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The Essentially Tame Jacquet-Langlands Correspondence for Inner Forms of $GL(n)$

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To Jacques Tits, on his eightieth birthday

Abstract: Let F be a non-Archimedean local field, let $n \geq 1$ be an integer and $G = GL_n(F)$. Let G' be an inner form of G , so that G' is isomorphic to $GL_m(D)$, for a central F -division algebra of dimension d^2 , $md = n$. Using the structure theory of Sécherre and Stevens, we define a concept of *parametric degree* for irreducible cuspidal representations of G' . We show that the image, under the Jacquet-Langlands correspondence, of the set of equivalence classes of irreducible cuspidal representations of G is the set of equivalence classes of irreducible cuspidal representations of G' of parametric degree n . In earlier papers, we defined a notion of *essential tameness* for irreducible cuspidal representations of G . We generalize this to representations of G' , and show that it is preserved by the Jacquet-Langlands correspondence. As in the earlier papers, the irreducible, essentially tame, cuspidal representations of G admit an explicit parametrization in terms of *admissible pairs*, which can be explicitly related to the local Langlands correspondence. Here, we generalize this construction to the case of irreducible, essentially tame, cuspidal representations of G' of parametric degree n . We determine completely the behaviour of the two parametrizations relative to the Jacquet-Langlands correspondence. As a consequence, we prove a conjecture of Bushnell and Fröhlich of 1983.

Keywords: Explicit local Jacquet-Langlands correspondence, admissible pair, inner form, essentially tame representation, parametric degree, character formula

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Let F be a non-Archimedean local field with finite residue field of characteristic p . In a series of papers [11], [12], [14], we gave an explicit description of the local Langlands correspondence in the *essentially tame* case. In this paper, we give a parallel description of the *Jacquet-Langlands correspondence*. The results illuminate those of the earlier papers. They finally answer the most interesting of the questions left open in [6].

We emphasize that, in this paper, the characteristic of F is arbitrary. The technical issues which forced us to restrict, in [14] in particular, to the characteristic zero case do not intervene here. The parts of [14] to which we appeal in this paper are valid in all characteristics.

1. Let $n \geq 1$ be an integer and let G be an inner form of $\mathrm{GL}_n(F)$. Thus there is a central F -division algebra D , of dimension d^2 , $d \geq 1$, such that $G \cong \mathrm{GL}_m(D)$, $md = n$. Let $\mathcal{A}_m^\square(D)$ denote the set of equivalence classes of irreducible smooth representations of G which are *essentially square-integrable*, modulo the centre F^\times of G . Thus, in particular, $\mathcal{A}_n^\square(F)$ denotes the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(F)$ which are essentially square-integrable modulo centre.

The *Jacquet-Langlands correspondence* gives a canonical bijection

$$(1.1) \quad j : \mathcal{A}_n^\square(F) \xrightarrow{\approx} \mathcal{A}_m^\square(D),$$

specified by a character relation on elliptic regular elements.

The idea of the correspondence, and the basic method for proving it, both originate with Jacquet, Langlands [27] which treats the case $n = 2$ when F has characteristic zero. For the general case with F of characteristic zero, see Deligne, Kazhdan, Vignéras [21], also Rogawski [37]. The positive characteristic case is treated by Badulescu [2], with a supplement in [16] required for this paper.

2. The set $\mathcal{A}_m^\square(D)$ contains the set $\mathcal{A}_m^0(D)$ of equivalence classes of irreducible *cuspidal* representations of $\mathrm{GL}_m(D)$. An early step in this paper identifies the image of $\mathcal{A}_n^0(F)$ in $\mathcal{A}_m^\square(D)$ under the map (1.1).

This relies on a notion of *parametric degree* for representations $\pi \in \mathcal{A}_m^\square(D)$, generalizing the cases of $\mathcal{A}_n^\square(F)$, $\mathcal{A}_1^\square(D)$ treated in [9]. The definition in [9] uses the classification theory for cuspidal representations of $\mathrm{GL}_n(F)$ in [17] and the analogous results of Broussous [4] or Zink [43] for $\mathrm{GL}_1(D)$. The necessary

elements of the classification are now available for $GL_m(D)$ in the paper [41] of Sécherre and Stevens, the culmination of Sécherre’s programme [38], [39], [40].

