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Can An Anisotropic Reductive Group Admit A Tits System?

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Dedicated to Jacques Tits in honour of his 80th birthday

Abstract: Seeking for a converse to a well-known theorem by Borel–Tits, we address the question whether the group of rational points $G(k)$ of an anisotropic reductive k -group can admit a split spherical BN-pair. We show that if k is a perfect field or a local field, then such a BN-pair must be virtually trivial. We also consider arbitrary compact groups and show that the only abstract BN-pairs they can admit are spherical, and even virtually trivial provided they are split.

Keywords: BN-pair, building, anisotropic reductive group, compact group

1. INTRODUCTION

In a seminal paper [5], Armand Borel and Jacques Tits established — amongst other things — that the group $G(k)$ of k -rational points of a (connected) reductive linear algebraic k -group G always possesses a canonical BN-pair, where k is an arbitrary ground field. More precisely, they showed that if P is a minimal parabolic k -subgroup of G , and if N is the normalizer in G of some maximal k -split

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torus contained in P , then $(P(k), N(k))$ is a BN-pair for $G(k)$. This result constitutes a cornerstone in understanding the abstract group structure of the group of k -rational points $G(k)$. As an application, it yields for example the celebrated simplicity result of Tits [19]. Of course, the aforementioned BN-pair is trivial when G is **anisotropic** over k . (Abusing slightly the standard conventions, we shall say that G is anisotropic if it has no proper k -parabolic subgroup, *i.e.* if $P = G$. As is well-known, this definition coincides with the standard one in case G is semi-simple (see [4, 11.21])). In fact, the abstract group structure of $G(k)$ remains intriguing and mysterious to a large extent in the anisotropic case. In this context, we propose the following.

Conjecture (Converse to Borel–Tits). *Let G be a reductive algebraic k -group which is anisotropic over k . Then every split spherical BN-pair for $G(k)$ is virtually trivial.*

Recall that a BN-pair (B, N) for a group G is called **spherical** if the associated Weyl group $W := N/T$ is finite, where $T := B \cap N$. It is said to be **split** if it is saturated (*i.e.* $T = \bigcap_{w \in W} wBw^{-1}$), and if there exists a nilpotent normal subgroup $U \triangleleft B$ such that $B \cong U \rtimes T$. This implies that the associated building enjoys the Moufang property (see *e.g.* [11]). The BN-pair is called **virtually trivial** if the associated building is finite or, equivalently, if B has finite index in G . The BN-pair $(P(k), N(k))$ for $G(k)$ described above is always split ([4, 14.19]). It is virtually trivial if and only if either k is finite or G is k -anisotropic. In particular, over infinite ground fields the conjecture can really be thought as a converse to the Borel–Tits theorem.

Besides the natural search for a converse to Borel–Tits, a motivation to consider the above conjecture is provided by the recent work of Peter Abramenko and Ken Brown [1], who constructed Weyl transitive actions on trees for certain anisotropic groups over global function fields. We refer to [2, Ch. 6] for more details on the relations and distinctions between BN-pairs, strong transitivity and Weyl transitivity.

Our first contribution concerns the special case when the ground field k is a local field. The k -anisotropy of G is then equivalent to the compactness of $G(k)$ (see [13]). In fact, our first step will be to establish the following two results, which concern arbitrary compact topological groups (not necessarily associated with algebraic groups).

Theorem 1. *Let G be a compact group. Then every BN-pair for G is spherical.*

Theorem 2. *Let G be a compact group. Then every split spherical BN-pair for G is virtually trivial.*

We emphasize that the BN-pairs appearing in these statements are *abstract*: The corresponding subgroups B and N are *not* supposed to be closed in G . Specializing to anisotropic groups over local fields, we deduce the following immediate corollary.

Theorem 3. *Let k be a local field and G be a connected semi-simple algebraic k -group which is anisotropic over k . Then:*

- (1) *Every BN-pair for $G(k)$ is spherical.*
- (2) *Every split spherical BN-pair $G(k)$ is virtually trivial.*

Finally, we consider the case of perfect ground fields.

Theorem 4. *Let k be a perfect field and G be a reductive algebraic k -group which is anisotropic over k . Then every split spherical BN-pair for $G(k)$ is virtually trivial.*

Notice that Theorems 3 and 4 are logically independent, since there exist local fields which are not perfect and vice-versa.

It would be very interesting to sharpen the conclusion of Theorems 3 and 4, that is, to show that, under suitable assumptions, the BN-pair must be trivial, and not only virtually trivial. However, we expect this to be quite difficult, since it is closely related to a conjecture due to Andrei Rapinchuk and Gopal Prasad (see [14]), which may be stated as follows: “*Let G be a reductive k -group which is anisotropic over k . Then, every finite quotient of $G(k)$ is solvable.*” As of today, this conjecture was confirmed only when G is the multiplicative group of a finite dimensional division algebra (see [15]). We now sketch informally how these two problems are related.

On one side, if $G(k)$ possesses a BN-pair with finite associated building Δ , and if $K := \ker(G(k) \curvearrowright \Delta)$ is the kernel of the corresponding action, then $G(k)/K$ is a finite group whose action on Δ is faithful, and thus $G(k)/K$ is a finite group which possesses a faithful BN-pair. But these groups have been classified: they are simple Chevalley groups, and in particular are not solvable (up to two exceptions).

