix} 1 + 2^{k-1} & 0 \\ 2^{k-1} & 1 + 2^{k-1} \end{pmatrix}, \begin{pmatrix} 1 & 2^{k-1} \\ 2^{k-1} & 1 \end{pmatrix} \right\rangle.$$

In particular, there is exactly one conjugacy class of subgroups of G isomorphic to $\mathrm{Alt}(4)$.

Proof. We first note that $t = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ is an element of order 3 in G and, since $|G|_3 = 3$, we may suppose that $t \in H$.

We proceed by induction on k. Suppose that $k = 2$. Then $O_2(G)$ has order 16 and is abelian. Since $O_2(H) \le O_2(G)$ and $Z(G) \ne 1$, we then have

$$O_2(H) = [O_2(G), t] = \left\langle \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Now suppose that $k > 2$. Let $B = 2A \cong \mathbb{Z}_{2^{k-1}} \times \mathbb{Z}_{2^{k-1}}$. Then

$$C_G(B) = \left\{ \begin{pmatrix} 1 + 2^{k-1}a & 2^{k-1}c \\ 2^{k-1}d & 1 + 2^{k-1}b \end{pmatrix} \mid a, b, c, d \in \{0, 1\} \right\}$$

which has order 2^4. Furthermore, $G/C_G(B) \cong \mathrm{Aut}(B)$. If $H \cap C_G(B) = O_2(H)$, then

$$O_2(H) = [C_G(B), t] = \left\langle \begin{pmatrix} 1 + 2^{k-1} & 0 \\ 2^{k-1} & 1 + 2^{k-1} \end{pmatrix}, \begin{pmatrix} 1 & 2^{k-1} \\ 2^{k-1} & 1 \end{pmatrix} \right\rangle$$

and we are done. Hence $HC_G(B)/C_G(B) \cong \mathrm{Alt}(4)$. By induction, we have

$$O_2(H) \le R = \left\langle \begin{pmatrix} 1 + 2^{k-2} & 0 \\ 2^{k-2} & 1 + 2^{k-2} \end{pmatrix}, \begin{pmatrix} 1 & 2^{k-2} \\ 2^{k-2} & 1 \end{pmatrix} \right\rangle C_G(B).$$

However, an easy calculation then shows that every element of $R \setminus B$ has order 4 and this contradicts $O_2(H)$ having exponent 2. This proves the lemma. $\square$

Lemma 2.23. *Let G be a group, $T \le G$ be a 2-group and $V \le Z(T)$ be a fours group. Assume that $t, \rho \in N_G(V) \cap N_G(T)$ are elements of order 2 and 3 respectively with $[t, \langle \rho, V \rangle] = 1$. If $[V, \rho] = C_T(t) = V$, then T is isomorphic to one of the following groups.*

(i) *An elementary abelian group of order 16.*
(ii) *A homocyclic group of rank 2 and order 2^{2n} for some $n \ge 1$.*
(iii) *A Sylow 2-subgroup of* $\mathrm{PSL}_3(4)$.
(iv) *A Sylow 2-subgroup of* $\mathrm{PSU}_3(4)$.

Proof. Because $C_T(t) = V$ and $[t, \rho] = 1$, we have that $C_T(\rho) = 1$. If $T = V$, then (ii) holds with $n = 1$. So assume that $T > V$ and let W be the preimage of $C_{T/V}(t)$. Then, since $C_T(\rho) = 1$ and $C_T(t) = V$, we have that $|W : V| = 4$. In particular, W/V has rank 2 which means that $|W'| \leq 2$. Thus, as ρ acts on W', we have that W is abelian and so W is either elementary abelian or homocyclic $\mathbb{Z}_4 \times \mathbb{Z}_4$

Assume first that W is elementary abelian. Then all involutions in Wt are conjugate in $\langle W, t \rangle$. As $C_T(t) = V$, this means that $T = W$, which is (i).

Assume next that W is homocyclic. Then, as $W/V = C_{T/V}(t)$ is elementary abelian of order 4, T/V satisfies the hypothesis of the lemma. Hence we may assume by induction T/V is isomorphic to one of the groups in (i) - (iv).

Assume that T/V is elementary abelian of order 16. If $V = \Omega_1(T)$, then, as T/V can be considered as a direct some of two irreducible 2-dimensional modules for $\langle \rho \rangle$, we have that $T = WW_1$, where W_1 is also homocyclic of order 16 and is normalized by ρ. Furthermore, $C_{W_1}(W) = C_W(T) = V$. Using Lemma 2.22 shows that the action of W_1 on W is uniquely determined. As $W = [t, W_1]$, we see that T is uniquely determined and so we have that T is isomorphic to a Sylow 2-subgroup of $\mathrm{PSU}_3(4)$. If $\Omega_1(T) \neq V$, then we have $T = WW_1$, where W_1 is elementary abelian of order 4 and is normalized by ρ. If $C_{W_1}(W) = 1$, then the action of W_1 on W is uniquely determined as above. We therefore have that VW_1 and $(VW_1)^t$ are elementary abelian of order 16 and the action of W_1 on $(VW_1)^t$ is uniquely determined. It follows that T is isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$. Hence we have $C_{W_1}(W) = W_1$ and consequently T is abelian. Then $\Omega_1(T)$ is elementary abelian. But this forces $W = \Omega_1(T)$ which is a contradiction as $W \cong \mathbb{Z}_4 \times \mathbb{Z}_4$.

Assume next that T/V is isomorphic to a Sylow 2-subgroup of $\mathrm{PSU}_3(4)$ or $\mathrm{PSL}_3(4)$. Then there is a homocyclic group $W_1 \cong \mathbb{Z}_8 \times \mathbb{Z}_8$ on which $\langle T/W_1, \rho \rangle$ acts as $\mathrm{Alt}(4)$. Using Lemma 2.22 we get $[W_1, T] = V$ and this contradicts $Z(T/V) = W/V$. Hence we finally have T/V is homocyclic. As $T' = \langle [x, y] \rangle$ where $T = \langle x, y \rangle$, the fact that $C_{T'}(\rho) = 1$ implies that T is abelian. As $\Omega_1(T) = V$ and $C_T(\rho) = 1$, we have that T is homocyclic. □

Corollary 2.24. *Assume the hypothesis of Lemma 2.23. If the coset Tt contains more than one T-conjugacy class of involutions, then T is homocyclic and Tt contains exactly four T-conjugacy classes of involutions.*

Proof. We consider each of the possibilities for T given in Lemma 2.23. If T is elementary abelian, then $C_T(t) = V$ has index 4 in T and so Tt contains exactly 4 involutions and they are all T-conjugate. Assume now that T is non-abelian. Let $W = C_{T/V}(t)$. We note that $|t^T| = |T/V| = |T|/4$ and, if $x \in T$ and $(xt)^2 = 1$, then t inverts x. As T is non-abelian, then T has exponent 4 and so $x^2 \in V$. We infer that, if $x \in T$ is inverted by t, then $x \in W$. It follows that Tt contains at most $|W|$ involutions. Since $|W| = |T/V|$, we have that Tt contains exactly one T-conjugacy class of involutions in this case. Finally, if T is homocyclic, we have that t inverts every element of T. Thus Tt consists of involutions and it follows that Tt contains exactly four T-conjugacy classes of involutions. $\square$

Lemma 2.25. *Let $t \in G$ be an involution and T be a Sylow 2-subgroup of $C_G(t)$ such that $T \cong \langle t\rangle \times D$, where D is dihedral of order 4 or 8. Assume that there is a fours group $V \leq D$, such that $C_G(t)$ contains a 3-element ρ which acts non-trivially on V. Then (for a possibly different choice of the element ρ) there is a Sylow 2-subgroup R of $C_G(V)$ which is normalized by $\langle T, \rho\rangle$ such that $R = U\langle t\rangle$ where U is isomorphic to one of the groups listed in the conclusion of Lemma 2.23.*

Proof. By considering a minimal counter example to the lemma, we may assume that $G = N_G(V) = C_G(V)T\langle\rho\rangle$. Set $E = \langle V, t\rangle$. Then E is elementary abelian of order 2^3. As G is a counter example to the lemma, we have $E \notin \mathrm{Syl}_2(C_G(V))$. As $T \in \mathrm{Syl}_2(C_G(t))$, we have $E \in \mathrm{Syl}_2(C_G(E))$. Thus $C_G(E) = E \times O_{2'}(C_G(E))$. Assume that $S \in \mathrm{Syl}_2(C_G(V))$ is normalized by T. Then $W = N_S(E) > E$ and since $W = N_S(E)$ centralizes V, we deduce that $\langle\rho\rangle W C_G(E)/C_G(E) \cong \mathrm{Alt}(4)$. In particular, we have $|W| = 2^5$ and $W \in \mathrm{Syl}_2(N_G(E))$. If $W^\rho \neq W$, then there exists $c \in O_{2'}(C_G(E))$ such that ρc normalizes W. Let ρ^* be the 3-part of ρc. Then ρ^* acts non-trivially on V and centralizes t. Thus we may as well suppose that ρ normalizes W. Set $\overline{G} = G/V$. Then $\overline{\langle T\rangle W} \in \mathrm{Syl}_2(C_{\overline{N_G(V)}}(\overline{t}))$ and $\overline{\rho}$ is a 3-element which acts non-trivially on $\overline{[W,\rho]}$ and centralizes $\overline{t}$. Therefore, by induction, there is a subgroup $U \geq V$ such that $\overline{U}\langle\overline{t}\rangle \in \mathrm{Syl}_2(C_{\overline{G}}(\overline{[W,\rho]}))$ where $\overline{U}$ is one of the groups listed in the conclusion of Lemma 2.23 and $\overline{\langle\rho, T\rangle}$ normalizes $\overline{U}$. By considering $U\langle T, \rho\rangle$, we have that U is also listed in the conclusion of Lemma 2.23. We may assume that $U\langle t\rangle \leq S$ and, as G is a counter example to the lemma, $S \neq U\langle t\rangle$. Since t is not centralized by an abelian group of order 16 and $U \neq V$, we see that $t^G \cap U = \emptyset$.

If U is either elementary abelian or is isomorphic to a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$ or $\mathrm{PSU}_3(4)$, then, by Corollary 2.24, every involution in $U\langle t\rangle \setminus U$ is conjugate to t by an element of U. Setting $X = N_S(U\langle t\rangle)$, we then have $U\langle t\rangle < X = C_X(t)U\langle t\rangle = U\langle t\rangle$, a contradiction. Thus U is homocyclic of rank 2. Since $|U| \geq 16$, we have that U is a characteristic subgroup of $U\langle t\rangle$. It follows that X normalizes $\Omega_2(U) = [W, \rho]$. Since X centralizes V, we have $[[W, \rho], X] \leq V$ and thus $\overline{X} \leq C_{\overline{G}}(\overline{[W, \rho]})$ and this is our final contradiction as $\overline{U\langle t\rangle} \in \mathrm{Syl}_2(C_{\overline{G}}(\overline{[W, \rho]}))$. This concludes the verification of Lemma 2.25. □

Lemma 2.26. *Assume that p is an odd prime, W is a 2-group and that $E \leq \mathrm{Aut}(W)$ is a non-cyclic abelian p-subgroup. If $C_W(e)$ contains at most one involution for each $e \in E^\#$, then $|E| = 9$, $W \cong 2^{1+4}_+$, 2^{1+6}_- or 2^{1+8}_+ and $C_W(E) = Z(W)$.*

Proof. Let W be a minimal counter example to the claim. Since E is abelian and non-cyclic, $W = \langle C_W(e) \mid e \in E^\#\rangle$. Choose e such that $C_W(e) \neq C_W(E)$. Then, as $C_W(e)$ contains exactly one involution, $C_W(e)$ is cyclic or generalized quaternion. Because E normalizes and does not centralize $C_W(e)$, it follows that $C_W(e)$ is a quaternion group of order 8 and $p = 3$. In particular, we have that $C_W(E) = Z(C_W(e))$ has order 2 and if $C_W(f) > C_W(E)$ for some $f \in E^\#$, then $C_W(f)$ is a quaternion group of order 8. We also have that W is non-abelian and $Z(W) = C_W(E)$. We now choose $e \in E^\#$ such that $C_W(e)$ is contained in the second centre of W. Then $C_W(e)$ is normal in W. Since the automorphism group of $C_W(e)$ is isomorphic to $\mathrm{Sym}(4)$, we get that $W = C_W(e)C_W(C_W(e))$. Set $W_0 = C_W(C_W(e))$. Then $W_0 \not\leq C_W(e)$, for otherwise e would be the trivial automorphism of W. Thus W_0 is a non-trivial 2-group. Suppose that $W_0 \leq C_W(f)$ for some $f \in E^\#$. Then, as $C_W(f)$ is quaternion of order 8 and W_0 is normalized by E, we infer that $C_W(f) = W_0$ and $W = C_W(e)C_W(f) \cong 2^{1+4}_+$. Furthermore, we have that E is elementary abelian of order 9. Thus we may suppose that every element of E induces a non-trivial automorphism of W_0. As $W_0 < W$, we have that $|E| = 9$ and that $W_0 \cong 2^{1+4}_+$, 2^{1+6}_- or 2^{1+8}_+ by induction. The first two cases immediately deliver $W \cong 2^{1+6}_-$ or $W \cong 2^{1+8}_+$ and the theorem then holds. So suppose that $W_0 \cong 2^{1+8}_+$. Then, for each $f \in E^\#$, $C_{W_0}(f)$ is a quaternion group of order 8. But $e \in E^\#$ and $C_W(e) \cap W_0 = Z(W_0)$ and so we have a contradiction. This completes the proof of the lemma. □

Lemma 2.27. *Suppose that $X \cong \mathrm{Aut}(\mathrm{PSL}_3(2^a))$ with $a \geq 2$, $S \in \mathrm{Syl}_2(X)$ and $T = S \cap F^*(X)$. If $U \leq S$ and $U \cong T$, then $U = T$.*

Proof. Set $K = F^*(X)$. Suppose that $U \le S$ with $U \cong T$ and $U \ne T$. Let F_1 and F_2 be the two elementary abelian subgroups of T of order 2^{2a} and note that every involution of T is contained in $F_1 \cup F_2$ and that $F_1 \cap F_2 = Z(T)$. Since the group of diagonal outer automorphisms of K has order 3, $\mathrm{Out}(K)$ has abelian Sylow 2-subgroups. Therefore $Z(U) = \Phi(U) = U' \le T$ and UK/K is elementary abelian of order at most 4. Let E_1 and E_2 be the elementary abelian subgroups of U of order 2^{2a}. As $U = E_1E_2$, we may suppose that $E_1 \not\le T$. As $|UK/K| \le 4$, $|E_1 \cap T| \ge 2^{2a-2}$. Assume that $(E_1 \cap T)Z(T) \ne Z(T)$. Then we may suppose that $(E_1 \cap T)Z(T) \le F_1$. But then $F_1^u \ge (E_1 \cap T)Z(T) > Z(T)$ for all $u \in U$. Hence U normalizes F_1 and therefore UK/K consists of field outer automorphisms. Hence $|E_1 \cap T| \ge 2^{2a-1}$. Now let $e \in E_1 \setminus T$. Then e centralizes either $(E_1 \cap T)Z(T)/Z(T) = F_1/Z(T)$ or $Z(T) \le E_1$ and e centralizes $Z(T)$. In either case e cannot be a field automorphism of K and we have a contradiction. Thus $E_1 \cap T = Z(T)$, $|UT/T| = 4$ and $a = 2$. But then $Z(U) = E_1 \cap T = Z(T)$ and we see that U centralizes $Z(T)$. This means that UK/K does not contain non-trivial field outer automorphisms. It follows that $|UK/K| \le 2$ and this is a contradiction. □

3. Basic properties of groups with a strongly p-embedded subgroups

In this section we reveal the basic structural properties of groups with strongly p-embedded subgroups. The first lemma is one which we already alluded to in Section 2 and states the equivalence between the two definitions of strongly p-embedded subgroups which we have given.

Lemma 3.1. *Assume that G is a group, p is a prime, $H \le G$ and $S \in \mathrm{Syl}_p(H)$. Then $\Gamma_{S,1}(G) \le H$ if and only if p divides $|H|$ and $|H \cap H^g|$ is not divisible by p for all $g \in G \setminus H$.*

Proof. See [9, Lemma 17.11]. □

The fundamental lemmas which get us started in the proof of Theorem 1.4 are as follows.

Lemma 3.2. *Suppose that G is a group, p is a prime, H is a strongly p-embedded subgroup of G and $K \le G$ such that $H \cap K$ has order divisible by p. Then the following statements hold.*

(i) $\mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$;
(ii) *if* $K \not\leq H$ *and , then* $H \cap K$ *is a strongly* p*-embedded subgroup of* K;
(iii) $\mathrm{Syl}_p(H \cap K) \subseteq \mathrm{Syl}_p(K)$;
(iv) $\bigcap H^G \leq O_{p'}(G)$; *and*
(v) *if* $m_p(H) \geq 2$, *then* $O_{p'}(G) = \bigcap H^G$.

Proof. Let $S \in \mathrm{Syl}_p(H)$. Then, as H is strongly p-embedded in G, $N_G(S) \leq H$. Thus $S \in \mathrm{Syl}_p(G)$ and (i) holds.

Suppose that S is chosen so that $S \cap K \in \mathrm{Syl}_p(H \cap K)$. Let T be a non-trivial subgroup of $S \cap K$. Then, as H is strongly p-embedded, $N_K(T) \leq N_G(T) \leq H$, thus $N_K(T) \leq H \cap K$ and so we have that $H \cap K$ is strongly p-embedded in K. Thus (ii) holds.

Part (iii) follows from (i) and (ii).

Since, by Lemma 3.1, p does not divide $|H \cap H^g|$, we have that $\bigcap H^G \leq O_{p'}(G)$. So (iv) holds.

Finally, assume that $m_p(H) \geq 2$ and suppose that $A \leq S$ is elementary abelian of order p^2. Since H is strongly p-embedded, for $a \in A^\#$, we have $C_{O_{p'}(G)}(a) \leq C_G(a) \leq H$. Therefore, from coprime action,

$$O_{p'}(G) = \langle C_{O_{p'}(G)}(a) \mid a \in A^\# \rangle \leq \langle C_G(a) \mid a \in A^\# \rangle \leq H.$$

Thus (v) follows from (iv). $\square$

Lemma 3.3. *Suppose that* G *is a group,* p *is a prime and* H *is a strongly* p*-embedded subgroup of* G. *Set* $\overline{G} = G/O_{p'}(G)$ *and assume further that* $\overline{H} \neq \overline{G}$. *Then*

(i) $\overline{H}$ *is strongly* p*-embedded in* $\overline{G}$.
(ii) $F^*(\overline{G})$ *is a non-abelian simple group; and*
(iii) *if* G *is a* $\mathcal{K}$*-group,* p *is odd, and* $m_p(G) \geq 2$, *then* $(F^*(\overline{G}), p) \in \mathcal{E}$.

Proof. Let $S \in \mathrm{Syl}_p(H)$. Then $\overline{S} \in \mathrm{Syl}_p(\overline{H})$. Choose X a non-trivial subgroup of $\overline{S}$. Then there exists $T \leq S$ such that $X = \overline{T}$. Therefore, as H is strongly p-embedded in G, we have

$$HO_{p'}(G) \geq N_G(T)O_{p'}(G) = N_G(X)$$

by the Frattini Argument. Hence $\overline{H} \geq N_{\overline{G}}(X)$. Since $\overline{H} \neq \overline{G}$, we conclude that $\overline{H}$ is strongly p-embedded in $\overline{G}$. Hence (i) holds.

Since $\overline{H}$ is strongly p-embedded in $\overline{G}$ and $O_{p'}(\overline{G}) = 1$, we have $F(\overline{G}) = 1$. Assume that $E(\overline{G})$ is not simple. Let K_1 be an arbitrary component of $\overline{G}$ and let K_2 be a component with $K_1 \neq K_2$. Then, as $O_{p'}(\overline{G}) = 1$, p divides $|K_2|$. Let $T = \overline{S} \cap K_2$. Then $T \in \mathrm{Syl}_p(K_2)$. Thus, as $\overline{H}$ is strongly p-embedded in $\overline{G}$ and $[K_1, K_2] = 1$, $K_1 \leq N_{\overline{G}}(T) \leq \overline{H}$. Since K_1 was chosen arbitrarily, we have that $E(\overline{G}) \leq \overline{H}$. Then Lemma 3.2 (iv) implies that $E(\overline{G}) = 1$ which is impossible. Thus $F^*(\overline{G}) = F(\overline{G})E(\overline{G}) = E(\overline{G})$ is a simple group and (ii) holds

Finally (iii), follows directly from the definition of $\mathcal{E}$ and Proposition 2.5.