For $\pi \in \mathcal{A}_m^\square(D)$, the parametric degree $\delta(\pi)$ is a positive integer dividing n . If $\delta(\pi) = n$, then π is cuspidal. The converse holds for $GL_n(F)$ but not in general. Just as in [9], the parametric degree is determined by the normalized formal degree. Since the Jacquet-Langlands correspondence preserves the normalized formal degree [21], [37], [16], it must also preserve the parametric degree. Thus

$$(2.1) \quad j(\mathcal{A}_n^0(F)) = \{\pi \in \mathcal{A}_m^\square(D) : \delta(\pi) = n\} \subset \mathcal{A}_m^0(D).$$

For $\pi \in \mathcal{A}_m^\square(D)$, let $t(\pi)$ denote the number of unramified characters χ of F^\times such that $\chi\pi \cong \pi$. (Here we use the customary notation: $\chi\pi$ is the representation $g \mapsto \chi(\text{Nrd } g)\pi(g)$, where Nrd denotes the reduced norm map $GL_m(D) \rightarrow F^\times$.) The integer $t(\pi)$ divides $\delta(\pi)$. One says that π is *essentially tame* if p does not divide $\delta(\pi)/t(\pi)$. In particular, if p does not divide n , every $\pi \in \mathcal{A}_m^\square(D)$ is essentially tame.

We denote by $\mathcal{A}_m^{\text{et}}(D)$ the set of classes of representations $\pi \in \mathcal{A}_m^\square(D)$ which have $\delta(\pi) = n$ and are essentially tame. This generalizes the definition of $\mathcal{A}_n^{\text{et}}(F)$ in [11].

Because it preserves both the parametric degree and the t -invariant, the Jacquet-Langlands correspondence (1.1) induces a canonical bijection

$$(2.2) \quad j : \mathcal{A}_n^{\text{et}}(F) \xrightarrow{\approx} \mathcal{A}_m^{\text{et}}(D).$$

The aim of the paper is to make this map explicit, in a manner parallel to that of [11], [12], [14].

3. Such an aim, of course, requires an explicit description of both sets $\mathcal{A}_n^{\text{et}}(F)$, $\mathcal{A}_m^{\text{et}}(D)$. For the first of them, we have the “naïve correspondence” of [11]. This is a canonical bijection

$$(3.1) \quad \begin{aligned} P_n(F) &\longrightarrow \mathcal{A}_n^{\text{et}}(F), \\ (E/F, \xi) &\longmapsto {}_F\Pi_\xi, \end{aligned}$$

where $P_n(F)$ is the set of F -isomorphism classes of *admissible pairs* $(E/F, \xi)$ of degree n (the definition is recalled in 4.1 below). The map (3.1) generalizes

the constructions of Howe's early paper [26]. Our first task is to produce an analogous bijection

$$(3.2) \quad \begin{aligned} P_n(F) &\longrightarrow \mathcal{A}_m^{\text{et}}(D), \\ (E/F, \xi) &\longmapsto {}_D\Pi_\xi. \end{aligned}$$

4. Here, and indeed throughout the paper, we rely on [38–41]. These papers contain all the necessary elements of a complete theory of types [19] for the groups $\text{GL}_m(D)$, directly generalizing [17] and [4]. In particular, the cuspidal representations of $\text{GL}_m(D)$ are explicitly presented as induced representations. The description extends in essence to all of $\mathcal{A}_m^\square(D)$. However, the theory exhibits some novel technical difficulties, as compared with the “split” case of $\text{GL}_n(F)$, and it has not yet reached the same level of completeness. Consequently, we cannot simply follow [11] to construct the bijection (3.2): our approach must be more indirect.

We start in §§3,4 by attaching to $\pi \in \mathcal{A}_m^{\text{et}}(D)$ an admissible pair, using the classification in [41]. The process involves many choices, and there is no guarantee, at first, that it has any intrinsic meaning. (However, in the case of $\text{GL}_1(D)$ with p not dividing $\dim_F D$, it is easy to see that it gives the bijection of [20], [28], on which [6] is based.)

Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ be *totally ramified*, in the sense that $t(\pi) = 1$. An admissible pair $(E/F, \xi)$ attached to π then has E/F totally ramified. Some simple calculations show that $(E/F, \xi)$ can be recovered from the values of the character function $\text{tr } \pi$ of π at certain elliptic regular elements. The pair is therefore uniquely determined by π , up to F -isomorphism. Further calculation shows that π is matched, via the Jacquet-Langlands correspondence (2.2), with the representation ${}_F\Pi_{\nu\xi} \in \mathcal{A}_n^{\text{et}}(F)$, where ν is an unramified quadratic character of F^\times depending only on the integers m, d . This step is to be found in §5, the formula for ν being given in 5.3.

