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On Iwahori–Hecke Algebras with Unequal Parameters and Lusztig's Isomorphism Theorem

Meinolf Geck

Dedicated to Professor Jacques Tits on his 80th birthday

Abstract: By Tits' deformation argument, a generic Iwahori–Hecke algebra $\mathbf{H}$ associated to a finite Coxeter group W is abstractly isomorphic to the group algebra of W. Lusztig has shown how one can construct an explicit isomorphism, provided that the Kazhdan–Lusztig basis of $\mathbf{H}$ satisfies certain deep properties. If W is crystallographic and $\mathbf{H}$ is a one-parameter algebra, then these properties are known to hold thanks to a geometric interpretation. In this paper, we develop some new general methods for verifying these properties, and we do verify them for two-parameter algebras of type $I_2(m)$ and F_4 (where no geometric interpretation is available in general). Combined with previous work by Alvis, Bonnafé, DuCloux, Iancu and the author, we can then extend Lusztig's construction of an explicit isomorphism to all types of W, without any restriction on the parameters of $\mathbf{H}$.
Keywords: Coxeter groups, Hecke algebras, Kazhdan–Lusztig cells

1. Introduction

Let (W,S) be a Coxeter system where W is finite. Let F be a field of characteristic zero and $A=F[v_s^{\pm 1} \mid s \in S]$ the ring of Laurent polynomials over F, where $\{v_s \mid s \in S\}$ is a collection of indeterminates such that $v_s=v_t$ whenever $s,t \in S$ are conjugate in W. Let $\mathbf{H}$ be the associated "generic" Iwahori–Hecke

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algebra. This is an associative algebra over A, which is free as an A-module with basis $\{T_w \mid w \in W\}$. The multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$; here, $l\colon W \to \mathbb{Z}_{\geqslant 0}$ is the usual length function on W.

Let K be the field of fractions of A. By scalar extension, we obtain a K-algebra $\mathbf{H}_K = K \otimes_A \mathbf{H}$, which is well-known to be separable. On the other hand, there is a unique ring homomorphism $\theta_1\colon A \to F$ such that $\theta_1(v_s) = 1$ for all $s \in S$. Then we can regard F as an A-algebra (via θ_1) and obtain $F \otimes_A \mathbf{H} = F[W]$, the group algebra of W over F. By a general deformation argument due to Tits (see [5, Chap. IV, §2, Exercise 27]), one can show that $\mathbf{H}_{K'}$ and $K'[W]$ are abstractly isomorphic where $K' \supseteq K$ is a sufficiently large field extension.

One of the purposes of this paper is to prove the following finer result which was first obtained by Lusztig [17] for finite Weyl groups in the case where all v_s $(s \in S)$ are equal.

Theorem 1.1. *There exists an algebra homomorphism $\psi\colon \mathbf{H} \to A[W]$ with the following properties:*

(a) *If we extend scalars from A to F (via θ_1), then ψ induces the identity map.*
(b) *If we extend scalars from A to K, we obtain an isomorphism $\psi_K\colon \mathbf{H}_K \xrightarrow{\sim} K[W]$.*

In particular, (b) implies that, if F is a splitting field for W, then $\mathbf{H}_K \cong K[W]$ is a split semisimple algebra. Recall that it is known that $F_0 = \mathbb{Q}\big(\cos(2\pi/m_{st}) \mid s,t \in S\big) \subseteq \mathbb{R}$ is a splitting field for W; see [14, Theorem 6.3.8]. (Here, m_{st} denotes the order of st in W.) Note that $F_0 = \mathbb{Q}$ if W is a finite Weyl group, that is, if $m_{st} \in \{2,3,4,6\}$ for all $s,t \in S$.

The above result shows that, when W is finite, the algebra $\mathbf{H}_K$ and its representation theory can be understood, at least in principle, via the isomorphism $\mathbf{H}_K \xrightarrow{\sim} K[W]$; see [14] and [22, §20–24] where this is further developped.

This paper is organised as follows. In Section 2, we recall the basic facts about Kazhdan–Lusztig bases and cells. We present Lusztig's conjectures **P1**–**P15** and explain, following [22], how the validity of these conjectures leads to a proof of

Theorem 1.1. In this argument, a special role is played by Lusztig's asymptotic ring **J** which is defined using the leading coefficients of the structure constants of the Kazhdan–Lusztig basis.

Now, **P1**–**P15** are known to hold for finite Weyl groups in the equal parameter case, thanks to a deep geometric interpretation of the Kazhdan–Lusztig basis; see Kazhdan–Lusztig [16], Lusztig [22], Springer [23]. The case of non-crystallographic finite Coxeter groups is covered by Alvis [1] and DuCloux [6]. So it remains to consider the case of unequal parameters where W is of type B_n, F_4 or $I_2(m)$ (m even). Type B_n (with two independent parameters and a certain monomial order on them) has been dealt with by Bonnafé, Iancu and the author; see [4], [3], [13], [9]. In Sections 3 and 4, we develop new general methods for verifying **P1**–**P15**, based on the "leading matrix coefficients" introduced in [7]. In Section 5, we show how this can be used to deal with W of type F_4 and $I_2(m)$, for all choices of parameters. We also indicate how our methods lead to a new proof of **P1**–**P15** for type H_4, which is based on the results of Alvis [1] and Alvis–Lusztig [2] but which does not rely on DuCloux's computation [6] of all structure constants of the Kazhdan–Lusztig basis.

Finally, we put all the pieces into place and complete the proof of Theorem 1.1.

2. The Kazhdan–Lusztig basis

It will be convenient to slightly change the setting of the introduction. So let (W, S) be a Coxeter system and $l\colon W \to \mathbb{Z}_{\geqslant 0}$ be the usual length function. Throughout this paper, W will be finite. Let Γ be an abelian group (written additively). Following Lusztig [22], a function $L\colon W \to \Gamma$ is called a *weight function* if $L(ww') = L(w) + L(w')$ whenever $w, w' \in W$ are such that $l(ww') = l(w) + l(w')$. Note that L is uniquely determined by the values $\{L(s) \mid s \in S\}$. Furthermore, if $\{c_s \mid s \in S\}$ is a collection of elements in Γ such that $c_s = c_t$ whenever $s, t \in S$ are conjugate in W, then there is (unique) weight function $L\colon W \to \Gamma$ such that $L(s) = c_s$ for all $s \in S$.

Let $R \subseteq \mathbb{C}$ be a subring and $A = R[\Gamma]$ be the free R-module with basis $\{\varepsilon^g \mid g \in \Gamma\}$. There is a well-defined ring structure on A such that $\varepsilon^g \varepsilon^{g'} = \varepsilon^{g+g'}$ for all $g, g' \in \Gamma$. We write $1 = \varepsilon^0 \in A$. Given $a \in A$ we denote by a_g the coefficient of ε^g, so that $a = \sum_{g \in \Gamma} a_g \, \varepsilon^g$. Let $\mathbf{H} = \mathbf{H}_A(W, S, L)$ be the *generic Iwahori–Hecke algebra* over A with parameters $\{v_s \mid s \in S\}$ where $v_s := \varepsilon^{L(s)}$

for $s \in S$. This an associative algebra which is free as an A-module, with basis $\{T_w \mid w \in W\}$. The multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. The element T_1 is the identity element.

Example 2.1. Assume that $\Gamma = \mathbb{Z}$. Then A is nothing but the ring of Laurent polynomials over R in an indeterminate ε; we will usually denote $v = \varepsilon$. Then $\mathbf{H}$ is an associative algebra over $A = R[v, v^{-1}]$ with relations:

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v^{c_s} - v^{-c_s})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. This is the setting of Lusztig [22].

Example 2.2. (a) Assume that $\Gamma = \mathbb{Z}$ and L is constant on S; this case will be referred to as the *equal parameter case*. Note that we are automatically in this case when W is of type A_{n-1}, D_n, $I_2(m)$ where m is odd, H_3, H_4, E_6, E_7 or E_8 (since all generators in S are conjugate in W).

(b) Assume that W is finite and irreducible. Then unequal parameters can only arise in types B_n, $I_2(m)$ where m is even, and F_4.

Example 2.3. A "universal" weight function is given as follows. Let Γ_0 be the group of all tuples $(n_s)_{s\in S}$ where $n_s \in \mathbb{Z}$ for all $s \in S$ and $n_s = n_t$ whenever $s, t \in S$ are conjugate in W. (The addition is defined componentwise). Let $L_0 \colon W \to \Gamma_0$ be the weight function given by sending $s \in S$ to the tuple $(n_t)_{t\in S}$ where $n_t = 1$ if t is conjugate to s and $n_t = 0$, otherwise. Let $A_0 = R[\Gamma_0]$ and $\mathbf{H}_0 = \mathbf{H}_{A_0}(W, S, L_0)$ be the associated Iwahori–Hecke algebra, with parameters $\{v_s \mid s \in S\}$. Then $A_0 = R[\Gamma_0]$ is nothing but the ring of Laurent polynomials in indeterminates v_s ($s \in S$) with coefficients in R, where $v_s = v_t$ whenever $s, t \in S$ are conjugate in W. Furthermore, if $S' \subseteq S$ is a set of representatives for the classes of S under conjugation, then $\{v_s \mid s \in S'\}$ are algebraically independent.

Remark 2.4. Let k be any commutative ring (with 1) and assume we are given a collection of elements $\{\xi_s \mid s \in S\} \subseteq k^\times$ such that $\xi_s = \xi_t$ whenever $s, t \in S$ are conjugate in W. Then we have an associated Iwahori–Hecke algebra $H =$

$H_k(W, S, \{\xi_s\})$ over k. Again, this is an associative algebra; it is free as a k-module with basis $\{T_w \mid w \in W\}$. The multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (\xi_s - \xi_s^{-1})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. Now let A_0 be as in Example 2.3, where $R = \mathbb{Z}$. Then we can certainly find a (unique) unital ring homomorphism $\theta_0 \colon A_0 \to k$ such that $\theta_0(v_s) = \xi_s$ for all $s \in S$. Regarding k as an A_0-module (via θ_0), we find that H is obtained by extension of scalars from $\mathbf{H}_0$:

$$H_k(W, S, \{\xi_s\}) \cong k \otimes_A \mathbf{H}_0.$$

We conclude that $H_k(W, S, \{\xi_s\})$ can always be obtained by "specialisation" from the "universal" generic Iwahori–Hecke algebra $\mathbf{H}_0$.

We now recall the basic facts about the Kazhdan–Lusztig basis of $\mathbf{H}$, following Lusztig [18], [22]. For this purpose, we need to assume that Γ admits a total ordering $\leqslant$ which is compatible with the group structure, that is, whenever $g, g', h \in \Gamma$ are such that $g \leqslant g'$, then $g + h \leqslant g' + h$. Such an order on Γ will be called a *monomial order*. One readily checks that this implies that $A = R[\Gamma]$ is an integral domain; we usually reserve the letter K to denote its field of fractions. We will assume throughout that

$$L(s) > 0 \quad \text{for all } s \in S.$$

Now, there is a unique ring involution $A \to A$, $a \mapsto \bar{a}$, such that $\overline{\varepsilon^g} = \varepsilon^{-g}$ for all $g \in \Gamma$. We can extend this map to a ring involution $\mathbf{H} \to \mathbf{H}$, $h \mapsto \overline{h}$, such that

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1} \qquad (a_w \in A).$$

We define $\Gamma_{\geqslant 0} = \{g \in \Gamma \mid g \geqslant 0\}$ and denote by $\mathbb{Z}[\Gamma_{\geqslant 0}]$ the set of all integral linear combinations of terms ε^g where $g \geqslant 0$. The notations $\mathbb{Z}[\Gamma_{>0}]$, $\mathbb{Z}[\Gamma_{\leqslant 0}]$, $\mathbb{Z}[\Gamma_{<0}]$ have a similar meaning.

Theorem 2.5 (Kazhdan–Lusztig [15], Lusztig [18], [22]). *For each $w \in W$, there exists a unique $C'_w \in \mathbf{H}$ (depending on $\leqslant$) such that*

- $\overline{C}'_w = C'_w$ *and*
- $C'_w = T_w + \sum_{y \in W} p_{y,w} T_y$ *where $p_{y,w} \in \mathbb{Z}[\Gamma_{<0}]$ for all $y \in W$.*

The elements $\{C'_w \mid w \in W\}$ *form an* A*-basis of* $\mathbf{H}$*, and we have* $p_{y,w} = 0$ *unless* $y < w$ *(where* $<$ *denotes the Bruhat–Chevalley order on* W*).*

Here we follow the original notation in [15], [18]; the element C'_w is denoted by c_w in [22, Theorem 5.2]. As in [22], it will be convenient to work with the following alternative version of the Kazhdan–Lusztig basis. We set $\mathbf{C}_w = (C'_w)^\dagger$ where $\dagger\colon \mathbf{H} \to \mathbf{H}$ is the A-algebra automorphism defined by $T_s^\dagger = -T_s^{-1}$ $(s \in S)$; see [22, 3.5]. Note that $\overline{h} = j(h)^\dagger = j(h^\dagger)$ for all $h \in \mathbf{H}$ where $j\colon \mathbf{H} \to \mathbf{H}$ is the ring involution such that $j(a) = \bar{a}$ for $a \in A$ and $j(T_w) = (-1)^{l(w)} T_w$ for $w \in W$. Thus, we have

- $\overline{\mathbf{C}}_w = j(C'_w) = \mathbf{C}_w$ and
- $\mathbf{C}_w = (-1)^{l(w)} T_w + \sum_{y \in W} (-1)^{l(y)} \overline{p}_{y,w} T_y$ where $\overline{p}_{y,w} \in \mathbb{Z}[\Gamma_{>0}]$.