□

Corollary 3.4. *If $m_p(H) \geq 2$ and $O_{p'}(H) = 1$, then $F^*(G)$ is a non-abelian simple group.*

Proof. We have $m_p(H) = m_p(G) \geq 2$. Therefore, Lemma 3.2(iv) implies $O_{p'}(G) = \bigcap H^G \leq O_{p'}(H) = 1$. Then Lemma 3.3(ii) implies that $F^*(G)$ is a non-abelian simple group as claimed. □

Lemma 3.5. *Suppose that p is an odd prime and H is strongly p-embedded in G. Assume that for all involutions $t \in H$, p divides $|C_H(t)|$. Then for all involutions $t \in H$, $t^G \cap H = t^H$.*

Proof. Suppose that $t \in H$ is an involution. Obviously $t^H \subseteq t^G \cap H$. Assume that $g \in G$ and $t^g \in H$. Let $X \in \mathrm{Syl}_p(C_H(t))$, $Y \in \mathrm{Syl}_p(C_H(t^g))$ and note that by assumption X and Y are non-trivial. As H is strongly p-embedded in G, X and Y are Sylow p-subgroups of $C_G(t)$ and $C_G(t^g)$ respectively. Thus $X^g, Y \in \mathrm{Syl}_p(C_G(t^g))$ and so there exists $c \in C_G(t^g)$ such that $Y = X^{gc}$. Therefore $Y \leq H \cap H^{gc}$ and, since H is strongly p-embedded in G, we get $gc \in H$. But then $t^g = t^{gc} \in t^H$ and we are done. □

Lemma 3.6. *Suppose that p is an odd prime and H is strongly p-embedded in G. Assume that $t \in H$ is an involution, $t^G \cap H = t^H$ and F is a subgroup of H which contains t. If $N_G(F) \not\leq H$, then $C_G(t) \not\leq H$.*

Proof. Aiming for a contradiction, we assume that $C_G(t) = C_H(t)$. Let $k \in N_G(F) \setminus H$. Then $t^k \in F \leq H$ and, as $t^G \cap H = t^H$, there exists $h \in H$ such that $t^{kh} = t$. But then $kh \in C_G(t) = C_H(t) \leq H$. Hence $k \in H$, which is a contradiction. Thus $C_G(t) \not\leq H$ as claimed. □

4. Involutions in H

In this section we initiate the investigation of groups satisfying the hypotheses of Theorems 1.1 and 1.2. Specifically throughout the remainder of this article we assume that the following hypothesis holds.

Hypothesis 4.1. *p is an odd prime, H is a strongly p-embedded subgroup in G and $H \cap K$ has even order for each non-trivial normal subgroup K of G. Furthermore we assume that the following hold.*

(i) $O_{p'}(H) = 1$.
(ii) *$m_p(C_H(t))$ is at least 2 for each involution t in H.*
(iii) *$N_G(T)$ is a $\mathcal{K}$-group, for all non-trivial 2-subgroups T of G.*
(iv) $O^2(G) = G$.

Because of Hypothesis 4.1 (i) and (ii), we can apply Corollary 3.4 to see that $F^*(G)$ is a non-abelian simple group. The Frattini Argument then shows $G = HF^*(G)$. Furthermore, we have that $F^*(G) \cap H < F^*(G)$ and has even order.

Lemma 4.2. *Suppose that K is a subgroup of G, $K \not\leq H$, $m_p(H \cap K) \geq 2$ and K is a $\mathcal{K}$-group. Then the following hold.*

(i) $O_{p'}(K) \leq H$.
(ii) $(F^*(K/O_{p'}(K)), p) \in \mathcal{E}$.
(iii) *Either*
  (a) $O_{p'}(H \cap K) = O_{p'}(K)$; *or*
  (b) $(F^*(K/O_{p'}(K)), p) = (\mathrm{PSL}_3(4), 3)$ *or* $(\mathrm{Fi}_{22}, 5)$ *and* $|O_{p'}(H \cap K)/O_{p'}(K)| = 2$.

Proof. As H is strongly p-embedded in G, we can use Lemma 3.2 (ii) and (v) to see that $O_{p'}(K) \leq O_{p'}(H \cap K) \leq H$. Then, as K is a $\mathcal{K}$-group and $m_p(H \cap K) \geq 2$, Lemma 3.3(iii) implies $(F^*(K/O_{p'}(K)), p) \in \mathcal{E}$. Thus (ii) holds. Part (iii) follows from (ii) and Corollary 2.11. □

Lemma 4.3. *There exists an involution $t \in H$ such that $C_G(t) \not\leq H$.*

Proof. Assume the lemma is false. Set $K = F^*(G)$. Then K is a non-abelian simple group and $C_K(t) \le H \cap K$ for all involutions t in $H \cap K$. It follows from [9, Lemma 17.13] that K contains a strongly 2-embedded subgroup. Therefore, by [2], we have $K \cong \mathrm{SL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$ or ${}^2\mathrm{B}_2(2^a)$ for some a. In particular, K is a $\mathcal{K}$-group and as $m_p(G) \ge 2$, Proposition 2.5 delivers the contradiction. Thus there exists an involution t in H such that $C_G(t) \not\le H$. □

We can now present our first significant result.

Lemma 4.4. *If $t \in H$ is an involution, then*

(i) *$C_G(t) \not\le H$ and $C_H(t)$ is strongly p-embedded in $C_G(t)$;*
(ii) *$O_{p'}(C_G(t)) \le H$;*
(iii) *$(F^*(C_G(t)/O_{p'}(C_G(t))), p) \in \mathcal{E}$; and*
(iv) *$O(C_G(t)) = O(O_{p'}(C_G(t)))$.*

Proof. By Lemma 4.3, there exists an involution $t \in G$ such that $C_G(t) \not\le H$. Choose t with $|C_H(t)|_2$ maximal. Set $K = C_G(t)$ and note that $C_H(t)$ is strongly p-embedded in $C_G(t)$ by Lemma 3.2 (ii). Then, by Hypothesis 4.1(ii) and Lemma 4.2(i) and (ii), $O_{p'}(K) \le H$ and $(F^*(K/O_{p'}(K)), p) \in \mathcal{E}$. Let $T \in \mathrm{Syl}_2(H \cap K)$. We will show that $C_G(s) \not\le H$, for all involutions $s \in T$. Assume first that $T \notin \mathrm{Syl}_2(K)$. Then $N_G(T) \not\le H$. Hence Hypothesis 4.1 (ii) and Lemmas 3.5 and 3.6 combine to give us that $C_G(s) \not\le H$ for all involutions $s \in T$ which is our claim. So assume that $T \in \mathrm{Syl}_2(K)$. Then, as $(H \cap K)/O_{p'}(K)$ is strongly p-embedded in $K/O_{p'}(K)$, $(F^*(K/O_{p'}(K), p) \in \mathcal{E}$ and $|K : H \cap K|$ is odd, Lemma 2.14 implies that $F^*(K/O_{p'}(K)) = K/O_{p'}(K) \cong \mathrm{M}_{11}$. Let $F \le T$ be such that $F \cap O_{p'}(K)$ is a Sylow 2–subgroup of $O_{p'}(K)$ and $|F : F \cap O_{p'}(K)| = 2$. Assume that $s \in T$ is an involution with $C_G(s) \le H$. Then, as $K/O_{p'}(K) \cong \mathrm{M}_{11}$ has exactly one conjugacy class of involutions, all the involutions of T are G-conjugate to involutions in F. Thus $\emptyset \ne s^G \cap F \le s^G \cap H = s^H$ by Lemma 3.5 and we may therefore assume that $s \in F$. However, from the structure of M_{11}, $N_K(FO_{p'}(K))/O_{p'}(K) \not\le (H \cap K)/O_{p'}(K)$ and so we have $N_G(F) \not\le H$ by the Frattini Argument. Thus Lemma 3.6 implies that $C_G(s) \not\le H$ which is a contradiction. Hence $C_G(s) \not\le H$ for all involutions $s \in T$ and our claim is proved.

Let z be a 2-central involution of H which centralizes t. Then we may suppose that $z \in T$. It follows that $C_G(z) \not\le H$. By maximality of $|C_H(t)|_2$ we may assume $t = z$ and so T is a Sylow 2-subgroup of H and this proves (i).

Suppose that t is an involution in H. Then by Hypothesis 4.1(ii), $m_p(C_H(t)) \geq 2$ and $C_G(t) \not\leq H$ by (i). Hence Lemma 4.2 (i) and (ii) gives parts (ii) and (iii). Finally, as $O(C_G(t))$ is soluble, (iii) implies (iv). □

As a first application of Lemma 4.4, we show that if $E(H)$ is non-trivial then it is quasisimple.

Lemma 4.5. *If $E(H) \neq 1$, then $E(H)$ is quasisimple.*

Proof. Set $E = E(H)$ and suppose that E is the product of at least two components of H. Assume that L_1 is a component of H and let L_2 be the product of all the components of H which are distinct from L_1. Then $E = L_1L_2$ and, since $O_2(H) \leq O_{p'}(H) = 1$ by Hypothesis 4.1 (i), there exists an involution $t \in L_1 \setminus L_2$. We have $C_G(t) \geq L_2$ and, as p divides the order of each component in L_2, we have that $L_2/Z(L_2)$ is isomorphic to a subnormal section of $C_H(t)/O_{p'}(C_H(t))$. Using Lemma 4.4 (ii) and (iii), Proposition 2.7 and the fact that $O_2(H) = 1$, we deduce that (L_2, p) is $(\Omega_8^+(2), 5)$, $(\mathrm{Alt}(p), p)$ or $(\mathrm{Alt}(p) \times \mathrm{Alt}(p), p)$ where in the latter two cases we have $p \geq 5$. Furthermore, we have that $O_p(C_H(t)/O_{p'}(C_H(t))) = 1$ and thus, as $O_p(H) \leq C_G(t)$, we have $O_p(H) = 1$. By applying the above argument to L_1 with an involution taken from L_2, we have that (L_1, p) is $(\Omega_8^+(2), 5)$ or $(\mathrm{Alt}(p), p)$. The possibilities for the isomorphism type of E when $p = 5$ are thus $\Omega_8^+(2) \times \Omega_8^+(2)$, $\Omega_8^+(2) \times \mathrm{Alt}(5) \times \mathrm{Alt}(5)$, $\Omega_8^+(2) \times \mathrm{Alt}(5)$, $\mathrm{Alt}(5) \times \mathrm{Alt}(5)$ or $\mathrm{Alt}(5) \times \mathrm{Alt}(5) \times \mathrm{Alt}(5)$ and, for $p > 5$, we have $E \cong \mathrm{Alt}(p) \times \mathrm{Alt}(p) \times \mathrm{Alt}(p)$ or $\mathrm{Alt}(p) \times \mathrm{Alt}(p)$. Since $O_{p'}(H) = 1 = O_p(H)$ we have $E = F^*(H)$. Therefore, as $p \geq 5$, the structure of E and the fact that the given components have no outer automorphisms of order p implies that $E(H)$ contains a Sylow p-subgroup of H. Select an involution $d \in L_1L_2$ which projects non-trivially onto each component as a 2-central involution. Then p does not divide $C_E(d)$ and we have a contradiction to Hypothesis 4.1(ii). Thus E is quasisimple as claimed. □

5. Centralizers of involutions in H

In this section we work under Hypothesis 4.1 and aim to uncover the basic structure of the centralizers of involutions from H under the additional hypothesis that H has no components and that none of the involutions in H are classical involutions (see Definition 2.1). We formalize the configuration we shall be investigating in the following hypothesis.

Hypothesis 5.1. *Hypothesis 4.1 holds,*

(i) $E(H) = 1$; *and*
(ii) G *does not contain a classical involution.*

Assume Hypothesis 5.1. Then, as $O_{p'}(H) = 1$ by Hypothesis 4.1 (i), we have that $F^*(H) = O_p(H)$. Set $Q = O_p(H)$ and note that, as H is strongly p-embedded in G, we have $H = N_G(Q)$ and, as $Q = F^*(H)$, $C_G(Q) \leq Q$.

Lemma 5.2. *Suppose that $s \in H$ is an involution with $C_Q(s) > 1$. Then there exists a normal component L in $C_G(s)$ such that*

$$LO_{p'}(C_G(s))/O_{p'}(C_G(s)) = F^*(C_G(s)/O_{p'}(C_G(s))).$$

Furthermore, if $Z(L)$ has order divisible by 2, then $L/Z(L) \cong \mathrm{PSL}_3(4)$.

Proof. Since $C_Q(s) > 1$ and $O_{p'}(C_G(s)) \leq H$ by Lemma 4.4(ii), we have that $[C_Q(s), O_{p'}(C_G(s))] = 1$. Hence $F^*(C_G(s)) \not\leq O_{p'}(C_G(s))$. Thus, as by Lemma 4.4 (iii) $C_G(s)/O_{p'}(C_G(s))$ is an almost simple group , we have $E(C_G(s)) \not\leq O_{p'}(C_G(s))$ and so there is a component L of $C_G(s)$ such that $LO_{p'}(C_G(s))/O_{p'}(C_G(s)) = F^*(C_G(s)/O_{p'}(C_G(s)))$. Since L is the unique component of $C_G(s)$ which has order divisible by p, we deduce that L is normal in $C_G(s)$. If 2 divides $|Z(L)|$, then Lemma 2.12, the fact that $C_Q(s)O_{p'}(C_G(s))/O_{p'}(C_G(s))$ is a non-trivial normal subgroup of $C_H(s)/O_{p'}(C_H(s))$ and Proposition 2.7, together imply that $L \cong \mathrm{SL}_2(p^a)$ or $L/Z(L) \cong \mathrm{PSL}_3(4)$. Since s is not a classical involution by Hypothesis 5.1 (ii), we must have $L/Z(L) \cong \mathrm{PSL}_3(4)$ and thus the lemma is established. □

Lemma 5.3. *Let $u \in H$ be an involution with $C_Q(u) = 1$. Then there is a normal 2-component L_u in $C_G(u)$ such that L_u is not contained in $O_{p'}(C_G(u))$.*

Proof. Let W be a Sylow 2-subgroup of $O_{p'}(C_G(u))$. Then we see that $C_G(u) = N_{C_G(u)}(W)O_{p'}(C_G(u))$. Set $\overline{C_G(u)} = C_G(u)/O_{\{p,2\}'}(C_G(u))$ and let further $\overline{F} = F^*(\overline{C_G(u)})$. Since, by Lemma 4.4(iii), $C_G(u)/O_{p'}(C_G(u))$ is an almost simple group, we may assume that $\overline{F} \leq \overline{O_{p'}(C_G(u))}$ for otherwise there would be a component of $\overline{C_G(u)}$ not contained in $\overline{O_{p'}(C_G(u))}$. Note that, as $C_H(u)$ is strongly p-embedded in $C_G(u)$, $O_p(\overline{F}) = 1$. Suppose that $C_{\overline{C_G(u)}}(\overline{W}) \not\leq \overline{O_{p'}(C_G(u))}$. Then $\overline{F}$ is not a 2-group and consequently $E(\overline{C_G(u)}) \neq 1$. Since $\overline{W}$ intersects each component of $\overline{C_G(u)}$ in a Sylow 2-subgroup, we see that $C_{\overline{C_G(u)}}(\overline{W})$ normalizes each component of $\overline{C_G(u)}$. Therefore, as $C_{\overline{C_G(u)}}(\overline{W})\overline{F}/\overline{F}$ is not soluble, the Schreier property of simple groups delivers a contradiction to $\overline{F}$ being self-centralizing. Hence $C_{\overline{C_G(u)}}(\overline{W}) \le \overline{O_{p'}(C_G(u))}$.

As $C_Q(u) = 1$, we have that $H = QC_H(u)$ and Q is abelian. Assume that $s \in W$ is an involution such that $s \neq u$ and $C_Q(s) \notin \mathrm{Syl}_p(C_G(s))$. Since $s \neq u$, we have $C_Q(s) > 1$. Therefore Lemma 5.2 implies that $C_G(s)$ has a normal component L_s with $L_s \not\le O_{p'}(C_G(s))$. Let $P \in \mathrm{Syl}_p(H \cap C_G(s))$. Then, as $H \cap C_G(s)$ normalizes $C_Q(s)$ and $uQ \in Z(H/Q)$, $C_Q(s) \le P$ and $[P, u] \le C_Q(s)$. Since $C_Q(s) < P$ and $C_Q(s)$ is normal in $H \cap C_G(s)$, Corollary 2.8 implies that $C_Q(s) \le \Phi(P)$ or $(L_s, p) \cong (\mathrm{PSL}_2(8), 3)$ or $({}^2\mathrm{B}_2(32), 5)$. In the former case, we have that $[P, u] \le C_Q(s) \le \Phi(P)$ and so $P \le C_G(u)$, which contradicts $[C_Q(s), u] \neq 1$. Thus we have the latter situation, and, as $P \neq C_Q(s)$, this means that $C_Q(s)$ is of order either p or p^2. As $u \in N_{C_G(s)}(u)$, we see that $C_Q(s) \le L_s$ and so $C_Q(s)$ is cyclic. As $[P, u] \le P$ also $[P, us] \le Q$ and so also $C_Q(s)$ is not a Sylow p–subgroup of $C_G(us)$. This implies that also $C_Q(us)$ is cyclic of order p or p^2. In particular, since $\langle u, s\rangle$ acts on Q, we get, using coprime action, that $|Q/\Phi(Q)| = p^2$. But then H/Q is isomorphic to a subgroup of $\mathrm{GL}_2(p)$. It follows that $m_p(C_H(u)) \le 1$ and this contradicts Hypothesis 4.1(ii) which asserts that $m_p(C_H(u)) \ge 2$. We have seen that for any involution $s \neq u$ in W we have that $C_Q(s)$ is a Sylow p-subgroup of $C_G(s)$.

As, by Hypothesis 4.1 (ii), $m_p(H/Q) = m_p(C_H(u)) \ge 2$, there is a non-cyclic abelian p-subgroup E contained in $C_H(u)$. Suppose that $e \in E^\#$ and $w \in C_W(e)$ is an involution. Then $C_Q(w)\langle e\rangle$ is a p-subgroup of $C_G(w)$. If $w \neq u$, then $C_Q(w) \in \mathrm{Syl}_p(C_G(w))$. Since this is not the case, we see that for all $e \in E^\#$, u is the unique involution in $C_W(e)$. Therefore using Lemma 2.26 we have $p = 3$ and $W \cong 2^{1+4}_+$, 2^{1+6}_- or 2^{1+8}_+. By Lemma 4.4 (iii) $F^*(C_G(u)/O_{p'}(C_G(u)))$ is a non-abelian simple group and so, since $C_G(W) \le O_{p'}(C_G(u))$, we have that $N_G(W)/C_G(W)$ is non-soluble. In particular, we have that W is not isomorphic to 2^{1+4}_+ which has soluble automorphism group. Let $s \in W$ be an involution with $s \neq u$. Then $C_W(s) \cong 2 \times 2^{1+4}_-$ if $W \cong 2^{1+6}_-$ and $C_W(s) \cong 2 \times 2^{1+6}_+$ if $W \cong 2^{1+8}_+$. Since u inverts Q, it inverts $C_Q(s)$. Thus $C_Q(s)$ is abelian and $C_Q(s)$ admits a faithful action of $C_W(s)/\langle s\rangle$. By Lemma 4.4 (iii), $(F^*(C_G(s)/O_{p'}(C_G(s))), p) \in \mathcal{E}$. Since $C_Q(s) \in \mathrm{Syl}_p(C_G(s))$ and $C_W(s)/\langle s\rangle$ is extraspecial of order 2^5 or 2^7, we have a contradiction to Proposition 2.7 (i), (iii), (iv) and (v). This finally proves that there is a component in $C_G(u)/O_{\{2,p\}'}(C_G(u))$. $\square$

We let $\mathcal{I}$ be the set of involutions $t \in H$ with $C_Q(t) > 1$ and such that $|C_Q(t)| \geq |C_Q(s)|$ for all involutions $s \in H$. For any involution $x \in H$ with $C_Q(x) > 1$, we let L_x denote the normal component of $C_G(x)$ described in Lemma 5.2.

Lemma 5.4. *Suppose that s is an involution in H with $C_Q(s) > 1$. Then $C_{C_G(s)}(L_sC_Q(s)) = C_{C_G(s)}(L_s) = O_{p'}(C_G(s))$.*

Proof. We clearly have $[C_Q(s), O_{p'}(C_G(s))] = 1$. Therefore, we also have $[\langle C_Q(s)^{C_G(s)}\rangle, O_{p'}(C_G(s))] = 1$. Hence, as $L_sC_Q(s) \leq \langle C_Q(s)^{C_G(s)}\rangle$, $O_{p'}(C_G(s)) \leq C_{C_G(s)}(L_sC_Q(s)) \leq C_{C_G(s)}(L_s)$. On the other hand, because $F^*(C_G(s)/O_{p'}(C_G(s))) = L_sO_{p'}(C_G(s))/O_{p'}(C_G(s))$, we also have $C_{C_G(s)}(L_s) \leq O_{p'}(C_G(s))$. This proves the lemma. □

Lemma 5.5. *Let $s \in H$ be an involution such that $C_Q(s) > 1$. Then $C_H(s)/O_{p'}(C_G(s))$ is p-closed. In particular, $F^*(C_G(s)/O_{p'}(C_G(s)))$ is not isomorphic to* $\mathrm{Alt}(2p)$, $p \geq 5$ *or to* Fi_{22}.

Proof. As $C_Q(s)O_{p'}(C_G(s))/O_{p'}(C_G(s))$ is a non-trivial normal p-subgroup of $C_H(s)/O_{p'}(C_G(s))$, the assertion follows from Proposition 2.7. □

Lemma 5.6. *Assume that $s, t \in H$ are involutions, $C_Q(t) = 1$ and $t \neq s \in C_H(t)$. Then $C_G(\langle s, t\rangle)$ is a p'-group. In particular, if $P \in \mathrm{Syl}_p(C_G(t))$, then $m_2(N_{C_G(t)}(P)) \leq 2$.*

Proof. Since $C_Q(t) = 1$, we have $H = QC_H(t)$ and $C_H(s) = C_Q(s)C_H(\langle s, t\rangle)$. Let $P \in \mathrm{Syl}_p(C_H(\langle s, t\rangle))$. Then by Lemma 3.2 (i) $C_Q(s)P \in \mathrm{Syl}_p(C_H(s)) \subseteq \mathrm{Syl}_p(C_G(s))$. Since t normalizes $C_Q(s)P$ and t centralizes P but inverts $C_Q(s)$, we must have that $C_Q(s) \not\leq \Phi(C_Q(s)P)$. Assume that $P \neq 1$. Then we get $C_Q(s)\Phi(C_Q(s)P) \neq C_Q(s)P$. In particular, $C_H(s)/O_{p'}(C_H(s))$ does not act irreducibly on $C_Q(s)P/\Phi(C_Q(s)P)$. Since $C_H(s)/O_{p'}(C_H(s))$ is p-closed by Lemma 5.5, Corollary 2.8 implies that (L_s, p) is either $(\mathrm{PSL}_2(8), 3)$ or $({}^2\mathrm{B}_2(32), 5)$. As t inverts $C_Q(s)$, we see that $C_Q(s) \leq L_s$ and then $C_Q(s)$ is a cyclic group of order dividing p^2. Since $Q = C_Q(st)C_Q(s)$, we have that $|Q/\Phi(Q)| \leq p^2$. This means that H/Q is isomorphic to a subgroup of $\mathrm{GL}_2(p)$ and implies that $C_H(t)$ has cyclic Sylow p-subgroups. This of course contradicts Hypothesis 4.1(ii). Thus $C_G(\langle s, t\rangle)$ is a p'-group as claimed. □

Lemma 5.7. *Suppose that $t \in \mathcal{I}$ or $C_Q(t) = 1$. Then $O_{p'}(C_G(t))$ has cyclic Sylow 2-subgroups. In particular, $O_{p'}(C_G(t))$ has a normal 2-complement.*

Proof. Assume that $F = \langle t, s\rangle$ is a fours group in $O_{p'}(C_G(t))$. Then $Q = C_Q(t)C_Q(s)C_Q(ts)$.