5. The core of the paper is §6. We take a general representation $\pi \in \mathcal{A}_m^{\text{et}}(D)$, with an attached admissible pair $(E/F, \xi)$ (possibly one of many). We calculate the character value $\text{tr } \pi(\zeta u)$ at suitable elements ζu near a root of unity $\zeta \in E$, that generates the maximal unramified sub-extension K/F of E/F . The outcome is an expression for $\text{tr } \pi(\zeta u)$ in terms of $(E/F, \xi)$ and character values

$\text{tr } \rho(u)$, for various essentially tame, *totally ramified*, cuspidal representations ρ of the G -centralizer of K . We know, from §5, how such representations relate to representations of split groups and to admissible pairs. We are able to deduce that the process, of attaching an admissible pair to $\pi \in \mathcal{A}_m^{\text{et}}(D)$, yields the inverse of a *canonical bijection* (3.2). This is the Parametrization Theorem of 6.1.

We further show that, if $(E/F, \xi) \in P_n(F)$, then ${}_D\Pi_\xi$ is of the form $\mathbf{j}({}_F\Pi_{\nu\xi})$, for a canonically determined, tame, quadratic character ν of E^\times : this is the First Comparison Theorem 6.1.

We calculate the character ${}_D\nu_\xi = \nu_\xi$ in terms of the “symplectic signs” ubiquitous in [14]. We interpret these signs in terms of m , D and $(E/F, \xi)$ to give an explicit formula for the restriction ${}_D\nu_\xi \mid U_E$: see 6.9 Corollary and (6.7.4).

It remains only to calculate the value of the tamely ramified character ${}_D\nu_\xi$ at one prime element of E . This is the subject of the Second Comparison Theorem of 7.1. We leave the answer in terms of symplectic signs: a calculation along the lines of the proof of 8.4 Theorem of [14] is necessary to reduce it to numerical form.

Together, the two Comparison Theorems show:

Theorem A. *Let D be a central F -division algebra of dimension d^2 , $d \geq 1$, let $m \geq 1$ be an integer and set $n = md$. If $(E/F, \xi)$ is an admissible pair with $[E:F] = n$, there is a canonically determined, tamely ramified character $\nu = {}_D\nu_\xi$ such that $(E/F, \nu\xi)$ is admissible, $\nu^2 = 1$, and*

$${}_D\Pi_\xi = \mathbf{j}({}_F\Pi_{\nu\xi}),$$

where $\mathbf{j} : \mathcal{A}_n^{\text{et}}(F) \rightarrow \mathcal{A}_m^{\text{et}}(D)$ is the Jacquet-Langlands correspondence.

The determination of ${}_D\nu_\xi$ in the theorems of 6.1, 7.1 implies immediately:

Theorem B. *For $i = 1, 2$, let $(E/F, \xi_i)$ be an admissible pair with $[E:F] = n$. If $\xi_1^{-1}\xi_2$ is tamely ramified, then ${}_D\nu_{\xi_1} = {}_D\nu_{\xi_2}$.*

A closer examination of the form of the character ${}_D\nu_\xi$, which we omit here, yields:

Theorem C. *For $i = 1, 2$, let D_i be a central F -division algebra and suppose that $\dim_F D_1 = \dim_F D_2 = d^2$, $d \geq 1$. If $(E/F, \xi)$ is an admissible pair with $[E:F]$ divisible by d , then ${}_{D_1}\nu_\xi = {}_{D_2}\nu_\xi$.*