Thus, if the BN-pair for $G(k)$ were nontrivial, there would exist (modulo these two exceptions) a non-solvable finite quotient of $G(k)$.

Conversely, suppose that $G(k)$ possesses a nontrivial and non-solvable finite quotient $F' := G(k)/K$. Let $R \leq F'$ be the solvable radical of F' , that is, its largest solvable normal subgroup. Going to the quotient $F := F'/R$, we thus know that $G(k)$ surjects onto a nontrivial finite group with trivial solvable radical (namely, F). Let now M be a minimal normal subgroup of F . Then M is a direct product of non-Abelian simple groups which are pairwise isomorphic, say $M \cong S_1 \times \cdots \times S_k$ with $S_i \cong S$ for all $i \in \{1, \dots, k\}$. By the classification of finite simple groups, S is very likely to be a Chevalley group. Such a group possesses a root datum, and thus also a nontrivial BN-pair whose associated (finite) building is in bijection with S/B . Repeating this construction for each S_i , we then get a finite building $\Delta = \Delta_1 \times \cdots \times \Delta_k$ on which $M = S_1 \times \cdots \times S_k$ acts strongly transitively. Finally, the action of $\text{Aut}(M)$ on the set of p -Sylow subgroups of M (where $p = \text{char } k$) induces an action of $\text{Aut}(M)$ on Δ making the diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & \text{Aut}(M) \\ \iota \uparrow & & \downarrow \\ M & \xrightarrow{\text{strongly tr.}} & \text{Aut}(\Delta) \end{array}$$

commute, where $\alpha(f)$ denotes the conjugation by f for all $f \in F$. In particular, we get a strongly transitive action of F , and thus also of $G(k)$, on the finite building Δ . This yields a nontrivial and virtually trivial BN-pair for $G(k)$.

General conventions. All algebraic groups considered here are supposed to be affine, all topological groups are assumed Hausdorff and all BN-pairs have finite rank.

2. PROOF OF THEOREM 1

2.1. Heuristic sketch. Let G be a compact group and let (B, N) be a BN-pair for G . Let also Δ be the associated building. We consider the Davis realization of Δ , noted $|\Delta|_{\text{CAT}(0)}$ in this paper, and which is a complete CAT(0) space, as well as a simplicial complex, on which G acts by simplicial isometries. The key step in the proof of Theorem 1 is to establish that this action is elliptic (Theorem 2.5 below). To do so, we use a result of Martin Bridson stating that such an action is always

semi-simple, and we then argue by contradiction, assuming that G possesses an element with no fixed point. Such an element would then generate a subgroup Q of G which acts by translations on $|\Delta|_{\text{CAT}(0)}$. Moreover, the structure of simplicial complex of $|\Delta|_{\text{CAT}(0)}$ implies that the set of translation lengths of the elements of Q is discrete at 0. The contradiction now comes from divisibility properties of compact and procyclic groups, which we apply to Q .

2.2. Procyclic groups. Let G be a profinite group. Recall that G is said to be **procyclic** if there exists a $g \in G$ such that the subgroup generated by g is dense in G , that is, $G = \overline{\langle g \rangle}$. Moreover G is said to be **pro- p** for some prime p if every finite Hausdorff quotient of G is a p -group.

The following basic properties of procyclic groups can be found in [16, 2.7]. The symbol \mathbb{P} denotes the set of all primes.

Proposition 2.1. *Let G be a procyclic group. Then,*

- (i) G is the direct product $G = \prod_{p \in \mathbb{P}} G_p$ of its p -Sylow subgroups, and each G_p is a pro- p procyclic group.
- (ii) G is, in a unique way, a quotient of $\hat{\mathbb{Z}} := \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. If G is pro- p for some $p \in \mathbb{P}$, then it is a quotient of \mathbb{Z}_p .

2.3. Divisible groups. Recall that an element $g \in G$ is said to be **n -divisible** for some $n \in \mathbb{N}$ if there exists an $h \in G$ such that $h^n = g$. We say that g is **divisible** if it is n -divisible for each $n \geq 1$. The group G is called **n -divisible** (respectively **divisible**) when all its elements are.

Now, every prime q different from p is invertible in \mathbb{Z}_p since its p -adic valuation is zero. Hence, the additive group \mathbb{Z}_p is q -divisible for each $q \in \mathbb{P} \setminus \{p\}$. In particular, Proposition 2.1 implies that if a procyclic group G has trivial q -Sylow subgroups, then G is q -divisible.

We conclude this paragraph by stating the following characterization of divisibility for compact groups (see [12, Corollaire 2]).

Proposition 2.2. *Let G be a compact topological group. Then, G is divisible if and only if it is connected.*

2.4. Semi-simple actions on CAT(0) spaces. Let G be a group acting on a metric space (X, d) . For every $g \in G$, we define the **translation length** of

g by $|g| := \inf\{d(x, g \cdot x) \mid x \in X\} \in [0, \infty)$ and the **minimal set** of g by $\text{Min}(g) := \{x \in X \mid d(x, g \cdot x) = |g|\}$. An element $g \in G$ is said to be **semi-simple** when $\text{Min}(g)$ is nonempty. In that case, we say that g is **elliptic** if it fixes some point, that is, if $|g| = 0$; otherwise, if $|g| > 0$, we call g **hyperbolic**.