Assume further that $t \in \mathcal{I}$. Then, as $F \le O_{p'}(C_G(t)) \le H$ and $[F, C_Q(t)] \le [O_{p'}(C_G(t)), C_Q(t)] = 1$, the maximality of $C_Q(t)$ implies that $C_Q(t) = C_Q(F) = C_Q(s)$. But then $Q = C_Q(t)$, which contradicts $C_G(Q) \le Q$. Hence no such F exists and so $O_{p'}(C_G(t))$ has either quaternion or cyclic Sylow 2-subgroups.

Assume that $C_Q(t) = 1$. Then, by Lemma 5.3, there is a 2-component L in $C_G(t)$ with $LO_{p'}(C_G(t))/O_{p'}(C_G(t)) = F^*(C_G(t)/O_{p'}(C_G(t)))$. Now we have $[L, F] \le O(C_G(t))$. In particular, $C_H(F)$ contains a Sylow p-subgroup P of L and, as P is non-trivial, this contradicts Lemma 5.6. Thus once again, as t is not a classical involution, we have that $O_{p'}(C_G(t))$ has cyclic Sylow 2-subgroups and a normal 2-complement.

Since, by Hypothesis 5.1 (ii), t is not a classical involution, we deduce that $O_{p'}(C_G(t))$ has cyclic Sylow 2-subgroups and a normal 2-complement. □

6. Centralizers of involutions with $F^*(C_G(t)/O_{p'}(C_G(t))) \not\cong \mathrm{PSL}_2(p^f)$

In this section we will prove the following theorem.

Theorem 6.1. *Assume that Hypotheses 4.1 and 5.1 hold. Then either*

(i) *there exists $t \in \mathcal{I}$ with $F^*(C_G(t)/O_{p'}(C_G(t))) \cong \mathrm{PSL}_2(p^f)$ for some $f > 1$; or*
(ii) *for all involutions $t \in H$, $C_Q(t) = 1$ and $F^*(C_G(t)/O_{p'}(C_G(t))) \cong \mathrm{PSL}_2(p^f)$ with $f > 1$ and $p \equiv 3 \pmod 4$.*

We prove Theorem 6.1 via a sequence of lemmas the first of which shows that the theorem holds if $\mathcal{I}$ is empty. We continue to use the notation of Section 5. In particular, $Q = O_p(H)$.

Lemma 6.2. *If $\mathcal{I} = \emptyset$, then Theorem 6.1 (ii) holds.*

Proof. Suppose that $\mathcal{I} = \emptyset$. Then $C_Q(t) = 1$ for all involutions $t \in H$ and, in particular, $m_2(H) = 1$. Therefore the Sylow 2-subgroups of H are either cyclic or quaternion. Let $t \in H$ be an involution. Then $C_Q(t) = 1$. By Lemma 5.3 there is a normal 2-component L of $C_G(t)$ with $L \not\le O_{p'}(C_G(t))$. Note that Lemma 4.4 (ii) and (iv) imply that $O(L) \le H$. Since H has cyclic or quaternion Sylow

$(L_t/Z(L_t),p)$	$C_{L_t/Z(L_t)}(s)$
$(\mathrm{M}_{11},3)$	$\mathrm{GL}_2(3)$
$({}^2\mathrm{F}_4(2)',5)$	$2^9.\mathrm{Frob}(20)$
$(\mathrm{McL},5)$	$2{\cdot}\mathrm{Alt}(8)$
$(\mathrm{J}_4,11)$	$2_+^{1+12}.3{\cdot}\mathrm{Aut}(\mathrm{M}_{22})$
$(\mathrm{PSU}_3(p^a),p)$	$O^{p'}(C_{L_t/Z(L_t)}(s))\cong \mathrm{SL}_2(p^a)$

Table 1. Centralizers of s in $L_t/Z(L_t)$

2-subgroups, so does $C_H(t)$. If $t\in L$, then $tO(L)\in Z(L/O(L))$ is non-trivial. Therefore, as the Sylow 2-subgroups of $L\cap H$ are either cyclic or quaternion, Corollary 2.13 implies that $L/O(L)\cong \mathrm{SL}_2(p^f)$ for some f. But then G contains a classical involution, and this contradicts Hypothesis 5.1 (ii). Hence $t\notin L$. Since t is the unique involution in $C_H(t)$, we deduce that $|H\cap L|$ is odd. Now Lemma 2.15 shows that $L/O(L)\cong \mathrm{PSL}_2(p^f)$ where $f>1$ is odd and $p\equiv 3 \pmod 4$. $\square$

We may now assume that $\mathcal{I}\neq\emptyset$ and so we begin to restrict the possibilities for the structure of $C_G(t)$, where $t\in H$ is an involution with $C_Q(t)>1$. By Lemma 4.4 $C_G(t)\not\leq H$ and $O_{p'}(C_G(t))\leq H$ and by Lemma 5.2, there is a normal component L_t of $C_G(t)$ such that $L_t\not\leq O_{p'}(C_G(t))$. By Lemma 4.4 (iii) we then have $(L_t/Z(L_t),p)\in\mathcal{E}$. Our aim now is to show that there is an involution $t\in\mathcal{I}$ with $L_t\cong \mathrm{PSL}_2(p^f)$.

Lemma 6.3. *If $t\in H$ is an involution with $C_Q(t)>1$, then $(L_t/Z(L_t),p)$ is one of $(\mathrm{PSL}_2(p^a),p)$, with p arbitrary and $a\geq 2$, $({}^2\mathrm{G}_2(3^{2a-1}),3)$ with $a\geq 2$, $(\mathrm{PSL}_2(8),3)$, $(\mathrm{PSL}_3(4),3)$ or $({}^2\mathrm{B}_2(32),5)$.*

Proof. Assume the claim is false. Then, as $(L_t/Z(L_t),p)\in\mathcal{E}$ and $O_p(H\cap L_t)\neq 1$, we have that $(L_t/Z(L_t),p)$ is one of $(\mathrm{PSU}_3(p^a),p)$ with $a\geq 2$, $(\mathrm{M}_{11},3)$, $(\mathrm{McL},5)$, $({}^2\mathrm{F}_4(2)',5)$ or $(\mathrm{J}_4,11)$. Furthermore, by Lemma 5.2, $Z(L_t)$ has odd order. In particular, in each of the cases we need to consider there is an involution $s\in L_t$ such that $C_{L_t}(s)$ has structure as indicated in the Table 1. In particular, we note that in each case $s\in O^{p'}(O^p(C_{L_t}(s)))$. By Lemma 3.2 (i) and (ii), $H\cap L_t$ is strongly p-embedded in L_t. Since in each of the groups under consideration as L_t, the centralizer of s has order divisible by p we may assume that $s\in H\cap L_t$. Since $Q\cap L_t=C_Q(t)$ is normalized by $C_H(t)$, Proposition 2.7 implies that we either have that $C_Q(t)\in\mathrm{Syl}_p(L_t)$ or $p=11$, $L_t\cong J_4$ and $C_Q(t)$ is the centre of the Sylow 11-subgroup of $H\cap L_t$ or $L_t\cong\mathrm{PSU}_3(p^a)$. In each case, we have

$C_Q(s) \geq C_{C_Q(t)}(s) > 1$. Therefore Lemma 5.2 indicates that $C_G(s)$ has a normal component L_s such that $L_s \not\leq O_{p'}(C_G(s))$. Since $[O_{p'}(C_G(s)), L_s] = 1$, we have that $L_s = O^{p'}(O_{p'}(C_G(s))L_s)$. Hence, we either have $O^{p'}(O^p(C_G(s))) = L_s$, or p divides $|C_G(s)/O_{p'}(C_G(s))L_s|$. In the latter case, Proposition 2.5 shows that $C_G(s)/O_{p'}(C_G(s)) \cong \mathrm{PSL}_2(8):3$ and $p = 3$ or $C_G(s)/O_{p'}(C_G(s)) \cong {}^2\mathrm{B}_2(32):5$ and $p = 5$. In this case we have $L_s = O^{p'}(O^p(C_G(s)))$ as well. It follows that in all cases $s \in L_s$, as $s \in O^{p'}(O^p(C_{L_t}(s)))$. Therefore Lemma 5.2 implies that $L_s/Z(L_s) \cong \mathrm{PSL}_3(4)$ and that $p = 3$. Furthermore, the Sylow 3-subgroups of $C_G(s)$ have order 9 and so after consulting Table 1 we get $L_t \cong \mathrm{M}_{11}$, $\mathrm{PSU}_3(3)$ or $\mathrm{PSU}_3(9)$. If $t \in O_{p'}(C_G(s))L_s$, then either $C_G(\langle t, s\rangle)$ would have order coprime to 3 or would contain a component K such that $K/Z(K)$ is isomorphic to $\mathrm{PSL}_3(4)$, from the choice of s we see that both of these scenarios are impossible. Hence t induces an outer automorphism of L_s and so with Lemma 2.10 we get that $C_{L_s/Z(L_s)}(t) \cong 3^2.\mathrm{Q}_8$, $\mathrm{PSL}_2(7)$ or $\mathrm{Alt}(5)$. Since $C_{L_s}(t)$ projects to a subgroup of $C_G(t)/O_{p'}(C_G(t))$ which is normal in the centralizer of the image of s in $C_G(t)/O_{p'}(C_G(t))$, this then gives us our final contradiction. $\square$

Lemma 6.4. *If $t \in \mathcal{I}$, then $L_t/Z(L_t)$ is not isomorphic to ${}^2\mathrm{G}_2(3^{2n+1})$, $n \geq 1$.*

Proof. Suppose that $L_t/Z(L_t) \cong {}^2\mathrm{G}_2(3^{2n+1})$ with $n \geq 1$. Note that by [10, Theorem 2.5.12] the outer automorphisms of L_t have odd order and $Z(L_t) = 1$. Let $R \in \mathrm{Syl}_2(L_tO_{p'}(C_G(t))) = \mathrm{Syl}_2(C_G(t))$ be such that $R \cap H \in \mathrm{Syl}_2(C_H(t))$ and set $T = R \cap O_{p'}(C_G(t))$ and $E = T \cap L_t$. Then, by Lemma 5.7, T is a cyclic group and, as $L_t \cong {}^2\mathrm{G}_2(3^{2n+1})$, E is elementary abelian of order 8. Since $[L_t, O_{p'}(C_G(t))] = 1$, $R = T \times E$ is abelian and so $R \in \mathrm{Syl}_2(C_G(R))$. By Lemma 2.17, t is conjugate in G to an element $t^x \in R \setminus \{t\}$ and so Lemma 2.20 implies that T has order 2. Therefore R is elementary abelian of order 16.

From the structure of ${}^2\mathrm{G}_2(3^{2n+1})$ we have that $N_{L_t}(E)/E \cong \mathrm{Frob}(21) \cong N_{L_t^x}(E^x)/E^x$. Therefore $N_G(R)/C_G(R)$ contains distinct subgroups isomorphic to $\mathrm{Frob}(21)$ and $N_G(R)$ has orbits of lengths either 8 and 7 or 15 on $R^\#$. As $\mathrm{GL}_4(2)$ has no subgroups of order $3^2 \cdot 5 \cdot 7$ it is impossible for $N_G(R)$ to be transitive on $R^\#$. Hence $|t^{N_G(R)}| = |t^G \cap R| = 8$, $N_G(R)/C_G(R) \cong 2^3.\mathrm{Frob}(21)$ or $\mathrm{PSL}_3(2)$ and E is the unique normal subgroup of $N_G(R)$ of order 2^3. Let $s \in E^\#$ be chosen so that s normalizes a Sylow 3-subgroup D of $C_H(t)$, then s and t are not G-conjugate and $s \in H$. Furthermore $C_{L_t}(s) \cong 2 \times \mathrm{PSL}_2(3^{2n+1})$ and $C_{N_G(R)}(s)$ has non-abelian Sylow 2-subgroups. In particular, as 3 divides

$|C_G(\langle s,t\rangle)|$, Lemma 5.6 implies that $C_Q(s) \neq 1$. By Lemma 5.2, there is a normal component L_s in $C_G(s)$ with $L_s \not\leq O_{p'}(C_G(s))$. We then have that t induces an automorphism of L_s which centralizes a subgroup of L_s which is isomorphic to $\mathrm{PSL}_2(3^{2n+1})$. By Lemmas 2.10 and 6.3, we see that $L_s/Z(L_s) \cong {}^2\mathrm{G}_2(3^{2m+1})$ for some $m \geq 1$ or $L_s/Z(L_s) \cong \mathrm{PSL}_2(3^m)$ for some $m \geq 2n+1$. In the former case we have $C_{L_s}(t) \cong C_{L_t}(s)$ and so we infer that $L_t \cong L_s$ (but perhaps $s \notin \mathcal{I}$). Let $B \in \mathrm{Syl}_2(C_G(s))$ be chosen so that $R \leq B$. Then, as $\mathrm{Out}(L_s)$ has odd order and $Z(L_s) = 1$, $B = (B \cap O_{p'}(C_G(s))) \times (B \cap L_s)$ where $B \cap L_s$ is elementary abelian of order 8. Notice that $t \in B$ and $C_B(t) \geq \langle s\rangle(B \cap L_s)$. Since $R \in \mathrm{Syl}_2(C_G(t))$ and $|R| = 2^4$, we get that $R = \langle s\rangle(B \cap L_s)$. However, we then have $N_{L_s}(B \cap L_s) \leq N_G(R)$ and so $N_{L_s}(B \cap L_s)$ leaves both E and $B \cap L_s$ invariant. Hence $E = (B \cap L_s)$ and consequently $s \in E \subset L_s$ which is a contradiction. Assume that $L_s \cong \mathrm{PSL}_2(3^m)$. Then, by Proposition 2.5 and Lemma 4.4, $L_s = O^{3'}(C_G(s))$. Since $O^{3'}(C_{N_G(R)}(s))C_G(R)/C_G(R) \cong \mathrm{Alt}(4)$ and $|O^{3'}(C_{N_G(R)}(s)) \cap R| = 2^2$ or 2^3, we see that $O^{3'}(C_{N_G(R)}(s))$ contains a noncyclic abelian group of order at least 8. Hence L_s contains a noncyclic abelian group of order 8. So we have a contradiction to the Sylow 2-subgroup structure of $\mathrm{PSL}_2(3^m)$. This completes the proof of the lemma. □

Lemma 6.5. *If $t \in \mathcal{I}$, then $(L_t/Z(L_t), p)$ is not either $(\mathrm{PSL}_2(8), 3)$ or $({}^2\mathrm{B}_2(32), 5)$.*

Proof. Assume $(L_t/Z(L_t), p)$ is either $(\mathrm{PSL}_2(8), 3)$ or $({}^2\mathrm{B}_2(32), 5)$. Then $C_G(t)/O_{p'}(C_G(t)) \cong \mathrm{PSL}_2(8) : 3$ or ${}^2\mathrm{B}_2(32) : 5$ respectively. In both cases, there is an involution $s \in L_t$ which normalizes $C_Q(t)$. Thus $s \in H$. Let R be a Sylow 2-subgroup of $C_G(t)$ which contains $\langle s\rangle$. Then $R = T \times (R \cap L_t)$, where T is a Sylow 2-subgroup of $O_{p'}(C_G(t))$. By Lemma 5.7 we have that T is cyclic. By Lemma 2.17, t is conjugate to some element t^x in $R \setminus \{t\}$. Since $Z(R \cap L_t)$ is elementary abelian, Lemma 2.20 implies that $T = \langle t\rangle$ has order 2.

Assume that $(L_t, p) = (\mathrm{PSL}_2(8), 3)$. Then R is elementary abelian of order 2^4 and $N_{C_G(t)}(R)/C_{C_G(t)}(R) \cong \mathrm{Frob}(21) \cong N_{C_G(t^x)}(R)/C_{C_G(t^x)}(R)$. Just as in the proof of Lemma 6.4, we get $N_G(R)/C_G(R) \cong 2^3.\mathrm{Frob}(21)$ or $\mathrm{PSL}_3(2)$ and the $N_G(R)$-orbits on R have lengths 8 and 7 with t in the orbit of length 8. Since s normalizes a Sylow 3-subgroup of $C_H(t)$ and since s is contained in a unique Sylow 2-subgroup of L_t, we can choose $P \in \mathrm{Syl}_3(N_G(R) \cap H)$ so that $t \in C_R(P)$.

Let $F = \langle t, r\rangle = C_R(P)$. We may choose notation so that $|r^{N_G(R)}| = 7$. Since 3 divides $|C_G(F)|$, Lemma 5.6 implies that $C_Q(r) > 1$. Furthermore, as $C_{N_G(R)}(r)/R$ contains a subgroup isomorphic to Alt(4), we have $|O^{3'}(C_{N_G(R)}(r))| \geq 2^4.3$.

By Lemmas 5.2 and 6.3, we have that $L_r/Z(L_r)$ is isomorphic to one of $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_2(3^m)$, $m \geq 2$, or ${}^2\mathrm{G}_2(3^{2n+1})$, $n \geq 1$. If $L_r = O^{3'}(C_G(r))$, then $L_r \geq O^{3'}(C_{N_G(R)}(r))$. As ${}^2\mathrm{G}_2(3^{2n+1})$ and $\mathrm{PSL}_2(8)$ have Sylow 2-subgroups of order 8 and in $\mathrm{PSL}_2(3^m)$ the normalizer of a four group is isomorphic to either Alt(4) or Sym(4), these possibilities cannot arise. If $L_r/Z(L_r) \cong \mathrm{PSL}_3(4)$, then $|C_Q(r)| = 3^2$. Now in $C_G(t)$, we see that $|C_{C_G(t)}(r)|_3 = 3$ and $C_{C_G(t)}(r)O_{3'}(C_G(t))/O_{3'}(C_G(t))$ is soluble. Since t induces an automorphism of L_r, we now deduce a contradiction from Lemma 2.10. Thus $L_r \neq O^{3'}(C_G(r))$. Then $L_r \cong \mathrm{PSL}_2(8)$. Since R is abelian, $R \leq C_G(r)$. Let $D \in \mathrm{Syl}_2(C_G(r))$ with $R \leq D$. Then $D = (D \cap O_{3'}(C_G(r))) \times (D \cap L_r)$. Thus $C_D(t) \geq \langle r\rangle \times (D \cap L_r) \leq Z(D)$. It follows that $D \leq C_G(t)$ and so $R = D$. Since $N_G(R)$ normalizes a unique subgroup of order 2^3 in R, it follows that $r \in R \cap L_t = R \cap L_r$ which is of course impossible. This shows that $L_r \not\cong \mathrm{PSL}_2(8)$ and so $L_t/Z(L_t) \not\cong \mathrm{PSL}_2(8)$.

Assume that $(L_t, p) = ({}^2\mathrm{B}_2(32), 5)$. Let $P \in \mathrm{Syl}_5(C_H(t))$ and let $s \in L_t$ be such that $|C_G(s) \cap P| = 5$. Then $s \in H$ and $|C_{L_tP}(s)| = 2^{10} \cdot 5$. Therefore $O^{5'}(C_G(s))$ has a subgroup of order $2^9 \cdot 5$ and 2-rank 5. By Lemmas 5.2, 5.5 and 6.3, we have that $L_s/Z(L_s)$ is isomorphic to ${}^2\mathrm{B}_2(32)$ or $\mathrm{PSL}_2(5^m)$ for some $m \geq 1$. In the latter case, Proposition 2.5 implies that $L_s = O^{5'}(C_G(s))$. Since the 2-rank of $L_s/Z(L_s)$ is 2, we have a contradiction. Thus $L_s \cong {}^2\mathrm{B}_2(32)$. Since $s \in Z(R)$, $R \leq C_G(s)$. Let $D \in \mathrm{Syl}_2(C_G(s))$ with $R \leq D$. Then $D = (D \cap O_{5'}(C_G(s))) \times (D \cap L_s)$. As $t \in \Omega_1(D)$ and $\Omega_1(D) \leq (D \cap O_{5'}(C_G(s)))Z(D \cap L_s)$, we have that $D \cap L_s \leq R$. Therefore $R = (D \cap L_s)\langle s\rangle$. But then $s \in R' = (D \cap L_s)'$ and this is of course impossible. $\square$

Lemma 6.6. *There exists $t \in \mathcal{I}$ such that $(L_t/Z(L_t), p)$ is $(\mathrm{PSL}_2(p^f), p)$.*

Proof. Suppose the lemma is false and let $t \in \mathcal{I}$. Then by Proposition 2.5 and Lemmas 5.2, 5.5, 6.3, 6.4 and 6.5, we may assume that $(L_t/Z(L_t), p)$ is $(\mathrm{PSL}_3(4), 3)$.