Since the elements $\{\mathbf{C}_w \mid w \in W\}$ form a basis of $\mathbf{H}$, we can write

$$\mathbf{C}_x \mathbf{C}_y = \sum_{z \in W} h_{x,y,z}\, \mathbf{C}_z \qquad \text{for any } x, y \in W,$$

where $h_{x,y,z} = \overline{h}_{x,y,z} \in A$ for all $x, y, z \in W$. The structure constants $h_{x,y,z}$ can de described more explicitly in the following special case. Let $s \in S$ and $w \in W$. Then we have

$$\mathbf{C}_s\, \mathbf{C}_w = \begin{cases} \mathbf{C}_{sw} + \displaystyle\sum_{\substack{y \in W \\ sy < y < w}} \mu^s_{y,w}\, \mathbf{C}_y & \text{if } sw > w, \\ (v_s + v_s^{-1})\, \mathbf{C}_w & \text{if } sw < w, \end{cases}$$

where $\mu^s_{y,w} \in A$; see [22, Theorem 6.6].

Remark 2.6. We refer to [22, Chap. 8] for the definition of the preorders $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ and the corresponding equivalence relations $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$, $\sim_{\mathcal{LR}}$ on W. (Note that these depend on the weight function L and the monomial order on Γ.) The equivalence classes with respect to these relations are called left, right and two-sided cells of W, respectively.

Each left cell $\mathfrak{C}$ gives rise to a representation of $\mathbf{H}$ (and of W). This is constructed as follows (see [18, §7]). Let $[\mathfrak{C}]_A$ be an A-module with a free A-basis $\{e_w \mid w \in \mathfrak{C}\}$. Then the action of $\mathbf{C}_w$ $(w \in W)$ on $[\mathfrak{C}]_A$ is given by the Kazhdan–Lusztig structure constants, that is, we have

$$\mathbf{C}_w . e_x = \sum_{y \in \mathfrak{C}} h_{w,x,y}\, e_y \qquad \text{for all } x \in \mathfrak{C} \text{ and } w \in W.$$

Furthermore, let $\theta_1 \colon A \to R$ be the unique ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. Extending scalars from A to R (via θ_1), we obtain a module $[\mathfrak{C}]_1 := R \otimes_A [\mathfrak{C}]_A$ for $R[W] = R \otimes_A \mathbf{H}$.

Following Lusztig [22], given $z \in W$, we define

$$\boldsymbol{a}(z) := \min\{g \in \Gamma_{\geqslant 0} \mid \varepsilon^g\, h_{x,y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}] \text{ for all } x, y \in W\}.$$

Thus, we obtain a function $\boldsymbol{a} \colon W \to \Gamma$. (If $\Gamma = \mathbb{Z}$ with its natural order, then this reduces to the function first defined by Lusztig [20].) Given $x, y, z \in W$, we define $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ to be the constant term of $\varepsilon^{\boldsymbol{a}(z)}\, h_{x,y,z}$, that is, we have

$$\varepsilon^{\boldsymbol{a}(z)}\, h_{x,y,z} \equiv \gamma_{x,y,z^{-1}} \mod \mathbb{Z}[\Gamma_{>0}].$$

Next, recall that $p_{1,z}$ is the coefficient of T_1 in the expansion of C'_w in the T-basis. By [22, Prop. 5.4], we have $p_{1,z} \neq 0$. As in [22, 14.1], we define $\Delta(z) \in \Gamma_{\geqslant 0}$ and $0 \neq n_z \in \mathbb{Z}$ by the condition that $\varepsilon^{\Delta(z)} p_{1,z} \equiv n_z \bmod \mathbb{Z}[\Gamma_{<0}]$. We set

$$\mathcal{D} = \{z \in W \mid \boldsymbol{a}(z) = \Delta(z)\}.$$

Now Lusztig [22, Chap. 14] has formulated the following 15 conjectures:

- **P1.** For any $z \in W$ we have $\boldsymbol{a}(z) \leqslant \Delta(z)$.
- **P2.** If $d \in \mathcal{D}$ and $x, y \in W$ satisfy $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.
- **P3.** If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.
- **P4.** If $z' \leqslant_{\mathcal{LR}} z$ then $\boldsymbol{a}(z') \geqslant \boldsymbol{a}(z)$. Hence, if $z' \sim_{\mathcal{LR}} z$, then $\boldsymbol{a}(z) = \boldsymbol{a}(z')$.
- **P5.** If $d \in \mathcal{D}$, $y \in W$, $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.
- **P6.** If $d \in \mathcal{D}$, then $d^2 = 1$.
- **P7.** For any $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x}$.
- **P8.** Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$. Then $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, $z \sim_{\mathcal{L}} x^{-1}$.
- **P9.** If $z' \leqslant_{\mathcal{L}} z$ and $\boldsymbol{a}(z') = \boldsymbol{a}(z)$, then $z' \sim_{\mathcal{L}} z$.
- **P10.** If $z' \leqslant_{\mathcal{R}} z$ and $\boldsymbol{a}(z') = \boldsymbol{a}(z)$, then $z' \sim_{\mathcal{R}} z$.
- **P11.** If $z' \leqslant_{\mathcal{LR}} z$ and $\boldsymbol{a}(z') = \boldsymbol{a}(z)$, then $z' \sim_{\mathcal{LR}} z$.
- **P12.** Let $I \subseteq S$ and W_I be the parabolic subgroup generated by I. If $y \in W_I$, then $\boldsymbol{a}(y)$ computed in terms of W_I is equal to $\boldsymbol{a}(y)$ computed in terms of W.
- **P13.** Any left cell $\mathfrak{C}$ of W contains a unique element $d \in \mathcal{D}$. We have $\gamma_{x^{-1},x,d} \neq 0$ for all $x \in \mathfrak{C}$.
- **P14.** For any $z \in W$, we have $z \sim_{\mathcal{LR}} z^{-1}$.

P15. If $x, x', y, w \in W$ are such that $\boldsymbol{a}(w) = \boldsymbol{a}(y)$, then

$$\sum_{y' \in W} h_{w,x',y'} \otimes h_{x,y',y} = \sum_{y' \in W} h_{y',x',y} \otimes h_{x,w,y'} \quad \text{in } \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma].$$

(The above formulation of **P15** is taken from Bonnafé [3].)

Remark 2.7. Assume that we are in the equal parameter case; see Example 2.2. In this case, $A = \mathbb{Z}[\Gamma]$ is nothing but the ring of Laurent polynomials in one variable v. Suppose that all polynomials $p_{x,y} \in \mathbb{Z}[v^{-1}]$ and all structure constants $h_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$ have non-negative coefficients. Then Lusztig [22, Chap. 15] shows that **P1**–**P15** follow.

Now, if (W, S) is a finite Weyl group, that is, if $m_{st} \in \{2, 3, 4, 6\}$ for all $s, t \in S$, then the required non-negativity of the coefficients is shown by using a deep geometric interpretation of the Kazhdan–Lusztig basis; see Kazhdan–Lusztig [16], Springer [23]. Thus, **P1**–**P15** hold for finite Weyl groups in the equal parameter case. If (W, S) is of type $I_2(m)$ (where $m \notin \{2, 3, 4, 6\}$), H_3 or H_4, the non-negativity of the coefficients has been checked explicitly by Alvis [1] and DuCloux [6].

Note that simple examples show that the coefficients of the polynomials $p_{y,w}$ or $h_{x,y,z}$ may be negative in the presence of unequal parameters; see Lusztig [18, p. 106], [22, §7].

We now use **P1**–**P15** to perform the following constructions, following Lusztig [22]. Let $\mathbf{J}$ be the free $\mathbb{Z}$-module with basis $\{t_w \mid w \in W\}$. We define a bilinear product on $\mathbf{J}$ by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}}\, t_z \qquad (x, y \in W).$$

Remark 2.8. By [22, 5.6], the map $\mathbf{H} \to \mathbf{H}$ defined by $\mathbf{C}_w \mapsto \mathbf{C}_{w^{-1}}$ $(w \in W)$ is an anti-involution; so we have $h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}$ for all $w, x, y, z \in W$. In particular, this implies that $\boldsymbol{a}(z) = \boldsymbol{a}(z^{-1})$ for all $z \in W$. By [22, 13.9], the map $\mathbf{J} \to \mathbf{J}$ defined by $t_w \mapsto t_{w^{-1}}$ $(w \in W)$ also is an anti-involution of $\mathbf{J}$; so we have $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$ for all $x, y, z \in W$.

Theorem 2.9 (Lusztig [22, Chap. 18]). *Assume that* **P1**–**P15** *hold. Then* $\mathbf{J}$ *is an associative ring with identity element* $1_{\mathbf{J}} = \sum_{d \in \mathcal{D}} n_d t_d$. *Let* $\mathbf{J}_A = A \otimes_{\mathbb{Z}} \mathbf{J}$.

Then we have a unital homomorphism of A-algebras

$$\phi\colon \mathbf{H} \to \mathbf{J}_A, \qquad \mathbf{C}_w \mapsto \sum_{\substack{z\in W, d\in\mathcal{D}\\ \boldsymbol{a}(z)=\boldsymbol{a}(d)}} h_{w,d,z}\, n_d\, t_z,$$

The ring $\mathbf{J}$ will be called the *asymptotic algebra* associated to $\mathbf{H}$ (with respect to $\leqslant$). It first appeared in [21] in the equal parameter case.

Remark 2.10. In [22, Theorem 18.9], the formula for ϕ looks somewhat different: instead of the factor n_d, there is a factor $\hat{n}_z$ which is defined as follows. Given $z \in W$, there is a unique element of $\mathcal{D}$ such that $\gamma_{z,z^{-1},d} \neq 0$; then $\hat{n}_z = n_d = \pm 1$ (see **P3**, **P5**, **P13**). Now one easily checks, using **P1**–**P15**, that the map $t_w \mapsto \hat{n}_w \hat{n}_{w^{-1}} t_w$ defines a ring involution of $\mathbf{J}$. Composing Lusztig's homomorphism in [22, 18.9] with this involution, we obtain the above formula (which seems more natural; see, e.g., the discussion in [11, §5]).

The structure of $\mathbf{J}$ is to some extent clarified by the following remark, which is taken from [22, 20.1].

Remark 2.11. Assume that **P1**–**P15** hold. Recall that $A = R[\Gamma]$ where $R \subseteq \mathbb{C}$ is a subring. Now assume that R is a field. Let $\theta_1\colon A \to R$ be the unique ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. Then $R \otimes_A \mathbf{H} = R[W]$. Via θ and extension of scalars, we obtain an induced homomorphism of R-algebras

$$\phi_1\colon R[W] \to \mathbf{J}_R = R \otimes_{\mathbb{Z}} J, \qquad \mathbf{C}_w \mapsto \sum_{\substack{z\in W, d\in\mathcal{D}\\ \boldsymbol{a}(z)=\boldsymbol{a}(d)}} \theta(h_{w,d,z})\, n_d\, t_z.$$

Now, the kernel of ϕ_1 is a nilpotent ideal in $R[W]$; see [22, Prop. 18.12(a)]. Since $R[W]$ is a semisimple algebra, we conclude that ϕ_1 is injective and, hence, an isomorphism. In particular, we can now conclude that

- $\mathbf{J}_R \cong R[W]$ *is a semisimple algebra*;
- $\mathbf{J}_R$ *is split if R is a splitting field for W*.

We can push this discussion even further. Let P be the matrix of $\phi\colon \mathbf{H} \to \mathbf{J}_A$ with respect to the standard bases of $\mathbf{H}$ and $\mathbf{J}_A$. Let P_1 be the matrix obtained by applying θ_1 to all entries of P. Then P_1 is the matrix of ϕ_1 with respect to the standard bases of $R[W]$ and $\mathbf{J}_R$. We have seen above that $\det(P_1) \neq 0$. Hence, clearly, we also have $\det(P) \neq 0$. Consequently, we obtain an induced isomorphism $\phi_K\colon \mathbf{H}_K \xrightarrow{\sim} \mathbf{J}_K$ where K is the field of fractions of A. In particular, if R is a splitting field for W, then $\mathbf{J}_R$ is split semisimple and, hence, $\mathbf{H}_K \cong \mathbf{J}_K$ will be split semisimple, too.

We now obtain the following result which was first obtained by Lusztig [17] (for finite Weyl groups in the equal parameter case).

Theorem 2.12 (Lusztig). *Assume that R is a field and that* **P1**–**P15** *hold. Then there exists an algebra homomorphism $\psi\colon \mathbf{H} \to A[W]$ with the following properties:*

(a) *Let $\theta_1\colon A \to R$ be the unique ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. If we extend scalars from A to R (via θ_1), then ψ induces the identity map.*

(b) *If we extend scalars from A to K (the field of fractions of A), then ψ induces an isomorphism $\psi_K\colon \mathbf{H}_K \xrightarrow{\sim} K[W]$. In particular, $\mathbf{H}_K$ is a semisimple algebra, which is split if R is a splitting field for W.*

Proof. As in Remark 2.11, we have an isomorphism $\phi_1\colon R[W] \xrightarrow{\sim} \mathbf{J}_R$. Let $\alpha := \phi_1^{-1}\colon \mathbf{J}_R \xrightarrow{\sim} R[W]$. By extension of scalars, we obtain an isomorphism of A-algebras $\alpha_A\colon \mathbf{J}_A \xrightarrow{\sim} A[W]$. Now set $\psi := \alpha_A \circ \phi\colon \mathbf{H} \to A[W]$.

(a) If we extend scalars from A to R via θ_1, then $\mathbf{H}_R = R[W]$. Furthermore, $\phi\colon \mathbf{H} \to \mathbf{J}_A$ induces the map ϕ_1 already considered at the beginning of the proof. Hence ψ induces the identity map.