A **geodesic line** (respectively, **geodesic segment**) in X is an isometry $f: \mathbb{R} \rightarrow X$ (respectively, $f: [0; 1] \rightarrow X$); by abuse of language, we will identify f with its image in X .

The following lemma follows from Proposition 2.4 in [6].

Lemma 2.3. *Let (X, d) be a complete CAT(0) metric space, and let C be a closed convex nonempty subset of X . Then:*

- (i) *For every $x \in X$, there is a unique $y \in C$ such that $d(x, y) = d(x, C)$, where $d(x, C) := \inf_{z \in C} d(x, z)$. We call y the **projection** of x on C and we write $y = \text{proj}_C x$.*
- (ii) *For all $x_1, x_2 \in X$, we have $d(\text{proj}_C x_1, \text{proj}_C x_2) \leq d(x_1, x_2)$.*

Suppose now that (X, d) is a cell complex. We then say that G acts by **cellular isometries** on X if it preserves the metric, as well as the cell decomposition of X .

The following result is due to Martin Bridson [7].

Proposition 2.4. *Let X be a locally Euclidean CAT(0) cell complex with finitely many isometry types of cells, and G be a group acting on X by cellular isometries. Then every element of G is semi-simple. Moreover, $\inf\{|g| \neq 0 \mid g \in G\} > 0$.*

We now establish the following result, which is the key ingredient for the proof of Theorem 1:

Theorem 2.5. *Let X be a locally Euclidean CAT(0) cell complex with finitely many isometry types of cells, and G be a compact group acting on X by cellular isometries (not necessarily continuously). Then every element of G is elliptic.*

Proof. Suppose for a contradiction there exists a $g \in G$ without fixed point. Proposition 2.4 then implies that g is hyperbolic. Let $Q = \overline{\langle g \rangle}$ be the closure of the subgroup generated by g in G . So, Q is compact.

Claim 1: Q is Abelian.

This is clear since it contains a dense Abelian (in fact cyclic) subgroup.

Claim 2: For every $h \in Q$, the minimal set $\text{Min}(h)$ is a closed convex subset of X which is stabilized by Q .

This follows from [6, Proposition II.6.2].

Claim 3: For every $h \in Q$ and every nonempty closed convex subset C of X stabilized by Q , the set $C \cap \text{Min}(h)$ is nonempty.

Note first that $\text{Min}(h)$ is nonempty by Proposition 2.4. Let $x \in \text{Min}(h)$ and consider the projections $y := \text{proj}_C x$ and $z := \text{proj}_C hx$ provided by Lemma 2.3. Since $hC = C$, we then obtain

$$d(x, y) = \inf_{c \in C} d(x, c) = \inf_{c \in C} d(hx, hc) = \inf_{c \in C} d(hx, c) = d(hx, z).$$

Hence $d(hx, hy) = d(x, y) = d(hx, z)$, and so $z = hy = \text{proj}_C hx$ by uniqueness of projections. Since in addition $d(y, z) \leq d(x, hx) = |h|$ by Lemma 2.3, we finally get $d(y, hy) = |h|$ and therefore $y \in C \cap \text{Min}(h)$.

Claim 4: For all $h_1, h_2 \in Q$, the set $\text{Min}(h_1) \cap \text{Min}(h_2)$ is nonempty.

As $\text{Min}(h_1)$ and $\text{Min}(h_2)$ are nonempty by Proposition 2.4, the claim follows from Claims 2 and 3.

Claim 5: Let $h \in Q$ and let C be a nonempty closed convex subset of X stabilized by Q . We may thus consider the action of h on C . Denote by $|h|_C$ the translation length of h for this action. Then, h is semi-simple in C and $|h| = |h|_C$.

Claim 3 yields that if $x \in \text{Min}(h)$, then $y := \text{proj}_C x \in \text{Min}(h)$. Since $\text{Min}(h)$ is nonempty by Proposition 2.4, the claim follows.

Claim 6: For every $h \in Q$ and $n \geq 1$, we have $|h^n| = n|h|$.

By Claim 4, we may choose an $x \in \text{Min}(h) \cap \text{Min}(h^n)$. Note that h is elliptic (respectively hyperbolic) if and only if h^n is so (see [6, II.6.7 and II.6.8]). In particular, if h is hyperbolic, then x belongs to some h -axis, which is also an h^n -axis. In any case, we obtain $d(x, h^n x) = nd(x, hx)$, whence $|h^n| = d(x, h^n x) = nd(x, hx) = n|h|$.

Claim 7: *Every divisible element of Q is elliptic.*

Let $h \in Q$ be divisible and suppose for a contradiction it is not elliptic. Then h is hyperbolic by Proposition 2.4. For each natural number $n \geq 1$, choose an $h_n \in Q$ such that $h_n^n = h$. In particular, all h_n are hyperbolic. Moreover, $|h_n^n| = n|h_n|$ by Claim 6. Therefore, we obtain a sequence (h_n) of elements of Q such that $|h_n| = |h|/n > 0$, contradicting the second part of Proposition 2.4.