Let R be a Sylow 2-subgroup of $C_G(t)$ and set $T = R \cap O_{3'}(C_G(t))$. By Lemma 5.7, T is cyclic. By Proposition 2.7 (iv), $(L_t \cap H)/Z(L_t) \cong 3^2 : \mathrm{Q}_8$ and so $|C_Q(t)O_{3'}(C_H(t))/O_{3'}(C_H(t))| = |C_Q(t)| = 9$. We also remark that as T

is cyclic, $Z(L_t)$ is cyclic and if $Z(L_t)$ is non-trivial then $t \in Z(L_t)$.

We remark that the normalizer of an elementary abelian group E of order 2^4 in $L_t/Z(L_t)$ has shape $2^4 : \mathrm{Alt}(5) \cong 2^4 : \mathrm{SL}_2(4)$. Further $N_{L_t/Z(L_t)}(E)$ acts transitively on $E^\sharp$. Suppose that $|Z(L_t)| = 2$. Then because of the transitive action of L_t on $E^\sharp$, we have that E lifts to an elementary abelian group F of order 32. In particular involutions lift to involutions. Suppose that $N_{L_t}(F)$ acts decomposably on F. Then $N_{L_t}(F)/E$ is a direct product of a group of order two by $\mathrm{PSL}_2(4)$. But then a Sylow 2–subgroup splits over $Z(L_t)$, which is not possible. Hence $N_{L_t}(F)$ acts indecomposably on F. Assume now that $Z(L_t)$ is cyclic of order four. If the preimage F_1 of E would be abelian, then it is of type $(4, 2, 2, 2, 2)$. Now $|\Omega_1(F_1)| = 32$ and $|\Phi(F_1)| = 2$. But then $N_{L_t}(F_1)$ cannot act indecomposably on $F_1/\Phi(F_1)$, which contradicts the fact just proved for $|Z(L_t)| = 2$. Hence F_1 is nonabelian and so E lifts to a group of symplectic type.

Assume that TL_t contains an elementary abelian subgroup F of order 32. Then we have that $|Z(L_t)| \le 2$. We select F such that $F \cap H$ has order exactly 4 and set $F \cap H = \langle s, t\rangle$ where $s \in L_t$ with $s \ne t$. As $s \in L_t$, s and st invert $C_Q(t)$. As $[t, Q] \ne 1$ there is $x \in \{s, st\}$ such that $C_Q(x) \ne 1$. If, additionally, $x \notin \mathcal{I}$, then, as $|C_Q(t)| = 9$, $|C_Q(x)| = 3$ and, by Lemma 5.2 and Corollary 2.8, $L_x \cong \mathrm{PSL}_2(8)$ and $C_G(x)/O_{3'}(C_G(x)) \cong \mathrm{PSL}_2(8) : 3$. Since $C_G(x)/O_{3'}(C_G(x)) \cong \mathrm{PSL}_2(8) : 3$ and $F \le C_G(x)$, we have that $|F \cap O_{3'}(C_G(x))| \ge 4$. Since $O_{3'}(C_G(x)) \le H$, we get $F \cap O_{3'}(C_G(x)) = F \cap H$. But then $t \in O_{3'}(C_G(x))$ and as $[O_{3'}(C_G(x)), C_Q(x)] = 1$, we have $C_Q(x) \le C_Q(t)$ whereas x inverts $C_Q(t)$. Hence, if $C_Q(x) \ne 1$, then $x \in \mathcal{I}$. Hence we also have $L_x/Z(L_x) \cong \mathrm{PSL}_3(4)$. Let U be a Sylow 2-subgroup of $O_{3'}(C_G(x))$. Then, as $x \in \mathcal{I}$, U is cyclic and U commutes with L_x. We have that $TF \cap O_{3'}(C_G(x))$ is cyclic and TF is abelian. This implies that $|TFO_{3'}(C_G(x))/O_{3'}(C_G(x))| = 2^4$ and hence $F \le UL_x$ by Lemma 2.10. In particular, U and F commute. As TF is a Sylow 2-subgroup of $C_G(F)$, we have $TF = UF$ and $|U| = |T|$. Assume that $|T| > 2$. Then, on the one hand t is the unique involution in $\Phi(TF)$, while on the other hand, x is the unique involution in UF and so, as $x \ne t$, we must have $T = \langle t\rangle$ has order 2. As $N_{L_t}(F)$ centralizes only t in F and $N_{L_x}(F)$ centralizes only x in F, we have $N_G(F) \not\le C_G(t)$. Of course we also have that $N_G(F)/C_G(F)$ is isomorphic to a subgroup of $\mathrm{GL}_5(2)$. From the structure of L_t, we know that $N_{C_G(t)}(F)$ induces orbits of length 1,15,15 on $F^\sharp$ and so t has 31 or 16 $N_G(F)$-conjugates in F. If all involutions in F are conjugate, we get that $N_G(F)/C_G(F)$ has order $2^2 \cdot 3 \cdot 5 \cdot 31$ if $N_{C_G(t)}(F)$ induces $\mathrm{Alt}(5)$ on F or $2^3 \cdot 3 \cdot 5 \cdot 31$ if $N_{C_G(t)}(F)$ induces $\mathrm{Sym}(5)$ on F. As in both cases, by Sylow's Theorem, the Sylow 31-subgroup would be normal, this contradicts the structure of the normalizer of a Sylow 31-subgroup of $\mathrm{GL}_5(2)$. So t has 16 $N_G(F)$-conjugates in F. Since $x \in \mathcal{I}$, we may argue in a similar way to see that there are 16 $N_G(F)$-conjugates of x in F. Recall that x was chosen in $\{s, st\}$ and $s \not\sim st$ in L_t, at least one of s and st is not $N_G(F)$ conjugate to t. It follows that either $C_Q(s) = 1$ or $C_Q(st) = 1$. Assume that $y \in \{s, st\}$ is such that $C_Q(y) = 1$. Then $Q = C_Q(t)C_Q(x)$ and, as $|C_Q(x)| = |C_Q(t)| = 9$, Q is elementary abelian of order 3^4 and H/Q is isomorphic to a subgroup of $\mathrm{GL}_4(3)$. By Lemmas 5.3 and 5.7 there is a 2-component L in $C_G(y)$ and $O_{3'}(C_G(y))$ has a normal 2-complement and a cyclic Sylow 2-subgroup. Since $O(L) \le H$ and as $|O(L)|$ has order coprime to both 2 and 3 and divides $|\mathrm{GL}_4(3)|$, we have that $O(L)$ is cyclic of order dividing 65. Since $p = 3$ and $C_H(y)$ is strongly 3-embedded in $C_G(y)$, we have that $|O_3(C_H(y)/O(C_G(y)))| \ge 9$ from Proposition 2.7. But then, as $O(L)$ is cyclic, $O_3(C_H(y)) \ne 1$, but this is impossible as $H = C_H(y)Q$, $Q = O_3(H)$ and $Q \cap C_H(y) = 1$. This contradiction shows that TL_t does not contain an elementary abelian subgroup of order 32. Therefore L_t is a central extension of $\mathrm{PSL}_3(4)$ by a cyclic group of order 4. In particular, $t \in L_t$ and L_t has precisely two conjugacy classes of involutions.

Assume that there is an $s \in t^G \cap C_G(t) \setminus L_tO_{p'}(C_G(t))$. Then, by Lemma 2.10 we may assume that $C_{C_Q(t)}(s) \ne 1$ which means that $s \in H$. If $C_Q(t) = C_Q(s)$, then, as $t \in \mathcal{I}$, we get $[Q, s] = 1$, a contradiction. So we have that $|C_Q(s) \cap C_Q(t)| = 3$. As $t \in \mathcal{I}$, we see by coprime action that $|Q| = 3^4$. Furthermore, we have that Q is elementary abelian. Now choose $u \in L_t$ an involution such that u inverts $C_Q(t)$. By Proposition 2.7 (iv), we have that u is a square in H. In particular, u acts on Q as an element of $\mathrm{PSL}_4(3)$. Since $u \sim_{L_t} ut$, Lemma 3.5 implies that $u \sim_H ut$. Therefore both $C_Q(u)$ and $C_Q(ut)$ are non-trivial. Because u and ut act on Q as elements of $\mathrm{SL}_4(3)$, we see that $|C_Q(u)| = |C_Q(ut)| = 3^2$. Finally, we have $C_Q(\langle u, t\rangle) = 1$ and so by coprime action $|Q| = 3^6$, a contradiction. We have proved that $t^G \cap C_G(t) \subseteq L_tO_{p'}(C_G(t))$. In particular, by Lemma 2.17 all involutions in $C_G(L_t)L_t$ are conjugate to t.

Let $u \in L_t$ be an involution such that $uZ(L_t)/Z(L_t) \in Z(RZ(L_t)/Z(L_t))$. We have shown that $u \in t^G$. Set $R_1 = C_R(u)$ and note that as $\langle t, u\rangle$ is a fours group and $u \notin Z(R)$, we have $|R : R_1| = 2$. Since $u \sim_G t$, there exists $R_0 \in \mathrm{Syl}_2(C_G(u))$ such that $R_0 > R_1$. Obviously R_1 is normal in R_0. If $R_0 \le C_G(t)$, then $\Omega_1(Z(R_0)) = \langle t, u\rangle$ and this contradicts the fact that $|\Omega_1(Z(R))| = 2$. Hence R_0 doesn't centralize t. Let $W \cong \mathbb{Z}_4 * \mathrm{Q}_8 * \mathrm{Q}_8$ be the preimage in L_t of an elementary abelian subgroup of $L_t/Z(L_t)$ of order 16 which is contained in R. Then $|W : C_W(u)| = 2$ and $C_W(u) = \langle u\rangle \times V$, where $V \cong \mathbb{Z}_4 * \mathrm{Q}_8$. Set $R_2 = \langle t^G \cap R_1\rangle$. Then $R_2 \le R \cap L_t$ and R_0 normalizes R_2. As $C_W(u)$ is generated by involutions, we have that $C_W(u) \le R_2$. As $C_{R/Z(L_t)}(C_W(u)/Z(L_t)) = W/Z(L_t)$, we see that $Z(R_2) = Z(C_W(u)) = \langle u, Z(L_t)\rangle$. In particular, R_0 normalizes $\langle u, Z(L_t)\rangle$ and hence R_0 normalizes $\langle t\rangle$ which is the characteristic subgroup of $\langle u, Z(L_t)\rangle$ generated by squares. Since $R_0 \not\le C_G(t)$ we have our final contradiction. Hence there exists $t \in \mathcal{I}$ such that $L_t/Z(L_t) \cong \mathrm{PSL}_2(p^f)$ for some $f \ge 2$. □

Proof of Theorem 6.1. This is a consequence of a combination of all the lemmas in this section culminating in Lemma 6.6. □

7. Centralizers of involutions with $F^*(C_G(t)/O_{p'}(C_G(t))) \cong \mathrm{PSL}_2(p^f)$

We continue to assume that Hypotheses 4.1 and 5.1 hold. We use the notation established in the Sections 4, 5 and 6. In particular, $Q = O_p(H)$. Because of Theorem 6.1, we may assume that the following hypothesis is satisfied:

Hypothesis 7.1. *Hypotheses 4.1 and 5.1 hold and either*

(i) *there exists* $t \in \mathcal{I}$ *and* $F^*(C_G(t)/O_{p'}(C_G(t))) \cong \mathrm{PSL}_2(p^f)$ *for some* $f > 1$*;*
(ii) *for all involutions* $t \in H$*,* $C_Q(t) = 1$ *and* $F^*(C_G(t)/O_{p'}(C_G(t))) \cong \mathrm{PSL}_2(p^f)$ *with* $f > 1$ *and* $p \equiv 3 \pmod 4$*.*

Our objective is to prove that if Hypothesis 7.1 holds, then $F^*(G) \cong {}^2\mathrm{G}_2(3^{2a+1})$ for some $a \ge 1$. We fix the following notation throughout this section. Let $t \in H$ be an involution such that either $t \in \mathcal{I}$ or $C_Q(t) = 1$ and such that $E(C_G(t)/O_{p'}(C_G(t))) \cong \mathrm{PSL}_2(p^f)$ with $f > 1$. By Lemmas 5.2 and 5.3 there is a normal 2-component of $C_G(t)$, which we denote by L, with

$$LO_{p'}(C_G(t))/O_{p'}(C_G(t)) = F^*(C_G(t)/O_{p'}(C_G(t))).$$

Since t is not a classical involution by Hypothesis 5.1 (ii), we have $L/O(L) \cong \mathrm{PSL}_2(p^f)$.

We fix $T \in \mathrm{Syl}_2(O_{p'}(C_G(t)))$, $S \in \mathrm{Syl}_2(C_G(t))$ such that $T \leq S$, $D = S \cap L$ and $U \in \mathrm{Syl}_2(G)$ with $S \leq U$.

Obviously $t \in T$ and, by Lemma 5.7, we have that T is cyclic and $O_{p'}(C_G(t)) = TO(C_G(t))$. Finally we note that

$$F^*(C_G(t)/O(C_G(t))) \cong T \times L/O(L) \cong \mathbb{Z}_{2^k} \times \mathrm{PSL}_2(p^f)$$

for some $k \geq 1$ and $f > 1$.

Lemma 7.2. *There is no involution in $C_G(t)$ which induces a non-trivial field automorphism on $L/O(L)$.*

Proof. Assume that there is an involution $x \in S$ which induces a non-trivial field automorphism on $L/O(L)$. Then $C_{L/O(L)}(x) \cong \mathrm{PGL}_2(p^r)$ where $2r = f$. In particular, p divides $|C_L(x)|$ and so, as H is strongly p-embedded, there is a $C_G(t)$-conjugate s of x contained in $C_H(t)$. Since $C_G(\langle s, t\rangle)$ is not a p'-group, Lemma 5.6 implies that $C_Q(t) \neq 1 \neq C_Q(s)$. In particular, Lemma 5.2 implies that $O(L) = Z(L) = 1$ and that $C_G(s)$ has a normal component L_s with $L_s \not\leq O_{p'}(C_G(s))$. Since $C_{C_Q(t)}(s) = C_{C_Q(t)}(st)$ and $C_G(Q) \leq Q$, we may assume that $C_Q(s) \not\leq C_Q(t)$. Therefore, $|C_Q(s)| > p^r$ and so $L_s \not\cong \mathrm{PSL}_2(p^r)$. Observe that t induces a non-trivial automorphism on L_s which centralizes a subgroup of L_s which is isomorphic to $\mathrm{PSL}_2(p^r)$. Suppose that $p^r \neq 3$. Then Lemma 6.3 shows that either $(L_s/Z(L_s), p) = (\mathrm{PSL}_2(p^f), p)$, $({}^2\mathrm{G}_2(3^{2a-1}), 3)$, $(\mathrm{PSL}_2(8), 3)$, $(\mathrm{PSL}_3(4), 3)$ or $({}^2\mathrm{B}_2(32), 5)$. As $C_{L_s}(t)$, contains $\mathrm{PSL}_2(p^r)$, we see that $(L_s/Z(L_s), p) = (\mathrm{PSL}_2(p^f), p)$ or $({}^2\mathrm{G}_2(3^r), 3)$. But in the latter we have that $C_{C_G(s)/O_{p'}(C_G(s))}(t) = \langle t\rangle \times \mathrm{PSL}_2(3^r)$, while $C_G(t) \cap C_G(s)$ involves $\mathrm{PGL}_2(p^r)$, a contradiction. Thus, if $p^r > 3$, we deduce that $L_s \cong \mathrm{PSL}_2(p^f)$ and t induces a field automorphism on L_s. Assume now that $p^r = 3$. Then we have that $L_t \cong \mathrm{PSL}_2(9)$ and $C_{L_t}(s) \cong \mathrm{Sym}(4)$. Furthermore, we have $C_{C_G(t)}(s)$ is soluble. Since $L_s \not\leq L_t$, t does not centralize L_s. Thus t induces an automorphism of L_s and, as $[C_Q(s), O_{p'}(C_G(s))] = 1$, $C_Q(s) \cap C_{L_t}(s) > 1$ and $C_{L_t}(s) \cong \mathrm{Sym}(4)$,

$$C_{L_t}(s) \cap O_{p'}(C_G(s)) \leq C_{C_{L_t}(s)}(C_Q(s) \cap C_{L_t}(s)) \leq Z(C_{L_t}(s)) = 1.$$

Thus t centralizes a subgroup of $C_G(s)/O_{p'}(C_G(s))$ which is isomorphic to $\mathrm{Sym}(4)$.

Using Lemma 6.3, we have the following possibilities for the isomorphism type of $L_s/Z(L_s)$: $L_s/Z(L_s) \cong \mathrm{PSL}_2(3^m)$, $m \geq 2$, ${}^2\mathrm{G}_2(3^{2a+1})$, $a \geq 1$, $\mathrm{PSL}_3(4)$ or $\mathrm{PSL}_2(8)$. In the first case, we get $|C_Q(s)| = 3^m$ and so, as $t \in \mathcal{I}$, we must have $L_s \cong \mathrm{PSL}_2(9)$ as well, with t inducing a non-trivial field automorphism on L_s. If $L_s \cong {}^2\mathrm{G}_2(3^{2a+1})$, then t acts as an inner automorphism of L_s and so t centralizes a subgroup of L_s isomorphic to $\mathrm{PSL}_2(3^{2a+1})$. This subgroup would of course have to be contained in $\mathrm{PSL}_2(9)$ which is impossible. Thus L_s is not a Ree group. Suppose that $L_s/Z(L_s) \cong \mathrm{PSL}_3(4)$. We now use the fact that $C_{C_G(s)/O_{p'}(C_G(s))}(t)$ contains a subgroup isomorphic to $\mathrm{Sym}(4)$ and is soluble. If t acts as an inner automorphism of L_s, we have $C_{L_s}(t)$ is a 2-group which is not the case. Thus t acts as an outer automorphism of L_s. Hence Lemma 2.10 indicates that $C_{C_G(s)/O_{p'}(C_G(s))}(t)$ is not soluble which is also impossible. Finally, if $L_s \cong \mathrm{PSL}_2(8)$, we see $C_{C_G(s)/O_{p'}(C_G(s))}(t) \cong 2 \times \mathrm{Alt}(4)$ which does not contain a subgroup isomorphic to $\mathrm{Sym}(4)$. Hence this case is also impossible. Thus we have shown that, if $L_t \cong \mathrm{PSL}_2(9)$, then $L_s \cong \mathrm{PSL}_2(9)$ and t induces a field automorphism on L_s.

We have shown that, if $L_t \cong \mathrm{PSL}_2(p^f)$ and s induces a non-trivial field automorphism on L_t, then $L_s \cong \mathrm{PSL}_2(p^f)$ and t induces a non-trivial field automorphism on L_s. It follows that $L_s \cap L_t \cong \mathrm{PGL}_2(p^r)$ and that $L_s \cap L_t \cap H = N_{L_s\cap L_t}(C_Q(\langle s,t\rangle))$. Therefore there is an involution $u \in L_s \cap L_t \cap H$, which centralizes $\langle s,t\rangle$ and inverts $C_Q(x)$ for $x \in \{s,t\}$. If $C_Q(st) \leq C_Q(t)$, then u inverts $C_Q(st)$ while, if $C_Q(st) \not\leq C_Q(t)$, we have that $L_{st} \cong \mathrm{PSL}_2(p^f)$ by the argument above. Since $L_s \cap L_t = C_{L_t}(s) = C_{L_t}(st)$, we deduce that $L_s \cap L_t = L_s \cap L_{st}$. In particular, $u \in L_{st}$ inverts $C_Q(st)$ in this case as well. Therefore, in any case we have that $Q = C_Q(t)C_Q(s)C_Q(st)$ is inverted by u. Hence $C_Q(u) = 1$. Let L_u be the 2-component in $C_G(u)$ given by Lemma 5.3.

If $(L_u \cap H)/O(L_u)$ is p-closed, then $\langle s,t\rangle$ normalizes a Sylow p-subgroup of $L_u \cap H$. Thus p divides $|C_G(\langle u,x\rangle)|$ for some $x \in \langle s,t\rangle^{\#}$. This of course contradicts Lemma 5.6. Thus $(L_u \cap H)/O(L_u)$ is not p-closed and hence, either $p > 3$ and $L_u/O(L_u) \cong \mathrm{Alt}(2p)$ or $2{\cdot}\mathrm{Alt}(2p)$, or $p = 5$ and $L_u \cong \mathrm{Fi}_{22}$ or $2{\cdot}\mathrm{Fi}_{22}$. Notice that $C_{L_t}(u)$ is a dihedral group of order $2(p^f \pm 1)$. Thus $C_{C_G(t)}(u) = C_{C_G(u)}(t)$ is soluble. Hence t induces a non-trivial automorphism on $L_u/O(L_u)$. If $L_u/O_{p'}(L_u) \cong \mathrm{Alt}(2p)$, then by considering the possibilities for t we see that $p = 5$ and $L_u/O_{p'}(L_u) \cong \mathrm{Alt}(10)$. Furthermore, in this case we have that the image of t corresponds to a permutation of cycle shape $1^2.2^4$ or $1^4.2^3$. In each case $C_{L_u}(t)$ contains a section which is isomorphic to $\mathrm{Sym}(4)$. Since T is cyclic, we see that $C_{C_G(t)}(u)$ contains no such section. So assume that $L_u/O_{p'}(L_u) \cong \mathrm{Fi}_{22}$.

In this case we select an involution w of L_u which centralizes a non-trivial 5-subgroup of L_u (see [4, page 160] to see that this is possible). Then w may be chosen in H and we have $C_G(\langle u, w\rangle)$ has order divisible by 5 in contradiction to Lemma 5.6. This eliminates all the possibilities for $L_u/O(L_u)$ and so we conclude that no involution induces a field automorphism on L_t as claimed. □

Lemma 7.3. *We have* $|T| = 2$.