(b) This immediately follows from (a) by a formal argument: Let Q be the matrix of the A-linear map ψ with respect to the standard A-bases of $\mathbf{H}$ and $A[W]$. We only need to show that $\det(Q) \neq 0$. But, by (a), we have $\theta_1(\det(Q)) = 1$; in particular, $\det(Q) \neq 0$.

Finally, note that, if R is a splitting field for W, then so is K. Hence, in this case, $\mathbf{H}_K \cong K[W]$ is a split semisimple algebra. □

Note that the *statement* of the above result does not make any reference to the monomial order $\leqslant$ on Γ or the corresponding Kazhdan–Lusztig basis; these are only needed in the proof.

Remark 2.13. Assume that **P1**–**P14** hold. Then the partitions of W into left, right and two-sided cells can be recovered from the structure of $\mathbf{J}$. Indeed, given $x, y \in W$, write $x \leftrightarrow_{\mathcal{L}} y$ if there exists some $z \in W$ such that $\gamma_{x,y^{-1},z} \neq 0$. Then one easily checks that $\sim_{\mathcal{L}}$ is the transitive closure of $\leftrightarrow_{\mathcal{L}}$. (Note that, by [22, Prop. 18.4(a)], the relations $\sim_{\mathcal{L}}$ and $\leftrightarrow_{\mathcal{L}}$ are actually the same when we are in the equal parameter case.) Thus, the left cells are determined by **J**. Furthermore, we have $x \sim_{\mathcal{R}} y$ if and only if $x^{-1} \sim_{\mathcal{L}} y^{-1}$. Finally, by **P4**, **P9**, the two-sided cells are the smallest subsets of W which are at the same time unions of left cells and unions of right cells.

3. The $\boldsymbol{a}$-function and orthogonal representations

The aim of this and the following section is to develop some new methods for verifying **P1**–**P15** for a given group W and weight function L. These methods should not rely on any positivity properties or geometric interpretations as mentioned in Remark 2.7, so that we may hope to be able to apply them in the general case of unequal parameters.

One of the main problems in the verification of **P1**–**P15** is the determination of the $\boldsymbol{a}$-function. Note that, if we just wanted to use the definition of $\boldsymbol{a}(z)$, then we would have to compute all structure constants $h_{x,y,z}$ where $x, y \in W$—which is very hard to get a hold on. We shall now describe a situation in which this problem can be solved by a different approach, which is inspired by [13, §4].

For the rest of this section, let us assume that $R = \mathbb{R}$. Then R is a splitting field for W; see [14, 6.3.8]. The set of irreducible representations of W (up to isomorphism) will be denoted by

$$\mathrm{Irr}(W) = \{E^\lambda \mid \lambda \in \Lambda\}$$

where Λ is some finite indexing set and E^λ is an R-vectorspace with a given $R[W]$-module structure. We shall also write

$$d_\lambda = \dim E^\lambda \qquad \text{for all } \lambda \in \Lambda.$$

Let K be the field of fractions of A. By extension of scalars, we obtain a K-algebra $\mathbf{H}_K = K \otimes_A \mathbf{H}$. This algebra is known to be split semisimple; see [14, 9.3.5]. Furthermore, by Tits' Deformation Theorem, the irreducible representations of $\mathbf{H}_K$ (up to isomorphism) are in bijection with the irreducible representations of W; see [14, 8.1.7]. Thus, we can write

$$\mathrm{Irr}(\mathbf{H}_K) = \{E_\varepsilon^\lambda \mid \lambda \in \Lambda\}.$$

The correspondence $E^\lambda \leftrightarrow E^\lambda_\varepsilon$ is uniquely determined by the following condition:

$$\operatorname{trace}\big(w, E^\lambda\big) = \theta_1\big(\operatorname{trace}(T_w, E^\lambda_\varepsilon)\big) \qquad \text{for all } w \in W,$$

where $\theta_1 \colon A \to F$ is the unique ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. Note also that $\operatorname{trace}\big(T_w, E^\lambda_\varepsilon\big) \in A$ for all $w \in W$. Note that all these statements can be proved without using **P1**–**P15**.

The algebra $\mathbf{H}$ is *symmetric*, with trace from $\tau \colon \mathbf{H} \to A$ given by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. The sets $\{T_w \mid w \in W\}$ and $\{T_{w^{-1}} \mid w \in W\}$ form a pair of dual bases. Hence we have the following orthogonality relations for the irreducible representations of $\mathbf{H}_K$:

$$\sum_{w \in W} \operatorname{trace}\big(T_w, E^\lambda_\varepsilon\big)\operatorname{trace}\big(T_{w^{-1}}, E^\mu_\varepsilon\big) = \begin{cases} d_\lambda\, \boldsymbol{c}_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu; \end{cases}$$

see [14, 8.1.7]. Here, $0 \neq \boldsymbol{c}_\lambda \in A$ and, following Lusztig, we can write

$$\boldsymbol{c}_\lambda = f_\lambda\, \varepsilon^{-2\boldsymbol{a}_\lambda} + \text{combination of terms } \varepsilon^g \text{ where } g > -2\boldsymbol{a}_\lambda,$$

where $\boldsymbol{a}_\lambda \in \Gamma_{\geqslant 0}$ and f_λ is a strictly positive real number; see [14, 9.4.7]. These invariants are explicitly known for all types of W; see Lusztig [22, Chap. 22].

We shall also need the basis which is dual to the Kazhdan–Lusztig basis. Let $\{\mathbf{D}_w \mid w \in W\} \subseteq \mathbf{H}$ be such that $\tau(\mathbf{C}_x\, \mathbf{D}_{y^{-1}}) = \delta_{xy}$ for all $x, y \in W$. Then

$$h_{x,y,z} = \tau(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_{z^{-1}}) \quad \text{for all } x, y, z \in W.$$

One also shows that $\mathbf{D}_w$ can be written as a sum of $(-1)^{l(w)} T_w$ and a $\mathbb{Z}[\Gamma_{>0}]$-linear combination of terms T_y $(y \in W)$; see [22, Chap. 10] or [7, 2.4]

We now recall the basic facts concerning the leading matrix coefficients introduced in [7]. Let us write

$$A_{\geqslant 0} = \text{set of } R\text{-linear combinations of terms } \varepsilon^g \text{ where } g \geqslant 0,$$
$$A_{>0} = \text{set of } R\text{-linear combinations of terms } \varepsilon^g \text{ where } g > 0.$$

Note that $1 + A_{>0}$ is multiplicatively closed. Furthermore, every element $x \in K$ can be written in the form

$$x = r_x\, \varepsilon^{\gamma_x} \frac{1+p}{1+q} \qquad \text{where } r_x \in R,\ \gamma_x \in \Gamma \text{ and } p, q \in A_{>0};$$

note that, if $x \neq 0$, then r_x and γ_x indeed are *uniquely determined* by x; if $x = 0$, we have $r_0 = 0$ and we set $\gamma_0 := +\infty$ by convention. We set

$$\mathcal{O} := \{x \in K \mid \gamma_x \geqslant 0\} \qquad \text{and} \qquad \mathfrak{p} := \{x \in K \mid \gamma_x > 0\}.$$

Then it is easily verified that $\mathcal{O}$ is a valuation ring in K, with maximal ideal $\mathfrak{p}$. Note that we have

$$\mathcal{O} \cap A = A_{\geqslant 0} \qquad \text{and} \qquad \mathfrak{p} \cap A = A_{>0}.$$

We have a well-defined R-linear ring homomorphism $\mathcal{O} \to R$ with kernel $\mathfrak{p}$. The image of $x \in \mathcal{O}$ in R is called the *constant term* of x. Thus, the constant term of x is 0 if $x \in \mathfrak{p}$; the constant term equals r_x if $x \in \mathcal{O}^\times$.

By [7, Prop. 4.3], each E_ε^λ affords a so-called *orthogonal representation.* By [7, Theorem 4.4 and Remark 4.5], this implies that there exists a basis of E_ε^λ such that the corresponding matrix representation $\rho^\lambda \colon \mathbf{H}_K \to M_{d_\lambda}(K)$ has the following properties. Let $\lambda \in \Lambda$ and $1 \leqslant i, j \leqslant d_\lambda$. For any $h \in \mathbf{H}_K$, we denote by $\rho_{ij}^\lambda(h)$ the (i,j)-entry of the matrix $\rho^\lambda(h)$. Then

$$\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(T_w) \in \mathcal{O}, \qquad \varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{C}_w) \in \mathcal{O}, \qquad \varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{D}_w) \in \mathcal{O}$$

for any $w \in W$ and

$$(-1)^{l(w)} \varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(T_w) \equiv \varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{C}_w) \equiv \varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{D}_w) \bmod \mathfrak{p}.$$

Hence, the above three elements of $\mathcal{O}$ have the same constant term which we denote by $c_{w,\lambda}^{ij}$. The constants $c_{w,\lambda}^{ij} \in R$ are called the *leading matrix coefficients* of ρ^λ. Given $w \in W$, there exists some $\lambda \in \Lambda$ and $i, j \in \{1, \ldots, d_\lambda\}$ such that $c_{w,\lambda}^{ij} \neq 0$. We use this fact to define the following relation.

Definition 3.1. Let $\lambda \in \Lambda$ and $w \in W$. We write $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$ if $c_{w,\lambda}^{ij} \neq 0$ for some $i, j \in \{1, \ldots, d_\lambda\}$.

(This is in analogy to Lusztig [22, 20.2] or [19, p. 139]; see Lemma 3.2 below.)

One can show that "$\leftsquigarrow\!\!\rightsquigarrow_L$" does not depend on the choice of the orthogonal representations ρ^λ (see [11, Remark 3.10]), but we don't need this here. For our purposes, the characterisation of "$\leftsquigarrow\!\!\rightsquigarrow_L$" given in the following result will be sufficient.

Recall from Remark 2.6 that every left cell $\mathfrak{C}$ of W gives rise to a left $R[W]$-module denoted by $[\mathfrak{C}]_1$.

Lemma 3.2. *Let $\lambda \in \Lambda$ and $\mathfrak{C}$ be a left cell of W. Then $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$ for some $w \in \mathfrak{C}$ if and only if E^λ is a constituent of $[\mathfrak{C}]_1$.*

Proof. Let $i \in \{1, \ldots, d_\lambda\}$. The assertion immediately follows from the identity

$$\frac{1}{f_\lambda} \sum_{k=1}^{d_\lambda} \sum_{w \in \mathfrak{C}} (c_{w,\lambda}^{ik})^2 = \text{multiplicity of } E^\lambda \text{ in } [\mathfrak{C}]_1.$$

which was proved in [7, Prop. 4.7]. □

Remark 3.3. Let $w, w' \in W$ and $\lambda \in \Lambda$ be such that $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$ and $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w'$. Let $\mathfrak{C}$, $\mathfrak{C}'$ be the left cells such that $w \in \mathfrak{C}$ and $w' \in \mathfrak{C}'$. By Lemma 3.2, E^λ is a constituent of both $[\mathfrak{C}]_1$ and $[\mathfrak{C}']_1$. Hence, $\mathrm{Hom}_W([\mathfrak{C}]_1, [\mathfrak{C}']_1) \neq 0$ and so $\mathfrak{C}, \mathfrak{C}'$ are contained in the same two-sided cell. In particular, $w \sim_{\mathcal{LR}} w'$.

This argument also implies **P14**, i.e., the assertion that $w \sim_{\mathcal{LR}} w^{-1}$ for all $w \in W$. Indeed, choose $\lambda \in \Lambda$ such that $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$, that is, $c_{w,\lambda}^{ij} \neq 0$ for some $i, j \in \{1, \ldots, d_\lambda\}$. By [7, Theorem 4.4], we also have $c_{w^{-1},\lambda}^{ji} = c_{w,\lambda}^{ij} \neq 0$ and so $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w^{-1}$. Hence, the previous discussion shows that $w \sim_{\mathcal{LR}} w^{-1}$, as claimed.

(This was first proved by Lusztig [19, Lemma 5.2] in the equal parameter case. One can check that Lusztig's proof also carries over to the case of unequal parameters.)

Lemma 3.4. *Let $z \in W$ and $\lambda \in \Lambda$ be such $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L z$. Then $\boldsymbol{a}(z) \geqslant \boldsymbol{a}_\lambda$.*

(A similar result was proved in [13, Prop. 4.1], but under additional assumptions. See also Lusztig [20, Prop. 6.4] where this result was obtained in the equal parameter case, based on the geometric interpretation which is available there.)