6. We put these results in the context of the Langlands correspondence: since we now appeal to essential features of [14], we have to assume that F has characteristic zero. Let \mathcal{W}_F denote the Weil group of F , and let $\mathfrak{G}_n^0(F)$ denote the set of equivalence classes of irreducible smooth representations of \mathcal{W}_F of dimension n . If $(E/F, \xi)$ is an admissible pair with $[E:F] = n$, we may view ξ as a character of \mathcal{W}_E via class field theory and form the induced representation

$${}_F\Sigma_\xi = \text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \xi$$

of \mathcal{W}_F . This representation is irreducible, and we so obtain a bijection

$$(E/F, \xi) \longmapsto {}_F\Sigma_\xi$$

of $P_n(F)$ with the subset $\mathfrak{G}_n^{\text{et}}(F)$ of classes of representations $\sigma \in \mathfrak{G}_n^0(F)$ which are essentially tame, in an obvious sense [11]. The Langlands correspondence then induces a canonical bijection

$${}_F\mathcal{L} : \mathfrak{G}_n^{\text{et}}(F) \longrightarrow \mathcal{A}_n^{\text{et}}(F).$$

This is of the form ${}_F\Sigma_\xi \mapsto {}_F\Pi_{\mu\xi}$, for a canonically determined, tamely ramified character $\mu = {}_F\mu_\xi$ of E^\times .

Composing with the Jacquet-Langlands correspondence

$$j : \mathcal{A}_n^{\text{et}}(F) \longrightarrow \mathcal{A}_m^{\text{et}}(D),$$

we obtain the Langlands correspondence

$${}_D\mathcal{L} : \mathfrak{G}_n^{\text{et}}(F) \longrightarrow \mathcal{A}_m^{\text{et}}(D).$$

This is bijective. Combining the results of this paper with those of [12], [14], we get:

Theorem D.

- (1) *Let $(E/F, \xi)$ be an admissible pair in which $[E:F] = n$. There is a canonically determined, tamely ramified character $\lambda = {}_D\lambda_\xi$ of E^\times such that*

$${}_D\mathcal{L} : {}_F\Sigma_\xi \longmapsto {}_D\Pi_{\lambda\xi}.$$

Indeed, ${}_D\lambda_\xi = {}_F\mu_\xi \cdot {}_D\nu_\xi$.

- (2) If $(E/F, \xi_i)$, $i = 1, 2$, are admissible pairs such that $\xi_1^{-1}\xi_2$ is tamely ramified, then

$${}_D\lambda_{\xi_1} = {}_D\lambda_{\xi_2}.$$

- (3) The character $\lambda = {}_D\lambda_\xi$ satisfies $\lambda^4 = 1$ and $\lambda^2|_{U_E} = 1$.

This result was conjectured, for the case $m = 1$, in [6]. However, the values for λ proposed there (and in [34]) are no closer to reality than those proposed by Moy [33] for the character ${}_F\mu_\xi$: see the discussion in [12] 2.2.

7. A number of discrete technical issues, of possibly wider interest, arise in the course of the paper. The first of these, already mentioned, is the relation between the (algebraic) parametric degree and the (analytic) formal degree. This is general in nature, independent of any consideration of characteristic and of our underlying tameness hypotheses. We have therefore given it separate treatment in Appendix 1.

At a critical point in the character calculations of §7, we have to evaluate the character of a “wide extension” (aka “ β -extension”) of the Heisenberg representation containing a simple character. The same problem arose at the corresponding point of [14], where we returned to the original definition in [17]. The extra complexity of the general case in [39] makes that a daunting prospect here. We have substituted a general argument, based on the interaction between the Glauberman correspondence and intertwining of representations. This simplifies a parallel result of Stevens (the Principal Lemma 2.4 of [42]), so we have included a proof in Appendix 2.

Notation

Throughout, F denotes a non-Archimedean local field. The discrete valuation ring in F is denoted \mathfrak{o}_F , and the maximal ideal of \mathfrak{o}_F is \mathfrak{p}_F . We write $\mathbb{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$, $q = q_F = |\mathbb{k}_F|$, and we let p denote the characteristic of \mathbb{k}_F . We set $U_F = U_F^0 = \mathfrak{o}_F^\times$, and $U_F^k = 1 + \mathfrak{p}_F^k$, $k \geq 1$. We write v_F for the canonical (surjective) valuation map $F^\times \rightarrow \mathbb{Z}$. We denote by μ_F the group of roots of unity in F of order relatively prime to p .

We use the obvious analogues of these conventions for a finite field extension E/F . Additionally, we denote by $N_{E/F}$ and $\text{Tr}_{E/F}$ the relative norm and trace maps $E \rightarrow F$ respectively.