Proof. Suppose that $|T| > 2$. We have that D is a dihedral group and, setting $R = T \times D$, we have $R \in \mathrm{Syl}_2(F^*(C_G(t))/O(C_G(t)))$. Let $v \in S$ denote a PGL_2-automorphism of L (which may be trivial). We remark that T is normal in S, $R\langle v\rangle/T$ is a dihedral group and we note that we do not know the action of v on T. Finally, let z be an involution in $C_D(\langle v\rangle)$. Then, by Lemma 7.2, $\Omega_1(S) \le (T \times D)\langle v\rangle$ and $\Omega_1(C_S(\Omega_1(S))) \le (T \times D)\langle v\rangle$.

If $\Omega_2(T) \le C_S(\Omega_1(S))$, we see that

$$\langle t\rangle = \Omega_1(\Phi(C_S(\Omega_1(S)))).$$

On the other hand, if $\Omega_2(T)$ does not centralize $\Omega_1(S)$, then v is an involution and $[\Omega_2(T), v] \ne 1$. In particular $\Omega_2(T) \le \Omega_1(S)$. Set

$$\mathcal{W} = \{W \mid W \le \Omega_1(S), W \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2\}.$$

Suppose that there is a $W \in \mathcal{W}$ with $W \not\le R$. Then $W \cap R$ has order 8 and is either elementary abelian or is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. In the former case, $W \cap D$ is elementary abelian of order 4 and as $R\langle v\rangle/T$ is a dihedral group, we have that $C_{\Omega_1(S)}(W \cap D) \le R$, which is a contradiction as $W \not\le R$. Therefore $W \cap R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and we may suppose that $v \in W$. Since $C_{\Omega_1(S)}(v) = \langle t, v, z\rangle$, this case cannot happen either. Hence every member of $\mathcal{W}$ is contained in R and consequently $R_1 = \langle \mathcal{W}\rangle$ char S. Thus $\langle t\rangle = \Omega_1(\Phi(Z(R_1)))$ is a characteristic subgroup of S. We have shown in both cases that

$$\langle t\rangle \text{ char } S.$$

Hence S is a Sylow 2-subgroup of G and Burnside's Lemma implies

$$t^G \cap Z(S) = \{t\}.$$

Since every involution in $TL/O(L)$ is conjugate to an involution in $Z(R)O(L)/O(L)$, we have

$$t^G \cap R = \{t\}.$$

On the other hand, by the Z^*-Theorem, $t^G \cap S \neq \{t\}$ and so we may assume that $t \sim_G v$. As $v \notin \Phi(C_S(t))$, we have that also $t \notin \Phi(C_S(v))$. This shows that $[\Omega_2(T), v] \neq 1$, which means that $v \sim_G vt$. Furthermore, conjugating by elements from D, we see that $v \sim_G vz$. Hence also $v \sim_G vtz$. Thus z and tz are the only involutions in $\Omega_1(Z(C_S(v))) = \langle t, v, z\rangle$ which are not conjugate to t in G. But then $N_G(C_S(v))$ normalizes $\langle z, tz\rangle$ and consequently $N_G(C_S(v)) \leq C_G(t)$. This contradicts $t \sim_G v$. Thus $T = \langle t\rangle$ has order 2 as claimed. □

Lemma 7.4. *No element of $C_G(t)$ induces a non-trivial field automorphism of even order on $L/O(L)$.*

Proof. We start by establishing some notation. If $C_G(t)/O_{p'}(C_G(t))$ contains a subgroup isomorphic to $\mathrm{PGL}_2(p^f)$, we select $v \in S$ of minimal order such that $LT\langle v\rangle/O_{p'}(LT) \cong \mathrm{PGL}_2(p^f)$ and otherwise we define $v = 1$. We also pick $y \in S$ such that y induces a field automorphism on $L/O(L)$ and $S = TD\langle v, y\rangle$. Set $Y = \langle y\rangle$.

Assume that Y is non-trivial. Then, by Lemmas 7.2 and 7.3, we have $\langle t\rangle = T \leq Y$. Let $R = TD$. Then $\Omega_1(S) = R$ if v is not an involution and otherwise $\Omega_1(S) = R\langle v\rangle$. Since Y is non-trivial, we have f is even and hence there is an $m \geq 3$ such that $\Omega_1(S) \cong \mathbb{Z}_2 \times \mathrm{Dih}(2^{m+1})$ if v is an involution and otherwise $\Omega_1(S) \cong \mathbb{Z}_2 \times \mathrm{Dih}(2^m)$. Let $z \in Z(D)^\#$. Then $\langle z\rangle = Z(\Omega_1(S)) \cap \Omega_1(S)'$. Hence

$$\langle z\rangle \text{ char } S.$$

Assume that $t^G \cap R = \{t\}$. Then since $\Omega_1(Z(S)) = \langle t, z\rangle \leq R$, we have $S \in \mathrm{Syl}_2(G)$. By the Z^*-Theorem, $t^G \cap S \neq \{t\}$ and so, as there are no involutions in $S \setminus R\langle v\rangle$, we may assume that $t \sim_G v$. In particular v is an involution. If $D\langle v\rangle$ is normal in S, then $S = D\langle v\rangle Y$ with Y cyclic, $\Omega_1(Y) = \langle t\rangle$ and $Y \cap D\langle v\rangle = 1$. Thus by the Thompson Transfer Lemma 2.18, we have that t is conjugate to a 2-central involution in $D\langle v\rangle$ and so $t^G \cap R \neq \{t\}$, a contradiction. Therefore, y does not normalizes $D\langle v\rangle$. As there are exactly two $LT\langle v\rangle$-classes of involutions in $R\langle v\rangle \setminus D$, we deduce that $v \sim_G vt$. Then conjugating by elements of D, we obtain $v \sim_G vz$ and $vt \sim_G vtz$. Hence $v \sim_G vz \sim_G vt \sim_G tvz$. This shows that $\{z, tz\}$ are the only involutions in $\langle t, v, z\rangle$, which are not conjugate to t in G. Since $\Omega_1(Z(C_S(v))) = \langle t, v, z\rangle$, $N_G(C_S(v)) \leq C_G(t)$, and this contradicts $t \sim_G v$ and proves that $t^G \cap R \neq \{t\}$.

So we have that $t^G \cap R \neq \{t\}$. Hence t is conjugate to either z or tz. If $S \in \mathrm{Syl}_2(G)$, then, as $N_G(S)$ controls fusion in $\Omega_1(Z(S)) = \langle t, z\rangle$, we must have that t is conjugate to both tz and z. However this is impossible as $z \in \Omega_(S)'$ and t is not. Therefore $S \notin \mathrm{Syl}_2(G)$. In particular there is $S_1 \leq G$ with $|S_1 : S| = 2$. Choose $s \in S_1 \setminus S$. Note that as t is not central in S_1 and $\langle z\rangle$ is characteristic in S, $t^s = tz$.

Let $y_1 \in Y$ be such that $y_1^2 = t$. If $|S_1 : C_{S_1}(s)| \leq 2$, then $t = y_1^2 \in C_{S_1}(s)$, and we have a contradiction since $t^s = tz$. Thus, as $R\langle v\rangle/\langle t\rangle \cong \mathrm{Dih}(2^{m+1})$, we can characterize $\langle z, t\rangle$ as

$$\langle z, t\rangle = \langle i \mid i^2 = 1, i \in S, |S_1 : C_{S_1}(i)| \leq 2\rangle.$$

This means that $\langle z, t\rangle$ is a characteristic subgroup of S_1. Therefore $S_1 \in \mathrm{Syl}_2(G)$ and z is 2-central. Additionally, as $y_1^2 = t$ and $S/R\langle v\rangle$ is cyclic, we have $\Omega_1(S/\langle z\rangle) = \Omega_1(S)/\langle z\rangle$. Hence $m_2(\Omega_1(S/\langle z\rangle)) = m_2(S) = 3$ and, in particular, $m_2(S_1/\langle z\rangle) \leq 4$. Furthermore, if E is an elementary abelian subgroup of S_1 of order 16, then $E \cap S \leq \Omega_1(S) \cong \mathbb{Z}_2 \times \mathrm{Dih}(2^{m+1})$ and so $E > E \cap S$ contains $Z(S)$. But $Z(S) > Z(S_1)$. Thus no such E exists and $m_2(S_1) = 3$.

Now, as $p^f \equiv 1 \pmod 4$, z inverts $O_p(C_H(t))$ and hence $z \in H$. Therefore, by Lemma 4.4 we have that $C_G(z)$ has a 2-component L_z. As z is 2-central and $|Z(S_1)| = 2$, we have that z is in the Schur multiplier of $L_z/O(L_z)$. From Lemmas 2.12 and 5.7 we get that $L_z/\langle z\rangle O(L_z) \cong \mathrm{PSL}_2(p^a)$, $\mathrm{Alt}(2p)$, $\mathrm{PSL}_3(4)$, $2{\cdot}\mathrm{PSL}_3(4)$ or Fi_{22}. The first case is impossible for otherwise z would be a classical involution. In the remaining cases there is either an elementary abelian subgroup of order 16 in S_1 (see [10, Propositions 5.2.10, 5.6.1 and 6.4.4]) or $L_z/\langle z\rangle O(L_z) \cong 2{\cdot}\mathrm{PSL}_3(4)$ and $m_2(S_1/\langle z\rangle) = 5$ (see [10, Proposition 6.4.4]). This contradicts the structure of S_1 described in the previous paragraph and thus completes the proof of the lemma. $\square$

We now summarize what we have discovered about the structure of $C_G(t)$. By Lemma 7.3, we have that $T = \langle t\rangle$ and Lemmas 7.2 and 7.4 imply that no element of S induces a field automorphism on $L/O(L)$. Thus we have

$$F^*(C_G(t)/O(C_G(t))) \cong \mathbb{Z}_2 \times \mathrm{PSL}_2(p^f)$$

and $O^{2'}(C_G(t)/O_{p'}(C_G(t)))$ is isomorphic to either $\mathrm{PSL}_2(p^f)$, $\mathrm{PGL}_2(p^f)$ or to $\mathrm{PSL}_2(p^f){\cdot}2$ where the last extension is by a product of a field and diagonal automorphism (which is necessarily non-split). In particular, we have that $S/\langle t\rangle$ is either a dihedral or semi-dihedral group.

Lemma 7.5. *Assume Hypothesis 7.1. Suppose that* $\mathrm{Syl}_2(C_G(t)) \subseteq \mathrm{Syl}_2(G)$. *Then* G *has elementary abelian Sylow* 2*-subgroups.*

Proof. Aiming for a contradiction, suppose that $S \in \mathrm{Syl}_2(C_G(t)) \subseteq \mathrm{Syl}_2(G)$ and that S is not elementary abelian. Then $Z(S) = \langle t, z\rangle$ where $\langle z\rangle = Z(S)\cap D$. Since z is a commutator in S and t is not, z and t are not G-conjugate and so, as $S \in \mathrm{Syl}_2(G)$, Burnside's Lemma [9, 6.2] implies that $Z(S)$ contains representatives from three distinct G-conjugacy classes. By Hypothesis 4.1 (iv), $G = O^2(G)$ and so we must have that $t \in \Phi(S)$ by Lemma 2.19. Since $t \in \Phi(S)$, S/D is cyclic of order 4. In particular, $\Omega_1(S) = TD$. Since TL has exactly three conjugacy classes of involutions with representatives z, t and zt, we deduce that $t^G \cap TL = \{t\}$. Therefore $t^G \cap S \subseteq t^G \cap TD = \{t\}$. Finally the Glauberman Z^*-Theorem [6] implies that $t \in Z^*(G) = 1$ and we have our contradiction. Hence, if $S \in \mathrm{Syl}_2(G)$, then S is abelian and consequently is elementary abelian. □

We recall that U is a Sylow 2-subgroup of G containing S.

Lemma 7.6. *U has a normal elementary abelian subgroup of order 4.*

Proof. As U is not dihedral or semi-dihedral, this follows from [9, 10.11]. □

Lemma 7.7. *S is either elementary abelian or* $S/T \cong \mathrm{Dih}(8)$. *Furthermore, if S is not abelian, then there is a fours group in S which is not contained in $Z(S)$ but is normal in U.*

Proof. We may suppose that S is non-abelian. Let $\langle z\rangle = Z(S) \cap D$. Since S is non-abelian, Lemma 7.5 implies that $U \neq S$. By Lemma 7.6, there exists a normal elementary abelian subgroup V of U of order 4. As $G = O^2(G)$, the Thompson Transfer Lemma 2.18 implies that t is conjugate to some involution $s \in C_G(V)$ such that U contains a Sylow 2-subgroup of $C_G(s)$. Hence we may assume that $V \le S$.

Suppose that $V = Z(S) = \langle t, z\rangle$. Then, as $U \neq S$, $|U : S| = 2$ and so we can write $U = S\langle x\rangle$ for some $x \in U$. Since z is a commutator in S and t is not, t and z are not U-conjugate. Therefore, as $U \neq S$, $t^x = tz$.

Let $\langle z, s\rangle$ be a fours group in D. Note that, as S/T contains a dihedral subgroup of order at least 8, $N_{SL}(\langle z, s\rangle)/T \cong \mathrm{Sym}(4)$ and that $N_{SL}(\langle z, s\rangle)$ normalizes $E = \langle s, t, z\rangle$. Since $S/\langle t\rangle$ is either dihedral or semi-dihedral, $E \in \mathrm{Syl}_2(C_G(E))$. By considering the action of $N_L(E)$ on E, we see that $tz \sim_G ts \sim_G tsz$ and $z \sim_G zs \sim_G z$ and by assumption we have $t^x = tz$. So the involutions in E are partitioned into two sets $t^G \cap E$ of size 4 and $z^G \cap E$ of size 3. Suppose that $|E \cap L^x| = 4$, then $N_{(SL)^x}(E \cap L_x) \cong \mathrm{Sym}(4)$. Then we must have $E \cap L^x = \langle z, s\rangle$. Since tz is centralized by L^x, we infer that $N_G(E)/C_G(E) \cong \mathrm{Sym}(4)$.

Let $R \in \mathrm{Syl}_2(N_G(E))$. Then $R/E \cong \mathrm{Dih}(8)$. We claim U contains no subgroup R_0 which is isomorphic to R. Assume this is false and let F be the subgroup of R_0 with $R_0/F \cong \mathrm{Dih}(8)$. Since S has a cyclic subgroup of index 4, R_0 has a cyclic subgroup C of index 8 with $C \leq S$. Since $|R_0| = 2^6$, $|C| \geq 8$. We conclude that CF/F is cyclic of order 4, $Z(CF)$ has order 2 and $|C| = 8$.

If $F \leq S$, then, as the 2-rank of S is 3, we have that $\langle t, z\rangle = Z(S) \leq F$ and as $C \leq S$, we have a contradiction to $|Z(CF)| = 2$. Thus $F \not\leq S$ and $(R_0 \cap S)/(F \cap S) \cong R_0/F \cong \mathrm{Dih}(8)$. Since C is inverted in $R_0 \cap S$, we have that $[R_0 \cap S, C]$ has order 4. Because, $|F \cap S| = 2^2$, $[R_0 \cap S, C]$ centralizes $F \cap S$ and so the structure of S indicates that $F \cap S \leq Z(S) = \langle t, z\rangle$. But then $F \cap S \leq Z(R_0 \cap S)$ which has order 2 and we have a contradiction. This contradiction arose from the assumption that R was isomorphic to a subgroup of U and in turn this followed from the supposition that $|E \cap L^x| = 4$. Hence we must have $|E \cap L^x| \leq 2$.

It follows that x does not normalize E and $EL^x/O(L^x) \cong \mathrm{PGL}_2(p^f)$. Therefore $S = \langle t\rangle \times D_0$ where D_0 is a dihedral group and x can be chosen so that $S = \langle E, E^x\rangle$ and $D_0 = \langle z, s, s^x\rangle$. In particular, D_0 is normalized by x and, as all the involutions in D_0 are contained in $L \cup L^x$ (and consequently G-conjugate to z), t is not G-conjugate to an element of D_0. As $C_G(t)$ involves $\mathrm{PGL}_2(p^f)$, we see that $C_H(t)$ contains a fours group. Then by 7.1 we have that $C_Q(t) \neq 1$ and $O(L) = 1$.

Let $u \in N_{E^xL}(C_Q(t))$, be an involution conjugate into D_0. Then $u \in H$ and, as u is conjugate to z, $C_G(u)$ has Sylow 2-subgroups isomorphic to U. By Lemmas 5.2 and 5.3 there is a 2-component M in $C_G(u)$. Since $|Z(U)| = 2$, we infer that $|Z(M/O(M))| \geq 2$. It follows from Lemma 2.12 that $M/O_{p'}(M) \cong \mathrm{PSL}_2(p^a)$,

$\mathrm{Alt}(2p)$, $\mathrm{PSL}_3(4)$ with $p = 3$ or Fi_{22} with $p = 5$. In the first case, we have that u is a classical involution which is impossible. In the remaining cases, we have incompatible Sylow 2-structure, as the sectional 2-rank of U is 3 whereas the sectional 2-rank of the possible components is at least 4.

Thus we have shown that there is a fours group different from $\langle t, z\rangle$ which is normal in S. This shows that S/T is dihedral of order at most 8. If S/T is abelian, then $S = DT$ is elementary abelian. This completes the proof of Lemma 7.7. $\square$

Lemma 7.8. *$U = S$ is elementary abelian.*

Proof. Let $V = \langle r, s\rangle$ be a fours group in D. Then there is some element of order three in L, which acts non-trivially on V and, by Lemma 7.7, V is normal in S. By Lemma 2.25, we have that $C_G(V)$ has a Sylow 2-subgroup, which is an extension of B by $\langle t\rangle$ where $B \cong \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$, a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$, a Sylow 2-subgroup of $\mathrm{SU}_3(4)$ or is elementary abelian of order 16. We write $S = \langle S \cap L_t, t, y\rangle$ where, if S is abelian, $y = 1$ and, if S is non-abelian, $y^2 \in T$. Set $U_1 = B\langle t, y\rangle$.

Assume first that $U = U_1 \in \mathrm{Syl}_2(G)$. If $B = V$, then $U = U_1 = S$ and we are done by Lemma 7.5. So we suppose further $B \neq V$ and seek a contradiction. Then t is not G-conjugate to any involution in B, as any such involution centralizes an abelian group of order 16. In particular, since $G = O^2(G)$, the Thompson Transfer Lemma 2.18 implies that $U_1 \neq BT$ and $U_1 \neq \langle y\rangle B$. Thus y is an involution and, using the Thompson Transfer Lemma again, we have that t is G-conjugate to some element y_1 in By and to $y_2 \in Byt$. For $i = 1, 2$, we have that $y_i \in N_G(V)$ and $[y_i, V] \neq 1$. As $N_G(V)/C_G(V) \cong \mathrm{Sym}(3)$, y_iB inverts some 3-element ρB. As y_i are conjugate to t, we see that $C_B(y_i)$ does not contain an elementary abelian subgroup of order 8. Hence if B is elementary abelian or a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$, we see that all involutions in By_i are conjugate. In the other cases we have that $V = \Omega_1(B)$ and so $C_B(\rho) = 1$. Therefore we may apply Lemma 2.21 to get that all the involutions in $\langle y_i, \rho\rangle B \setminus B$ are conjugate to y_i again. In all cases we have that $|C_B(y_i)|^2 = |B|$. In particular, we may suppose that $y_1 = y$ and that $y_2 = yt$. Since y is conjugate to t, $|C_G(y)|_2 = 2^4$. As $\langle y, t\rangle$ is abelian, it follows that $|C_B(y)| \leq 2^2$. Thus $|B| = |C_B(y)|^2 \leq 2^4$ and so B is either elementary abelian of order 16 or is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$. We summarize the conclusions about fusion of involutions in U_1 as follows: all the involutions in $By \cup Byt$ are G-conjugate to t and $t^G \cap B$ is empty. The coset Bt may or may not contain G-conjugates of

z, where $z \in C_V(y)^{\#}$. Since z and t are not G-conjugate, we have that $D = V$, $LS \cong \mathbb{Z}_2 \times \mathrm{PGL}_2(p^f)$ and $S \cong \mathbb{Z}_2 \times \mathrm{Dih}(8)$. By considering $N_U(S) \neq S$, we see that t and tz are G-conjugate. Therefore $z^G \cap S = V^{\#}$.

Suppose that B is elementary abelian. Then $|C_B(y)| = 2^2$ and $C_B(y)$ contains no conjugates of t. Since y is conjugate to t, we see that $C_B(y)$ consists of conjugates of z. It follows that V and $C_B(y)$ are conjugate in G. Since $|C_B(U)| = 2$, B is the Thompson subgroup of U. Therefore $C_B(y)$ and V are conjugate in $N_G(B)$. Hence $yC_G(B)$, $tC_G(B)$ and, by an argument similar to the one above, $tyC_G(B)$ are conjugate in $N_G(B)/C_G(B)$. In particular, as $tC_G(B)$ commutes with an element $\rho C_G(B)$ of order 3 which acts fixed-point-freely on B, $yC_G(B)$ centralizes $\rho_y C_G(B)$ and $tyC_G(B)$ centralizes $\rho_{ty} C_G(B)$ where $\rho_y C_G(B)$ and $\rho_{ty} C_G(B)$ both have order 3 and both act fixed-point-freely on B. The isomorphism between $\mathrm{GL}_4(2)$ and $\mathrm{Alt}(8)$ maps such elements of order 3 to 3-cycles. Up to conjugacy, $S\langle\rho\rangle C_G(B)/C_G(B) = \langle (1,2,3), (1,2)(4,5), (1,2)(6,7)\rangle$ where $t = (4,5)(6,7)$ and, say, $y = (1,2)(4,5)$. The elements of order 3 which commute with y and are inverted by t either move 3 or 8. In the former case we see that $N_G(B)/C_G(B)$ contains a subgroup isomorphic to $\mathrm{Sym}(5)$ and this contradicts the fact that $|N_G(B)/C_G(B)|_2 = 4$. Thus ρ and ρ_y commute. Since ρ and ρ_y cannot both commute with ρ_{ty}, we have a contradiction. Hence B is not elementary abelian.

