Pure and Applied Mathematics Quarterly
Volume 7, Number 3
(*Special Issue: In honor of Jacques Tits*)
783—796, 2011

Piecewise Linear Parametrization of Canonical Bases

G. Lusztig

To Jacques Tits on the occasion of his 80th birthday

Abstract: We extend the known piecewise linear parametrization of the canonical basis of the plus part of an enveloping algebra of type ADE to the nonsimplylaced case.

Keywords: canonical basis, quantized enveloping algebra, piecewise linear, Cartan datum

Introduction

In [L1] the author introduced the canonical basis for the plus part of a quantized enveloping algebra of type A, D or E. (The same method applies for non-simplylaced types, see [L3, 12.1].) Another approach to the canonical basis was later found in [Ka]. In [L1] we have also found that the set parametrizing the canonical basis has a natural piecewise linear structure that is, a collection of bijections with $\mathbf{N}^N$ such that any two of these bijections differ by composition with a piecewise linear automorphism of $\mathbf{N}^N$ (an automorphism which can be expressed purely in terms of operations of the form $a+b, a-b, \min(a,b)$). This led to the first purely combinatorial formula (involving only counting) for the dimension of a weight space of an irreducible finite dimensional representation [L1] or the dimension of the space of coinvariants in a triple tensor product [L2,

Received July 30, 2008.
Supported in part by the National Science Foundation.

6.5(f)]. (Later, different formulas in the same spirit were obtained by Littelmann [Li].) The construction of an analogous piecewise linear structure for the canonical basis in the nonsimplylaced case (based on a reduction to the simplylaced case) was only sketched in [L3] partly because it involved an assertion whose proof only appeared later (in [L4, 14.4.9]): as Berenstein and Zelevinsky write in [BZ, Proof of Theorem 5.2], "Lusztig (implicitly) claims that the transition map R_{2121}^{1212} for B_2 is obtained from the transition map R_{132132}^{213213} for type A_3...". In this paper we fill the gap in [L3] by making use of [L4, 14.4.9] which gives a relation between the canonical basis for a nonsimplylaced type and the canonical basis for a simplylaced type with a given (admissible) automorphism. At the same time we slightly extend [L4, 14.4.9] by allowing type A_{2n} with its non-admissible involution.

As an application we show that the canonical basis has a natural monoid structure and we define certain "Frobenius" endomorphisms of this monoid.

1. Parametrization

1.1. In this paper a Cartan datum is understood to be a pair $(I, \cdot)$ where I is a finite set and $(i \cdot j)$ is a symmetric positive definite matrix of integers indexed by $I \times I$ such that

$i \cdot i \in 2\mathbf{N}_{>0}$ for any $i \in I$;

$2\frac{i \cdot j}{i \cdot i} \in -\mathbf{N}$ for any $i \neq j$ in I;

We say that $(I, \cdot)$ as above is

-simply laced if $i \cdot j \in \{0, -1\}$ for any $i \neq j$ in I and $i \cdot i = 2$ for any $i \in I$;

-irreducible if $I \neq \emptyset$ and there is no partition $I = I' \sqcup I''$ with $I' \neq \emptyset$, $I'' \neq \emptyset$, $i' \cdot i'' = 0$ for all $i' \in I'$, $i'' \in I''$.

Let $(I, \cdot)$ be a Cartan datum. For $i \neq j$ in I we have $\frac{2i \cdot j}{i \cdot i} \frac{2j \cdot i}{j \cdot j} = 0, 1, 2$ or 3; accordingly, we set $h(i,j) = 2, 3, 4$ or 6. The Weyl group W of $(I, \cdot)$ is the group with generators $s_i (i \in I)$ and relations $s_i^2 = 1$ for $i \in I$ and $s_i s_j s_i \cdots = s_i s_j s_i \ldots$ (both products have $h(i,j)$ factors) for $i \neq j$ in I. Let $l : W \longrightarrow \mathbf{N}$ be the standard legth function relative to the generators s_i. Let w_0 be the unique element of W such that $l(w_0)$ is maximal. Let $N = l(w_0)$ and let $\mathcal{X}$ be the set of sequences $i_* = (i_1, i_2, \ldots, i_N)$ in I such that $s_{i_1} s_{i_2} \ldots s_{i_N} = w_0$ (in W). We regard $\mathcal{X}$ as the set of vertices of a graph in which i_*, i'_* are joined if the sequences i_*, i'_* coincide except at the places $k, k+1, k+2, \ldots, k+r-1$ where

$(i_k, i_{k+1}, \dots, i_{k+r-1}) = (p, p', p, \dots)$, $(i'_k, i'_{k+1}, \dots, i'_{k+r-1}) = (p', p, p', \dots)$, with $p \neq p'$ in I, $h(p, p') = r$. By a theorem of Matsumoto and Tits,

(a) this graph is connected.

1.2. Let $I = \{1, 2, \dots, 2n\}$, $n \geq 1$. For $i, j \in I$ we set $i \cdot j = 2$ if $i = j$, $i \cdot j = -1$ if $i - j = \pm 1$ and $i \cdot j = 0$ otherwise. Then $(I, \cdot)$ is a simply laced irreducible Cartan datum. Define a permutation $\sigma : I \longrightarrow I$ by $\sigma(i) = 2n + 1 - i$ for all i. We have $\sigma(i) \cdot \sigma(j) = i \cdot j$ for any i, j in I.

1.3. Let $I = \{1, 2, \dots, n-1, n, n'\}$, $n \geq 1$. For $i, j \in [1, n]$ we set $i \cdot j = 2$ if $i = j$, $i \cdot j = -1$ if $i - j = \pm 1$ and $i \cdot j = 0$ otherwise; we also set $n' \cdot n' = 2$, $(n-1) \cdot n' = n' \cdot (n-1) = -1$, $i \cdot n' = n' \cdot i = 0$ if $i < n-1$ or if $i = n$. Then $(I, \cdot)$ is a simply laced irreducible Cartan datum. It is irreducible if $n \geq 2$. Define a permutation $\sigma : I \longrightarrow I$ by $\sigma(i) = i$ for $i \in [1, n-1]$, $\sigma(n) = n'$, $\sigma(n') = n$. We have $\sigma(i) \cdot \sigma(j) = i \cdot j$ for any i, j in I.

1.4. Let $\underline{I} = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$, $n \geq 1$. For $i, j \in [1, n-1]$ we set $\bar{i} \circ \bar{j} = 2$ if $i = j$, $\bar{i} \circ \bar{j} = -1$ if $i - j = \pm 1$ and $\bar{i} \circ \bar{j} = 0$ otherwise; we also set $\bar{n} \circ \bar{n} = 4$, $\overline{n-1} \circ \bar{n} = \bar{n} \circ \overline{n-1} = -2$, $\bar{i} \circ \bar{n} = \bar{n} \circ \bar{i} = 0$ if $i < n-1$. Then $(\underline{I}, \circ)$ is an irreducible Cartan datum.

1.5. Let $(I, \cdot)$ be a simply laced Cartan datum and let $\sigma : I \longrightarrow I$ be a permutation such that $\sigma(i) \cdot \sigma(j) = i \cdot j$ for any i, j in I. Let $\underline{I}$ be the set of orbits of σ on I. For $\eta \in \underline{I}$ we set $\delta_\eta = 1$ if $\sigma(i) \cdot \sigma(j) = 0$ for any $i \neq j$ in η and $\delta_\eta = 2$, otherwise. We set $\delta = \max_{\eta \in \underline{I}} \delta_\eta \in \{1, 2\}$. We will assume that

(a) either $\delta = 1$ or $(I, \cdot)$ is irreducible.