We now establish the desired contradiction to the hyperbolicity of g . First note that the component group $P := Q/Q^0$ of Q is a profinite group. In fact, it is even procyclic, since the subgroup generated by the projection of g in P is dense in P , the natural mapping $\pi: Q \rightarrow Q/Q^0$ being continuous. In particular, it follows from Proposition 2.1 that P is the product of its p -Sylow subgroups P_p . Moreover, each P_p is a pro- p group and is therefore q -divisible for every $q \in \mathbb{P} \setminus \{p\}$. For each $p \in \mathbb{P}$, let Q_p be the subgroup of Q which is the pre-image of P_p under π .

Claim 8: *If $h, a, d \in Q$ with $ha = d^n$ for some $n \geq 1$ and a is elliptic, then $|h| = n|d|$.*

Write $C := \text{Min}(h) \cap \text{Min}(a)$. Then C is nonempty by Claim 4. Since d^n stabilizes C , Claim 5 implies that it is semi-simple in C with translation length $|d^n|_C = |d^n|$. Thus, $|d^n|_C = |d^n| = n|d|$ by Claim 6. Note also that ha is semi-simple in C with translation length $|ha|_C = |h|$. Therefore, $|h| = |ha|_C = |d^n|_C = n|d|$, as desired.

Claim 9: *Let $h \in Q$ be hyperbolic. Suppose that $ha_i = d_i^{n_i}$ for all $i \geq 1$, where $a_i, d_i \in Q$, each a_i is elliptic and where $n_i \geq 1$. Then the set $\{n_i \mid i \geq 1\}$ is bounded.*

Indeed, by Claim 8, the sequence (d_i) of elements of Q is such that $|d_i| = |h|/n_i > 0$. The claim now follows from the second part of Proposition 2.4.

Claim 10: *Let $p \in \mathbb{P}$. Then all elements of Q_p are elliptic.*

Suppose for a contradiction there exists an $h \in Q_p$ which is not elliptic, and is thus hyperbolic by Proposition 2.4. Let $q \in \mathbb{P} \setminus \{p\}$. Since $P_p = \pi(Q_p)$ is q -divisible, there exists an $h_q \in Q$ such that $h_q^q Q^0 = hQ^0$. Let $a \in Q^0$ such that $ha = h_q^q$. By Proposition 2.2, since Q^0 is compact and connected, it is divisible, and so a

is elliptic by Claim 7. Since the set of natural prime numbers distinct from p is unbounded, the desired contradiction now comes from Claim 9.

Let now $gQ^0 = (g_p)_{p \in \mathbb{P}}$ be the decomposition of $\pi(g)$ in $P = \prod_{p \in \mathbb{P}} P_p$ (that is, each $g_p \in P_p$). Let $q \in \mathbb{P}$, and choose an $a_q \in Q_p$ such that $\pi(a_q) = g_q^{-1}$. Then $\pi(ga_q)$ has no component in the q -Sylow of P , and is therefore q -divisible in P . Hence, there exist an $h_q \in Q$ and an $a \in Q^0$ such that $ga_qa = h_q^q$. By Claim 10, we know that a_q is elliptic. But so is a , and hence the product $a' := a_qa$ is also elliptic by Claim 4. Since q is an arbitrary prime, Claim 9 again yields the desired contradiction. \square

2.5. The Davis realization of a building. We recall from [10] that any building Δ admits a metric realization, denoted by $|\Delta|_{\text{CAT}(0)}$, which is a locally Euclidean CAT(0) cell complex with finitely many types of cells. Moreover any group of type-preserving automorphisms of Δ acts in a canonical way by cellular isometries on $|\Delta|_{\text{CAT}(0)}$. Finally, the cell supporting any point of $|\Delta|_{\text{CAT}(0)}$ determines a unique spherical residue of Δ . In particular, an automorphism of Δ which fixes a point in $|\Delta|_{\text{CAT}(0)}$ must stabilize the corresponding spherical residue in Δ .

Here is a reformulation of Theorem 1.

Theorem 2.6. *Let G be a compact group acting strongly transitively by type-preserving automorphisms on a thick building Δ . Then, Δ is spherical.*

Proof. Let (W, S) be the Coxeter system associated to Δ , and let Σ be the fundamental apartment of Δ . Then, the action of the stabilizer in G of Σ can be identified with the action of W on this apartment ([20, 2.8]).

Claim 1: $|\Sigma|_{\text{CAT}(0)}$ is a closed convex subset of $|\Delta|_{\text{CAT}(0)}$.

A basic fact about buildings is the existence, for each pair (Σ, C) consisting of an apartment Σ and of a chamber $C \in \Sigma$, of a *retraction of Δ onto Σ centered at C* , that is, of a simplicial map $\rho = \rho_{\Sigma, C}: \Delta \rightarrow \Sigma$ preserving minimal galleries from C and such that $\rho|_{\Sigma} = \text{id}_{\Sigma}$. The induced mapping $\bar{\rho}: |\Delta|_{\text{CAT}(0)} \rightarrow |\Sigma|_{\text{CAT}(0)}$ then maps every geodesic segment of $|\Delta|_{\text{CAT}(0)}$ onto a piecewise geodesic segment of $|\Sigma|_{\text{CAT}(0)}$ of same length. In particular, the mapping $\bar{\rho}$ is distance decreasing (see [10, Lemme 11.2]). Hence, if x and y are two points in $|\Sigma|_{\text{CAT}(0)}$, then the geodesic segment from x to y is entirely contained in $|\Sigma|_{\text{CAT}(0)}$ since its image by

$\bar{\rho}$ is also a geodesic from x to y . This proves that $|\Sigma|_{\text{CAT}(0)}$ is convex. To see it is closed, it suffices to note that it is complete as a metric space since it is precisely the Davis realization of the building Σ .