Proof. We begin by considering the structure constant $h_{x,y,z}$ for $x, y \in W$. We have $h_{x,y,z} = \tau(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_{z^{-1}})$. Now, by the general theory of symmetric algebras (see [14, Chap. 7]), we have

$$\tau(h) = \sum_{\lambda \in \Lambda} \boldsymbol{c}_\lambda^{-1} \mathrm{trace}(h, E^\lambda) = \sum_{\lambda \in \Lambda} \boldsymbol{c}_\lambda^{-1}\, \mathrm{trace}\big(\rho^\lambda(h)\big) = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i \leqslant d_\lambda} \boldsymbol{c}_\lambda^{-1}\, \rho_{ii}^\lambda(h),$$

for any $h \in \mathbf{H}$. Since $\rho^\lambda(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_{z^{-1}}) = \rho^\lambda(\mathbf{C}_x)\rho^\lambda(\mathbf{C}_y)\rho^\lambda(\mathbf{D}_{z^{-1}})$, we obtain

$$h_{x,y,z} = \sum_{\mu \in \Lambda} \sum_{1 \leqslant i,j,k \leqslant d_\mu} \boldsymbol{c}_\mu^{-1}\, \rho_{ij}^\mu(\mathbf{C}_x)\, \rho_{jk}^\mu(\mathbf{C}_y)\, \rho_{ki}^\mu(\mathbf{D}_{z^{-1}}).$$

We multiply this identity on both sides by $\rho^\lambda_{ls}(\mathbf{D}_{x^{-1}})\,\rho^\lambda_{rl}(\mathbf{D}_{y^{-1}})$ (where $\lambda \in \Lambda$ and $1 \leqslant l, r, s \leqslant d_\lambda$) and sum over all $x, y \in W$. Now, since $\{\mathbf{C}_w \mid w \in w\}$ and $\{\mathbf{D}_{w^{-1}} \mid w \in W\}$ form a pair of dual bases for $\mathbf{H}$, we have the following Schur relations (see [14, Chap. 7]):

$$\sum_{w \in W} \rho^\lambda_{ij}(\mathbf{C}_w)\,\rho^\mu_{kl}(\mathbf{D}_{w^{-1}}) = \delta_{il}\delta_{jk}\delta_{\lambda\mu}\boldsymbol{c}_\lambda,$$

where $\lambda, \mu \in \Lambda$, $1 \leqslant i, j \leqslant d_\lambda$ and $1 \leqslant k, l \leqslant d_\mu$. Then a straightforward computation yields that

$$\rho^\lambda_{rs}(\mathbf{D}_{z^{-1}}) = \sum_{x,y \in W} \boldsymbol{c}_\lambda^{-1}\,\rho^\lambda_{ls}(\mathbf{D}_{x^{-1}})\,\rho^\lambda_{rl}(\mathbf{D}_{y^{-1}})\,h_{x,y,z}.$$

Further multiplying by $\varepsilon^{\boldsymbol{a}(z)}$ and noting that $\boldsymbol{c}_\lambda^{-1} = f_\lambda^{-1}\,\varepsilon^{2\boldsymbol{a}_\lambda}/(1+g_\lambda)$ where $g_\lambda \in F[\Gamma_{>0}]$, we obtain

$$\varepsilon^{\boldsymbol{a}(z)}\,\rho^\lambda_{rs}(\mathbf{D}_{z^{-1}}) = \sum_{x,y \in W} \frac{f_\lambda^{-1}}{1+g_\lambda}\big(\varepsilon^{\boldsymbol{a}_\lambda}\rho^\lambda_{ls}(\mathbf{D}_{x^{-1}})\big)\,\big(\varepsilon^{\boldsymbol{a}_\lambda}\rho^\lambda_{rl}(\mathbf{D}_{y^{-1}})\big)\big(\varepsilon^{\boldsymbol{a}(z)}\,h_{x,y,z}\big).$$

Now all terms in the above sum lie in $\mathcal{O}$, hence the whole sum will lie in $\mathcal{O}$ and so $\varepsilon^{\boldsymbol{a}(z)}\,\rho^\lambda_{rs}(\mathbf{D}_{z^{-1}}) \in \mathcal{O}$.

Now assume, if possible, that $\boldsymbol{a}(z) < \boldsymbol{a}_\lambda$. Then we could conclude that the constant term of $\varepsilon^{\boldsymbol{a}_\lambda}\,\rho^\lambda_{rs}(\mathbf{D}_{z^{-1}})$ is zero, that is, $c^{rs}_{z^{-1},\lambda} = 0$, and this holds for all $1 \leqslant r, s \leqslant d_\lambda$. Since ρ^λ is an orthogonal representation, [7, Theorem 4.4] shows that then we also have $c^{rs}_{z,\lambda} = 0$ for all $1 \leqslant r, s \leqslant d_\lambda$, a contradiction. $\square$

We would like to find conditions which ensure that equality holds in Lemma 3.4. Consider the following property:

E1. Let $x, y \in W$ and $\lambda, \mu \in \Lambda$ be such that $E^\lambda \leftrightsquigarrow_L x$ and $E^\mu \leftrightsquigarrow_L y$. If $x \leqslant_L y$, then $\boldsymbol{a}_\mu \leqslant \boldsymbol{a}_\lambda$. In particular, if $x \in W$ and $\lambda, \mu \in \Lambda$ are such that $E^\lambda \leftrightsquigarrow_L x$ and $E^\mu \leftrightsquigarrow_L x$, then $\boldsymbol{a}_\lambda = \boldsymbol{a}_\mu$.

Assume that **E1** holds and let $z \in W$. Then we define $\tilde{\boldsymbol{a}}(z) = \boldsymbol{a}_\lambda$ where $\lambda \in \Lambda$ is such that $E^\lambda \leftrightsquigarrow_L z$. Note that $\tilde{\boldsymbol{a}}(z)$ is well-defined by **E1**. Furthermore, we have:

E1'. If $x, y \in W$ are such that $x \leqslant_{\mathcal{LR}} y$, then $\tilde{\boldsymbol{a}}(y) \leqslant \tilde{\boldsymbol{a}}(x)$. In particular, $\tilde{\boldsymbol{a}}$ is constant on two-sided cells.

Thus, Lemma 3.2 shows that, letting $\mathfrak{C}$ be the left cell containing $z \in W$, then

$$\tilde{\boldsymbol{a}}(z) = \boldsymbol{a}_\lambda \quad \text{if } E^\lambda \text{ is a constituent of } [\mathfrak{C}]_1.$$

Now Lusztig [22, 20.6, 20.7] shows that, if **P1**–**P15** hold, then **E1** holds and we have $\boldsymbol{a}(z) = \tilde{\boldsymbol{a}}(z)$ for all $z \in W$. Our aim is to show that **E1** is sufficient to prove the equality $\boldsymbol{a}(z) = \tilde{\boldsymbol{a}}(z)$ for all $z \in W$; see Proposition 3.6 below. This will be one of the key steps in our verification of **P1**–**P15** for W of type F_4 and $I_2(m)$.

Lemma 3.5. *Assume that* **E1** *holds. Let* $w \in W$ *and* $\lambda \in \Lambda$.

(a) *If* $\rho^\lambda(\mathbf{C}_w) \neq 0$ *then* $\tilde{\boldsymbol{a}}(w) \leqslant \boldsymbol{a}_\lambda$.
(b) *If* $\rho^\lambda(\mathbf{D}_{w^{-1}}) \neq 0$ *then* $\tilde{\boldsymbol{a}}(w) \geqslant \boldsymbol{a}_\lambda$.
(c) *We have* $\varepsilon^{\tilde{\boldsymbol{a}}(w)}\rho^\lambda_{ij}(\mathbf{D}_{w^{-1}}) \in \mathcal{O}$ *for all* $i, j \in \{1, \ldots, d_\lambda\}$.

Proof. (a) Let $\mathfrak{C}$ be a left cell such that E^λ occurs as a constituent of $[\mathfrak{C}]_1$. Now, if $\rho^\lambda(\mathbf{C}_w) \neq 0$, then $\mathbf{C}_w$ cannot act as zero in $[\mathfrak{C}]_A$. Hence, there exist $x, y \in \mathfrak{C}$ such that $h_{w,x,y} \neq 0$. We have $\tilde{\boldsymbol{a}}(x) = \tilde{\boldsymbol{a}}(y) = \boldsymbol{a}_\lambda$ by **E1'** and Lemma 3.2. Since, $h_{w,x,y} \neq 0$, we have $y \leqslant_{\mathcal{R}} w$ and so $\tilde{\boldsymbol{a}}(w) \leqslant \tilde{\boldsymbol{a}}(y) = \boldsymbol{a}_\lambda$ by **E1'**.

(b) Again, let $\mathfrak{C}$ be a left cell such that E^λ occurs as a constituent of $[\mathfrak{C}]_1$. Now, if $\rho^\lambda(\mathbf{D}_{w^{-1}}) \neq 0$, then $\mathbf{D}_{w^{-1}}$ cannot act as zero in $[\mathfrak{C}]_A$. Hence, there exists some $x \in \mathfrak{C}$ such that $\mathbf{D}_{w^{-1}}\mathbf{C}_x \neq 0$. We have $\tilde{\boldsymbol{a}}(x) = \boldsymbol{a}_\lambda$ by **E1'** and Lemma 3.2. Now, since τ is non-degenerate, there exists some $y \in W$ such that $\tau(\mathbf{D}_{w^{-1}}\mathbf{C}_x\mathbf{C}_y) \neq 0$. Then we also have $h_{x,y,w} = \tau(\mathbf{C}_x\mathbf{C}_y\mathbf{D}_{w^{-1}}) = \tau(\mathbf{D}_{w^{-1}}\mathbf{C}_x\mathbf{C}_y) \neq 0$ and so $w \leqslant_{\mathcal{R}} x$. This implies $\boldsymbol{a}_\lambda = \tilde{\boldsymbol{a}}(x) \leqslant \tilde{\boldsymbol{a}}(w)$ by **E1'**.

(c) Since ρ^λ is an orthogonal representation, we have $\varepsilon^{\boldsymbol{a}_\lambda}\rho^\lambda_{ij}(\mathbf{D}_{w^{-1}}) \in \mathcal{O}$ for all $i, j \in \{1, \ldots, d_\lambda\}$. Hence the assertion follows from (b). ☐

Proposition 3.6. *Assume that* **E1** *holds. Then* $\boldsymbol{a}(z) = \tilde{\boldsymbol{a}}(z)$ *for all* $z \in W$. *Furthermore, for* $x, y, z \in W$, *we have*

$$\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y} = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} f_\lambda^{-1}\, c^{ij}_{x,\lambda}\, c^{jk}_{y,\lambda}\, c^{ki}_{z,\lambda}.$$

Proof. Assume that $c^{ij}_{z,\lambda} \neq 0$. Now recall that $c^{ij}_{z,\lambda}$ is the constant term of $\varepsilon^{\boldsymbol{a}_\lambda}(T_z)$, $\varepsilon^{\boldsymbol{a}_\lambda}(\mathbf{C}_z)$ and $\varepsilon^{\boldsymbol{a}_\lambda}(\mathbf{D}_z)$. Hence, we have $\rho^\lambda(\mathbf{C}_x) \neq 0$ and $\rho^\lambda(\mathbf{D}_z) \neq 0$. So Lemma 3.5 yields that $\boldsymbol{a}(z) = \boldsymbol{a}_\lambda = \tilde{\boldsymbol{a}}(z)$.

Let $x, y, z \in W$. As in the proof of Lemma 3.4, we find that

$$\begin{aligned}\varepsilon^{\tilde{\boldsymbol{a}}(z)} h_{x,y,z^{-1}} &= \varepsilon^{\tilde{\boldsymbol{a}}(z)} \tau(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_z) = \varepsilon^{\tilde{\boldsymbol{a}}(z)} \sum_{\lambda \in \Lambda} \boldsymbol{c}_\lambda^{-1} \mathrm{trace}(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_z, E^\lambda) \\ &= \sum_{\lambda \in \Lambda} \frac{f_\lambda^{-1}}{1+g_\lambda} \, \varepsilon^{2\boldsymbol{a}_\lambda + \tilde{\boldsymbol{a}}(z)} \, \mathrm{trace}\big(\rho^\lambda(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_z)\big).\end{aligned}$$

Now $\rho^\lambda(\mathbf{C}_x \mathbf{C}_y \mathbf{D}_z) = \rho^\lambda(\mathbf{C}_x)\rho^\lambda(\mathbf{C}_y)\rho^\lambda(\mathbf{D}_z)$ and so the above expression equals

$$\sum_{\lambda \in \Lambda} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} \frac{f_\lambda^{-1}}{1+g_\lambda} \big(\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{C}_x)\big) \big(\varepsilon^{\boldsymbol{a}_\lambda} \rho_{jk}^\lambda(\mathbf{C}_y)\big) \big(\varepsilon^{\tilde{\boldsymbol{a}}(z)} \rho_{ki}^\lambda(\mathbf{D}_z)\big).$$

Furthermore, by Lemma 3.5(b), we have $\tilde{\boldsymbol{a}}(z) \geqslant \boldsymbol{a}_\lambda$ for all non-zero terms in the above sum. So the above sum can be rewritten as

$$\sum_{\lambda \in \Lambda \,:\, \boldsymbol{a}_\lambda \leqslant \tilde{\boldsymbol{a}}(z)} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} \frac{f_\lambda^{-1} \varepsilon^{\tilde{\boldsymbol{a}}(z) - \boldsymbol{a}_\lambda}}{1+g_\lambda} \big(\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{C}_x)\big) \big(\varepsilon^{\boldsymbol{a}_\lambda} \rho_{jk}^\lambda(\mathbf{C}_y)\big) \big(\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ki}^\lambda(\mathbf{D}_z)\big).$$

Since each ρ^λ is an orthogonal representation, the terms $\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ij}^\lambda(\mathbf{C}_x)$, $\varepsilon^{\boldsymbol{a}_\lambda} \rho_{jk}^\lambda(\mathbf{C}_y)$, $\varepsilon^{\boldsymbol{a}_\lambda} \rho_{ki}^\lambda(\mathbf{D}_z)$ all lie in $\mathcal{O}$. Hence, the whole sum lies in $\mathcal{O}$. First of all, this shows that $\varepsilon^{\tilde{\boldsymbol{a}}(z)} h_{x,y,z^{-1}} \in \mathcal{O} \cap \mathbb{Z}[\Gamma] = \mathbb{Z}[\Gamma_{\geqslant 0}]$ and so $\boldsymbol{a}(z) = \boldsymbol{a}(z^{-1}) \leqslant \tilde{\boldsymbol{a}}(z)$ (where the first equality holds by Remark 2.8). The reverse inequality holds by Lemma 3.4. Thus, we have shown that $\tilde{\boldsymbol{a}}(z) = \boldsymbol{a}(z)$.