For any $\eta \in \underline{I}$ we set $\eta \circ \eta = 2\delta^{-1}\delta_\eta |\eta|$. For any $\eta \neq \eta'$ in $\underline{I}$ we set

$\eta \circ \eta' = -\delta^{-1}\delta_\eta \delta_{\eta'} |\{(i, j) \in \eta \times \eta'; i \cdot j \neq 0\}|$.

We show that $(\underline{I}, \circ)$ is a Cartan datum. Assume first that $\delta = 1$. Let $\{x_\eta; \eta \in \underline{I}\}$ be a collection of real numbers, not all zero. Let $m = \sum_{\eta, \eta' \in \underline{I}} x_\eta x_{\eta'} \eta \circ \eta'$. it is enough to show that $m > 0$. For $i \in I$ let $y_i = x_\eta$ where $i \in \eta$. From the definitions we have $m = \sum_{i, i' \in I} y_i y_{i'} i \cdot i'$ and this is > 0 since $(i \cdot i')$ is positive definite. Assume next that $\delta = 2$. We can assume that $(I, \cdot), \sigma$ are as in 1.2. Denoting by $\bar{i}$ the subset $\{i, 2n+1-i\}$ of I $(i \in [1, n])$ we see that $(\underline{I}, \circ)$ is the same as that in 1.4 hence it is a Cartan datum.

1.6. Let $(I, \cdot), \sigma$ be as in 1.3. We define $(\underline{I}, \circ)$ in terms of $(I, \cdot), \sigma$ as in 1.5.

Denoting by $\bar{i}$ the subset $\{i\}$ of I ($i \in [1, n-1]$) and the subset $\{n, n'\}$ if $i = n$, we see that $(\underline{I}, \circ)$ is the same as that in 1.4.

1.7. Let $(I, \cdot)$ be a simply laced Cartan datum. Let W, l, w_0, N be as in 1.1. Let K be either:

(i) a subgroup of the multiplicative group of a field which is closed under addition in that field;

(ii) a set with a given bijection $\iota : \mathbf{Z} \xrightarrow{\sim} K$ with the operations $a + b, ab, a/b$ (on K) obtained by transporting to K the operations $\min(a, b), a + b, a - b$ on $\mathbf{Z}$;

(iii) the subset $\iota(\mathbf{N})$ of the set in (ii); this is stable under the operations $a + b, ab, a/(a + b)$.

Note that operations on K in (i) and (ii) have similar properties; they are both examples of "semifields". (See http://en.wikipedia.org/wiki/semifield.) The analogy between K in (ii) and K in (i) has been pointed out in [L4, 42.2.7] in connection with observing the analogy of the combinatorics of canonical bases and the geometry involved in total positivity.

Let $\tilde{\mathcal{X}}$ be the set of all objects $i_1^{c_1} i_2^{c_2} \dots i_N^{c_N}$ (also denoted by $i_*^{c_*}$) where $i_* = (i_1, i_2, \dots, i_N) \in \mathcal{X}$, $c_* = (c_1, c_2, \dots, c_N) \in K^N$. We regard $\tilde{\mathcal{X}}$ as the set of vertices of a graph in which two vertices $i_*^{c_*}$, ${i'_*}^{c'_*}$ are joined if either

the sequences i_*, i'_* coincide except at two places $k, k+1$ where $i'_k = i_{k+1}, i'_{k+1} = i_k$ and $i_k \cdot i_{k+1} = 0$; the sequences c_*, c'_* coincide except at the places $k, k+1$ where $c'_k = c_{k+1}, c'_{k+1} = c_k$; or

the sequences i_*, i'_* coincide except at three places $k, k+1, k+2$ where

$(i_k, i_{k+1}, i_{k+2}) = (p, p', p)$, $(i'_k, i'_{k+1}, i'_{k+2}) = (p', p, p')$,

with $p \cdot p' = -1$; the sequences c_*, c'_* coincide except at the places $k, k+1, k+2$ where

$(c_k, c_{k+1}, c_{k+2}) = (x, y, z)$, $(c'_k, c'_{k+1}, c'_{k+2}) = (x', y', z')$

with $x' = yz/(x+z)$, $y' = x+z$, $z' = xy/(x+z)$ or equivalently $x = y'z'/(x'+z')$, $y = x' + z'$, $z = x'y'/(x' + z')$.

We shall write $R_{i_*}^{i'_*}(c_*) = c'_*$ whenever $i_*^{c_*}$, ${i'_*}^{c'_*}$ are joined in the graph $\tilde{\mathcal{X}}$. Then $R_{i_*}^{i'_*} : K^N \to K^N$ can be viewed as a bijection defined whenever i_*, i'_* are joined in the graph $\mathcal{X}$.

Let $\mathcal{B}$ be the set of connected components of the graph $\tilde{\mathcal{X}}$. For any $i_* \in \mathcal{X}$ we define $\alpha_{i_*} : K^N \to \mathcal{B}$ by $c_* \mapsto$ connected component of $i_*^{c_*}$. Note that:

(a) α_{i_*} is a bijection.
If K is as in 1.7(i) this follows from the proof of [L4, 42.2.4]. If K is as in 1.7(ii) then, as in [L4, 42.2.7], it can be viewed as a homomorphic image of a K as in 1.7(ii) so that (a) holds in this case. The case where K is as in 1.7(iii) follows immediately from the case where K is as in 1.7(ii), or it can be obtained directly from [L4, 42.1.9].

For any i_*, i'_* in $\mathcal{X}$ we define a bijection $R_{i_*}^{i'_*} : K^N \xrightarrow{\sim} K^N$ as the composition $R_{i_*^t}^{i_*^{t-1}} \dots R_{i_*^1}^{i_*^2} R_{i_*^0}^{i_*^1}$ where $i_*^0, i_*^1, \dots, i_*^t$ is a sequence of vertices in $\tilde{\mathcal{X}}$ such that $(i_*^0, i_*^1), (i_*^1, i_*^2), \dots, (i_*^{t-1}, i_*^t)$ are edges of the graph $\mathcal{X}$ and $i_*^0 = i_*$, $i_*^t = i'_*$ (such a sequence exists by 1.1(a)). This agrees with the earlier definition in the case where i_*, i'_* are joined in $\mathcal{X}$. Note that $R_{i_*}^{i'_*}$ is independent of the choice above; it is equal to $\alpha_{i'_*}^{-1}\alpha_{i_*}$.