Claim 2: *If $g \in G$ is elliptic in $X = |\Delta|_{\text{CAT}(0)}$ and stabilizes $|\Sigma|_{\text{CAT}(0)}$, then g is also elliptic in $|\Sigma|_{\text{CAT}(0)}$.*

This follows from Claim 5 in the proof of Theorem 2.5.

Theorem 2.5 now implies that the induced action of W on $|\Sigma|_{\text{CAT}(0)}$ is elliptic, that is, every $w \in W$ is elliptic. Notice that the W -action on $|\Sigma|_{\text{CAT}(0)}$ is proper, since by construction, it is cellular and the stabilizer of every point is a spherical (in particular finite) parabolic subgroup of W . Recalling now that every infinite finitely generated Coxeter group contains elements of infinite order (in fact, so do all finitely generated infinite linear groups by a classical result of Schur [17]; in the special case of Coxeter groups, a direct argument may be found in [2, Proposition 2.74]), we deduce that W is finite. In other words Δ is spherical. \square

3. PROOF OF THEOREM 2

3.1. Heuristic sketch. Let G be a compact group possessing a split spherical BN-pair, and let Δ be the associated building. We first establish Theorem 2 when G acts continuously on Δ . In that case, 2-transitive actions (which are closely related to strongly transitive actions) of G on subspaces X of Δ are easily seen to be possible only for finite X . The second step is then to show that the action of G on Δ has to be continuous. This uses the fact that buildings arising from split spherical BN-pairs are Moufang (see Proposition 3.3 below).

3.2. Continuous actions on buildings. Recall that a topological space X is said to satisfy the T_1 separation axiom when all its singletons are closed. The following is probably well-known.

Lemma 3.1. *Let G be a compact group. If G admits a continuous 2-transitive action on a T_1 topological space X , then X is finite.*

Proof. Define $Y := \{(x, y) \in X \times X \mid x \neq y\} \subset X \times X$, and fix $x, y \in X$ with $x \neq y$. Since the orbit map $\alpha_x: G \rightarrow X: g \mapsto g \cdot x$ is continuous, so is

$\alpha_x \times \alpha_y: G \rightarrow X \times X : g \mapsto (g \cdot x, g \cdot y)$. By 2-transitivity, we get $Y = (\alpha_x \times \alpha_y)(G)$, and so Y is compact.

Note also that the mapping $f: X \times X \rightarrow X \times X : (a, b) \mapsto (x, b)$ is continuous. Setting $Z := X \setminus \{x\}$, we then get $Z \times \{x\} = f^{-1}(\{(x, x)\}) \cap Y$, so that $Z \times \{x\}$ is closed in Y , and hence compact. It follows that Z is compact, being the image of $Z \times \{x\}$ by the projection on the first factor $X \times X \rightarrow X$, which is of course continuous.

In particular, Z is closed, and hence $\{x\}$ is open. It follows that X is discrete, and therefore finite since $X = \alpha_x(G)$ is compact. □

Let Δ be a building of type (W, S) , and denote by $\text{Ch } \Delta$ the set of its chambers. Consider the chamber system Γ of Δ , which is the labelled graph with vertex set $\text{Ch } \Delta$ and with an edge labelled by $s \in S$ for each pair of s -adjacent chambers of Δ (see [8, Ch.I Appendix D]). Let $J \subset S$. A **J -gallery** in Γ between two chambers x and y of Δ is a sequence $(x = x_0, x_1, \dots, x_l = y)$ of chambers of Δ such that for each $i \in \{1, \dots, l\}$, there exists an $s \in J$ such that x_{i-1} is s -adjacent to x_i . The natural number l is called the **length** of the gallery. A **minimal** gallery is a gallery of minimal length. The **distance** in Δ between two chambers $x, y \in \text{Ch } \Delta$ is the length of a minimal gallery joining x to y . The **diameter** of Γ is the supremum (in $\mathbb{N} \cup \{\infty\}$) of the distances between its vertices.

Let $J \subset S$. The **J -residue** $R = R_J(x)$ of some chamber $x \in \text{Ch } \Delta$ is the set of chambers of Δ which are connected to x by a J -gallery. When J has cardinality 1, we call R a **panel**.

In this paper, we will say that a group G acts **continuously** on Δ if the stabilizers of the residues of Δ are closed in G . Note that we can of course restrict our attention to the maximal proper residues, the others being obtained as intersections of those.

Lemma 3.2. *Let G be a compact group acting continuously and strongly transitively by type-preserving automorphisms on a spherical thick building Δ . Then Δ is finite.*

Proof. The stabilizer H in G of a panel P of Δ is a closed and thus compact subgroup of G .

Claim 1: H acts 2-transitively on $\text{Ch}(P)$.