If D is a central F -division algebra of finite dimension, then \mathfrak{o}_D is the discrete valuation ring in D and \mathfrak{p}_D is the unique maximal ideal of \mathfrak{o}_D . We denote by \mathbb{k}_D the field $\mathfrak{o}_D/\mathfrak{p}_D$: this is finite, with $q_D = q_F^d$ elements, where $d^2 = \dim_F D$. We set $U_D = \mathfrak{o}_D^\times$ and $U_D^k = 1 + \mathfrak{p}_D^k$, $k \geq 1$.

If A is a finite-dimensional, central simple F -algebra, then $\text{Nrd}_A : A^\times \rightarrow F^\times$ is the reduced norm map.

Most notation concerning the internal structure of representations is introduced as it arises, but is kept closely parallel to the standard usage of [17].

1. Classical character relations

We fix an integer $n \geq 1$. We are concerned with locally profinite groups of the form $G = \text{GL}_m(D)$, where D is a central F -division algebra of finite dimension d^2 , $d \geq 1$, and $n = md$. We set $A = M_m(D)$, so that $G = A^\times$.

To reach the effective starting point of the paper, we have to gather threads from several sources and weave them together. In this section, we concentrate on ideas from harmonic analysis, some elementary, some deriving ultimately from the trace formula. The main point is to recall the Jacquet-Langlands correspondence and its main properties. In characteristic zero, we can largely rely on the original sources [21], [37]. In positive characteristic, the situation is more involved. The principal result is given in [2], relying on [1], [29], [30]. A crucial point for us comes from the recent [16].

One proof has been deferred to the end of the next section, as it will be easier to treat with the machinery of hereditary orders to hand.

1.1. We recall some basic results from the Appendix to [8]. Those pages were written for the split case $G = \text{GL}_n(F)$, but no attention was paid to the characteristic of F . The proofs apply unchanged to the present situation.

Let $g \in G$ and denote by $f_g^A(t) \in F[t]$ the *reduced characteristic polynomial* of g . For us, g is *regular* if $f_g^A(t)$ has no repeated root in an algebraic closure of F . It is *elliptic regular* if it is regular and $f_g^A(t)$ is irreducible over F . We denote by G_{reg} , $G_{\text{reg}}^{\text{ell}}$ the sets of regular or elliptic regular elements of G respectively.

Remark. In [8], we introduced the notion of a *quasi-regular element* of G : an element $g \in G$ is quasi-regular if $f_g^A(t)$ has no repeated irreducible factor over F . An element is *elliptic quasi-regular* if it is quasi-regular and $f_g^A(t)$ is irreducible over F . Everything we do applies equally, on replacing “regular” by “quasi-regular” and “elliptic regular” by “elliptic quasi-regular”.

We fix a Haar measure μ_G on G and use it to define the structure of a convolution algebra on the space $\mathcal{H}(G)$ of locally constant, compactly supported functions $G \rightarrow \mathbb{C}$. If (π, V) is an irreducible smooth representation of G , we extend π to an algebra homomorphism $\pi : \mathcal{H}(G) \rightarrow \text{End}_{\mathbb{C}}(V)$ in the standard way [13] §4. Since (π, V) is admissible, each operator $\pi(\varphi)$, $\varphi \in \mathcal{H}(G)$, has finite-dimensional range and so its trace $\text{tr } \pi(\varphi)$ is defined. As in [8] A.11, we have:

Proposition. *There is a locally constant function $\text{tr } \pi : G_{\text{reg}} \rightarrow \mathbb{C}$ such that*

$$\text{tr } \pi(\varphi) = \int_G \text{tr } \pi(g) \varphi(g) d\mu_G(g),$$

for all $\varphi \in \mathcal{H}(G)$ with support contained in G_{reg} .

One refers to the function $\text{tr } \pi$ as the *character* of π .

1.2. We need more detail in one particular case. We take an irreducible *cuspidal* representation π of G and assume it is given as an induced representation

$$\pi \cong c\text{-Ind}_{\mathbf{J}}^G \Lambda,$$

where \mathbf{J} is an open subgroup of G , compact modulo the centre F^\times of G , and Λ is an irreducible smooth representation of \mathbf{J} . In particular, $\dim \Lambda$ is finite. If K is a compact open subgroup of G , A.14 of [8] gives the expression

$$(1.2.1) \quad \text{tr } \pi(g) = \sum_{x \in K \backslash G / \mathbf{J}} \sum_{y \in Kx\mathbf{J} / \mathbf{J}} \text{tr } \Lambda(y^{-1}gy), \quad g \in G_{\text{reg}}.$$

Here, we regard $\text{tr } \Lambda$ as a function on G , vanishing outside of \mathbf{J} . If we confine g to a fixed compact subset C of G_{reg} , only finitely many of the inner terms

$$\sum_{y \in Kx\mathbf{J} / \mathbf{J}} \text{tr } \Lambda(y^{-1}gy)$$

can be non-zero *loc. cit.*

This uniform convergence property of (1.2.1) is stronger than we need for the purely algebraic manipulations in this paper. The following non-uniform version is easier to use here.