1.8. Let $(I, \cdot), \sigma$ be as in 1.5. Define $(\underline{I}, \circ)$ as in 1.5. Define $W, l, w_0, N, \mathcal{X}$ as in 1.1. Let $\underline{W}, \underline{l}, \underline{w}_0, \underline{N}, \underline{\mathcal{X}}$ be the analogous objects defined in terms of $(\underline{I}, \circ)$. The generators of W are denoted by $s_i (i \in I)$ as in 1.1; similarly let $\underline{s}_\eta (\eta \in \underline{I})$ be the generators of $\underline{W}$. For any $\eta \in \underline{I}$ let $w_\eta \in W$ be the longest element in the subgroup of W generated by $\{s_i; i \in \eta\}$; let $N_\eta = l(w_\eta)$. We can identify $\underline{W}$ with the subgroup of W generated by $\{w_\eta; \eta \in \underline{I}\}$ by sending $\underline{s}_\eta$ to w_η. Then $\underline{w}_0 = w_0$ and $\underline{\mathcal{X}}$ becomes the set of sequences $\eta_* = (\eta_1, \eta_2, \dots, \eta_{\underline{N}})$ in $\underline{I}^{\underline{N}}$ such that $w_{\eta_1} w_{\eta_2} \dots w_{\eta_{\underline{N}}} = w_0$. We have $\underline{W} = \{w \in W; \sigma(w) = w\}$ where $\sigma : W \longrightarrow W$ is the automorphism given by $\sigma(s_i) = s_{\sigma(i)}$ for all i. For any $\eta \in \underline{I}$ let $\mathcal{X}^\eta$ be the set of sequences $(h_1, h_2, \dots, h_{N_\eta})$ in η^{N_η} such that $s_{h_1} s_{h_2} \dots s_{h_{N_\eta}} = w_\eta$.

Let $\tilde{\mathcal{X}}$ be as in 1.7. Let $\underline{\tilde{\mathcal{X}}}$ be the set of all objects $\eta_1^{\mathfrak{c}_1} \eta_2^{\mathfrak{c}_2} \dots \eta_{\underline{N}}^{\mathfrak{c}_{\underline{N}}}$ (also denoted by $\eta_*^{\mathfrak{c}_*}$) where $\eta_* = (\eta_1, \eta_2, \dots, \eta_{\underline{N}}) \in \underline{\mathcal{X}}$, $\mathfrak{c}_* = (\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_{\underline{N}}) \in K^{\underline{N}}$.

Let $\hat{\mathcal{X}}$ be the set of all pairs $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*)$ where $\eta_*^{\mathfrak{c}_*} \in \underline{\tilde{\mathcal{X}}}$ and $\mathfrak{d}_* = (\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_{\underline{N}})$ is such that $\mathfrak{d}_j \in \mathcal{X}^{\eta_j}$ for $j \in [1, \underline{N}]$. Let $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*) \in \hat{\mathcal{X}}$. For $j \in [1, \underline{N}]$, $k \in [1, N_j]$ (where $N_j = N_{\eta_j}$) let $\epsilon_{j,k}$ be the number of $k' \in [1, N_j]$ such that $h_{k'} = h_k$ where $\mathfrak{d}_j = (h_1, h_2, \dots, h_{N_j})$. We have $\epsilon_{j,k} \in \{1, 2\}$. Let $\epsilon_j = \max_{k \in [1, N_j]} \delta_{j,k} \in \{1, 2\}$. Let $c_*^j = (\epsilon_j \epsilon_{j,1}^{-1} \mathfrak{c}_j, \epsilon_j \epsilon_{j,2}^{-1} \mathfrak{c}_j, \dots, \epsilon_j \epsilon_{j,N_j}^{-1} \mathfrak{c}_j) \in K^{N_j}$. Let $c_* = c_*^1 c_*^2 \dots c_*^{\underline{N}} \in K^N$ be the concatenation of $c_*^1, c_*^2, \dots, c_*^{\underline{N}}$. Let $i_* = \mathfrak{d}_1 \mathfrak{d}_2 \dots \mathfrak{d}_{\underline{N}}$ be the concatenation of $\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_{\underline{N}}$. We have $i_* \in \mathcal{X}$ and $i_*^{c_*} \in \tilde{\mathcal{X}}$.

We show that the connected component of $i_*^{c_*}$ in $\tilde{\mathcal{X}}$ depends only on $\eta_*^{\mathfrak{c}_*}$, not on $\mathfrak{d}_*$. Let $\mathfrak{d}'_* = (\mathfrak{d}'_1, \mathfrak{d}'_2, \dots, \mathfrak{d}'_{\underline{N}})$ be another sequence such that $\mathfrak{d}'_j \in \mathcal{X}^{\eta_j}$ for

$j \in [1, \underline{N}]$. Let i'_* be the concatenation of $\mathfrak{d}'_1, \mathfrak{d}'_2, \dots, \mathfrak{d}'_{\underline{N}}$. Define c'_* in terms of $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}'_*)$ in the same way as c_* was defined in terms of $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*)$. We must show that $i_*^{c_*}, i'^{\,c'_*}_*$ are in the same connected component of $\tilde{\mathcal{X}}$. We may assume that I is a single σ-orbit η. Assume first that $\eta = \{i, i'\}$ with $i \cdot i' = -1$. It is enough to show that $i^c i'^{2c} i^c$ and $i'^c i^{2c} i'^c$ are joined in $\tilde{\mathcal{X}}$ (where $c \in K$). This is clear since $c + c = 2c, c(2c)/(c+c) = c$. Next assume that η is not of the form $\{i, i'\}$ with $i \cdot i' = -1$. Then $\eta = \{i_1, i_2, \dots, i_k\}$ where $s_{i_1}, s_{i_2}, \dots, s_{i_k}$ commute with each other. It is enough to show that the connected component of $i_1^c i_2^c \dots i_k^c$ in $\tilde{\mathcal{X}}$ does not depend on the order in which $i_1, i_2, \dots, i_k$ are written (where $c \in K$); this is obvious.

We see that the map $\hat{\mathcal{X}} \to \mathcal{B}$ given by $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*) \mapsto$ connected component of $i_*^{c_*}$ factors through a map $s : \underline{\tilde{\mathcal{X}}} \to \mathcal{B}$ ($\mathcal{B}$ as in 1.7).

We define a permutation $\sigma : \mathcal{X} \to \mathcal{X}$ by

$i_* = (i_1, i_2, \dots, i_N) \mapsto (\sigma(i_1), \sigma(i_2), \dots, \sigma(i_N))$

and a permutation $\sigma : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ by $i_*^{c_*} \mapsto \sigma(i_*)^{c_*}$. This last permutation respects the graph structure of $\tilde{\mathcal{X}}$ hence induces a permutation (denoted again by σ) of $\mathcal{B}$.

We show that the image of $s : \underline{\tilde{\mathcal{X}}} \to \mathcal{B}$ is contained in the set $\mathcal{B}^\sigma$ of fixed points of $\sigma : \mathcal{B} \to \mathcal{B}$. Let $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*) \in \hat{\mathcal{X}}$; we associate to it $i_*^{c_*} \in \tilde{\mathcal{X}}$ as above. For $j \in [1, \underline{N}]$ we set $\mathfrak{d}'_j = (\sigma(h_1), \sigma(h_2), \dots, \sigma(h_{N_j}))$ (where $\mathfrak{d}_j = (h_1, h_2, \dots, h_{N_j})$, $N_j = N_{\eta_j}$) and $\mathfrak{d}'_* = (\mathfrak{d}'_1, \mathfrak{d}'_2, \dots, \mathfrak{d}'_{\underline{N}})$. Let i'_* be the concatenation of $\mathfrak{d}'_1, \mathfrak{d}'_2, \dots, \mathfrak{d}'_{\underline{N}}$. We have $i'_* \in \mathcal{X}$. Now $i'^{\,c_*}_*$ is associated to $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}'_*) \in \hat{\mathcal{X}}$ in the same way as $i_*{}^{c_*}$ is associated to $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*) \in \hat{\mathcal{X}}$; hence $i_*^{c_*}, i'^{\,c_*}_*$ are in the same connected component of $\tilde{\mathcal{X}}$ by an earlier argument. This verifies our claim.