Indeed, let C be a chamber of P and let $B := \text{Stab}_G(C) \subset H$. We first show that B , and thus also H , is transitive on the set $\mathcal{C} = P \setminus \{C\}$. Let $C_1, C_2 \in \mathcal{C}$ and let Σ_1 (respectively, Σ_2) be an apartment containing C and C_1 (respectively, C and C_2). By strong transitivity, B is transitive on the set of apartments containing C , and so there exists a $b \in B$ such that $b\Sigma_1 = \Sigma_2$. Hence $bC_1 = C_2$. It now remains to show that H is transitive on P . But if $C_1, C_2 \in P$, then since Δ is thick, we may choose a chamber C in P different from C_1, C_2 . The stabilizer B' of C in G is then contained in H and is transitive on $P \setminus \{C\}$ by the previous argument.

Now, identifying Δ with $\Delta(G, B)$, so that $H = B \cup BsB$ for some generator s of the corresponding Weyl group, we get a 2-transitive, continuous action by left translation of the compact group H on the topological space H/B . Moreover, this space is T_1 since B is closed in G by hypothesis. Lemma 3.1 then implies that P is finite. In other words, as P was arbitrary, the building Δ is *locally finite*, that is, every panel is finite. The following observation now allows us to conclude:

Claim 2: *Every locally finite spherical building is finite.*

Indeed, let $\Gamma = \text{Ch } \Delta$ be the graph whose vertices are the chambers of Δ , and such that two chambers of Δ are adjacent if they share a common panel. Since Δ is locally finite, so is Γ . Hence, fixing a vertex $x \in \Gamma$, each ball in Γ centered at x with radius n ($n \in \mathbb{N}$) possesses a finite number of vertices. Moreover, as Δ is spherical, the diameter of Δ is finite ([8, Ch.IV, 3]), and hence the diameter of Γ is also finite. Thus Γ is contained in such a ball, and is therefore finite. \square

3.3. Moufang buildings. Let $\Delta = \Delta(G, B)$ be the building associated to a split spherical BN-pair $(B = T \ltimes U, N)$ of type (W, S) . It is well-known (see the main result of [11]) that the existence of a splitting for the above BN-pair is equivalent to the fact that the building Δ enjoys the Moufang property, as defined in [20, Chapter 11].

Two chambers $x, y \in \text{Ch } \Delta$ are called **opposite** if they are at maximal distance in the chamber system of Δ . Similarly, one can define *opposite residues* (see for instance [2, 5.7]). The set of chambers (respectively, residues) of Δ which are opposite to a given chamber C (respectively, residue R) will be denoted by C^{op} (respectively, R^{op}).

Proposition 3.3. *Let $P = BW_J B$ be a proper standard parabolic subgroup of $\Delta = \Delta(G, B)$ for some proper subset J of S , let C be the fundamental chamber (i.e. the unique chamber fixed by B) and let R be the unique J -residue containing C . Define the subgroup $V := \bigcap_{p \in P} pUp^{-1}$ of G . Then V acts simply transitively on R^{op} .*

Proof. Let Σ be an apartment containing C . By [20, 9.11], there exists a minimal gallery $\gamma_{R'}$, one for each residue $R' \in R^{\text{op}}$, beginning at C and ending at a chamber C' in R' such that the type of $\gamma_{R'}$ is independent of the choice of R' and $C = \text{proj}_R C'$. Let $R' \in R^{\text{op}}$ be the unique residue of Σ opposite R and let C' be the last chamber of $\gamma_{R'}$. Let also α be a root of Σ containing C but not C' . By [20, 8.21], $R \cap \Sigma \subset \alpha$. By [20, 9.7], therefore, R is fixed pointwise by the root group U_α . Since P maps R to itself, we have $C \in R \subset \alpha^p$ and hence $p^{-1}U_\alpha p \subset U$ for all $p \in P$ by the definition of root subgroups (see [20, 11.1]) and the fact that the ‘radical’ U does not depend on the choice of the apartment Σ (see [20, Proposition 11.11(iii)]). Thus $U_\alpha \subset V$. Now, as in [2, 7.67], one shows that the subgroup of V generated by all U_α ’s of the latter form acts transitively on the set $\{\gamma_{R''} \mid R'' \in R^{\text{op}}\}$, and hence also on R^{op} .

Suppose $h \in V$ maps $R' \in R^{\text{op}}$ to itself. Then h acts trivially on R . Since the restriction of $\text{proj}_{R'}$ to R is a bijection from R to R' (by [20, 9.11] again), it follows that h acts trivially on R' . By [20, 9.8], therefore, h fixes two opposite chambers of Σ and hence h fixes Σ . By [20, 9.7] again, we conclude that $h = 1$. \square

In particular, we have the following (compare [8, Ch.IV, 5]).

Lemma 3.4. *Let C be the fundamental chamber of Δ . Then U acts simply transitively on C^{op} . Equivalently, U acts simply transitively on the set of apartments containing C .*

Lemma 3.5. *Let $P = BW_J B$ be a proper standard parabolic subgroup of $\Delta = \Delta(G, B)$ for some proper subset J of S , let C be the fundamental chamber and let R be the unique J -residue containing C . Then there exist two chambers in C^{op} which are opposite to one another. In particular, $|R^{\text{op}}| \geq 2$.*

Proof. The first assertion holds by [2, Proposition 4.104] and the second follows since no proper residue contains two opposite chambers. \square

We are now ready to complete the proof of Theorem 2.