Now let us return to the above sum. We have already noted that each term lies in $\mathcal{O}$, hence the constant term of the whole sum above can be computed term by term. Thus, the contant term of $\varepsilon^{\boldsymbol{a}(z)} h_{x,y,z^{-1}}$ equals

$$\sum_{\lambda \in \Lambda \,:\, \boldsymbol{a}_\lambda = \tilde{\boldsymbol{a}}(z)} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} f_\lambda^{-1} \, c_{x,\lambda}^{ij} \, c_{y,\lambda}^{jk} \, c_{z,\lambda}^{ki}.$$

We note that, in fact, the sum can be extended over all $\lambda \in \Lambda$. Indeed, if $c_{z,\lambda}^{ki} \neq 0$ for some λ, k, i, then $\tilde{\boldsymbol{a}}(z) = \boldsymbol{a}_\lambda$ by the definition of $\tilde{\boldsymbol{a}}(z)$. Thus, we have reached the conclusion that

$$\gamma_{x,y,z} = \sum_{\lambda \in \Lambda} \sum_{1 \leqslant i,j,k \leqslant d_\lambda} f_\lambda^{-1} \, c_{x,\lambda}^{ij} \, c_{y,\lambda}^{jk} \, c_{z,\lambda}^{ki}.$$

It remains to notice that the expression on the right hand side is symmetrical under cyclic permutations of x, y, z. This immediately yields that $\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y}$. □

Lemma 3.7. *Assume that* **E1** *holds. Let* $w \in W$*. Then* $\boldsymbol{a}(w) \leqslant \Delta(w)$*. Furthermore,*

$$\sum_{\lambda\in\Lambda}\sum_{1\leqslant i\leqslant d_\lambda} f_\lambda^{-1}\, c_{w,\lambda}^{ii} = \begin{cases} n_w & \text{if } w \in \mathcal{D}, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. We use an argument similar to that in the proof of [13, Lemma 4.6]. First note that $\tau(\mathbf{C}_w) = \overline{p}_{1,w}$. So we obtain the identity

$$\overline{p}_{1,w} = \sum_{\lambda\in\Lambda} \boldsymbol{c}_\lambda^{-1}\,\mathrm{trace}(\rho^\lambda(\mathbf{C}_w)) = \sum_{\lambda\in\Lambda}\sum_{1\leqslant i\leqslant d_\lambda} \frac{f_\lambda^{-1}}{1+g_\lambda}\varepsilon^{\boldsymbol{a}_\lambda}\big(\varepsilon^{\boldsymbol{a}_\lambda}\rho_{ii}^\lambda(\mathbf{C}_w)\big).$$

By Proposition 3.6 and Lemma 3.5(a), we have $\boldsymbol{a}(w) = \tilde{\boldsymbol{a}}(w) \leqslant \boldsymbol{a}_\lambda$ for all non-zero terms in the above sum. Thus, we obtain

$$\varepsilon^{-\boldsymbol{a}(w)}\overline{p}_{1,w} = \sum_{\lambda\in\Lambda\,:\,\boldsymbol{a}(w)\leqslant\boldsymbol{a}_\lambda}\;\sum_{1\leqslant i\leqslant d_\lambda} \frac{f_\lambda^{-1}}{1+g_\lambda}\varepsilon^{\boldsymbol{a}_\lambda-\boldsymbol{a}(w)}\big(\varepsilon^{\boldsymbol{a}_\lambda}\rho_{ii}^\lambda(\mathbf{C}_w)\big).$$

Since each ρ^λ is orthogonal, each term $\varepsilon^{\boldsymbol{a}_\lambda}\rho_{ii}^\lambda(\mathbf{C}_w)$ lies in $\mathcal{O}$. This shows, first of all, that $\varepsilon^{-\boldsymbol{a}(w)}\overline{p}_{1,w} \in \mathcal{O}\cap\mathbb{Z}[\Gamma] = \mathbb{Z}[\Gamma_{\geqslant 0}]$ and so $\boldsymbol{a}(w) \leqslant \Delta(w)$, as required. Furthermore, the constant term of the whole sum can be determined term by term. Thus, we have

$$\varepsilon^{-\boldsymbol{a}(w)}\overline{p}_{1,w} \equiv \sum_{\lambda\in\Lambda\,:\,\boldsymbol{a}(w)=\boldsymbol{a}_\lambda}\;\sum_{1\leqslant i\leqslant d_\lambda} f_\lambda^{-1}c_{w,\lambda}^{ii}.$$

But then the sum can be extended over all $\lambda \in \Lambda$ because we have $c_{w,\lambda}^{ii} = 0$ unless $\boldsymbol{a}(w) = \tilde{\boldsymbol{a}}(w) = \boldsymbol{a}_\lambda$. On the other hand, we have $\varepsilon^{-\boldsymbol{a}(w)}\overline{p}_{1,w} \equiv n_w$ if $\boldsymbol{a}(w) = \Delta(w)$, and $\varepsilon^{-\boldsymbol{a}(w)}\overline{p}_{1,w} \equiv 0$ if $\boldsymbol{a}(w) < \Delta(w)$. □

Corollary 3.8. *Assume that* **E1** *holds. Then* **P1***,* **P4***,* **P7** *and* **P8** *hold. Furthermore, for any* $z \in W$*, we have* $\boldsymbol{a}(z) = \boldsymbol{a}_\lambda$ *where* $\lambda \in \Lambda$ *is such that* $E^\lambda \leftrightsquigarrow_L z$*.*

Proof. By Proposition 3.6, we have $\boldsymbol{a}(z) = \tilde{\boldsymbol{a}}(z)$ and $\gamma_{x,y,z} = \gamma_{y,z,x}$ for all $x, y, z \in W$. Hence, by **E1'** and Lemma 3.7, we have that **P1**, **P4**, **P7** hold. Finally, note that **P8** is a formal consequence of **P7** and Remark 2.8; see [22, 14.8]. □

Remark 3.9. Assume that **E1** holds. Then Proposition 3.6 and Lemma 3.7 show that $\gamma_{x,y,z}$ and n_w ($w \in \mathcal{D}$) can be recovered from the knowledge of the leading matrix coefficients. Consequently, by Remark 2.13, the partition of W into left, right and two-sided cells is completely determined by the leading matrix coefficients.

This leads to a new approach to contructing Lusztig's asymptotic ring **J** and study its representation theory; see [11] for further details.

4. Methods for checking P1–P15

Our aim now is to formulate a set of conditions which, together with **E1** (formulated in the previous section), imply most of the properties **P1**–**P15**. Consider the following properties:

E2. Let $x, y \in W$ and $\lambda, \mu \in \Lambda$ be such that $E^\lambda \leftrightsquigarrow_L x$ and $E^\mu \leftrightsquigarrow_L y$. If $x \leqslant_{\mathcal{LR}} y$ and $\boldsymbol{a}_\lambda = \boldsymbol{a}_\mu$, then $x \sim_{\mathcal{LR}} y$.
E3. Let $x, y \in W$ be such that $x \leqslant_{\mathcal{L}} y$ and $x \sim_{\mathcal{LR}} y$, then $x \sim_{\mathcal{L}} y$.
E4. Let $\mathfrak{C}$ be a left cell of W. Then the function $\mathfrak{C} \to \Gamma_{\geqslant 0}$, $w \mapsto \Delta(w)$, reaches its minimum at exactly one element of $\mathfrak{C}$.

Note that, if **E1** is assumed to hold, then **E2** can be reformulated as follows:

E2'. If $x, y \in W$ are such that $x \leqslant_{\mathcal{LR}} y$ and $\tilde{\boldsymbol{a}}(x) = \tilde{\boldsymbol{a}}(y)$, then $x \sim_{\mathcal{LR}} y$.

Remark 4.1. The relevance of the above set of conditions is explained as follows.

Assume that, for a given group W and weight function $L\colon W \to \Gamma$, we can compute explicitly all polynomials $p_{y,w}$ where $y \leqslant w$ in W and all polynomials $\mu^s_{y,w}$ where $y, w \in W$ and $s \in S$ are such that $sy < y < w < sw$.

Then note that this information alone is sufficient to determine the pre-order relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ and the corresponding equivalence relations. Furthermore, we can construct the representations afforded by the various left cells of W. Finally, the irreducible representations of W and the invariants $\boldsymbol{a}_\lambda$ for $\lambda \in \Lambda$ are explicitly known in all cases. Thus, given the above information alone, we can verify that **E1**–**E4** hold.

Remark 4.2. Assume that **P1**–**P15** hold for W. Then **E1**–**E4** hold for W.

Indeed, by [22, 20.6, 20.7] (whose proofs involve **P1**–**P15**), we have $\boldsymbol{a}(z) = \boldsymbol{a}_\lambda$ if $E^\lambda \leftrightsquigarrow_L z$ (see also Lemma 3.2). Hence **P4** implies **E1** and **P11** implies **E2**. Furthermore, **E3** follows by a combination of **P4** and **P9**. Finally, **E4** follows from **P1** and **P13**, where the minimum of the Δ-function is reached at the unique element of $\mathcal{D}$ contained in a given left cell.

Lemma 4.3. *Assume that* **P1** *holds. Let* $\mathcal{D} = \{d \in W \mid \boldsymbol{a}(d) = \Delta(d)\}$. *Then*

$$\sum_{d\in\mathcal{D}} \gamma_{x^{-1},y,d}\, n_d = \delta_{xy} \qquad \text{for any } x, y \in W.$$

Proof. As in the proof of [22, 14.5], we compute the constant term of $\tau(\mathbf{C}_{x^{-1}}\mathbf{C}_y)$ in two ways. On the one hand, we have $\tau(\mathbf{C}_{x^{-1}}\mathbf{C}_y) \in \delta_{xy} + \mathbb{Z}[\Gamma_{>0}]$; hence $\tau(\mathbf{C}_{x^{-1}}\mathbf{C}_y)$ has constant term δ_{xy}. On the other hand, we have

$$\begin{aligned}\tau(\mathbf{C}_{x^{-1}}\mathbf{C}_y) &= \sum_{z\in W} h_{x^{-1},y,z}\tau(\mathbf{C}_z) = \sum_{z\in W} h_{x^{-1},y,z}\, \overline{p}_{1,z}\\ &= \sum_{z\in W} \varepsilon^{\Delta(z)-\boldsymbol{a}(z)} \left(\varepsilon^{\boldsymbol{a}(z)} h_{x^{-1},y,z}\right) \left(\varepsilon^{-\Delta(z)}\overline{p}_{1,z}\right).\end{aligned}$$

Now, by the definition of $\Delta(z)$, the term $\varepsilon^{-\Delta(z)}\overline{p}_{1,z}$ lies in $\mathbb{Z}[\Gamma_{\geqslant 0}]$ and has constant term n_z. The term $\varepsilon^{\boldsymbol{a}(z)}h_{x^{-1},y,z}$ also lies in $\mathbb{Z}[\Gamma_{\geqslant 0}]$ and has constant term $\gamma_{x^{-1},y,z^{-1}}$. Finally, by **P1**, we have $\boldsymbol{a}(z) \leqslant \Delta(z)$. Hence, the constant term of the whole sum can be computed term by term and we obtain

$$\delta_{xy} = \sum_{z\in W\,:\,\boldsymbol{a}(z)=\Delta(z)} \gamma_{x^{-1},y,z^{-1}}\, n_z.$$

Now, by [22, 5.6], we have $p_{1,z} = p_{1,z^{-1}}$ and so $n_z = n_{z^{-1}}$, $\Delta(z) = \Delta(z^{-1})$. Since we also have $\boldsymbol{a}(z) = \boldsymbol{a}(z^{-1})$ by Remark 2.8, we can rewrite the above expression as

$$\delta_{xy} = \sum_{z\in W\,:\,\boldsymbol{a}(z)=\Delta(z)} \gamma_{x^{-1},y,z}\, n_z = \sum_{d\in\mathcal{D}} \gamma_{x^{-1},y,d}\, n_d,$$

as desired. □

Proposition 4.4. *Assume that* **E1**–**E4** *hold for* W *and all parabolic subgroups of* W. *Then* **P1**–**P14** *hold for* W.

Proof. By Corollary 3.8, we already know that **P1**, **P4**, **P7**, **P8** hold. Now let us consider the remaining properties.

P2 Let $x, y \in W$ and assume that $\gamma_{x^{-1},y,d} \neq 0$ for some $d \in \mathcal{D}$. First we show that d is uniquely determined by this condition. Indeed, let $\mathfrak{C}$ be the left cell containing x. By **P8**, we have $d \sim_{\mathcal{L}} x$, i.e., $d \in \mathfrak{C}$. By **P1**, **P4**, we have $\Delta(d) = \boldsymbol{a}(d) = \boldsymbol{a}(w) \leqslant \Delta(w)$ for all $w \in \mathfrak{C}$. Thus, the Δ-function, restricted to $\mathfrak{C}$, reaches its minimum at d. Now **E4** shows that d is uniquely determined, as claimed.

Hence, the sum in Lemma 4.3 reduces to one term and we have $\gamma_{x^{-1},y,d}n_d = \delta_{xy}$. Since the left hand side is assumed to be non-zero, we deduce that $x = y$.

P3 Let $y \in W$. By Lemma 4.3, there exists some $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$. Arguing as in the proof of **P2**, we see that d is uniquely determined.

P5 is a formal consequence of **P1**, **P3**; see [22, 14.5].

P6 is a formal consequence of **P2**, **P3**; see [22, 14.6].

P9 Let $x, y \in W$ be such that $x \leqslant_{\mathcal{L}} y$ and $\boldsymbol{a}(x) = \boldsymbol{a}(y)$. In particular, we have $x \leqslant_{\mathcal{LR}} y$ and, by **E1** and Proposition 3.6, we have $\tilde{\boldsymbol{a}}(x) = \tilde{\boldsymbol{a}}(y)$. So **E2'** implies that $x \sim_{\mathcal{LR}} y$. Finally, **E3** yields $x \sim_{\mathcal{L}} y$, as required.

P10 is a formal consequence of **P9**; see [22, 14.10].