So we have shown that $B \cong \mathbb{Z}_4 \times \mathbb{Z}_4$. Suppose that H contains a conjugate z^g of z. Then $U^g \leq C_G(z^g)$. By Lemmas 5.2 and 5.3, $C_G(z^g)$ has a normal 2-component L_r. Since $U^g L_r$ has Sylow 2-subgroup U^g and $Z(U^g) = \langle z^g\rangle$, we infer that $Z(L_r/O(L_r))$ has order divisible by 2. Thus Lemma 2.12 implies that $L_r/O_{p'}(L_r)$ is isomorphic to one of $\mathrm{PSL}_2(p^e)$, $\mathrm{Alt}(2p)$, $\mathrm{PSL}_3(4)$ or Fi_{22}. Since $|U^g/Z(U^g)| = 2^5$, we can only have $L_r/O_{p'}(L_r) \cong \mathrm{PSL}_2(p^e)$. But then z^g is a classical involution which is impossible by hypothesis. Thus H does not contain G-conjugates of z. It follows that $H \cap L$ has odd order and so Corollary 2.9 implies that $p^f \equiv 3 \pmod 4$. On the other hand, $L\langle y\rangle/O(L) \cong \mathrm{PGL}_2(p^f)$ and so $L\langle y\rangle \cap H$ has order divisible by 2. Hence we may assume that $H \cap L\langle t, y\rangle$ contains $E = \langle t, y\rangle$ and $E \in \mathrm{Syl}_2(C_H(t))$. Since all the involutions in E are conjugate to t, we have $E \in \mathrm{Syl}_2(H)$. Furthermore, using Lemma 3.5, we have $N_H(E)/O(N_H(E)) \cong \mathrm{Alt}(4)$. Since all the involutions in E are conjugate, we also have $C_Q(t) \neq 1$. Thus Lemma 5.2 implies that L is a component and $[L, O_{p'}(C_G(t))] = [L, O(C_G(t))] = 1$.

In $C_G(t)$, E normalizes exactly two Sylow p-subgroups one of which is $C_Q(t)$. Since $C_G(E)$ acts on the set of p-subgroups of $C_G(t)$ which are normalized by E, we deduce that $|C_G(E) : C_H(E)| \le 2$. Noting that $C_{C_G(t)}(E)$ has order divisible by 2^3, we now have that $|C_G(E) : C_H(E)| = 2$. As $y \not\sim_{C_G(t)} yt$, we deduce that $|N_G(E)/C_G(E)| = 3$ and $O^2(N_G(E)) \le H$. In particular, $N_G(E)$ has elementary abelian Sylow 2-subgroups of order 8. Let $E_1 \in \mathrm{Syl}_2(N_G(E))$ be chosen so that

$$E_1 = E\langle z\rangle = \langle t, y, z\rangle \le S$$

where $z \in Z(S)^\#$. By the Frattini Argument, $N_{N_G(E)}(E_1)C_G(E) = N_G(E)$ and so there exists an element $\rho \in N_G(E)$ of 3-power order which normalizes but does not centralize E_1 and additionally $\rho^3 \in C_G(E_1)$. Because t, y, ty, tz, zy and zty are pairwise conjugate and t is not conjugate to z, we have that $z^G \cap E_1 = \{z\}$. Thus ρ centralizes z.

Since t inverts B, the preimage of $C_{B/\langle z\rangle}(t)$ in B is the subgroup $X = \{f \in B \mid f^2 \in \langle z\rangle\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Now y acts non-trivially on $\Omega_1(B) = V$ and, as y has order 2, a short calculation shows that y normalizes every subgroup of X of order 4. It follows that

$$C_{B/\langle z\rangle}(t) = C_{B/\langle z\rangle}(E_1) = X/\langle z\rangle.$$

Hence $N_U(E_1) = E_1X$ and $N_U(E_1)/E_1$ is the four group of $\mathrm{Aut}(E_1)$ which centralizes $E_1/\langle z\rangle$. This means that $N_U(E_1)$ is extraspecial of order 32 of $+$-type. Since y and ty are not conjugate in $C_G(t)$, we have $N_G(E_1)/C_G(E_1) \cong \mathrm{Alt}(4)$. In particular, $N_U(E_1) \in \mathrm{Syl}_2(N_G(E_1))$. Let $F \in \mathrm{Syl}_2(N_G(E_1))$ such that $F \le U$. Further we may choose ρ such that it normalizes F. Note that F has index 2 in the Sylow 2-subgroup U of G and that B intersects F in a subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. Also $C_G(F) = \langle z\rangle C_H(F)$ and $\rho^3 \in C_H(F)$. Since F is extraspecial of $+$-type and order 32, $N_G(F)/FC_G(F)$ is isomorphic to a subgroup of $\mathrm{O}_4^+(2) \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$. As $N_G(F)/FC_G(F)$ has Sylow 2-subgroups of order two and, as these subgroups consist of a non-trivial element which centralizes $\mathbb{Z}_2 \times \mathbb{Z}_4$, we see that $N_G(F)/FC_G(F) \cong \mathrm{Sym}(3)$ or $(\mathbb{Z}_3 \times \mathbb{Z}_3) : 2$. In both cases $O_3(N_G(F)/FC_G(F))$ is inverted. Suppose the latter. We now have that the 18 involutions of $F \setminus Z(F)$ are conjugate in $N_G(F)$. Since $V \le F$, we get that all the involutions in F are conjugate to z, but $t \in F$ and, as z and t are not conjugate, we have a contradiction. We have that $U = F\langle u\rangle$ where $u \in B$ has order 4. And $N_G(F) = \langle u, \rho\rangle FC_G(F)$, $C_F(u) = B \cap F \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. As t inverts B all elements in Bt are involutions. Hence there are involutions in $B\langle t\rangle \setminus F$. Choose such an involution w. Then w centralizes $\Omega_1(B)$. So $Z((B \cap F)\langle w \rangle) = \Omega_1(B)$. In particular, $B \cap F$ is not centralized by w, but $B \cap F$ is normalized by $\langle w \rangle$. It follows that $W = (B \cap F)\langle w \rangle \cong \mathrm{Dih}(8) \times 2$. Now $W = [F, w]\langle w \rangle$ and so W is normal in $U = F\langle w \rangle$. Since $\mathrm{Aut}(W)$ is a 2-group, $N_G(W) = C_G(W)U$. Set $\overline{C_G(z)} = C_G(z)/\langle z \rangle$. Then, as $N_G(W) = C_G(W)U$, we have $C_{\overline{F}}(\overline{w})\langle \overline{w} \rangle$ is a Sylow 2-subgroup of $C_{\overline{C_G(z)}}(\overline{w})$. In particular, $\overline{w}$ is not $\overline{C_G(z)}$-conjugate to a subgroup of $\overline{F}$. Therefore, the Thompson Transfer Lemma 2.18, implies that $\overline{C_G(z)}$ has a subgroup R of index two with elementary abelian Sylow 2-subgroup $\overline{F}$ of order 16. Recall that ρ acts fixed point freely on $E \leq F$ and so $[\overline{F}, \rho] = \overline{F}$. Set $\widetilde{R} = C_G(z)/O(C_G(z))\langle z \rangle$.

Suppose that $E(\widetilde{R}) \neq 1$ and recall that R is a $\mathcal{K}$-group by hypothesis. Then, as $\widetilde{R}$ has elementary abelian Sylow 2-subgroups of order 16 which admit a fixed-point-free element of order 3, we deduce that $R/O(R)\langle z \rangle$ has either one or two components isomorphic to $\mathrm{PSL}_2(l)$ for some prime $l \equiv 3, 5 \pmod 8$ or has a single component which is isomorphic $\mathrm{PSL}_2(16)$. Since $F \in \mathrm{Syl}_2(R)$ and $N_G(F)/FC_G(F) \cong \mathrm{Sym}(3)$, we see that just one component $\mathrm{PSL}_2(l)$, $l \equiv 3, 5 \pmod 8$ is possible. But then we have some element of order three, whose commutator with F is quaternion of order eigth, contradicting $[F, \rho] = F$. This contradiction finally shows that $E(\widetilde{R}) = 1$. In particular, $C_G(z) = O(C_G(z))N_G(F)$ and, as $\langle z \rangle \in \mathrm{Syl}_2(C_{C_G(z)}(\rho))$, $\langle z \rangle$ is a Sylow 2-subgroup of $C_G(\rho)$. Therefore $C_G(\rho)$ has a normal 2-complement. We claim that $\mathrm{Syl}_p(C_G(\rho)) \neq \{1\}$. We first investigate ρ^3 which we assume for now is non-trivial. Recall that $\rho \in H$ and $\rho^3 \in C_G(E_1) \leq C_G(t)$. Thus ρ^3 normalizes $C_Q(t)$. Since $C_L(z) \cap H = 1$, and $\rho^3 \in C_G(z)$, we have $\rho^3 \notin L$. We have $C_{C_G(t)}(L) = O_{p'}(C_G(t))$. If $L\langle \rho^3 \rangle$ is isomorphic to a direct product $A \times L$ (necessarily with $A \leq H$), then $A\langle \rho^3 \rangle \cap L \leq H \cap C_L(z) = 1$, which is impossible unless $\rho^3 \in A$. So in this case we have $C_Q(t) \leq C_Q(\rho^3)$. So suppose that $L\langle \rho^3 \rangle$ is not isomorphic to a direct product. Hence ρ^3 induces an outer automorphism of L. But then, as ρ is a 3-element, ρ^3 induces a field automorphism on L and $C_{C_Q(t)}(\rho^3) \neq 1$. Hence in each case we have $C_Q(\rho^3) \neq 1$. This of course is also true if $\rho^3 = 1$. Since $E \in \mathrm{Syl}_2(C_H(t))$, E commutes with ρ^3 and E inverts $C_Q(t)$, we have that, setting $J = C_Q(\rho^3)$, $J = C_J(t) \times C_J(ty) \times C_J(y)$. Hence ρ centralizes a diagonal element in this decomposition of J and so $C_J(\rho) \neq 1$. This then proves our claim. Since $\langle z \rangle$ is a Sylow 2-subgroup of $C_G(\rho)$ and $C_G(\rho)$ has a normal 2-complement, we now see that $\langle z \rangle$ is contained in a conjugate of H. But then $\langle z \rangle$ is conjugate to an element of E and this means that z and t are G-conjugate. We have proved that this is not the case and so at this stage we deduce $U \neq U_1$.

Assume $B \neq V$ and $U \neq U_1$. Set $U_0 = BT$. Then V is characteristic in U_0 as $V = \Omega_1(Z(U_0))$. If B is not elementary abelian of order 16, then we claim that V is a characteristic subgroup of U_1. If $U_1 = U_0$, we are done. Thus $y \neq 1$ and $|U_1 : U_0| = 2$. Suppose that $\alpha \in \mathrm{Aut}(U_1)$ with $V \neq \alpha(V)$. Then $U_0 \neq \alpha(U_0)$. Hence $|U_0 : U_0 \cap \alpha(U_0)| = 2$. Assume that B is homocyclic. Since $B \cap \alpha(B)$ is centralized by $B\alpha(B)$ and t inverts B, we either have that $tB \not\subset B\alpha(B)$ or $B \cap \alpha(B)$ is elementary abelian. In the latter case we have that $\alpha(V) = \Omega_1(\alpha(B)) = \Omega_1(B \cap \alpha(B)) = \Omega_1(B) = V$, which we supposed was not the case. Therefore, $B\alpha(B) \neq U_1$ and so $|B : B \cap \alpha(B)| = 2$ and again we have $\alpha(V) = V$. So we may assume that B is non-abelian. In particular, we have that $|B| = 2^6$. Then $B \cap \alpha(U_0)$ has order 2^5. Now $U_1/\alpha(B)$ is abelian and so $V = B' \leq \alpha(U_0)$ and $\alpha(V) = \alpha(B)' \leq B$. Thus $Z(B \cap \alpha(U_0)) \geq V\alpha(V) > V$, this contradicts the structure of B as the centre of every subgroup of B of index 2 is V. Thus V is a characteristic subgroup of U_1 and our claim is proved.

Set $U_2 = N_U(U_1)$. As $BT \in \mathrm{Syl}_2(C_G(V))$, we deduce that $|U_2 : U_1| = 2$ and that $U_1 = BT$. In particular, we have $y = 1$ and S is elementary abelian of order 2^3. Since $C_{U_2}(t) = S \leq U_1$, there must be at least two BT-conjugacy classes of involutions in Bt. It follows from Corollary 2.24 that B is homocyclic and there are four BT-conjugacy classes of involutions in Bt. Recall that there is an element $\rho \in N_L(V)$ of order 3 and that ρ normalizes B, centralizes t and $C_B(\rho) = 1$. Thus Bt contains two $B\langle t, \rho\rangle$ classes of involutions. Now U_2 does not centralize t and so we deduce $\langle U_2, \rho\rangle$ induces a transitive action on the four BT-conjugacy classes of involutions in Bt. It follows that $|N_G(U_1) : U_1|$ is divisible by 4. But this contradicts $BT \in \mathrm{Syl}_2(C_G(V))$ and V being normalized by $N_G(U_1)$.

So we have shown that B is elementary abelian of order 16. Note that B is characteristic in U_1. By Corollary 2.24, all the involutions in Bt are conjugate in BT. Therefore, as $U_1 < U$, we deduce that $U_1 = BS > BT$ and so, in particular, $|U_1| = 2^6$. By Lemma 7.7, there is a non-central fours group X of S which is normal in U. Since $U_1 \in \mathrm{Syl}_2(N_G(V))$ and $U > U_1$, $X \neq V$. We have that $X \leq S \leq BS$ and so X normalizes B and B normalizes X. Therefore $[B, X] \leq B \cap X \leq B \cap S = V$. Since $X \neq V$, we infer that $[B, X] = B \cap X = \langle z\rangle = Z(S)$. However, $|[B, x]| = 2^2$ for all $x \in BS \setminus B$ and so we have a contradiction.

So we are left with the case that $B = V$ and $U > U_1 = S = \langle V, t, y\rangle$. Let $U_2 = N_U(S) > S$. We claim that VT is normal in U_2. This is obviously true if $y = 1$. So suppose that S is non-abelian. Then, by Lemma 7.7, there is a fours group $X \leq S$ which is normal in U and is not contained in $Z(S)$. It follows that TX is an elementary abelian group of order 8. Since S contains exactly two such subgroups, we deduce that both TX and TV are normal in U_2. Let $E = VT$. Since U_2 does not centralize t and since E is normalized by an element of order 3 in L, we see that $|t^G \cap E| \geq 4$. Since $Z(U_2) \cap E \neq 1$, we get $|t^G \cap E| = 4$. Therefore, $|N_G(E)/C_G(E)| = 12$ or 24. As $N_G(E)/C_G(E)$ is a subgroup of $\mathrm{PSL}_3(2)$, this means $N_G(E)/C_G(E) \cong \mathrm{Alt}(4)$ or $\mathrm{Sym}(4)$. In both cases $C_E(O_2(N_G(E)/C_G(E)))$ is non-trivial and normal in $N_G(E)$ and so it must be V. But $E = BT \in \mathrm{Syl}_2(C_G(V))$ and we have a contradiction. This is our final contradiction and so Lemma 7.8 is proved. □

Theorem 7.9. *Assume Hypothesis 7.1. Then* $F^*(G) \cong {}^2\mathrm{G}_2(3^{2a+1})$.

Proof. By Lemma 7.8 the Sylow 2-subgroups of G are elementary abelian of order 8. Therefore Theorem 2.3 gives $F^*(G) \cong {}^2\mathrm{G}_2(3^{2a+1})$ for some $a \geq 1$. □

8. Components in H

Assume that Hypothesis 4.1 holds. From Lemma 4.5, we know that $E(H)$ is quasisimple if $E(H) \neq 1$. The objective of this section is to prove that $E(H) = 1$.

Set $E = E(H)$. So E is a quasisimple group. Since $O_{p'}(G) = 1$, we have that $F(H)$ is a p-group. In particular, E contains involutions. If $t \in H$ is an involution, we know by Lemma 4.4 that $C_G(t) \not\leq H$, $O_{p'}(C_G(t)) \leq H$ and $(F^*(C_G(t)/O_{p'}(C_G(t))), p) \in \mathcal{E}$. Whenever t is an involution from H, we use the following bar notation $\overline{C_G(t)} = C_G(t)/O_{p'}(C_G(t))$.

Lemma 8.1. *If* $N_G(X)$ *is a* $\mathcal{K}$*-group for all* p'*-subgroups* X *of* G*, then* $E = F^*(H)$ *is quasisimple.*

Proof. Set $Q = O_p(H)$ and assume that $Q \not\leq E(H)$. Of course $[E, Q] = 1$. Suppose that t is an involution in E. Then, by Lemma 4.4, $C_G(t) \not\leq H$, $O_{p'}(C_G(t)) \leq H$ and $(F^*(\overline{C_G(t)}), p) \in \mathcal{E}$. Since Q and $O_{p'}(C_H(t))$ commute, we have that $F^*(C_G(t)) \not\leq O_{p'}(C_G(t))$. As $F^*(\overline{C_G(t)})$ is almost simple, $C_G(t)$ has a unique component L_t which has order divisible by p and we have $\overline{L_t} = \overline{F^*(C_G(t))}$. Note that, as $\overline{Q} \neq 1$, Proposition 2.7 implies that $\overline{C_H(t)}$ is p-closed and soluble.

Assume that p divides $|C_E(t)/Z(E)|$. Then $O_p(\overline{C_H(t)})$ contains non-trivial normal subgroups $\overline{Q}$ and $O_p(\overline{C_E(t)})$ and they commute. This contradicts the structure of $\overline{C_H(t)}$ given in Proposition 2.7. Therefore, $C_E(t) \leq O_{p'}(C_H(t))$ for all involutions $t \in E$. Suppose that $C_E(t) \not\leq O_{p'}(C_G(t))$. Then using Corollary 2.11 and noting that $\overline{C_H(t)}$ is soluble, we get $|\overline{C_E(t)}| = 2$ and $(\overline{L}_t, p) = (\mathrm{PSL}_3(4), 3)$.

We consider the case $(\overline{L}_t, p) = (\mathrm{PSL}_3(4), 3)$ independently from the assumption that $C_E(t) \not\leq O_{p'}(C_G(t))$. In this case $\overline{Q}$ is a normal 3-subgroup of $\overline{C_H(t)}$ and so Proposition 2.7(iv) implies that $N_{L_t}(Q)/Q$ is a non-abelian 2-group. Suppose that $s \in C_H(E)$ is an involution. Then $EO_{3'}(C_G(s))/O_{3'}(C_G(s))$ is a quasisimple normal subgroup of $C_H(s)/O_{3'}(C_G(s))$ which by Lemma 4.4(iii) is strongly 3-embedded in $C_G(s)/O_{3'}(C_G(s))$. However, as $(F^*(C_G(s)/O_{3'}(C_G(s))), 3) \in \mathcal{E}$, we have a contradiction to Proposition 2.7 as, when $p = 3$, $\overline{C_H(t)}$ must be soluble. Thus $C_H(E)$ has odd order and, in particular, $N_{L_t}(Q)/Q$ maps isomorphically into $H/C_H(E)$ and so we deduce that $\mathrm{Out}(E)$ has non-abelian Sylow 2-subgroups. Now E is a $\mathcal{K}$-group and since $\mathrm{Out}(E)$ has non-abelian Sylow 2-subgroups, we infer that E is a Lie type group of Lie rank at least 4 defined over a field of characteristic r. But these groups all possess an involution s with $C_E(s)$ involving $\mathrm{PSL}_2(r)$. Since 3 divides $|\mathrm{PSL}_2(r)|$, this contradicts $C_E(s) \leq O_{3'}(C_H(s))$.

Thus $(\overline{L_t}, p) \neq (\mathrm{PSL}_3(4), 3)$ and we conclude further that $C_E(t) \leq O_{p'}(C_G(t))$ for all involutions $t \in E$.

Suppose now that $T \in \mathrm{Syl}_2(E)$ and $t \in Z(T)^\#$ is an involution. Then

$$T \leq C_E(t) \leq O_{p'}(C_G(t)) \leq C_G(L_t).$$

Assume that $s \in T$ is an involution. Then $C_E(s) \leq O_{p'}(C_G(s))$ and thus

$$[C_E(s), \langle Q^{C_G(s)}\rangle] \leq [O_{p'}(C_G(s)), \langle Q^{C_G(s)}\rangle] = \langle [O_{p'}(C_G(s)), Q]^{C_G(s)}\rangle = 1.$$

As $L_t \leq C_G(s)$, we see $\langle Q^{C_G(s)}\rangle \geq \langle Q^{L_t}\rangle = L_tQ$. Thus L_t centralizes $C_E(s)$. It follows that L_t centralizes $K = \langle C_E(x) \mid x \in T, x^2 = 1\rangle$. If p divides $|K|$, then we have the contradiction $L_t \leq H$ as H is strongly p-embedded. Hence p does not divide $|K|$ and so $K \neq E$. It follows from [9, 17.13] that $N_E(K)$ is a strongly 2-embedded subgroup of E. Hence $E \cong \mathrm{SL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$ or ${}^2\mathrm{B}_2(2^{2a-1})$ for some $a \geq 2$ and $K \leq N_G(T)$ by [2]. Since L_t centralizes T and L_t is normal in $C_G(t)$, we see that L_t is a characteristic subgroup of $C_G(T)$. Hence $N_G(T)$ normalizes L_t. In particular, $N_G(T) = N_H(T)L_t$ by the Frattini Argument. By Lemmas 3.2(v) and 3.3 (ii) and (iii), we have $F^*(N_G(T)/O_{p'}(N_G(T))) \cong L_t/Z(L_t)$. Thus using Corollary 2.11 and the fact that $L_t/Z(L_t) \not\cong \mathrm{PSL}_3(4)$, we have $O_{p'}(N_G(T)) = O_{p'}(N_H(T))$. If p divides $|N_E(T)|$, then $N_E(T)O_{p'}(N_G(T))/O_{p'}(N_G(T))$ commutes with $QO_{p'}(N_G(T))/O_{p'}(N_G(T))$ and as before we have a contradiction to the structure of L_t via Proposition 2.7. Hence $N_E(T) \le O_{p'}(N_H(T)) = O_{p'}(N_G(T))$. In particular, L_t centralizes $N_E(T)$. Let J be a complement to T in $N_E(T)$ and set $M = N_E(J)$. Then $E = \langle N_E(T), M\rangle$. Since L_t centralizes $N_E(T)$, $L_t \le C_G(J)$. Therefore, using J is a p'-group and $N_G(J)$ is a $\mathcal{K}$-group, Lemma 3.3 (ii) and (iii) imply that L_t is the unique component in $N_G(J)$. In particular, M normalizes L_t and so $E = \langle N_E(T), M\rangle$ normalizes L_t as well. But again by Lemma 3.3 (ii) and (iii), we have that $F^*(L_tE/O_{p'}(L_tE)) = L_tO_{p'}(L_tE)/O_{p'}(L_tE)$ which means that $E \le O_{p'}(L_tE)$ and contradicts $O_{p'}(H) = 1$. This contradiction shows that $Q \le E$ and so $F^*(H) = E$ is quasisimple. □

Throughout the remainder of this section we assume the following hypothesis.