Now let $\xi \in \mathcal{B}^\sigma$ and let $\eta_* \in \underline{\mathcal{X}}$. We show that $\xi = s(\eta_*^{\mathfrak{c}_*})$ for some $\mathfrak{c}_* \in K^{\underline{N}}$. We can find $\mathfrak{d}_* = (\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_{\underline{N}})$ such that $\mathfrak{d}_j \in \mathcal{X}^{\eta_j}$ for $j \in [1, \underline{N}]$. Let $i_* = \mathfrak{d}_1 \mathfrak{d}_2 \dots \mathfrak{d}_{\underline{N}}$ be the concatenation of $\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_{\underline{N}}$. We have $i_* \in \mathcal{X}$ and by 1.1(a) we can find $c_* \in K^N$ such that $i_*^{c_*} \in \xi$. Let $\mathfrak{d}'_*$ be obtained from $\mathfrak{d}_*$ as in the previous paragraph and let i'_* be the concatenation of $\mathfrak{d}'_1, \mathfrak{d}'_2, \dots, \mathfrak{d}'_{\underline{N}}$. We have $i'_* = \sigma(i_*) \in \mathcal{X}$. Since ξ is σ-stable we see that $i_*^{c_*}, i'^{\,c_*}_*$ are in the same connected component of $\tilde{\mathcal{X}}$. Now $c_* \in K^N$ can be viewed as the concatenation of $c_*^1, c_*^2, \dots, c_*^{\underline{N}}$ where $c_*^j = (c_1^j, c_2^j, \dots, c_{N_j}^j) \in K^{N_j}$, $N_j = N_{\eta_j}$. For $j \in [1, \underline{N}]$ we write $\mathfrak{d}_j = (h_1, h_2, \dots, h_{N_j}) \in \eta_j^{N_j}$ and we define $c'^{\,j}_* = (c'^{\,j}_1, c'^{\,j}_2, \dots, c'^{\,j}_{N_j}) \in K^{N_j}$

by

(i) ${c'}_k^j = c_{k'}^j$ where $\sigma(h_k) = h_{k'}$ if $s_{h_1}, s_{h_2}, \dots, s_{h_{N_j}}$ commute with each other and

(ii) ${c'}_1^j = c_2^j c_3^j/(c_1^j + c_3^j), {c'}_2^j = c_1^j + c_3^j, {c'}_3^j = c_1^j c_2^j/(c_1^j + c_3^j)$ if $h_1 \cdot h_2 = -1$, $h_1 = h_3$.

Let $c'_* \in K^N$ be the concatenation of ${c'}_*^1, {c'}_*^2, \dots, {c'}_*^{\underline{N}}$. From the definitions we see that $i_*^{c_*}, {i'}_*^{c'_*}$ are in the same connected component of $\tilde{\mathcal{X}}$. Hence ${i'}_*^{c_*}, {i'}_*^{c'_*}$ are in the same connected component of $\tilde{\mathcal{X}}$. Using the bijectivity of $\alpha_{i'_*} : K^N \longrightarrow \mathcal{B}$ (see 1.7(a)) we deduce that $c_* = c'_*$. Hence in (i) we have $c_k^j = c_{k'}^j$ whenever $\sigma(h_k) = h_{k'}$, hence c_k^j is a constant $\mathfrak{c}_j$ when k varies in $[1, N_j]$. Moreover in (ii) we have $c_1^j = c_2^j c_3^j/(c_1^j + c_3^j), c_2^j = c_1^j + c_3^j, c_3^j = c_1^j c_2^j/(c_1^j + c_3^j)$ hence $c_2^j = 2\mathfrak{c}_j, c_1^j = c_3^j = \mathfrak{c}_j$ for some $\mathfrak{c}_j \in K$. Let $\mathfrak{c}_* = (\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_{\underline{N}}) \in K^{\underline{N}}$. From the definitions we see that $s(\eta_*^{\mathfrak{c}_*})$ is the connected component of $i_*^{c_*}$. Our claim is verified.

Assume that $\eta_* \in \underline{\mathcal{X}}$, $\mathfrak{c}_* \in K^{\underline{N}}$, $\mathfrak{c}'_* \in K^{\underline{N}}$ are such that $s(\eta_*, \mathfrak{c}_*) = s(\eta_*, \mathfrak{c}'_*)$. We show that $\mathfrak{c}_* = \mathfrak{c}'_*$. We can find $\mathfrak{d}_* = (\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_{\underline{N}})$ such that $\mathfrak{d}_j \in \mathcal{X}^{\eta_j}$ for $j \in [1, \underline{N}]$. We define $i_*^{c_*} \in \tilde{\mathcal{X}}$ in terms of $(\eta_*^{\mathfrak{c}_*}, \mathfrak{d}_*)$ as above and we define similarly ${i'}_*^{c'_*} \in \tilde{\mathcal{X}}$ in terms of $(\eta_*^{\mathfrak{c}'_*}, \mathfrak{d}_*)$. Note that $i_* = i'_*$. By assumption, $i_*^{c_*}, i_*^{c'_*}$ are in the same connected component of $\tilde{\mathcal{X}}$. From 1.7(a) we see that $c_* = c'_*$. Now $c_* \in K^N$ is the concatenation of $c_*^1, c_*^2, \dots, c_*^{\underline{N}}$ where $c_*^j = (c_1^j, c_2^j, \dots, c_{N_j}^j) \in K^{N_j}$, $N_j = N_{\eta_j}$. Similarly, $c'_* \in K^N$ is the concatenation of ${c'}_*^1, {c'}_*^2, \dots, {c'}_*^{\underline{N}}$ where ${c'}_*^j = ({c'}_1^j, {c'}_2^j, \dots, {c'}_{N_j}^j) \in K^{N_j}$ for $j \in [1, \underline{N}]$. We see that for any j and any $k \in [1, N_j]$ we have $c_k^j = {c'}_k^j$. If $\epsilon_j = 1$ it follows that $\mathfrak{c}_j = \mathfrak{c}'_j$. If $\epsilon_j = 2$ it follows that $(\mathfrak{c}_j, 2\mathfrak{c}_j, \mathfrak{c}_j) = (\mathfrak{c}'_j, 2\mathfrak{c}'_j, \mathfrak{c}'_j)$ hence again $\mathfrak{c}_j = \mathfrak{c}'_j$. We see that $\mathfrak{c}_* = \mathfrak{c}'_*$ as required.

From the previous two paragraphs we see that for any $\eta_* \in \underline{\mathcal{X}}$ the map $\alpha_{\eta_*} : K^{\underline{N}} \longrightarrow \mathcal{B}^\sigma$ given by $\mathfrak{c}_* \mapsto s(\eta_*^{\mathfrak{c}_*})$ is a bijection.

For any η_*, η'_* in $\underline{\mathcal{X}}$ we define a bijection $R_{\eta_*}^{\eta'_*} : K^{\underline{N}} \longrightarrow K^{\underline{N}}$ by $R_{\eta_*}^{\eta'_*} = \alpha_{\eta'_*}^{-1} \alpha_{\eta_*}$.