P11 is a formal consequence of **P4**, **P9**, **P10**; see [22, 14.11].

P12 Since **E1**–**E4** are assumed to hold for W and for W_I, we already know that **P1**–**P11** hold for W and W_I. Now **P12** is a formal consequence of **P3**, **P4**, **P8** for W and W_I; see [22, 14.12].

P13 Let $\mathfrak{C}$ be a left cell. First we show that $\mathfrak{C}$ contains at most one element from $\mathcal{D}$. Let $d \in \mathfrak{C} \cap \mathcal{D}$. By **P1**, **P4**, we have $\Delta(d) = \boldsymbol{a}(d) = \boldsymbol{a}(w) \leqslant \Delta(w)$ for all $w \in \mathfrak{C}$. Thus, the Δ-function (restricted to $\mathfrak{C}$) reaches its minimum at d. So **E4** shows that d is uniquely determined, as claimed.

Now let $x \in \mathfrak{C}$. By Lemma 4.3, there exists some $d \in \mathcal{D}$ such that $\gamma_{x^{-1},x,d} \neq 0$. By **P8**, we have $d \in \mathfrak{C}$ and so $d \in \mathfrak{C} \cap \mathcal{D}$. By the previus argument, $\mathfrak{C} \cap \mathcal{D} = \{d\}$.

P14 is a formal consequence of **P6**, **P13**; see [22, 14.14]. □

Finally, we discuss the remaining property in Lusztig's list which is not covered by the above arguments: property **P15**.

Remark 4.5. Assume that we are in the equal parameter case. Then, by [22, 14.15 and 15.7], **P15** can be deduced once **P4**, **P9** and **P10** are known to hold. Hence, in this case, all of **P1**–**P15** are a consequence of **E1**–**E4**.

The following two results will be useful in dealing with **P15** in the case of unequal parameters.

Remark 4.6. Following [22, 14.15], we can reformulate **P15** as follows. Let $\breve{\Gamma}$ be an isomorphic copy of Γ; then L induces a weight function $\breve{L}\colon W \to \breve{\Gamma}$. Let $\breve{\mathbf{H}} = \mathbf{H}_{\breve{A}}(W,S,\breve{L})$ be the corresponding Iwahori–Hecke algebra over $\breve{A} = R[\breve{\Gamma}]$, with parameters $\{\breve{v}_s \mid s \in S\}$. We have a corresponding Kazhdan–Lusztig basis $\{\breve{\mathbf{C}}_w \mid w \in W\}$. We shall regard A and $\breve{A}$ as subrings of $\mathcal{A} = R[\Gamma \oplus \breve{\Gamma}]$. By extension of scalars, we obtain $\mathcal{A}$-algebras $\mathbf{H}_{\mathcal{A}} = \mathcal{A} \otimes_A \mathbf{H}$ and $\breve{\mathbf{H}}_{\mathcal{A}} = \mathcal{A} \otimes_{\breve{A}} \breve{\mathbf{H}}$. Let $\mathcal{E}$ be the free $\mathcal{A}$-module with basis $\{e_w \mid w \in W\}$. We have an obvious left $\mathbf{H}_{\mathcal{A}}$-module structure and an obvious right $\breve{\mathbf{H}}_{\mathcal{A}}$-module structure on $\mathcal{E}$ (induced by left and right multiplication). Now consider the following condition, where $s,t \in S$ and $w \in W$:

$$(\mathbf{C}_s.e_w).\breve{\mathbf{C}}_t - \mathbf{C}_s.(e_w.\breve{\mathbf{C}}_t) = \text{combination of } e_y \text{ where } y \leqslant_{\mathcal{LR}} w,\ y \not\sim_{\mathcal{LR}} w. \qquad (*)$$

As remarked in [22, 14.15], $(*)$ is already known to hold if $sw < w$ or $wt < w$. Hence, it is sufficient to consider $(*)$ for the cases where both $sw > w$ and $wt > w$.

The discussion in [22, 14.15] shows that **P15** is equivalent to $(*)$, provided that **P4**, **P11** are already known to hold.

By looking at the proof of Theorems 2.9, one notices that it only requires a property which looks weaker than **P15**; we called this property **P15'** in [10, §5]. The following result shows that, in fact, **P15** is equivalent to **P15'**.

Lemma 4.7. *Assume that* **P1**, **P4**, **P7**, **P8** *hold. Then* **P15** *is equivalent to the following property*

P15′. *If* $x, x', y, w \in W$ *satisfy* $\boldsymbol{a}(w) = \boldsymbol{a}(y)$, *then*

$$\sum_{u \in W} \gamma_{w,x',u^{-1}}\, h_{x,u,y} = \sum_{u \in W} h_{x,w,u}\, \gamma_{u,x',y^{-1}}.$$

Note that, on both sides, the sum needs only be extended over all $u \in W$ *such that* $\boldsymbol{a}(u) = \boldsymbol{a}(w) = \boldsymbol{a}(y)$ *(thanks to* **P4***).*

Proof. First note that **P15′** appears in [22, 18.9(b)], where it is deduced from **P4**, **P15**. Now we have to show that, conversely, **P1**, **P4**, **P7**, **P8** and **P15′** imply **P15**. First we claim that **P15′** implies the following statement (which appears in [22, 18.10]):

If $x, y, y' \in W$ are such that $\boldsymbol{a}(y) = \boldsymbol{a}(y')$, then

$$h_{x,y',y} = \sum_{\substack{d \in \mathcal{D}, z \in W \\ \boldsymbol{a}(d)=\boldsymbol{a}(z)}} h_{x,d,z}\, n_d\, \gamma_{z,y',y^{-1}}. \tag{$*$}$$

To see this, note that on the right hand side, we may replace the condition $\boldsymbol{a}(d) = \boldsymbol{a}(z)$ by the condition $\boldsymbol{a}(d) = \boldsymbol{a}(y')$; see **P4**, **P8**. Using also **P15′** (where $w = d \in \mathcal{D}$ and x' is replaced by y'), we see that the right hand side of $(*)$ equals

$$\sum_{d \in \mathcal{D} \,:\, \boldsymbol{a}(d)=\boldsymbol{a}(y')} n_d \Big(\sum_{z \in W} h_{x,d,z}\, \gamma_{z,y',y^{-1}} \Big) = \sum_{d \in \mathcal{D} \,:\, \boldsymbol{a}(d)=\boldsymbol{a}(y')} n_d \Big(\sum_{z \in W} \gamma_{d,y',z^{-1}}\, h_{x,z,y} \Big).$$

Now $\gamma_{d,y',z^{-1}} = 0$ unless $\boldsymbol{a}(d) = \boldsymbol{a}(y')$; see **P8**, **P4**. Using also **P7** and Lemma 4.3, the right hand side of the above equation can be rewritten as

$$\sum_{z \in W} h_{x,z,y} \Big(\sum_{d \in \mathcal{D}} \gamma_{y',z^{-1},d}\, n_d \Big) = \sum_{z \in W} h_{x,z,y}\, \delta_{zy'} = h_{x,y',y}.$$

Thus, $(*)$ is proved.

Now consider the left hand side in **P15** where $x, x,' y, w \in W$ are such that $a := \boldsymbol{a}(w) = \boldsymbol{a}(y)$. If $h_{w,x',y'} \neq 0$ then $y' \leqslant_{\mathcal{R}} w$ and so $a = \boldsymbol{a}(w) \leqslant \boldsymbol{a}(y')$ by **P4**; similarly, if $h_{x,y',y} \neq 0$, then $y \leqslant_{\mathcal{L}} y'$ and so $\boldsymbol{a}(y') \leqslant \boldsymbol{a}(y) = a$. Hence, $\boldsymbol{a}(y') = a$, and so we may assume that the sum only runs over all $y' \in W$ such that $\boldsymbol{a}(y') = a$. Inserting now $(*)$ into the left hand side of **P15**, we obtain the expression

$$\sum_{\substack{y' \in W \\ \boldsymbol{a}(y')=a}} \sum_{\substack{d \in \mathcal{D}, z \in W \\ \boldsymbol{a}(d)=\boldsymbol{a}(z)}} \gamma_{z,y',y^{-1}}\, h_{w,x',y'} \otimes h_{x,d,z}\, n_d$$

$$= \sum_{\substack{d \in \mathcal{D}, z \in W \\ \boldsymbol{a}(d)=\boldsymbol{a}(z)}} \Big(\sum_{\substack{y' \in W \\ \boldsymbol{a}(y')=a}} \gamma_{z,y',y^{-1}}\, h_{w,x',y'} \Big) \otimes h_{x,d,z}\, n_d.$$

Now, using Remark 2.8 and **P15′**, we can rewrite the interior sum as follows:

$$\sum_{\substack{y' \in W \\ \boldsymbol{a}(y')=a}} \gamma_{z,y',y^{-1}}\, h_{w,x',y'} = \sum_{\substack{y' \in W \\ \boldsymbol{a}(y')=a}} \gamma_{y'^{-1},z^{-1},y}\, h_{x'^{-1},w^{-1},y'^{-1}}$$

$$= \sum_{\substack{u \in W \\ \boldsymbol{a}(u)=a}} h_{x'^{-1},w^{-1},u}\, \gamma_{u,z^{-1},y} = \sum_{\substack{u \in W \\ \boldsymbol{a}(u)=a}} \gamma_{w^{-1},z^{-1},u^{-1}}\, h_{x'^{-1},u,y^{-1}}$$

$$= \sum_{\substack{u \in W \\ \boldsymbol{a}(u)=a}} \gamma_{z,w,u}\, h_{u^{-1},x',y} = \sum_{\substack{u \in W \\ \boldsymbol{a}(u)=a}} \gamma_{z,w,u^{-1}}\, h_{u,x',y}.$$

Inserting this back into the above expression, we find that

$$\sum_{\substack{d\in\mathcal{D},z\in W\\ \boldsymbol{a}(d)=\boldsymbol{a}(z)}} \Big(\sum_{\substack{y'\in W\\ \boldsymbol{a}(y')=a}} \gamma_{z,y',y^{-1}}\, h_{w,x',y'}\Big)\otimes h_{x,d,z}\, n_d$$
$$= \sum_{\substack{d\in\mathcal{D},z\in W\\ \boldsymbol{a}(d)=\boldsymbol{a}(z)}} \Big(\sum_{\substack{u\in W\\ \boldsymbol{a}(u)=a}} \gamma_{z,w,u^{-1}}\, h_{u,x',y}\Big)\otimes h_{x,d,z}\, n_d.$$

Using also $(*)$, we obtain the expression

$$\sum_{\substack{u\in W\\ \boldsymbol{a}(u)=a}} h_{u,x',y}\otimes\Big(\sum_{\substack{d\in\mathcal{D},z\in W\\ \boldsymbol{a}(d)=\boldsymbol{a}(z)}} \gamma_{z,w,u^{-1}}\, h_{x,d,z}\, n_d\Big) = \sum_{\substack{u\in W\\ \boldsymbol{a}(u)=a}} h_{u,x',y}\otimes h_{x,w,u},$$

which is the right hand side of **P15**. Note that, in the right hand side of **P15**, the sum need only be extended over all $y'\in W$ such that $\boldsymbol{a}(y')=a$. (The argument is similar to the one we used to prove the analogous statement for the left hand side.) $\square$

Example 4.8. Assume that (W,S) is of type H_4. Then we are in the equal parameter case. So, in order to verify **P1**–**P15**, it is sufficient to verify **E1**–**E4**; see Remark 4.5. Now Alvis [1] has computed all polynomials $p_{y,w}$ where $y\leqslant w$ in W. Since we are in the equal parameter case, this also determines all polynomials $\mu^s_{y,w}$ where $y,w\in W$ and $s\in S$ are such that $sy<y<w<sw$; see [22, 6.5]. In this way, Alvis explicitly determined the relations $\leqslant_{\mathcal{L}}$ and $\leqslant_{\mathcal{LR}}$; he also found the decomposition of the left cell representations into irreducibles.

It turns out that the partial order induced on the set of two-sided cells is a total order. (I thank Alvis for having verified this using the data in [1].) With the notation in [*loc. cit.*], this total order is given by:

$$G^*\leqslant_{\mathcal{LR}} F^*\leqslant_{\mathcal{LR}} E^*\leqslant_{\mathcal{LR}} D^*\leqslant_{\mathcal{LR}} C^*\leqslant_{\mathcal{LR}} B^*\leqslant_{\mathcal{LR}} A^*$$
$$= A\leqslant_{\mathcal{LR}} B\leqslant_{\mathcal{LR}} C\leqslant_{\mathcal{LR}} D\leqslant_{\mathcal{LR}} E\leqslant_{\mathcal{LR}} F\leqslant_{\mathcal{LR}} G.$$

Comparing with the information on the invariants $\boldsymbol{a}_\lambda$ provided by Alvis–Lusztig [2], we see that **E1** and **E2** hold. Furthermore, **E3** is already explicitly stated in [1, Cor. 3.3]. Finally, **E4** is readily checked using Alvis' computation of the left cells and the polynomials $p_{y,w}$.

In this way, we obtain an alternative proof of **P1**–**P15** for H_4, which does not rely on DuCloux's computation [6] of all structure constants $h_{x,y,z}$ $(x,y,z\in W)$.

Similar arguments can of course also be applied to (W,S) of type H_3.

5. Lusztig's homomorphism

We now use the methods developped in the previous section to verify **P1**–**P15** for type F_4 and $I_2(m)$. Then we are in a position to extend the construction of Lusztig's isomorphism to the general case of unequal parameters.