Theorem 3.6. *Let G be a compact topological group possessing a spherical split BN-pair $(B = T \rtimes U, N)$. Then the associated building is finite.*

Proof. Let $\Delta = \Delta(G, B)$ be the building associated to (B, N) , and let (W, S) be the corresponding Coxeter system.

We start with some basic observations in the case (W, S) is not irreducible. Suppose thus that S decomposes as $S = S_1 \amalg S_2$ with $s_1 s_2 = s_2 s_1$ for all $s_1 \in S_1$ and $s_2 \in S_2$. Then W splits as a direct product $W \cong W_1 \times W_2$, where $W_i = \langle S_i \rangle$, and the building Δ decomposes canonically as a product $\Delta = \Delta_1 \times \Delta_2$ of buildings of type (W_1, S_1) and (W_2, S_2) respectively (see [20, Proposition 7.33]).

In particular, we obtain induced actions of G on both Δ_1 and Δ_2 , which are obviously strongly transitive. The corresponding BN-pairs for G may be described as follows. Since each $s \in S$ can be written as a coset $nT \in N/T = W$, we may choose, for $i = 1, 2$, a set \overline{N}_i of representatives in N for the elements of S_i . For each $i = 1, 2$, consider now the subgroup N_i of N generated by \overline{N}_i and T , and set $B_i := \langle B \cup N_{3-i} \rangle = BN_{3-i}B \leq G$. Then (B_i, N_i) is a spherical BN-pair for G , and the associated building is nothing but $\Delta_i = \Delta(G, B_i)$.

We claim that the BN-pair (B_i, N_i) is split. This follows readily from the aforementioned equivalence between splittings of BN-pairs and the Moufang property for the associated buildings. More precisely, consider the group $U_i = \bigcap_{g \in B_i} gUg^{-1}$ which is the kernel of the U -action on Δ_{3-i} . Then U_i acts sharply transitively on the chambers of Δ_i which are opposite the standard chamber C , which by definition is the unique chamber fixed by B_i . Therefore we have $B_i \cong T_i \rtimes U_i$, where $T_i = \bigcap_{w \in W_i} wB_iw^{-1}$, and U_i induces a splitting of the BN-pair (B_i, N_i) as claimed.

This shows that the given split BN-pair for G yields various split BN-pairs for G corresponding to the various irreducible components of Δ . Since $\text{Ch } \Delta$ is naturally in one-to-one correspondence with the Cartesian product $\text{Ch } \Delta_1 \times \cdots \times \text{Ch } \Delta_n$ of the chamber sets of the various irreducible components of Δ , the desired finiteness result readily follows provided we establish it for each irreducible BN-pair (B_i, N_i) as above. In other words, there is no loss of generality in assuming that the building Δ is irreducible. We adopt henceforth this additional assumption.

Let now \mathcal{P} denote the set of maximal proper standard parabolic subgroups of G . Pick any $P \in \mathcal{P}$. Thus P is of the form $P = BW_JB$ for some maximal subset $J \subsetneq S$, where $W_J = \langle J \rangle$. In particular, P is a maximal subgroup of G (see [2, Lemma 6.43(1)]). Define the normal subgroup

$$V := \bigcap_{p \in P} pUp^{-1} \trianglelefteq P$$

of P . As V is contained in U , it is also nilpotent. Moreover, V acts faithfully on Δ . Indeed, the kernel $\ker(G \curvearrowright \Delta)$ of the action of G on Δ is obviously contained in the stabilizer of the chambers of the fundamental apartment Σ , that is, in $\bigcap_{w \in W} wBw^{-1} = T$, and so

$$V \cap \ker(G \curvearrowright \Delta) \subseteq U \cap T = \{1\}.$$

Now, since V is normal in P , we have $P \subseteq \mathcal{N}_G(V)$. Moreover, as the conjugation automorphism $\kappa_g: G \rightarrow G: x \mapsto gxg^{-1}$ is continuous, we get $\mathcal{N}_G(\overline{V}) \supseteq \mathcal{N}_G(V)$ and so $\mathcal{N}_G(\overline{V}) \supseteq P$. Hence, by maximality of P , we obtain that either $\mathcal{N}_G(\overline{V}) = P$ or $\mathcal{N}_G(\overline{V}) = G$.

Claim: $\mathcal{N}_G(\overline{V}) = P$ for all $P \in \mathcal{P}$.

Assume for a contradiction that $\mathcal{N}_G(\overline{V}) = G$ for some $P \in \mathcal{P}$. In other words, $\overline{V} \triangleleft G$. In particular, the center $\mathcal{Z}(\overline{V}) \subseteq \overline{V}$ of \overline{V} is also a normal subgroup of G . Moreover, V is nontrivial since, by Proposition 3.3, it acts transitively on R^{op} and since $|R^{\text{op}}| \geq 2$ by Lemma 3.5. As V is nilpotent, this implies that $\mathcal{Z}(V)$ is also nontrivial.