We regard $\underline{\tilde{\mathcal{X}}}$ as the set of vertices of a graph in which two vertices $\eta_*^{\mathfrak{c}_*}$, ${\eta'}_*^{\mathfrak{c}'_*}$ are joined if the sequences η_*, η'_* coincide except at the places $k, k+1, k+2, \dots, k+r-1$ where

$(\eta_k, \eta_{k+1}, \dots, \eta_{k+r-1}) = (p, p', p, \dots)$, $(\eta'_k, \eta'_{k+1}, \dots, \eta'_{k+r-1}) = (p', p, p', \dots)$,
with $p \neq p'$ in $\underline{I}$, $h(p, p') = r$ and $R_{\eta_*}^{\eta'_*}(\mathfrak{c}_*) = \mathfrak{c}'_*$ or equivalently $R_{\eta'_*}^{\eta_*}(\mathfrak{c}'_*) = \mathfrak{c}_*$. Here $h(p, p')$ is the analogue of $h(i, i')$ (see 1.1) for $(\underline{I}, \circ)$ instead of $(I, \cdot)$.

Let $\underline{\mathcal{B}}$ be the set of connected components of the graph $\underline{\tilde{\mathcal{X}}}$. From the definitions we see that the map $s : \underline{\tilde{\mathcal{X}}} \longrightarrow \mathcal{B}^\sigma$ factors through a map $\bar{s} : \underline{\mathcal{B}} \longrightarrow \mathcal{B}^\sigma$. We show that

(a) $\bar{s}$ is a bijection.

The surjectivity of $\bar{s}$ follows from the surjectivity of s. To prove that $\bar{s}$ is injective we assume that $\eta_*^{\mathfrak{c}_*}$, $\eta'^{\mathfrak{c}'_*}_*$ are two elements of $\underline{\tilde{\mathcal{X}}}$ such that $s(\eta_*^{\mathfrak{c}_*}) = s(\eta'^{\mathfrak{c}'_*}_*)$; we must show that $\eta_*^{\mathfrak{c}_*}$, $\eta'^{\mathfrak{c}'_*}_*$ are in the same connected component of $\underline{\tilde{\mathcal{X}}}$. By the connectedness of the graph $\underline{\mathcal{X}}$ (see 1.1(a)) we can find $\mathfrak{c}''_* \in K^{\underline{N}}$ such that $\eta'^{\mathfrak{c}'_*}_*$, $\eta_*^{\mathfrak{c}''_*}$ are in the same connected component of $\underline{\tilde{\mathcal{X}}}$. We have $s(\eta'^{\mathfrak{c}'_*}_*) = s(\eta_*^{\mathfrak{c}''_*})$ hence $s(\eta_*^{\mathfrak{c}''_*}) = s(\eta_*^{\mathfrak{c}_*})$. Using the bijectivity of α_{η_*} we deduce that $\mathfrak{c}_* = \mathfrak{c}''_*$. Thus, $\eta'^{\mathfrak{c}'_*}_*$, $\eta_*^{\mathfrak{c}_*}$ are in the same connected component of $\underline{\tilde{\mathcal{X}}}$ and our claim is verified.

Let $\eta \in \underline{I}$. We define a map $\underline{\lambda}_\eta : \underline{\mathcal{B}} \longrightarrow K$ by $\xi \mapsto \mathfrak{c}_1$ where $\eta_*^{\mathfrak{c}_*}$ is any element of ξ such that $\eta_1 = \eta$. (This map is well defined by an argument similar to that in [L4, 42.1.14].) Similarly we define a map $\underline{\rho}_\eta : \underline{\mathcal{B}} \longrightarrow K$ by $\xi \mapsto \mathfrak{c}_{\underline{N}}$ where $\eta_*^{\mathfrak{c}_*}$ is any element of ξ such that $\eta_{\underline{N}} = \eta$.

We define a map $\lambda_\eta : \mathcal{B}^\sigma \longrightarrow K$ by $\xi \mapsto c_1$ where $i_*^{c_*}$ is any element of ξ such that $i_1 \in \eta$. (This map is well defined.) Similarly we define a map $\rho_\eta : \mathcal{B}^\sigma \longrightarrow K$ by $\xi \mapsto c_N$ where $i_*^{c_*}$ is any element of ξ such that $i_N \in \eta$.

From the definitions we have $\lambda_\eta \bar{s} = \underline{\lambda}_\eta$, $\rho_\eta \bar{s} = \underline{\rho}_\eta$,

1.9. We apply the definitions in 1.8 to $(I, \cdot), \sigma$ as in 1.3 and to $(\underline{I}, \circ)$ as in 1.4, 1.6. Let $\eta_*^{\mathfrak{c}_*}, \eta'^{\mathfrak{c}'_*}_*$ be two joined vertices of $\underline{\tilde{\mathcal{X}}}$. We show:

(i) if η_*, η'_* coincide except at the places $k, k+1$ where $(\eta_k, \eta_{k+1}) = (\bar{i}, \bar{i}')$, $(\eta'_k, \eta'_{k+1}) = (\bar{i}', \bar{i})$, $i - i' \notin \{0, 1, -1\}$ then $\mathfrak{c}_*, \mathfrak{c}'_*$ coincide except at the places $k, k+1$ where $(\mathfrak{c}_k, \mathfrak{c}_{k+1}) = (x, y)$, $(\mathfrak{c}'_k, \mathfrak{c}'_{k+1}) = (y, x)$;

(ii) if η_*, η'_* coincide except at the places $k, k+1, k+2$ where $(\eta_k, \eta_{k+1}, \eta_{k+2}) = (\bar{i}, \bar{i}', \bar{i})$, $(\eta'_k, \eta'_{k+1}, \eta'_{k+2}) = (\bar{i}', \bar{i}, \bar{i}')$, $(i, i'$ in $[1, n-1]$, $i - i' = \pm 1$, then $\mathfrak{c}_*, \mathfrak{c}'_*$ coincide except at the places $k, k+1, k+2$ where $(\mathfrak{c}_k, \mathfrak{c}_{k+1}, \mathfrak{c}_{k+2}) = (x, y, z)$, $(\mathfrak{c}'_k, \mathfrak{c}'_{k+1}, \mathfrak{c}'_{k+2}) = (x', y', z')$ with $x' = yz/(x+z)$, $y' = x+z$, $z' = xy/(x+z)$ or equivalently $x = y'z'/(x'+z')$, $y = x'+z'$, $z = x'y'/(x'+z')$;

(iii) if η_*, η'_* coincide except at the places $k, k+1, k+2, k+3$ where

$(\eta_k, \eta_{k+1}, \eta_{k+2}, \eta_{k+3}) = (\overline{n-1}, \bar{n}, \overline{n-1}, \bar{n})$,

$(\eta'_k, \eta'_{k+1}, \eta'_{k+2}, \eta'_{k+3}) = (\bar{n}, \overline{n-1}, \bar{n}, \overline{n-1})$

then $\mathfrak{c}_*, \mathfrak{c}'_*$ coincide except at the places $k, k+1, k+2, k+3$ where

$(\mathfrak{c}_k, \mathfrak{c}_{k+1}, \mathfrak{c}_{k+2}, \mathfrak{c}_{k+3}) = (d, c, b, a)$, $(\mathfrak{c}'_k, \mathfrak{c}'_{k+1}, \mathfrak{c}'_{k+2}, \mathfrak{c}'_{k+3}) = (d', c', b', a')$
and

$d' = ab^2c/\epsilon$, $c' = \epsilon/\alpha$, $b' = \alpha^2/\epsilon$, $a' = bcd/\alpha$

(or equivalently

$d = a'b'^2c'/\epsilon'$, $c = \epsilon'/\alpha'$, $b = \alpha'^2/\epsilon'$, $a = b'c'd'/\alpha'$)

with the notation

$\alpha = ab + ad + cd$, $\epsilon = ab^2 + ad^2 + cd^2 + 2abd$,

$\alpha' = a'b' + a'd' + c'd'$, $\epsilon' = a'b'^2 + a'd'^2 + c'd'^2 + 2a'b'd'$.