Proposition 5.1. *Let $3 \leqslant m < \infty$ and (W, S) be of type $I_2(m)$, with generators s_1, s_2 such that $(s_1 s_2)^m = 1$. Then* **P1**–**P15** *hold for any weight function $L\colon W \to \Gamma$ and any monomial order $\leqslant$ such that $L(s_i) > 0$ for $i = 1, 2$.*

Proof. If $L(s_1) = L(s_2)$, this is proved by DuCloux [6], following the approach in [22, 17.5] (concerning the infinite dihedral group). Now assume that $L(s_1) \neq L(s_2)$; in particular, $m \geqslant 4$ is even. Without loss of generality, we can assume that $L(s_1) > L(s_2)$. It is probably possible to use arguments similar to those in [6] and [22, 17.5] (which essentially amount to computing all structure constants $h_{x,y,z}$). However, in the present case, it is rather straightforward to verify **E1**–**E4**. Indeed, by [14, §5.4], we have

$$\mathrm{Irr}(W) = \{1_W, \varepsilon, \varepsilon_1, \varepsilon_2, \rho_1, \rho_2, \ldots, \rho_{(m-2)/2)}\},$$

where 1_W is the trivial representation, ε is the sign representation, $\varepsilon_1, \varepsilon_2$ are two further 1-dimensional representations, and all ρ_j are 2-dimensional. We fix the notation such that s_1 acts as $+1$ in ε_1 and as -1 in ε_2. Using [14, 8.3.4], we find

$$\begin{aligned}
\boldsymbol{a}_{1_W} &= 0, & f_{1_W} &= 1, \\
\boldsymbol{a}_{\varepsilon_1} &= L(s_2), & f_{\varepsilon_1} &= 1, \\
\boldsymbol{a}_{\rho_j} &= L(s_1), & f_{\rho_j} &= \frac{m}{2 - \zeta^{2j} - \zeta^{-2j}} \qquad \text{for all } j, \\
\boldsymbol{a}_{\varepsilon_2} &= \frac{m}{2}\big(L(s_1) - L(s_2)\big) + L(s_2), & f_{\varepsilon_2} &= 1, \\
\boldsymbol{a}_{\varepsilon} &= \frac{m}{2}\big(L(s_1) + L(s_2)\big) & f_{\varepsilon} &= 1;
\end{aligned}$$

where $\zeta \in \mathbb{C}$ is a root of unity of order m. Observe that, in the above list, the $\boldsymbol{a}$-values are in strictly increasing order from top to bottom.

Now, by [22, 6.6, 7.5, 7.6] and [14, Exc. 11.3], we have the following multiplication rules for the Kazhdan–Lusztig basis. For any $k \geqslant 0$, write $1_k = s_1 s_2 s_1 \cdots$ (k factors) and $2_k = s_2 s_1 s_2 \cdots$ (k factors). Given $k, l \in \mathbb{Z}$, we define $\delta_{k>l}$ to be 1 if $k > l$ and to be 0 otherwise. Then

$$\begin{aligned}
\mathbf{C}_{1_1}\mathbf{C}_{1_{k+1}} &= (v_{s_1} + v_{s_1}^{-1})\mathbf{C}_{1_{k+1}},\\
\mathbf{C}_{2_1}\mathbf{C}_{2_{k+1}} &= (v_{s_2} + v_{s_2}^{-1})\mathbf{C}_{2_{k+1}},\\
\mathbf{C}_{2_1}\mathbf{C}_{1_k} &= \mathbf{C}_{2_{k+1}},\\
\mathbf{C}_{1_1}\mathbf{C}_{2_k} &= \mathbf{C}_{1_{k+1}} + \delta_{k>1}\zeta\mathbf{C}_{1_{k-1}} + \delta_{k>3}\mathbf{C}_{1_{k-3}},
\end{aligned}$$

for any $0 \leqslant k < m$, where $\zeta = v_{s_1}v_{s_2}^{-1} + v_{s_1}^{-1}v_{s_2}$. Using this information, the pre-order relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$ are easily and explicitly determined; see [22, 8.8]. The two-sided cells and the partial order on them are given by

$$\{1_m\} \leqslant_{\mathcal{LR}} \{1_{m-1}\} \leqslant_{\mathcal{LR}} W \setminus \{1_0, 2_1, 1_{m-1}, 1_m\} \leqslant_{\mathcal{LR}} \{2_1\} \leqslant_{\mathcal{LR}} \{1_0\}. \quad (\heartsuit)$$

The set $W \setminus \{1_0, 2_1, 1_{m-1}, 1_m\}$ consists of two left cells, $\{1_1, 2_2, 1_3, \ldots, 2_{m-2}\}$ and $\{1_2, 2_3, 1_4, \ldots, 2_{m-1}\}$, but these are not related by $\leqslant_{\mathcal{L}}$. (If they were, then, by [22, 8.6], the right descent set of the elements in one of them would have to be contained in the right descent set of the elements in the other one—which is not the case.) The other two-sided cells are just left cells. In particular, we see that **E3** holds.

Now we can also construct the representations given by the various left cells and decompose them into irreducibles; we obtain:

$$\begin{aligned}
\{1_0\} &\quad \text{affords} \quad 1_W,\\
\{2_1\} &\quad \text{affords} \quad \varepsilon_1,\\
\{1_1, 2_2, 1_3, \ldots, 2_{m-2}\} &\quad \text{affords} \quad \rho_1 + \rho_2 + \cdots + \rho_{(m-2)/2},\\
\{1_2, 2_3, 1_4, \ldots, 2_{m-1}\} &\quad \text{affords} \quad \rho_1 + \rho_2 + \cdots + \rho_{(m-2)/2},\\
\{1_{m-1}\} &\quad \text{affords} \quad \varepsilon_2,\\
\{1_m\} &\quad \text{affords} \quad \varepsilon.
\end{aligned}$$

Using this list and the above information on the $\boldsymbol{a}$-values and the partial order on the two-sided cells, we see that **E1** and **E2** hold.

Next, by [22, 7.4, 7.6] and [14, Exc. 11.3], the polynomials $p_{y,w}$ are explicitly known. Thus, we can determine the function $w \mapsto \Delta(w)$. We obtain

$$\begin{aligned} \Delta(1_{2k}) = \Delta(2_{2k}) &= kL(s_1) + kL(s_2) \quad \text{if } k \geqslant 0, \\ \Delta(2_1) &= L(s_2) \\ \Delta(1_{2k+1}) &= (k+1)L(s_1) - kL(s_2) \quad \text{if } k \geqslant 0 \\ \Delta(2_{2k+1}) &= kL(s_1) + (k-1)L(s_2) \quad \text{if } k \geqslant 1. \end{aligned}$$

Thus, we see that **E4** holds. In the left cell $\{1_1, 2_2, 1_3, \ldots, 2_{m-2}\}$, the function Δ reaches its minimum at 1_1; in the left cell $\{1_2, 2_3, 1_4, \ldots, 2_{m-1}\}$, the minimum is reached at 2_3. We see that

$$\mathcal{D} = \{1_0,\ 2_1,\ 1_1,\ 2_3,\ 1_{m-1},\ 1_m\},$$
$$n_{1_0} = n_{2_1} = n_{1_1} = n_{2_3} = n_{1_m} = +1,\ n_{1_{m-1}} = -(-1)^{m/2}.$$

Thus, we have verified that **E1**–**E4** hold for W. We also know that **P1**–**P15** hold for every proper parabolic subgroup of W. (Note that the only proper parabolic subgroups of W are $\langle s_1 \rangle$ and $\langle s_2 \rangle$.) Hence, by Remark 4.2 and Proposition 4.4, we can conclude that **P1**–**P14** hold for W.

It remains to verify **P15**. For this purpose, we must check that condition $(*)$ in Remark 4.6 holds for all $w \in W$ and $i, j \in \{1, 2\}$ such that $s_i w > w$, $ws_j > w$. A similar verification is done by Lusztig [22, 17.5] for the infinite dihedral group. We notice that the same arguments also work in our situation if w is such that we do not encounter the longest element $w_0 = 1_m = 2_m$ in the course of the verification. This certainly is the case if $l(w) < m - 2$. Thus, we already know that $(*)$ holds when $l(w) < m - 2$. It remains to verify $(*)$ when $l(w)$ equals $m - 2$ or $m - 1$, that is, when $w \in \{1_{m-2}, 2_{m-2}, 1_{m-1}, 2_{m-1}\}$.

Assume first that $w = 1_{m-2}$. The left descent set of w is $\{s_1\}$ and, since m is even, the right descent set of w is $\{s_2\}$. So we must check $(*)$ with $s = s_2$ and $t = s_1$. Using the above multiplication formulas, we find:

$$(\mathbf{C}_{2_1}.e_{1_{m-2}}).\breve{\mathbf{C}}_{1_1} = e_{2_{m-1}}.\breve{\mathbf{C}}_{1_1}.$$

Now, since m is even, $\{s_2\}$ is the right descent set of 1_{m-1}. Hence right-handed versions of the above multiplication rules imply that

$$(\mathbf{C}_{2_1}.e_{1_{m-2}}).\breve{\mathbf{C}}_{1_1} = e_{2_{m-1}}.\breve{\mathbf{C}}_{1_1} = e_{2_m} + \delta_{m>2}\breve{\zeta} e_{2_{m-2}} + \delta_{m>4} e_{2_{m-4}},$$

where $\breve{\zeta} = \breve{v}_{s_1}\breve{v}_{s_2}^{-1} + \breve{v}_{s_1}^{-1}\breve{v}_{s_2}$. On the other hand, we have

$$\begin{aligned}\mathbf{C}_{2_1}.(e_{1_{m-2}}.\breve{\mathbf{C}}_{1_1}) &= \mathbf{C}_{2_1}.\big(e_{1_{m-1}} + \delta_{m>3}\breve{\zeta}e_{1_{m-3}} + \delta_{m>5}e_{2_{m-5}}\big)\\ &= e_{2_m} + \delta_{m>3}\breve{\zeta}e_{2_{m-2}} + \delta_{m>5}e_{2_{m-4}}.\end{aligned}$$

Now note that, since m is even, we have $\delta_{m>3} = \delta_{m>2}$ and $\delta_{m>4} = \delta_{m>5}$. Hence, we actually see that the expression in $(*)$ is zero.

Now assume that $w = 2_{m-2}$. Then we must check $(*)$ with $s = s_1$ and $t = s_2$. Arguing as above, we find that

$$\begin{aligned}(\mathbf{C}_{1_1}.e_{2_{m-2}}).\breve{\mathbf{C}}_{2_1} &= \big(e_{1_{m-1}} + \delta_{m>3}\zeta e_{1_{m-3}} + \delta_{m>5}e_{m-5}\big).\breve{\mathbf{C}}_{2_1}\\ &= e_{1_m} + \delta_{m>3}\zeta e_{1_{m-2}} + \delta_{m>5}e_{m-4},\\ \mathbf{C}_{1_1}.(e_{2_{m-2}}.\breve{\mathbf{C}}_{2_1}) &= \mathbf{C}_{1_1}.e_{2_{m-1}} = e_{1_m} + \delta_{m>2}\zeta e_{1_{m-2}} + \delta_{m>4}e_{m-4}.\end{aligned}$$

Again, we see that the difference of these two expressions is zero.

Next, let $w = 1_{m-1}$. Then we must check $(*)$ with $s = t = s_2$. We obtain

$$\begin{aligned}(\mathbf{C}_{2_1}.e_{1_{m-1}}).\breve{\mathbf{C}}_{2_1} &= e_{2_m}.\breve{\mathbf{C}}_{2_1} = (\breve{v}_{s_2} + \breve{v}_{s_2}^{-1})e_{2_m},\\ \mathbf{C}_{2_1}.(e_{1_{m-1}}.\breve{\mathbf{C}}_{2_1}) &= \mathbf{C}_{2_1}.e_{1_m} = (v_{s_2} + v_{s_2}^{-1})e_{2_m}.\end{aligned}$$

Hence the difference of these two expressions is a scalar multiple of e_{2_m}. The description of $\leqslant_{\mathcal{LR}}$ in $(\heartsuit)$ now shows that $(*)$ holds.

Finally, let $w = 2_{m-1}$. Then we must check $(*)$ with $s = t = s_1$. We find

$$(\mathbf{C}_{1_1}.e_{2_{m-1}}).\breve{\mathbf{C}}_{1_1} = \big(e_{1_m} + \delta_{m>2}\zeta e_{1_{m-2}} + \delta_{m>4}e_{1_{m-4}}\big).\breve{\mathbf{C}}_{1_1}.$$

Furthermore, we obtain:

$$\begin{aligned}e_{1_m}.\breve{\mathbf{C}}_{1_1} &= (\breve{v}_{s_1} + \breve{v}_{s_1}^{-1})e_{1_m},\\ e_{1_{m-2}}.\breve{\mathbf{C}}_{1_1} &= e_{1_{m-1}} + \delta_{m>3}\breve{\zeta}e_{1_{m-3}} + \delta_{m>5}e_{1_{m-5}},\\ e_{1_{m-4}}.\breve{\mathbf{C}}_{1_1} &= e_{1_{m-3}} + \delta_{m>5}\breve{\zeta}e_{1_{m-5}} + \delta_{m>7}e_{1_{m-7}}.\end{aligned}$$

Inserting this into the above expression, we obtain

$$\begin{aligned}(\mathbf{C}_{1_1}.e_{2_{m-1}}).\breve{\mathbf{C}}_{1_1} &= (\breve{v}_{s_1} + \breve{v}_{s_1}^{-1})e_{1_m} + \delta_{m>2}\zeta e_{1_{m-1}}\\ &\quad + (\delta_{m>3}\zeta\breve{\zeta} + \delta_{m>4})e_{1_{m-3}} + (\zeta + \breve{\zeta})\delta_{m>5}e_{1_{m-5}} + \delta_{m>7}e_{1_{m-7}}.\end{aligned}$$

A similar computation yields

$$\begin{aligned}\mathbf{C}_{1_1}.(e_{2_{m-1}}.\check{\mathbf{C}}_{1_1}) &= (v_{s_1}+v_{s_1}^{-1})e_{1_m}+\delta_{m>2}\check{\zeta}e_{1_{m-1}}\\ &\quad+(\delta_{m>3}\zeta\check{\zeta}+\delta_{m>4})e_{1_{m-3}}+(\zeta+\check{\zeta})\delta_{m>5}e_{1_{m-5}}+\delta_{m>7}e_{1_{m-7}}\end{aligned}$$

and so

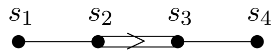
$$(\mathbf{C}_{1_1}.e_{2_{m-1}}).\check{\mathbf{C}}_{1_1}-\mathbf{C}_{1_1}.(e_{2_{m-1}}.\check{\mathbf{C}}_{1_1}) = (\check{v}_{s_1}+\check{v}_{s_1}^{-1}-v_{s_1}-v_{s_1}^{-1})e_{1_m}+\delta_{m>2}(\zeta-\check{\zeta})e_{1_{m-1}}.$$

The description of $\leqslant_{\mathcal{LR}}$ in (♡) now shows that $(*)$ holds.