Now, using again the continuity of the conjugation automorphism κ_h (for $h \in G$), we see that $\mathcal{Z}(V) = \mathcal{Z}_G(V) \cap V$ is contained in $\mathcal{Z}(\overline{V}) = \mathcal{Z}_G(\overline{V}) \cap \overline{V}$. Moreover, as V acts faithfully on Δ , so does $\mathcal{Z}(V)$. This implies in particular that $\mathcal{Z}(V)$, and thus also $\mathcal{Z}(\overline{V})$, act nontrivially on Δ .

Tits' transitivity Lemma (see [8, Lemma 6.61]) then guarantees that the group $\mathcal{Z}(\overline{V})$ is transitive on the chambers of Δ . In fact, this action is even simply transitive. Indeed, the stabilizers in $\mathcal{Z}(\overline{V})$ of the chambers of Δ are all conjugate by transitivity. They are thus all equal since $\mathcal{Z}(\overline{V})$ is Abelian, and are therefore contained in the kernel $\ker(G \curvearrowright \Delta)$ of the action of G on Δ . Since $\mathcal{Z}(V) \subseteq \mathcal{Z}(\overline{V})$, this implies that the action of $\mathcal{Z}(V)$ on $\text{Ch } \Delta$ is free. But since $\mathcal{Z}(V) \subseteq V \subseteq$

$U \subseteq B$, and as B stabilizes the fundamental chamber, it follows that $\mathcal{Z}(V)$ acts trivially on Δ . This contradiction establishes the Claim.

Since the normalizer of a closed subgroup is closed, we deduce from the Claim that every $P \in \mathcal{P}$ is closed. But this means that G acts continuously on Δ , and so Lemma 3.2 ensures that Δ is finite, as desired. \square

4. PROOF OF THEOREM 4

Let k be a perfect field and let $K = \bar{k}$ be its algebraic closure. In what follows, we identify an algebraic k -group G with its group of K -rational points.

The main tool for the proof of Theorem 4 is the following result due to Borel and Tits (see [3]).

Proposition 4.1. *Let G be a reductive algebraic k -group and let U be a unipotent k -subgroup of G . If k is perfect, then there exists a parabolic k -subgroup P of G whose unipotent radical $R_u(P)$ contains U .*

In particular, if G is anisotropic over k , then U must be trivial.

Proof of Theorem 4. Suppose for a contradiction that the split spherical BN-pair (B, N) for the reductive k -group G is such that B has infinite index in $G(k)$. Let $\Delta = \Delta(G(k), B)$ be the associated building, and let W be the corresponding (finite) Weyl group. Also, denote by \bar{B} the Zariski closure of B in G .

The Bruhat decomposition for G yields $G = \coprod_{w \in W} BwB$. Since $G(k)$ is Zariski dense in G by [4, 18.3], we have

$$G = \overline{G(k)} = \overline{\coprod_{w \in W} BwB} \subseteq \coprod_{w \in W} \overline{BwB}.$$

As G is connected, it cannot be written as a finite union of closed subsets in a nontrivial way. Therefore, we deduce that BwB is dense in G for some $w \in W$. In particular, so is \overline{BwB} .

Let now $A := (\bar{B})^0$ be the identity component of \bar{B} . Since A has finite index in \bar{B} , it follows that \overline{BwB} is a finite union of double cosets modulo A . As before, this implies that some double coset of the form AzA is dense in G .

Claim: $\bar{B} \neq G$.

Indeed, let U be the nilpotent normal subgroup of B arising from the splitting of the BN-pair, and suppose for a contradiction that B is dense in G . Then the Zariski closure \bar{U} of U in G is a nilpotent normal subgroup of $\bar{B} = G$ ([4, 2.1]). Its identity component \bar{U}^0 is thus contained in the radical of G , which coincides with the connected center $\mathcal{Z}(G)^0$ ([4, 11.21]). Hence, since \bar{U}^0 has finite index in \bar{U} , we get

$$[U : U \cap \mathcal{Z}(G)] \leq [U : U \cap \bar{U}^0] = [U\bar{U}^0 : \bar{U}^0] \leq [\bar{U} : \bar{U}^0] < \infty.$$

Now, if $u \in U \cap \mathcal{Z}(G)$, then u acts trivially on Δ since for any chamber gB , we have $ugB = guB = gB$. As U acts simply transitively on C^{op} by Lemma 3.4, where $C = 1_G B$ is the fundamental chamber of Δ , this implies that $u = 1$: otherwise, Δ would contain only one apartment, so that $[G(k) : B] < \infty$, a contradiction. So $U \cap \mathcal{Z}(G) = \{1\}$ and therefore U is finite. Using again the sharp transitivity of U on C^{op} , we deduce that Δ is the reunion of finitely many apartments, hence is finite, contradicting once more our initial hypothesis. The claim stands proven.

In particular A is a proper closed connected subgroup of G such that AzA is dense in G for some $z \in G$. The main result of [9] now implies that A is not reductive, *i.e.* the unipotent radical $R_u(A)$ is nontrivial. Moreover, since B is contained in $G(k)$ and is dense in \bar{B} , we know that \bar{B} is defined on k ([4, AG.14.4]). Hence, A is also k -defined ([4, 1.2]), and so is $R_u(A)$ since k is perfect ([18, 12.1.7(d)]). Thus $R_u(A)$ is a nontrivial unipotent k -subgroup of G . As remarked after Proposition 4.1 above, this contradicts the assumption that G is anisotropic over k . \square

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