Hypothesis 8.2. *Hypothesis 4.1 holds as well as*

(i) *$F^*(H) = E(H)$ is a quasisimple $\mathcal{K}$-group; and*
(ii) *G does not contain a classical involution.*

We begin with two lemmas which will be used frequently.

Lemma 8.3. *If $t \in H$ is an involution and $\overline{C_H(t)}$ is not soluble, then either*

(i) *$(F^*(\overline{C_G(t)}), p) = (\mathrm{Fi}_{22}, 5)$ and $\overline{C_H(t)} \cong \mathrm{Aut}(\Omega_8^+(2))$; or*
(ii) *$(F^*(\overline{C_G(t)}), p) = (\mathrm{Alt}(2p), p)$ and $F^*(\overline{C_H(t)}) \cong \mathrm{Alt}(p) \times \mathrm{Alt}(p)$ with $p \ge 5$.*

Moreover, in case (ii), the components of $\overline{C_H(t)}$ are not normal in $\overline{C_H(t)}$.

Proof. This follows from Lemma 4.4(iii) and Proposition 2.7. □

Lemma 8.4. *Assume that $t \in H$ is an involution and $|C_H(t)|_p \ne |C_E(t)|_p$. Then $(\overline{C_G(t)}, p) = (\mathrm{PSL}_2(8) : 3, 3)$ or $({}^2\mathrm{B}_2(32) : 5, 5)$. In particular, $C_G(t)$ has extraspecial Sylow p-subgroups of order p^3.*

Proof. Suppose that $|C_E(t)|_p \neq |C_H(t)|_p$. Then $\overline{C_H(t)} > \overline{C_E(t)}$ and, as $C_H(t)/C_E(t)$ is soluble and has order divisible by p, Proposition 2.7 implies that $\overline{C_H(t)}$ is p-closed. Let $T_1 \in \mathrm{Syl}_p(C_H(t))$ and let $T_2 = T_1 \cap E$. Then $\overline{T_1} > \overline{T_2}$ and $\overline{T_1}$ is normal in $\overline{C_H(t)}$. It follows from Corollary 2.8 that either $T_2 \leq \Phi(T_1)$ or $(\overline{C_G(t)}, p) = (\mathrm{PSL}_2(8) : 3, 3)$ or $({}^2\mathrm{B}_2(32) : 5, 5)$. Thus we may assume that $2 \leq m_p(T_2/T_1) \leq m_p(H/E)$. Since the p-rank of $\mathrm{Out}(E)$ is at most 2, we have that T_1/T_2 has p-rank 2 by Lemma 2.16. But then the structure of $\mathrm{Out}(E)$ in Lemma 2.16 shows that $C_H(t)$ cannot act irreducibly on $\overline{T_1}/\Phi(\overline{T_1})$ and Corollary 2.8 once again gives $p \in \{3, 5\}$ and the structure of $\overline{C_G(t)}$. $\square$

Lemma 8.5. *$E/Z(E)$ is not an alternating group of degree $n \geq 5$.*

Proof. Suppose that $E/Z(E) \cong \mathrm{Alt}(n)$ for some $n \geq 5$. Assume that $Z(E) = 1$. Let $n \geq 9$. Let t correspond to a product of two transpositions in E. Then $C_H(t)$ contains a normal subgroup isomorphic to $\mathrm{Alt}(n-4)$ with $n-4 \geq 5$. Thus $\overline{C_H(t)}$ contains such a normal subgroup and this contradicts Lemma 8.3. So $n < 9$. But then $C_E(t)$ is a $\{2, 3\}$–group with cyclic Sylow 3-group and so $m_r(C_H(t)) < 2$ for all odd primes r. This contradicts Hypothesis 4.1(ii).

This contradiction shows that $Z(E) \neq 1$. Since $Z(E)$ is a p-group, this means $E \cong 3{}^{\cdot}\mathrm{Alt}(6)$ or $3{}^{\cdot}\mathrm{Alt}(7)$ with $p = 3$. The first possibility fails as the centralizers of involutions in $\mathrm{Alt}(6)$ have order 8. Thus $E \cong 3{}^{\cdot}\mathrm{Alt}(7)$. In this case, $\overline{C_H(t)}$ has a normal Sylow 3-subgroup of order 9 and $C_H(t)$ does not act irreducibly on $O_3(\overline{C_H(t)})$, this contradicts Corollary 2.8. Hence we have shown $E/Z(E)$ is not an alternating group. $\square$

Next we show that $E/Z(E)$ cannot be a sporadic simple group.

Lemma 8.6. *$E/Z(E)$ is not a sporadic simple group.*

Proof. We use Lemmas 4.4 and 8.3 for all the sporadic groups. We first observe that the outer automorphism group of a sporadic simple group has order dividing 2 [10, Table 5.3]. Hence E has index at most 2 in H. In Table 2, for each possibility for $E/Z(E)$, we give the structure of the centralizer of some involution t in E (recall $O_2(E) = 1$). For each case, except $E \cong 3{}^{\cdot}\mathrm{M}_{22}$ with $p = 3$, we see that either p^2 does not divide $|C_E(t)|$ or that $C_H(t)/O_{p'}(C_H(t))$ has a simple section which is not isomorphic to $\mathrm{Alt}(p)$ or $\Omega_8^+(2)$. Assume that $E \cong 3{}^{\cdot}\mathrm{M}_{22}$ and $p = 3$. Then $C_H(t)$, with t as in Table 2, is soluble and has Sylow 3-subgroups of order 9.

$E/Z(E)$	Involution	$C_{E/Z(E)}(t)$	$E/Z(E)$	Involution	$C_{E/Z(E)}(t)$
M_{11}	$2A$	$\mathrm{GL}_2(3)$	M_{12}	$2A$	$2\times \mathrm{Sym}(5)$
M_{22}	$2A$	$2^4.\mathrm{Sym}(4)$	M_{23}	$2A$	$2^4:\mathrm{PSL}_3(2)$
M_{24}	$2A$	$2^{1+6}.\mathrm{PSL}_3(2)$	J_1	$2A$	$2\times \mathrm{Alt}(5)$
J_2	$2A$	$2_-^{1+4}.\mathrm{Alt}(5)$	J_3	$2A$	$2_-^{1+4}.\mathrm{Alt}(5)$
J_4	$2A$	$2^{1+12}.3{\cdot}\mathrm{M}_{22}.2$	Co_3	$2A$	$2{\cdot}\mathrm{PSp}_6(2)$
Co_2	$2A$	$2_+^{1+8}.\mathrm{PSp}_6(2)$	Co_1	$2C$	$2^{11}.\mathrm{M}_{12}$
HS	$2A$	$4 * 2_+^{1+4}.\mathrm{Sym}(5)$	McL	$2A$	$2{\cdot}\mathrm{Alt}(8)$
Suz	$2A$	$2_-^{1+6}.\Omega_6^-(2)$	He	$2A$	$2^2.\mathrm{PSL}_3(4).2$
Ly	$2A$	$2{\cdot}\mathrm{Alt}(11)$	Ru	$2A$	$2^{11}.\mathrm{Sym}(5)$
O'N	$2A$	$4.\mathrm{PSL}_3(4).2$	Fi_{22}	$2A$	$2{\cdot}\mathrm{PSU}_6(2)$
Fi_{23}	$2A$	$2{\cdot}\mathrm{Fi}_{22}$	Fi_{24}'	$2A$	$2{\cdot}\mathrm{Fi}_{22}.2$
HN	$2A$	$2{\cdot}\mathrm{HS}.2$	Th	$2A$	$2_+^{1+8}.\mathrm{Alt}(9)$
B	$2A$	$2.{}^2\mathrm{E}_6(2).2$	M	$2A$	$2{\cdot}\mathrm{B}$

Table 2. Centralizers of certain involutions in sporadic simple groups.

However, $\overline{C_H(t)}$ does not act irreducibly on $O_3(\overline{C_H(t)})$ and this then contradicts Corollary 2.8. Thus this possibility cannot occur either. Therefore $E/Z(E)$ is not a sporadic simple group. $\square$

We now begin our investigation of the case when $E/Z(E)$ is a group of Lie type.

Lemma 8.7. *Suppose that $E/Z(E)$ is a simple group of Lie type. Then p divides $|C_E(t)|$ for all involutions $t \in H$.*

Proof. Assume that $E/Z(E)$ is a Lie type group defined in characteristic r and that $C_E(t)$ is a p'-group. Then, as $m_p(C_H(t)) \geq 2$, $|C_E(t)|_p \neq |C_H(t)|_p$ and hence Lemma 8.4 implies that $p \in \{3,5\}$ and $C_H(t)$ has extraspecial Sylow p-subgroups. It follows that H/E and hence $\mathrm{Out}(E)$ has non-abelian Sylow p-subgroups. In particular, we see that E must admit diagonal automorphisms of order p. This shows that $E/Z(E)$ is $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$ and that p divides n. Since H/E has non-abelian Sylow p-subgroups, we further infer that p^2 divides n. Thus $n \geq 9$. But then the canonical form of t shows that the centralizer of t has a section isomorphic to $\mathrm{PSL}_4(r)$ or $\mathrm{PSU}_4(r)$. Both these groups have order divisible by 3 and 5, a contradiction. Thus p divides $|C_E(t)|$. $\square$

Lemma 8.8. *$E/Z(E)$ is not a rank 1 Lie type group.*

Proof. Suppose that $E/Z(E)$ is defined in characteristic 2. So $E/Z(E) \cong \mathrm{PSL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$ or ${}^2\mathrm{B}_2(2^{2a-1})$ for some $a \geq 2$. In the first and third case, the centralizer of an involution in $C_E(t)$ is a 2-group and so $m_p(\mathrm{Out}(E)) \geq 2$. Using [10, Theorem 2.5.12] we see that $\mathrm{Out}(E)$ is cyclic, which is a contradiction. Therefore $E/Z(E) \cong \mathrm{PSU}_3(2^a)$ for some $a \geq 2$. Let $t \in E$ be an involution and $S \in \mathrm{Syl}_p(C_H(t))$. Then $C_E(t)$ is 2-closed, contains a Sylow 2-subgroup T of E and T has a cyclic complement in $C_E(t)$ of order dividing $q+1$. In particular, $|C_E(t)|_p \neq |C_H(t)|_p$ and so S is extraspecial of order p^3 and $(\overline{C_G(t)}, p) = (\mathrm{PSL}_2(8)\colon 3, 3)$ or $({}^2\mathrm{B}_2(32)\colon 5, 5)$ by Lemma 8.4. Since $\mathrm{Out}(E)$ has abelian Sylow p-subgroups, p must divide $q+1$. Furthermore, the Sylow p-subgroups of $\mathrm{GU}_3(2^a)$ are abelian and so SE must involve field automorphisms of E. In particular, we have p divides a and $Z(T)$ is not centralized by S. From the structure of $\overline{C_G(t)}$, $T \leq O_2(C_H(t)) \leq O_{p'}(C_H(t)) = O_{p'}(C_G(t))$. Suppose that $T^g \neq T$ for some $g \in C_G(t)$. Then $T^g \leq O_2(C_H(t))$ and T^gE/E is a non-trivial 2-group of outer automorphisms of E. It follows that T^gE/E is cyclic and acts non-trivially on the cyclic group $C_E(t)/T$. However $T^g \leq O_2(C_H(t))$ and so this is impossible. Thus T is normalized by $C_G(t)$. Since $Z(T)$ is centralized by $C_E(t)$, $L = \langle C_E(t)^{C_G(t)}\rangle$ centralizes $Z(T)$. Since p divides $|C_E(t)|$, the structure of $\overline{C_G(t)}$ shows that $\overline{L} \geq O^p(\overline{C_G(t)})$. Suppose that $2^a \neq 8$. Then, as $|C_E(t)/T| = (2^a+1)/(2^a+1, 3)$, there is some element $\omega \in C_E(t)$, $o(\omega) = r$, r a Zsygmondi prime dividing 2^a+1. Note that r does not divide $|\overline{C_G(t)}|$ and that ω acts irreducibly on $T/Z(T)$. Thus $C_G(t)$ acts on $T/Z(T)$ irreducibly. Since $|T| = 2^{3a}$ with $|Z(T)| = 2^a$, using [14, Lemma 2.7.3] we have that $C_G(t)/C_G(T)T$ embeds into $\mathrm{SL}_2(2^a)\colon a$. Since $C_G(T) \leq O_{p'}(C_G(t))$ we have that $C_G(t)/C_G(T)T$ involves $\mathrm{SL}_2(8)\colon 3$ when $p = 3$ and involves ${}^2\mathrm{B}_2(32)\colon 5$ when $p = 5$ and these groups must be sections of $\mathrm{SL}_2(2^a) : a$. Furthermore, r divides $O_{p'}(C_G(t)/C_G(T)T)$ which is thus non-trivial. The structure of $\mathrm{SL}_2(2^a)$ now implies that $C_G(t)/C_G(T)T$ is soluble, a contradiction. So we are left with $2^a = 8$. Let U be a hyperplane of $Z(T)$. Then T/U is extraspecial of order 2^7. Since L centralizes $Z(T)$, L acts on T/U. Furthermore, $L/C_L(T)$ has a section isomorphic to $\mathrm{SL}_2(8)$ when $p = 3$ or to ${}^2\mathrm{B}_2(32)$ when $p = 5$. Since $\mathrm{Out}(T/U)$ is isomorphic to $\mathrm{O}_6^{\pm}(2)$, this is also impossible. Hence we have shown that E is not a rank 1 group defined in characteristic 2.

Assume that $E/Z(E)$ is $\mathrm{PSL}_2(r^a)$ with r odd. Since $\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$, we see with Lemma 8.5 that $r^a \neq 9$. Thus the Schur multiplier of E has order 2 and so $E \cong \mathrm{PSL}_2(r^a)$. Let t be an involution in E. Then $C_E(t)$ is a dihedral group and hence we have $|C_H(t)|_p > |C_E(t)|_p$ and Lemma 8.4 implies that either $p = 3$ and $\overline{C_G(t)} \cong \mathrm{PSL}_2(8) : 3$ or $p = 5$ and $\overline{C_G(t)} \cong {}^2\mathrm{B}_2(32) : 5$. Let $S \in \mathrm{Syl}_p(C_H(t))$. Then S is extraspecial of order p^3. Since $\mathrm{Out}(E)$ has cyclic Sylow p-subgroups, we must have $S \cap E$ is cyclic of order p^2. In particular, as $C_E(t)$ is a dihedral group, we have $Q = O_p(C_E(t)) > 1$. Thus, because $O_{p'}(C_G(t)) = O_{p'}(C_H(t))$, $[Q, O_{p'}(C_G(t))] = [Q, O_{p'}(C_H(t))] = 1$. Therefore $L = \langle Q^{C_G(t)} \rangle$ centralizes $O_{p'}(C_G(t))$ and so, as the Schur multipliers of $\mathrm{PSL}_2(8)$ and ${}^2\mathrm{B}_2(32)$ are both trivial and Q is inverted in $N_{C_G(t)}(Q)$, we have that L is a normal component of $C_G(t)$. Thus $C_G(t) = (L \times O_{p'}(C_H(t)))S$. Let $T \in \mathrm{Syl}_2(C_H(t))$. Then as E has one conjugacy class of involutions, $T \in \mathrm{Syl}_2(H)$ and T normalizes Q. Thus we can write $T = (T \cap L) \times (T \cap O_{p'}(C_G(t)))$. Let $T_L = T \cap L$. Then T_L is cyclic of order $p - 1$ (2 or 4) and $T_L \le Z(T)$. In particular, $Z(T) \ge \langle t \rangle T_L$, $t \notin T_L$ and $|Z(T)| \ge 4$. Assume that E has non-abelian Sylow 2-subgroups. Then $T \ne \langle t \rangle T_L$ and $T_L \cap E = 1$ for otherwise $|Z(T) \cap E| \ge 4$. Let $f \in \Omega_1(T_L)^\sharp$. Then f centralizes a Sylow 2-subgroup of E and so we have that f induces a non-trivial field automorphism of E. Thus, since $r^a \ne 3^2$, $C_H(f)$ contains a component F isomorphic to $\mathrm{PSL}_2(r^{a/2})$ and a is even. As a is even, we have that $C_E(t)$ is of order $r^a - 1$ [12, II 8.27]. Hence we have that p divides $r^a - 1$. In particular, p divides $|\mathrm{PSL}_2(r^{a/2})| = r^{a/2}(r^a - 1)/2$. Hence $\mathrm{PSL}_2(r^{a/2})$ is isomorphic to a subgroup of $C_H(f)/O_{p'}(C_G(f))$ which is consequently not soluble. Now Lemma 8.3 delivers a contradiction. It follows that E has abelian Sylow 2-subgroups and, as $|Z(T)| \ge 4$, $E \cong \mathrm{PSL}_2(r)$ where $r \equiv 3, 5 \pmod 8$. In particular, $T = \langle t \rangle T_L$. Therefore $T \le E$ and T_L has order $2 = p - 1$. Therefore $p = 3$ and $L \cong \mathrm{PSL}_2(8)$. Let $R \in \mathrm{Syl}_2(C_G(t))$ such that $R \ge T$. Then R is elementary abelian of order 16. Since $R \in \mathrm{Syl}_2(C_G(t))$, $R \in \mathrm{Syl}_2(C_G(R))$. Note that $N_L(R)/R$ is cyclic of order 7. Thus $N_L(R)$ induces orbits of length 1, 7 and 7 on the non-trivial elements of R. Since the non-trivial elements of T are all conjugate, $T_L \le L$, and $te \notin L$ where $e \in T_L$, we infer that all the involutions in R are conjugate to t. In particular, $R \in \mathrm{Syl}_2(C_G(x))$ for all $x \in R^\sharp$. Therefore $N_G(R)$ acts transitively on $R^\sharp$ and $N_G(R)/C_G(R)$ is a subgroup of $\mathrm{GL}_4(2)$ divisible by 3.5.7 and of odd order. There are no such subgroups in $\mathrm{GL}_4(2)$ and so we have our final contradiction to this configuration. Hence $E/Z(E) \not\cong \mathrm{PSL}_2(r^a)$ for odd r.

So we are left with $E/Z(E) \cong \mathrm{PSU}_3(r^a)$ or ${}^2\mathrm{G}_2(r^a)$, where $r = 3$, $a > 1$ in the latter. As in $\mathrm{Aut}(\mathrm{PSU}_3(3))$, no involution centralizes a group of order p^2, we get that $r^a \ne 3$. Therefore in all the cases, there is an involution t such that $E(C_E(t)/\langle t \rangle) \cong \mathrm{PSL}_2(r^a)$ is a non-abelian simple group. If p divides

$|\mathrm{PSL}_2(r^a)|$, then Lemmas 8.3 gives a contradiction. So we have that p does not divide $|\mathrm{PSL}_2(r^a)|$. As $C_E(t) = \mathrm{GU}_2(r^a)$ or $\langle t\rangle \times \mathrm{PSL}_2(r^a)$, we get that $C_E(t)$ is a p'-group and this contradicts Lemma 8.7. $\square$

Lemma 8.9. *$E/Z(E) \not\cong \mathrm{PSL}_3(2^a)$ for $a \geq 1$.*

Proof. Suppose that $E/Z(E) \cong \mathrm{PSL}_3(2^a)$, $a \geq 1$. Then E has exactly one conjugacy class of involutions. If $a = 1$, then $C_E(t)$ is dihedral of order 8 and $\mathrm{Out}(E)$ has order 2, so H does not satisfy our Hypothesis 4.1(ii). So we may assume that $a \geq 2$.

Let $t \in E$ be an involution. Then $T = O_2(C_E(t)) \in \mathrm{Syl}_2(C_E(t))$. Furthermore, a complement to T in $C_E(t)$ is cyclic of order dividing $2^a - 1$. Hence, again by Hypothesis 4.1 (ii), $|C_H(t)|_p > |C_E(t)|_p$. Thus Lemma 8.4 implies that $p \in \{3, 5\}$ and $\overline{C_G(t)}$ has extraspecial Sylow p-subgroups. Since $\mathrm{Out}(E)$ has abelian Sylow p-subgroups, we have $|C_E(t)|_p > 1$ and, in particular, p divides $2^a - 1$.

We have that $O_2(C_H(t)/O_{p'}(C_G(t)) \cap F^*(C_G(t)/O_{p'}(C_G(t)))) = 1$. In particular, T is normal in $O_{p'}(C_G(t))$ and, using Lemma 2.27, we now see that T is normal in $C_G(t)$.