Thus, we have verified that **P15** holds. □

Proposition 5.2. *Let (W,S) be of type F_4 with generators and diagram given by:*

F_4



Then **P1**–**P15** *hold for any weight function $L\colon W\to\Gamma$ and any monomial order $\leqslant$ such that $L(s_i)>0$ for $i=1,2,3,4$.*

Proof. The weight function L is specified by $a := L(s_1) = L(s_2) > 0$ and $b := L(s_3) = L(s_4) > 0$. We may assume without loss of generality that $b \geqslant a$. The preorder relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ and the corresponding equivalence relations on W have been determined in [8], based on an explicit computation of all the polynomials $p_{y,w}$ (where $y\leqslant w$ in W) and all polynomials $\mu^s_{y,w}$ (where $s\in S$ and $sy<y<w<sw$) using CHEVIE [12]. (The programs are available upon request.) Once all this information is available, it is also a straightforward matter to check that condition $(*)$ in Remark 4.6 is satisfied, that is, **P15** holds. Furthermore, **E3** and **E4** are explicitly stated in [8].

To check **E1** and **E2**, it is sufficient to use the information contained in Table 1 (which is taken from [10, p. 318]) and Table 2 (which is taken from [8, p. 362]). In these tables, the irreducible representations of W are denoted by d_i where d is the dimension and i is an additional index; for example, 1_1 is the trivial representation, 1_4 is the sign representation and 4_2 is the reflection representation.

Thus, by Proposition 4.4, **P1**–**P14** hold for W. (Note that, using similar computational methods, **E1**–**E4** are easily verified for all proper parabolic subgroups.) □

Table 1. The invariants f_λ and $\boldsymbol{a}_\lambda$ for type F_4

	$b>2a>0$		$b=2a>0$		$2a>b>a>0$		$b=a>0$	
E^λ	f_λ	$\boldsymbol{a}_\lambda$	f_λ	$\boldsymbol{a}_\lambda$	f_λ	$\boldsymbol{a}_\lambda$	f_λ	$\boldsymbol{a}_\lambda$
1_1	1	0	1	0	1	0	1	0
1_2	1	$12b-9a$	2	$15a$	1	$11b-7a$	8	$4a$
1_3	1	$3a$	2	$3a$	1	$-b+5a$	8	$4a$
1_4	1	$12b+12a$	1	$36a$	1	$12b+12a$	1	$24a$
2_1	1	$3b-3a$	2	$3a$	1	$2b-a$	2	a
2_2	1	$3b+9a$	2	$15a$	1	$2b+11a$	2	$13a$
2_3	1	a	1	a	1	a	2	a
2_4	1	$12b+a$	1	$25a$	1	$12b+a$	2	$13a$
4_1	2	$3b+a$	2	$7a$	2	$3b+a$	8	$4a$
9_1	1	$2b-a$	2	$3a$	1	$b+a$	1	$2a$
9_2	1	$6b-2a$	1	$10a$	1	$6b-2a$	8	$4a$
9_3	1	$2b+2a$	1	$6a$	1	$2b+2a$	8	$4a$
9_4	1	$6b+3a$	2	$15a$	1	$5b+5a$	1	$10a$
6_1	3	$3b+a$	3	$7a$	3	$3b+a$	3	$4a$
6_2	3	$3b+a$	3	$7a$	3	$3b+a$	12	$4a$
12_1	6	$3b+a$	6	$7a$	6	$3b+a$	24	$4a$
4_2	1	b	1	$2a$	1	b	2	a
4_3	1	$7b-3a$	1	$11a$	1	$7b-3a$	4	$4a$
4_4	1	$b+3a$	1	$5a$	1	$b+3a$	4	$4a$
4_5	1	$7b+6a$	1	$20a$	1	$7b+6a$	2	$13a$
8_1	1	$3b$	1	$6a$	1	$3b$	1	$3a$
8_2	1	$3b+6a$	1	$12a$	1	$3b+6a$	1	$9a$
8_3	1	$b+a$	2	$3a$	1	$3a$	1	$3a$
8_4	1	$7b+a$	2	$15a$	1	$6b+3a$	1	$9a$
16_1	2	$3b+a$	2	$7a$	2	$3b+a$	4	$4a$

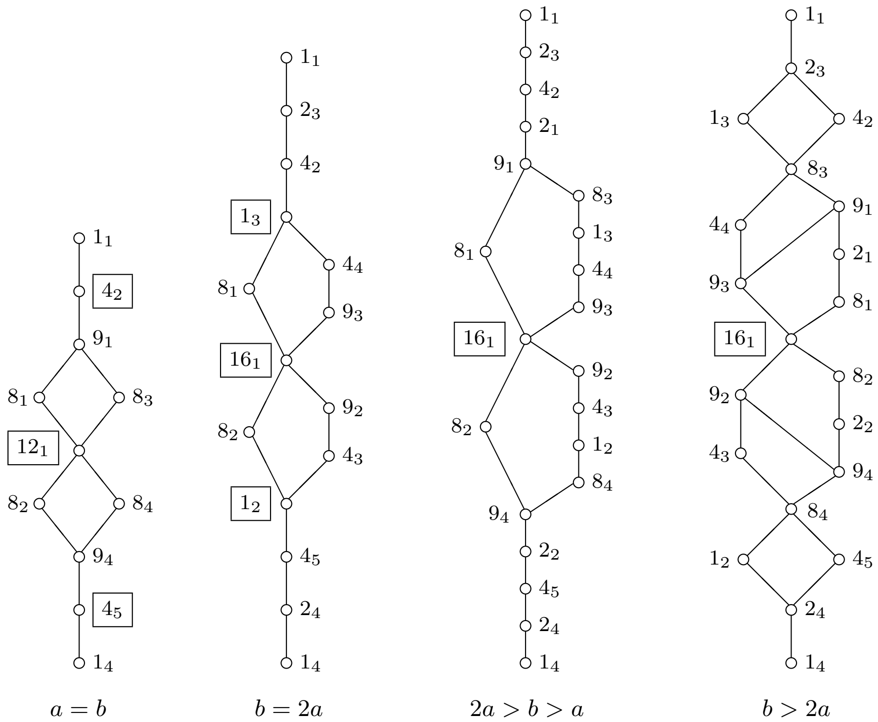
(This table corrects some errors concerning f_λ in [10, Table 1].)

Theorem 5.3. *Lusztig's conjectures* **P1**–**P15** *hold in the following cases.*

(a) *The* equal parameter case *where $\Gamma = \mathbb{Z}$ and $L(s) = a > 0$ for all $s \in S$ (where a is fixed).*

(b) *(W, S) of type B_n, F_4 or $I_2(m)$ (m even), with weight function $L\colon W \to \Gamma$ given by:*

TABLE 2. Partial order on two-sided cells in type F_4

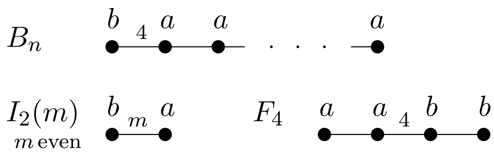


A box indicates a two-sided cell with several irreducible components, given as follows:

$\boxed{4_2} = \{2_1, 2_3, 4_2\}$, $\boxed{4_5} = \{2_2, 2_4, 4_5\}$, $\boxed{1_3} = \{1_3, 2_1, 8_3, 9_1\}$, $\boxed{1_2} = \{1_2, 2_2, 8_4, 9_4\}$,

$\boxed{12_1} = \{1_2, 1_3, 4_1, 4_3, 4_4, 6_1, 6_2, 9_2, 9_3, 12_1, 16_1\}$, $\boxed{16_1} = \{4_1, 6_1, 6_2, 12_1, 16_1\}$.

Otherwise, the two-sided cell contains just one irreducible respresentation.



where $a, b \in \Gamma_{>0}$ are such that $b > ra$ for all $r \in \mathbb{Z}_{\geqslant 1}$.

Proof. (a) See Remark 2.7. (b) For types $I_2(m)$ (m even) and F_4, see Propositions 5.1 and 5.2. Now let W be of type B_n with parameters as above. The left, right and two-sided cells are explicitly determined by Bonnafé and Iancu [4], [3]. A special feature of this case is that all left cells give rise to irreducible representations of W; see [4, Prop. 7.9]; furthermore, two left cells give rise to isomorphic irreducible representations of W if and only if they contained in the same two-sided cell; see [3, §3]. Based on these results, it is shown in [13, Theorem 1.3] that **P1**–**P15** hold except possibly **P9**, **P10**, **P15**. In [9, Theorem 5.13], the following implication is shown for all $x, y \in W$:

$$x \sim_{\mathcal{LR}} y \quad \text{and} \quad x \leqslant_{\mathcal{L}} y \quad \Rightarrow \quad x \sim_{\mathcal{L}} y. \tag{$\heartsuit$}$$

This then yields **P9**, **P10**; see [9, Cor. 7.12]. Finally, **P15**$'$ is shown in [9, Prop. 7.6] under the additional assumption that $y \sim_{\mathcal{L}} x' \sim_{\mathcal{R}} w^{-1}$. However, if this additional assumption is not satisfied, then one easily sees, using **P9** and **P10**, that both sides of **P15**$'$ are zero. Thus, **P15**$'$ holds in general and then Lemma 4.7 is used to deduce that **P15** also holds. □

Corollary 5.4. *Assume that W is finite and let $L_0 \colon W \to \Gamma_0$ be the "universal" weight function of Remark 2.3. Then* **P1**–**P15** *hold for at least one monomial order on Γ_0 where $L_0(s) > 0$ for all $s \in S$.*

Proof. By standard reduction arguments, we can assume that (W, S) is irreducible. If (W, S) if of type B_n, F_4 or $I_2(m)$ (m even), we choose a monomial order as in Theorem 5.3(b). Otherwise, we are automatically in the equal parameter case. Hence **P1**–**P15** hold by Theorem 5.3(a). □

Finally, we can show that Theorem 2.12 holds without using the hypothesis that **P1**–**P15** are satisfied!

Corollary 5.5. *Let $R \subseteq \mathbb{C}$ be a field. Then the statements in Theorem 2.12 hold for any weight function $L \colon W \to \Gamma$ where Γ is an abelian group such that $A = R[\Gamma]$ is an integral domain.*

Note that this implies Theorem 1.1, as stated in the introduction.

Proof. Let Γ_0, A_0 and $\mathbf{H}_0$ be as in Remark 2.3. To distinguish A_0 from A, let us write the elements of A_0 as R-linear combinations of ε_0^g where $g \in \Gamma_0$. By Corollary 5.4, we can choose a monomial order $\leqslant$ on Γ_0 such that **P1**–**P15** hold. Let $\psi_0 \colon \mathbf{H}_0 \to A_0[W]$ be the corresponding homomorphism of Theorem 2.12.

Let Q_0 be the matrix of the A_0-linear map ψ_0 with respect to the standard A_0-bases of $\mathbf{H}_0$ and $A_0[W]$. Let $\theta_0 \colon A_0 \to R$ be the unique ring homomorphism such that $\theta_0(\varepsilon_0^g) = 1$ for all $g \in \Gamma_0$. We denote by $\theta_0(Q_0)$ the matrix obtained by applying θ_0 to all entries of Q_0. By Theorem 2.12, $\theta_0(Q_0)$ is the identity matrix.

Now, there is a group homomorphism $\alpha\colon \Gamma_0 \to \Gamma$ such that $\alpha((n_s)_{s\in S}) = \sum_{s\in S} n_s L(s)$. This extends to a ring homomorphism $A_0 \to A$ which we denote by the same symbol. Extending scalars from A_0 to A (via α), we obtain $\mathbf{H} = A \otimes_{A_0} \mathbf{H}_0$ and $A[W] = A \otimes_{A_0} A_0[W]$. Furthermore, ψ_0 induces an algebra homomorphism $\bar{\psi}_0\colon \mathbf{H} \to A[W]$. Let $Q := \alpha(Q_0)$ be the matrix obtained by applying α to all entries of Q_0. Then, clearly, Q is the matrix of the A-linear map $\bar{\psi}_0$ with respect to the standard A-bases of $\mathbf{H}$ and $A[W]$.

Let $\theta_1\colon A \to R$ be the unique ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. As in the proof of Theorem 2.12, it remains to show that, if we apply θ_1 to all entries of Q, then we obtain the identity matrix. But, we certainly have $\theta_0 = \theta_1 \circ \alpha$ and, hence, $\theta_1(Q) = \theta_1(\alpha(Q_0)) = \theta_0(Q_0)$. So it remains to recall that the latter matrix is the identity matrix. □

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Meinolf Geck
Department of Mathematical Sciences
King's College, Aberdeen AB24 3UE
Scotland, UK
Email: m.geck@abdn.ac.uk