In case (i) and (ii) the result is obvious. In case (iii) we can assume that $n = 2$ and we consider the sequence of vertices of $\tilde{\mathcal{X}}$:

$$2^d 2'^d 1^c 2'^b 2^b 1^a$$

$$2^d 1^{\frac{bc}{b+d}} 2'^{b+d} 1^{\frac{cd}{b+d}} 2^b 1^a$$

$$2^d 1^{\frac{bc}{b+d}} 2'^{b+d} 2^{\frac{ab(b+d)}{\alpha}} 1^{\frac{\alpha}{b+d}} 2^{\frac{bcd}{\alpha}}$$

$$2^d 1^{\frac{bc}{b+d}} 2^{\frac{ab(b+d)}{\alpha}} 2'^{b+d} 1^{\frac{\alpha}{b+d}} 2^{\frac{bcd}{\alpha}}$$

$$1^{\frac{ab^2c}{\epsilon}} 2^{\frac{\epsilon}{\alpha}} 1^{\frac{dbc\alpha}{(b+d)\epsilon}} 2'^{b+d} 1^{\frac{\alpha}{b+d}} 2^{\frac{bcd}{\alpha}}$$

$$1^{\frac{ab^2c}{\epsilon}} 2^{\frac{\epsilon}{\alpha}} 2'^{\frac{\epsilon}{\alpha}} 1^{\frac{\alpha^2}{\epsilon}} 2'^{\frac{bcd}{\alpha}} 2^{\frac{bcd}{\alpha}}$$

in which any two consecutive lines represent an edge in $\tilde{\mathcal{X}}$. This proves our claim.

Note that the expressions appearing in the coordinate transformation (iii) first appeared in the case 1.7(iii) in a different but equivalent form in [L3, 12.5] and were later rewritten in the present form in [BZ, 7.1]. (In the last displayed formula in [L3, 12.5], $a + d - f$ should be replaced by $c + d - f$.) In the cases 1.7(ii), 1.7(iii) the coordinate transformation $K^4 \to K^4$ appearing in (iii) can be viewed as a coordinate transformation $\mathbf{N}^4 \to \mathbf{N}^4$, $(d, c, b, a) \mapsto (d', c', b', a')$, where

$d' = a + 2b + c - \min(a + 2b, a + 2d, c + 2d)$,

$c' = \min(a + 2b, a + 2d, c + 2d) - \min(a + b, a + d, c + d)$,

$b' = 2\min(a + b, a + d, c + d) - \min(a + 2b, a + 2d, c + 2d)$,

$a' = b + c + d - \min(a + b, a + d, c + d)$,

since $a + b + d \geq \min(a + 2b, a + 2d)$.

1.10. We apply the definitions in 1.8 to $(I,\cdot),\sigma$ as in 1.2. Then the associated $(\underline{I},\circ)$ is as in 1.4 (see 1.5). Let $\eta_*^{\mathfrak{c}_*},\eta_*'^{\mathfrak{c}'_*}$ be two joined vertices of $\underline{\tilde{\mathcal{X}}}$. We show that statements 1.9(i)-(iii) hold in the present case. In case (i) and (ii) the result is obvious. In case (iii) we can assume that $n=2$ and we consider the sequence of vertices of $\tilde{\mathcal{X}}$:

$$1^a4^a2^b3^{2b}2^b1^c4^c2^d3^{2d}2^d$$

$$1^a2^b4^a3^{2b}2^b1^c4^c2^d3^{2d}2^d$$

$$1^a2^b4^a3^{2b}2^b1^c2^d4^c3^{2d}2^d$$

$$1^a2^b4^a3^{2b}1^{\frac{cd}{b+d}}2^{b+d}1^{\frac{bc}{b+d}}4^c3^{2d}2^d$$

$$1^a2^b4^a1^{\frac{cd}{b+d}}3^{2b}2^{b+d}1^{\frac{bc}{b+d}}4^c3^{2d}2^d$$

$$1^a2^b4^a1^{\frac{cd}{b+d}}3^{2b}2^{b+d}4^c1^{\frac{bc}{b+d}}3^{2d}2^d$$

$$1^a2^b4^a1^{\frac{cd}{b+d}}3^{2b}4^c2^{b+d}1^{\frac{bc}{b+d}}3^{2d}2^d$$

$$1^a2^b4^a1^{\frac{cd}{b+d}}3^{2b}4^c2^{b+d}3^{2d}1^{\frac{bc}{b+d}}2^d$$

$$1^a2^b1^{\frac{cd}{b+d}}4^a3^{2b}4^c2^{b+d}3^{2d}1^{\frac{bc}{b+d}}2^d$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{\frac{ab(b+d)}{\alpha}}4^a3^{2b}4^c2^{b+d}3^{2d}1^{\frac{bc}{b+d}}2^d$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{\frac{ab(b+d)}{\alpha}}3^{\frac{2bc}{a+c}}4^{a+c}3^{\frac{2ab}{a+c}}T2^{b+d}3^{2d}1^{\frac{bc}{b+d}}2^d$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{\frac{ab(b+d)}{\alpha}}3^{\frac{2bc}{a+c}}4^{a+c}2^{\frac{d(b+d)(a+c)}{\alpha}}3^{\frac{2\alpha}{a+c}}2^{\frac{ab(b+d)}{\alpha}}1^{\frac{bc}{b+d}}2^d$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{\frac{ab(b+d)}{\alpha}}3^{\frac{2bc}{a+c}}4^{a+c}2^{\frac{d(b+d)(a+c)}{\alpha}}3^{\frac{2\alpha}{a+c}}1^{\frac{bcd\alpha}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{\frac{ab(b+d)}{\alpha}}3^{\frac{2bc}{a+c}}2^{\frac{d(b+d)(a+c)}{\alpha}}4^{a+c}3^{\frac{2\alpha}{a+c}}1^{\frac{bcd\alpha}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}3^{\frac{2bcd}{\alpha}}2^{b+d}3^{\frac{2ab^2c}{(a+c)\alpha}}4^{a+c}3^{\frac{2\alpha}{a+c}}1^{\frac{bcd\alpha}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}1^{\frac{\alpha}{b+d}}3^{\frac{2bcd}{\alpha}}2^{b+d}4^{\frac{\alpha^2}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}4^{\frac{ab^2c}{\epsilon}}1^{\frac{bcd\alpha}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{b+d}4^{\frac{\alpha^2}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}4^{\frac{ab^2c}{\epsilon}}1^{\frac{bcd\alpha}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{b+d}4^{\frac{\alpha^2}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}1^{\frac{bcd\alpha}{\epsilon}}4^{\frac{ab^2c}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{b+d}4^{\frac{\alpha^2}{\epsilon}}1^{\frac{bcd\alpha}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}4^{\frac{ab^2c}{\epsilon}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{b+d}4^{\frac{\alpha^2}{\epsilon}}1^{\frac{bcd\alpha}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}2^{\frac{\epsilon}{\alpha}}4^{\frac{ab^2c}{\epsilon}}1^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{b+d}4^{\frac{\alpha^2}{\epsilon}}1^{\frac{bcd\alpha}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}4^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}1^{\frac{\alpha}{b+d}}2^{b+d}1^{\frac{bcd\alpha}{\epsilon}}4^{\frac{\alpha^2}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}4^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}2^{\frac{bcd}{\alpha}}1^{\frac{\alpha^2}{\epsilon}}2^{\frac{\epsilon}{\alpha}}4^{\frac{\alpha^2}{\epsilon}}3^{\frac{2\epsilon}{\alpha}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}4^{\frac{ab^2c}{\epsilon}}$$