By considering T as the subgroup of lower unitriangular matrices in $\mathrm{SL}_3(2^a)$, we see that T contains exactly two elementary abelian subgroups F_1 and F_2 of order 2^{2a} (all the elements of T outside these two subgroups have order 4). Thus, as T is normal in $C_G(t)$, $C_G(t)$ permutes F_1 and F_2 and consequently $C_G(t)$ has a subgroup of index at most 2 which normalizes F_1. It follows that $N_G(F_1) \not\leq H$ and that $m_p(N_G(F_1)) \geq m_p(C_G(t)) = 2$. By Lemmas 3.2 and 3.3 (ii), we have that $L = F^*(N_G(F_1)/O_{p'}(N_G(F_1)))$ is a non-abelian simple group. Since $N_E(F_1)/O_2(N_E(F_1))$ has a section isomorphic to $\mathrm{SL}_2(2^a)$ with $a \geq 2$, and since p divides $|\mathrm{SL}_2(2^a)| = (2^a - 1)2^a(2^a + 1)$, we get with Proposition 2.7 that $F^*(N_G(F_1)/O_{p'}(N_G(F_1))) \cong \mathrm{Alt}(2p)$ or Fi_{22}. Furthermore, we have $N_H(F_1)/O_{p'}(N_G(F_1))$ is isomorphic to $(\mathrm{Alt}(p) \times \mathrm{Alt}(p)) : 2$ with no normal components or $\Omega_8^+(2)$, neither of these has a normal subgroup isomorphic to $\mathrm{PSL}_2(2^a)$ and so we have a contradiction. $\square$

The next two lemmas are needed to finally dispatch the Lie type groups as possibilities for E.

Lemma 8.10. *Suppose that K is a simple group of Lie type defined in characteristic 2 and p is an odd prime. Let $t \in K$ be an involution in a long root subgroup of K. If p divides $|C_K(t)|$, then either*

(i) *p divides $|O^{2'}(C_K(t))|$; or*
(ii) *K is isomorphic to one of $\mathrm{PSL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$, ${}^2\mathrm{B}_2(2^{2a+1})$ or $\mathrm{PSL}_3(2^a)$ for some $a \geq 1$.*

Proof. As $C_K(t) = O^{2'}(C_K(t))B$, where B is contained in a Borel subgroup of K, we may assume that p divides the order of a Borel subgroup. If the Lie rank of K is at least two and $K \not\cong \mathrm{PSL}_3(2^a)$, we have that t is centralized by a minimal parabolic subgroup and so p divides the order of this minimal parabolic. This is the assertion. □

Lemma 8.11. *Suppose that K is a simple group of Lie type defined in characteristic r, r odd, and of Lie rank at least 2. Let t be an involution in a fundamental subgroup of K and assume that p is an odd prime. If p divides $|C_K(t)|$, then p divides $|O^{2'}(C_K(t))|$.*

Proof. We have that $C_K(t) = L_1L_2S$, where L_1 is the fundamental subgroup, $L_2 = C_{C_K(t)}(L_1)$ and S is a Sylow 2-subgroup with $t \in Z(S)$. If $L_1L_2 = O^{2'}(L_1L_2)$, we are done. So we may assume that $r = 3$. If $L_2 = O^{2'}(L_2)$ and $L_2 \neq 1$, then, as 3 divides the order of L_2, we also are done. Hence we have $p = 3$ and L_2 is a central product of groups isomorphic to $\mathrm{SL}_2(3)$ or $\mathrm{PSL}_2(3)$. This now shows that K is isomorphic to one of the following $\mathrm{PSL}_3(3)$, $\mathrm{PSp}_4(3)$, $\mathrm{P\Omega}_6^{\pm}(3)$, $\Omega_7(3)$, $\mathrm{P\Omega}_8^{+}(3)$ or $\mathrm{G}_2(3)$. But in all cases we have some $\mathrm{GL}_2(3)$ involved, so 3 divides $|O^{2'}(C_K(t))|$. □

For X a Lie type group of Lie rank at least two in odd characteristic r we list $O^{r'}(C_X(t))$ for t a classical involution in Table 4. The information is taken from [10, Table 4.5.1, Theorem 4.5.5].

Lemma 8.12. *$E/Z(E)$ is not a simple Lie type group defined in characteristic 2.*

Proof. Suppose that $E/Z(E)$ is a Lie type group defined in characteristic 2. By Lemma 8.8 we have that the Lie rank of E is at least two. Further $E/Z(E) \not\cong \mathrm{PSL}_3(2^a)$, $a \geq 1$ by Lemma 8.9. Let $S \in \mathrm{Syl}_2(E)$ and let t be an involution in the centre of the long root group X_ρ contained in $Z(S)$. Then $C_E(t)$ is a subgroup

Simple group	Levi Factor of	Simple group	Levi Factor
$\mathrm{PSL}_n(q)$	$\mathrm{SL}_{n-2}(q)$	$\mathrm{PSU}_n(q)$	$\mathrm{SU}_{n-2}(q)$
$\mathrm{PSp}_{2n}(q)'$	$\mathrm{Sp}_{2(n-1)}(q)$	$\Omega_n^\pm(q)$	$\mathrm{SL}_2(q) * \mathrm{SO}_{n-4}^\pm(q)$
$\mathrm{G}_2(q)'$	$\mathrm{SL}_2(q)$	${}^3\mathrm{D}_4(q)$	$\mathrm{SL}_2(q^3)$
$\mathrm{F}_4(q)$	$\mathrm{Sp}_6(q)$	${}^2\mathrm{F}_4(q)$	${}^2\mathrm{B}_2(q)$
${}^2\mathrm{E}_6(q)$	$\mathrm{SU}_6(q)$	$\mathrm{E}_6(q)$	$\mathrm{SL}_6(q)/\mathbb{Z}_{(q-1,3)}$
$\mathrm{E}_7(q)$	$\mathrm{SO}_{12}^+(q)$	$\mathrm{E}_8(q)$	$\mathrm{E}_7(q)$

TABLE 3. Subgroups generated by root elements in the Levi complements of centralizers of involutions in the centre of long root subgroups in Lie type groups of rank at least 2 defined over a field of order $q = 2^a$.

Simple group X	$O^{r'}(C_X(t))$	Simple group X	$O^{r'}(C_X(x))$
$\mathrm{PSL}_n(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{SL}_{n-2}(r^a)$	$\mathrm{PSU}_n(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{SU}_{n-2}(r^a)$
$\mathrm{PSp}_{2n}(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{Sp}_{2(n-1)}(r^a)$	$\Omega_n^\pm(r^a)$	$(\mathrm{SL}_2(r^a) * \mathrm{SL}_2(r^a)) * \mathrm{SO}_{n-4}^\pm(r^a)$
$\mathrm{G}_2(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{SL}_2(r^a)$	${}^3\mathrm{D}_4(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{SL}_2(r^{3a})$
$\mathrm{F}_4(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{Sp}_6(r^a)$	${}^2\mathrm{E}_6(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{SU}_6(r^a)$
$\mathrm{E}_6(r^a)$	$\mathrm{SL}_2(r^a) * \mathrm{SL}_6(r^a)$	$\mathrm{E}_7(r^a)$	$\mathrm{SL}_2(r^a) \times \Omega_{12}^+(r^a)$
$\mathrm{E}_8(r^a)$	$\mathrm{SL}_2(r^a) \times \mathrm{E}_7(r^a)$		

TABLE 4. The group generated by the r-elements in the centralizer of a classical involution t in Lie type groups of rank at least 2 and odd characteristic r.

of $N_G(X_\rho)$ and t is centralized by the subgroup K_t of the Levi complement of $N_G(X_\rho)$ which is generated by root subgroups. These subgroups are given in Table 3 and we note that $\overline{K_t}$ is a characteristic subgroups of $\overline{C_E(t)}$ which in turn is normal in $\overline{C_H(t)}$. By Lemmas 8.10 and 8.7, we have that p divides $|O^{2'}(C_E(t))|$. Suppose that $K_t/O_{p'}(K_t)$ is not soluble. Then, by Lemma 8.3, we have that $F^*(\overline{C_H(t)}) \cong \mathrm{Alt}(p) \times \mathrm{Alt}(p)$ with $p \geq 5$ or $\Omega_8^+(2)$ with $p = 5$. Since $\overline{K_t}$ is normal in $\overline{C_H(t)}$, we infer that either $\overline{K_t} \cong \mathrm{Alt}(p) \times \mathrm{Alt}(p)$ and $p \geq 5$ or to $\Omega_8^+(2)$ and $p = 5$. As $\mathrm{Alt}(p)$, $p \geq 5$, is isomorphic to a Lie type group defined in characteristic 2 if and only if $p = 5$, we now have that $p = 5$ and $K_t \cong \mathrm{Alt}(5) \times \mathrm{Alt}(5)$ or $\Omega_8^+(2)$. Using Table 3 we get that $E/Z(E) \cong \Omega_{12}^+(2)$. Furthermore, we have $Z(E) = O_2(E) = 1$. Therefore there is a further involution $s \in E$ which has centralizer C with $C/O_2(C) \cong \mathrm{Sp}_8(2)$. Now using Lemma 8.3, we have a contradiction. Hence we must assume that $C_E(t)$ is soluble. By Lemmas 8.5 and 8.8, $E/Z(E) \not\cong \mathrm{PSp}_4(2)' \cong \mathrm{Alt}(6)$ or to $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$. Then using Table 3 again, we have $E/Z(E)$ is one of $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_5(2)$, ${}^2\mathrm{F}_4(2)'$, $\mathrm{PSL}_4(2)$ and $\Omega_8^+(2)$. In all the cases we check that $|C_H(t)|$ is only divisible by p^2 when $p = 3$ and $E/Z(E) \cong \mathrm{PSU}_4(2)$, $\mathrm{PSU}_5(2)$ or $\Omega_8^+(2)$. Suppose that $E/Z(E) \cong \Omega_8^+(2)$. Then $E \cong \Omega_8^+(2)$ and there is an involution $s \in E$ such that $C_E(s)/O_2(C_E(s))$ has a normal subgroup isomorphic to $\mathrm{Sp}_4(2)' \cong \mathrm{Alt}(6)$. This violates Lemma 8.3. If $E \cong \mathrm{PSU}_4(2)$, then there is an involution $s \in E$, which is not centralized by an elementary abelian group of order 9 contrary to Hypothesis 4.1 (ii). So we have $E \cong \mathrm{PSU}_5(2)$. In this case $\overline{C_H(t)} \cong 3_+^{1+2}.\mathrm{SL}_2(3)$ or $\overline{C_H(t)} \cong 3_+^{1+2}.\mathrm{GL}_2(3)$. Now Proposition 2.7 shows that this structure is impossible. Thus $E/Z(E)$ is not a Lie type group defined in characteristic 2. $\square$

Lemma 8.13. *$E/Z(E)$ is not a Lie type group in odd characteristic.*

Proof. Suppose that $E/Z(E)$ is a Lie type group defined in characteristic r, r an odd prime. By Lemma 8.8, we may assume that the Lie rank of E is at least two. Let t be a classical involution in E. By Lemmas 8.7 and 8.11, p divides $|O^{2'}(C_E(t))|$. Let K_1 be a subnormal subgroup of $C_E(t)$ containing t with $K_1 \cong \mathrm{SL}_2(r^a)$. Assume that p divides $|K_1|$ and that $r^a \neq 3$. Then Lemma 8.3 implies that $\overline{K}_1 \cong \mathrm{Alt}(p)$ or $\Omega_8^+(2)$. We conclude that $p = r^a = 5$ and that $F^*(\overline{C_G(t)}) \cong \mathrm{Alt}(10)$. In particular, K_1 is not normal in $C_H(t)$. Using Table 4 now shows that $E \cong \mathrm{PSL}_4(5)$, $\mathrm{PSp}_4(5)$, $\mathrm{PSU}_4(5)$, or $\mathrm{G}_2(5)$. The last case immediately fails, as $\mathrm{Out}(\mathrm{G}_2(5)) = 1$ means $E = H$ and K_1 is normal in $C_H(t)$.

If $E \cong \mathrm{PSp}_4(5)$, then there is an involution $s \in E$ with $O^{5'}(C_E(s)) \cong \mathrm{PSL}_2(5)$, which contradicts Hypothesis 4.1 (ii). If $E \cong \mathrm{PSU}_4(5)$, then there is an involution $s \in E$ with $F^*(C_E(x)/O_{5'}(C_E(x))) \cong \mathrm{PSL}_2(25)$ and this contradicts Lemma 8.3. So we must have that $E \cong \mathrm{PSL}_4(5)$. Since $O_{5'}(C_G(t)) \le H$ is normalized by $C_E(t)$, the structure of $\mathrm{Aut}(E)$ shows that $O_{5'}(C_G(t)) \cap E = \langle t\rangle$. It follows that $C_G(t)/O_{p'}(C_G(t))$ has a subgroup isomorphic to $C_E(t)/\langle t\rangle$. Since the subgroup M of even permutations of $\mathrm{Sym}(5) \wr \mathrm{Sym}(2)$ has $M/F^*(M)$ cyclic of order 4 and $C_E(t)/F^*(C_E(t))$ is a fours group, we see that $C_G(t)/O_{p'}(C_G(t)) \cong \mathrm{Sym}(10)$. But then $H > E$. In particular, there are involutions $s \in H \setminus E$ with $F^*(C_H(s)) \cong \mathrm{PSL}_3(5)$ in contradiction to Lemma 8.3.

So we may assume next that p does not divide the order of $\mathrm{SL}_2(r^a)$ or $r^a = 3$. Suppose that in the latter case we also have $p \ne 3$. Then in both cases p does not divide $r^a \pm 1$. Hence p divides the order of one of the factors listed in Table 4. Since p does not divide $|K_1|$, $C_E(t)$ has a component which is a group of Lie type in characteristic r. But these components are not isomorphic to either $\mathrm{Alt}(p)$ or $\Omega_8^+(2)$, and this contradicts Lemmas 4.4 and 8.3.

So we have that $p = 3 = r^a$. Suppose that $C_E(t)$ has a component F, then 3 divides $|F|$. Therefore Lemmas 4.4, 8.3 and Table 4 give a contradiction. So we have that $C_E(t)$ is soluble. Then reference once again to Table 4 gives $E/Z(E)$ is isomorphic to one of $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSp}_4(3)$, $\mathrm{PSU}_4(3)$, $\Omega_7(3)$, $\mathrm{P}\Omega_8^+(3)$, $\mathrm{G}_2(3)$. If $E \cong \Omega_7(3)$ or $\mathrm{P}\Omega_8^+(3)$, there is an involution s with $F^*(C_E(s)) \cong \Omega_6^-(3)$ and, if $E \cong \mathrm{PSL}_4(3)$ there is an involution $s \in E$ with $F^*(C_E(s)) \cong \mathrm{PSL}_2(9)$. Thus in these cases we obtain a contradiction via Lemma 8.3. If $E \cong \mathrm{PSL}_3(3)$ or $\mathrm{PSp}_4(3)$, then there is an involution $s \in E$ whose centralizer is not divisible by 9, in the second case $|C_E(s)| = 2^5 \cdot 3$, while in the first case $s = t$ and $C_E(t) \cong \mathrm{GL}_2(3)$. This contradicts Hypothesis 4.1 (ii). So we have that $E/Z(E) \cong \mathrm{PSU}_4(3)$ or $\mathrm{G}_2(3)$. Assume that $H = E$. Then, by Lemma 3.2 (v), $O_{3'}(C_G(t)) \le H$ and Corollary 2.11 gives $O_2(C_E(t)) = O_{3'}(C_G(t))$ which is extraspecial 2_+^{1+4}. It follows that $C_G(t)/C_{C_G(t)}(O_2(C_E(t))$ is soluble. But $\overline{C_G(t)}$ is an almost simple group and so we have that $O_2(C_E(t))$ is centralized by a Sylow 3-subgroup of $C_E(t)$ something which is impossible. Thus $E \ne H$. Hence there exists an involution $s \in H \setminus E$. If $E/Z(E) \cong \mathrm{G}_2(3)$, $C_E(s) \cong {}^2\mathrm{G}_2(3)$. If $E/Z(E) \cong \mathrm{PSU}_4(3)$, then $C_E(s)$ has a section isomorphic to $\mathrm{PSU}_3(3)$, $\mathrm{PSp}_4(3)$ or $\Omega_4^-(3)$. In all these cases we get a contradiction to Lemma 8.3 applied to $C_G(s)$. $\square$

9. Proofs of the Main Theorems

In this final section we assemble the proofs of our main theorems. We refer the reader to the introduction for their statements and continue the notation of the previous sections.

Proof of Theorem 1.1. Let $G_0 = O^2(G)$. Then $H_0 = H \cap G_0$ has even order and H_0 is strongly p-embedded in G_0 by Lemma 3.2(ii) and (iv). As H_0 is normal in H and as H_0 has even order by hypothesis, we have that G_0 and H_0 together satisfy Hypothesis 4.1. If G_0 contains a classical involution, then Theorem 2.2 implies that G_0 is a $\mathcal{K}$-group and then Proposition 2.5 implies that $F^*(G) = F^*(G_0) \cong \mathrm{PSU}_3(p^a)$ for some $a \geq 2$. Thus we may suppose that G_0 has no classical involutions. Since $F^*(H) = O_p(H)$, Hypothesis 5.1 is satisfied. Therefore Theorems 6.1 and 7.9 together imply that $F^*(G) = F^*(G_0) \cong {}^2\mathrm{G}_2(3^{2n-1})$ for some $n \geq 2$. $\square$

Proof of Theorem 1.2. Suppose that $F^*(H) \neq O_p(H)$. Then as $O_{p'}(H) = 1$, we have $E = E(H) \neq 1$. Let $G_0 = O^2(G)$. Then $G_0 \geq E(H)$ and so G_0 satisfies Hypothesis 4.1. By Lemma 8.1, we have $O_p(H) \leq E(H)$. Since we also have $O_{p'}(H) = 1$, we get $F^*(H) = E(H)$. $\square$

Proof of Theorem 1.3. Assume that $F^*(H) = E(H) = E$. Then Hypothesis 8.2 holds. We use the fact that E is a $\mathcal{K}$-group and use Lemmas 8.5, 8.6, 8.12 and 8.13 to deliver a contradiction. Hence we conclude that $F^*(H) = O_p(H)$. $\square$

Proof of Theorem 1.4. This is merely a combination of Theorems 1.1, 1.2 and 1.3. $\square$

Finally we prove Corollary 1.5. For this we require the following proposition about centralizers of involutions in Lie type groups defined in characteristic p.

Proposition 9.1. *Let p be an odd prime, X be a finite group and set $K = F^*(X)$. Suppose that K is a simple group of Lie type defined in characteristic p. Then either $m_p(C_X(x)) \geq 2$ for all involutions $x \in X$ or one of the following holds:*

(i) $K \cong \mathrm{PSL}_2(p^n)$ *with* $n \geq 1$*;*
(ii) $K \cong \mathrm{PSL}_3(p)$ *or* $\mathrm{PSU}_3(p)$*;*
(iii) $K \cong \mathrm{PSp}_4(p)$*; or*

(iv) $K \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$.

Proof. We may suppose that $K \not\cong \mathrm{PSL}_2(p^n)$. Let $x \in X$ be an involution and set $K_x = O^{p'}(C_K(x))$. Note that since the Lie type groups are generated by their (perhaps twisted) root subgroups the only Lie type groups L with $m_p(L) = 1$ are $L \cong \mathrm{SL}_2(p)$ for arbitrary p and $L \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$ with $p = 3$. Assume that $q = p^n$ and K has type ${}^d\Sigma(q)$ (using notation as in [10, Chapter 4]). Then, by [10, Definition 2.5.13], x induces either an inner-diagonal, graph, field or graph-field automorphism on K. If x induces a field automorphism, then [10, Proposition 4.9.1] gives that $K_x \cong {}^d\Sigma(q^{1/2})$. Since $m_p(C_X(x)) \leq 1$, we deduce that $K_x \cong \mathrm{PSL}_2(p)$ and consequently $K \cong \mathrm{PSL}_2(p^2)$ which is a contradiction. If x induces a graph-field automorphism of K, then [10, Proposition 4.9.1] implies $K_x \cong {}^2\Sigma(q^{1/2})$. Thus ${}^2\Sigma(q^{1/2})$ is $\mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$ and this is impossible. Therefore x is contained in the group generated by inner-diagonal and graph automorphisms of K. For each of the possibilities for K and x, the structure of K_x is described in [10, Table 4.5.1] and we avail ourselves of this information. Thus, if K has type $\mathrm{A}^\epsilon_m(q)$, then the only possibility is that $m = 2$ and $q = p$ and so (ii) holds. Similarly if K has type $\mathrm{B}_m(q)$ or $\mathrm{C}_m(q)$, we see again that $m = 2$ and $q = p$ and thus (iii) holds. The groups of type $\mathrm{D}^\epsilon_m(q)$ only need to be considered for $m \geq 4$, and in these cases $m_p(K_x) \geq 2$. The same is true for the groups of type $\mathrm{G}_2(q)$ and ${}^3\mathrm{D}_4(q)$. For $K \cong {}^2\mathrm{G}_2(3^{2n+1})'$, the only possibility is that $X \cong {}^2\mathrm{G}_2(3)$ and $K_x \cong \mathrm{PSL}_2(3)$ which gives (iv). The remaining exceptional groups are all easily seen to violate $m_p(K_x) \leq 1$. $\square$

Proof of Corollary 1.5. If $F^*(H)$ is a Lie type group defined in characteristic p and of rank at least 3, then Proposition 9.1 implies that $m_p(C_H(t)) \geq 2$ for all involutions $t \in H$. Thus Theorem 1.3 implies that H cannot be strongly p-embedded in any groups G. $\square$

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