$$2^{\frac{bcd}{\alpha}}3^{\frac{2bcd}{\alpha}}2^{\frac{bcd}{\alpha}}1^{\frac{\alpha^2}{\epsilon}}4^{\frac{\alpha^2}{\epsilon}}2^{\frac{\epsilon}{\alpha}}3^{\frac{2\epsilon}{\alpha}}2^{\frac{\epsilon}{\alpha}}1^{\frac{ab^2c}{\epsilon}}4^{\frac{ab^2c}{\epsilon}}$$

in which any two consecutive lines represent an edge in $\tilde{\mathcal{X}}$. This proves our claim.

1.11. Define $\mathcal{B}, \mathcal{B}^\sigma$ as in 1.7, 1.8 in terms of $(I, \cdot), \sigma$ as in 1.2. The objects analogous to $(I, \cdot), \sigma, \mathcal{B}, \mathcal{B}^\sigma$ when $(I, \cdot), \sigma$ are taken as in 1.3 are denoted by $(I', \cdot), \sigma', \mathcal{B}', \mathcal{B}'^{\sigma'}$.

Let $\underline{\tilde{\mathcal{X}}}$ be the graph attached to $(I, \cdot), \sigma$ as in 1.8 and let $\underline{\tilde{\mathcal{X}}}'$ be the analogous graph attached to $(I', \cdot), \sigma'$. From the results in 1.9, 1.10 we see that the graphs $\underline{\tilde{\mathcal{X}}}, \underline{\tilde{\mathcal{X}}}'$ are canonically isomorphic. Hence the sets $\underline{\mathcal{B}}$, $\underline{\mathcal{B}}'$ of connected components of $\underline{\tilde{\mathcal{X}}}, \underline{\tilde{\mathcal{X}}}'$ are in canonical bijection. Combining with the canonical bijection $\underline{\mathcal{B}} \leftrightarrow \mathcal{B}^\sigma$ (see 1.8(a)) and the analogous bijection $\underline{\mathcal{B}}' \leftrightarrow \mathcal{B}'^{\sigma'}$ we obtain a canonical bijection

(a) $\mathcal{B}^\sigma \leftrightarrow \mathcal{B}'^{\sigma'}$.

1.12. In this subsection we take K, ι as in 1.7(iii). Let $(I, \cdot)$ be a Cartan datum. Let $\mathfrak{f}$ be the $\mathbf{Q}$-algebra with 1 with generators $\theta_i (i \in I)$ and relations

$$\sum_{p,p' \in \mathbf{N}; p+p' = 1 - 2i\cdot j/(i\cdot i)} (-1)^{p'} (p!p'!)^{-1} \theta_i^p \theta_j \theta_i^{p'} = 0$$

for $i \neq j$ in I. Let $\mathbf{B}$ be the canonical basis of the $\mathbf{Q}$-vector space $\mathfrak{f}$ obtained by specializing under $v = 1$ the canonical basis of the quantum version of $\mathfrak{f}$ defined in [L1,L4]. For $i \in I$ and $b \in \mathbf{B}$ we define $l_i(b) \in \mathbf{N}$, by the requirement that $b \in \theta_i^{l_i(b)}\mathfrak{f}$, $b \notin \theta_i^{l_i(b)+1}\mathfrak{f}$; we define $r_i(b) \in \mathbf{N}$, by the requirement that $b \in \mathfrak{f}\theta_i^{l_i(b)}$, $b \notin \mathfrak{f}\theta_i^{l_i(b)+1}$.

If $(I, \cdot)$ is simply laced and $\mathcal{B}$ is as in 1.7 then we have a canonical bijection

(a) $\beta : \mathbf{B} \xrightarrow{\sim} \mathcal{B}$

such that $\lambda_i \beta = \iota l_i$, $\rho_i \beta = \iota r_i$ for all $i \in I$. (Here $\lambda_i, \rho_i : \mathcal{B} \longrightarrow K$ are defined as λ_η, ρ_η in 1.8 in the case where $\sigma = 1$.) See [L1,L2].

Now let $(I, \cdot), \sigma$ be as in 1.5. Let $(\underline{I}, \circ)$ be as in 1.5. Let $\underline{\mathbf{B}}$ be the analogue of $\mathbf{B}$ when $(I, \cdot)$ is replaced by $(\underline{I}, \circ)$ and let $\underline{l}_\eta : \underline{\mathbf{B}} \to \mathbf{N}$, $\underline{r}_\eta : \underline{\mathbf{B}} \to \mathbf{N}$ ($\eta \in \underline{I}$) be the functions analogous to l_i, r_i defined in terms of $(\underline{I}, \circ)$. The algebra automorphism $\theta_i \mapsto \theta_{\sigma(i)} (i \in I)$ of $\mathfrak{f}$ restricts to a permutation of $\mathbf{B}$ denoted again by σ. Let $\mathbf{B}^\sigma$ be the fixed point set of $\sigma : \mathbf{B} \to \mathbf{B}$. For $\eta \in \underline{I}$ we define $l_\eta : \mathbf{B}^\sigma \to \mathbf{N}$ by $l_\eta(b) = l_i(b)$ with $i \in \eta$; we define $r_\eta : \mathbf{B}^\sigma \to \mathbf{N}$ by $r_\eta(b) = r_i(b)$ with $i \in \eta$.

We have the following result:

(b) *there is a canonical bijection* $\gamma : \underline{\mathbf{B}} \xrightarrow{\sim} \mathbf{B}^\sigma$ *such that* $l_\eta \gamma = \iota \underline{l}_\eta$, $r_\eta \gamma = \iota \underline{r}_\eta$ *for any* $\eta \in \underline{I}$.

When $\delta = 1$ (see 1.5) this is established in [L4, 14.4.9]. Assume now that $\delta = 2$. Then $(I, \cdot), \sigma$ are as in 1.2. We shall use the notation in 1.11. Let $\mathbf{B}', \sigma' : \mathbf{B}' \to \mathbf{B}'$ be the analogues of $\mathbf{B}, \sigma : \mathbf{B} \to \mathbf{B}$ when $(I, \cdot), \sigma$ are replaced by $(I', \cdot), \sigma'$. Since $(\underline{I}, \circ)$ is the same when defined in terms of $(I, \cdot), \sigma$ or in terms of $(I', \cdot), \sigma'$ and since $\delta = 1$ for $(I', \cdot), \sigma'$ we see that we have a canonical bijection

(c) $\underline{\mathbf{B}} \leftrightarrow \mathbf{B}'^{\sigma'}$.

We now consider the following composition of bijections

$\underline{\mathbf{B}} \leftrightarrow \mathbf{B}'^{\sigma'} \leftrightarrow \mathcal{B}'^{\sigma'} \leftrightarrow \mathcal{B}^\sigma \leftrightarrow \mathbf{B}^\sigma$.

(The first bijection is given by (c). The fourth bijection is obtained from (a) which is compatible with the actions of σ by taking fixed point sets of σ. The second bijection is an analogue of the fourth bijection. The third bijection is given by 1.11(a).) This bijection has the required properties. This establishes (b) in our case.

2. The "Frobenius" endomorphism Φ_e of $\underline{\mathbf{B}}$

2.1. We assume that we are in the setup of 1.8 and that K, ι are as in 1.7(iii). Following [L5, 9.11] we consider the monoid M^+ (with 1) defined by the generators ξ_i^n $(i \in I, n \in \mathbf{Z})$ and the relations

(i) $\xi_i^a \xi_i^b = \xi_i^{\min(a,b)}$ for any $i \in I$ and a, b in $\mathbf{Z}$;

(ii) $\xi_i^a \xi_{i'}^b = \xi_{i'}^b \xi_i^a$ for any $i, i' \in I$ such that $i \cdot i' = 0$ and any a, b in $\mathbf{Z}$;

(iii) $\xi_i^a \xi_{i'}^b \xi_i^c = \xi_{i'}^{a'} \xi_i^{b'} \xi_{i'}^{c'}$ for any i, i' in I such that $i \cdot i' = -1$ and any integers a, b, c, a', b', c' such that $a' = b + c - \min(a, c)$, $b' = \min(a, c)$, $c' = a + b - \min(a, c)$, or equivalently $a = b' + c' - \min(a', c')$, $b = \min(a', c')$, $c = a' + b' - \min(a', c')$. (Here ξ_i^0 is not assumed to be 1.)

Remark. In the last line of [L5, 9.9] one should replace "adding c to the first entry of $\mathbf{c}$" by the text: "replacing the first entry c_1 of $\mathbf{c}$ by $\min(c, c_1)$". In [L5, 9.10(a)], $n + n'$ should be replaced by $\min(n, n')$.

For any $i_* \in \mathcal{X}$ we define a map $\zeta_{i_*} : K^N \longrightarrow M^+$ by
$c_* \mapsto \xi_{i_1}^{\iota^{-1}(c_1)} \xi_{i_2}^{\iota^{-1}(c_2)} \dots \xi_{i_N}^{\iota^{-1}(c_N)}$.
From [L5, 9.10] we see that ζ_{i_*} is injective. Clearly its image is independent of the choice of i_*; we denote it by M_0^+. Note that $\xi_i^n M_0^+ \subset M_0^+$, $M_0^+ \xi_i^n \subset M_0^+$ for any $i \in I$, $n \in \mathbf{N}$. In particular, M_0^+ is a submonoid (without 1) of M^+. We define $\zeta : \tilde{\mathcal{X}} \longrightarrow M_0^+$ by $i_*^{c_*} \mapsto \zeta_{i_*}(c_*)$. This map is constant on any connected connected component of $\tilde{\mathcal{X}}$ hence it induces a map $\bar{\zeta} : \mathcal{B} \longrightarrow M_0^+$ (necessarily a bijection).

Now $\xi_i^n \mapsto \xi_{\sigma(i)}^n$ (with $i \in I, n \in \mathbf{Z}$) defines a monoid automorphism $M^+ \longrightarrow M^+$ denoted again by σ. It restricts to a monoid automorphism $M_0^+ \longrightarrow M_0^+$ denoted again by σ. This is compatible with the bijection $\sigma : \mathcal{B} \longrightarrow \mathcal{B}$ via $\bar{\zeta}$. Note that the fixed points $M^{+\sigma}, M_0^{+\sigma}$ are submonoids of M^+, M_0^+. Consider the composite bijection $\underline{\mathbf{B}} \leftrightarrow \mathbf{B}^\sigma \leftrightarrow \mathcal{B}^\sigma \leftrightarrow M_0^{+\sigma}$. Here the first bijection is as in 1.12(b), the second bijection is induced by the one in 1.12(a) and the third bijection is induced by $\bar{\zeta}$. Via this bijection the monoid structure on $M_0^{+\sigma}$ becomes a monoid structure on $\underline{\mathbf{B}}$.

2.2. We show that the crystal graph structure on $\underline{\mathbf{B}}$ introduced in [Ka] is completely determined by the monoid structure of $M^{+\sigma}$. For simplicity we assume that $\sigma = 1$ so that $\underline{\mathbf{B}} = \mathbf{B}$. We identify $\mathbf{B} = M_0^+$ via $\bar{\zeta}$. As shown in [L2] giving the crystal graph structure on $\mathbf{B}$ is equivalent to giving for any $i \in I$, $n \in \mathbf{N}$ the subsets $l_i^{-1}(n)$ (see 1.12) of $\mathbf{B}$ and certain bijections $l_i^{-1}(0) \xrightarrow{\sim} l_i^{-1}(n)$. Now $l_i^{-1}(n)$ is exactly the set of $\xi \in M_0^+$ such that $\xi_i^a \xi = \xi$ for any $a \geq n$ and $\xi_i^a \xi \neq \xi$ for any $a \in [0, n-1]$. The inverse of the bijection $l_i^{-1}(0) \xrightarrow{\sim} l_i^{-1}(n)$ is given by $\xi \mapsto \xi_i^0 \xi$.

2.3. We fix an integer $e \geq 1$. There is a well defined endomorphism $\Phi_e : M^+ \longrightarrow M^+$ (as a monoid with 1) such that $\xi_i^n \mapsto \xi_i^{en}$ for any $i \in I, n \in \mathbf{Z}$. This restricts to a monoid endomorphism $M_0^+ \longrightarrow M_0^+$. Moreover, it commutes with $\sigma : M^+ \longrightarrow M^+$ hence it restricts to a monoid endomorphism $M_0^{+\sigma} \longrightarrow M_0^{+\sigma}$. Via the bijection $\underline{\mathbf{B}} \leftrightarrow M_0^{+\sigma}$ in 2.1 this becomes a monoid endomorphism $\underline{\mathbf{B}} \longrightarrow \underline{\mathbf{B}}$ denoted again by Φ_e. We call Φ_e the "Frobenius" endomorphism of the canonical basis $\underline{\mathbf{B}}$.

References

[BZ] A.Berenstein and A.Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), 77-128.

[Ka] M.Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465-516.

[Li] P.Littelmann, *Path and root operators in representation theory*, Ann.Math. **142** (1995), 449-525.

[L1] G.Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447-498.

[L2] G.Lusztig, *Canonical bases arising from quantized enveloping algebras, II, in "Common trends in mathematics and quantum field theories"*, Progr. of Theor. Phys. Suppl. **102** (1990), 175-201.

[L3] G.Lusztig, *Introduction to quantized enveloping algebras*, New developments in Lie theory and their applications, ed. J.Tirao, Progr. in Math., vol. 105, Birkhäuser Boston, 1992, pp. 49-65.

[L4] G.Lusztig, *Introduction to quantum groups*, Progr. in Math., vol. 110, Birkhäuser Boston, 1993.

[L5] G.Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr.in Math., vol. 123, Birkhäuser Boston, 1994, pp. 531-568.

G. Lusztig
Department of Mathematics,
M. I. T.,
Cambridge, MA02139
E-mail: gyuri@math.mit.edu