Lemma. *Let $g \in G_{\text{reg}}^{\text{ell}}$. There exist only finitely many cosets $x\mathbf{J} \in G/\mathbf{J}$ such that $x^{-1}gx \in \mathbf{J}$. In particular, only finitely many terms $\text{tr } \Lambda(x^{-1}gx)$, $x \in G/\mathbf{J}$, are non-zero.*

Proof. The F -subalgebra $F[g]$ of A is a field, and any two F -embeddings of $F[g]$ in A are G -conjugate, by the Skolem-Noether Theorem. It follows that the G -conjugacy class $C(g) = \{x^{-1}gx : x \in G\}$ of g is the set of zeros of the polynomial function $h \mapsto f_g^A(h)$, $h \in G$. That is, $C(g)$ is closed in G and hence locally compact. Consider the intersection $\mathbf{J} \cap C(g)$. The reduced norm map Nrd_A is constant on $C(g)$. On the other hand, \mathbf{J} has a unique maximal compact subgroup $J = \{x \in \mathbf{J} : \|\text{Nrd}_A x\| = 1\}$ and \mathbf{J}/J is cyclic. It follows that $\mathbf{J} \cap C(g)$ is compact.

The G -centralizer of g is the group $F[g]^\times$, and $F[g]^\times/F^\times$ is compact. The set S of $x \in G$ for which $x^{-1}gx \in \mathbf{J}$ satisfies $\mathbf{J}SF[g]^\times = S$. The local compactness of $C(g)$ implies, via Arens' Theorem [32] 2.13, that the canonical map $G/F[g]^\times \rightarrow C(g)$ is a homeomorphism. Consequently, the image $S/F[g]^\times$ of S in $G/F[g]^\times$ is homeomorphic to $\mathbf{J} \cap C(g)$ and is compact. Hence S is compact modulo F^\times . The quotient space $\mathbf{J} \backslash S$ is therefore finite, as required. \square

The lemma allows us to use the Mackey Formula (1.2.1) in the form

$$(1.2.2) \quad \text{tr } \pi(g) = \sum_{x \in G/\mathbf{J}} \text{tr } \Lambda(x^{-1}gx), \quad g \in G_{\text{reg}}^{\text{ell}},$$

without having to be concerned about convergence issues.

1.3. The group G acts on the set $G_{\text{reg}}^{\text{ell}}$ by conjugation. The map $g \mapsto f_g^A(t)$ induces a bijection between $G \backslash G_{\text{reg}}^{\text{ell}}$ and the set of monic, irreducible, separable polynomials over F of degree n (cf. 2.1 below).

The same applies if we replace A by another n^2 -dimensional, central simple F -algebra A' and set $G' = A'^\times$. Thus we have a canonical bijection $G \backslash G_{\text{reg}}^{\text{ell}} \cong G' \backslash G'_{\text{reg}}^{\text{ell}}$. We say that elements $g \in G_{\text{reg}}^{\text{ell}}$, $g' \in G'_{\text{reg}}^{\text{ell}}$ are *associate* if their conjugacy classes correspond under this bijection, that is, if $f_{g'}^{A'} = f_g^A$.

If $g \in G_{\text{reg}}^{\text{ell}}$, it is in practice safe to use the same notation g for an element of $G'_{\text{reg}}^{\text{ell}}$ which is associate to g .

1.4. Let $A = M_m(D)$ and $G = A^\times = \mathrm{GL}_m(D)$. If π is a smooth representation of G and χ is a character of F^\times , we denote by $\chi\pi$ the representation

$$\chi\pi : g \longmapsto \chi(\mathrm{Nrd}_A g) \pi(g), \quad g \in G.$$

Let $\mathcal{A}_m^\square(D)$ denote the set of equivalence classes of irreducible smooth representations of G which are *essentially square-integrable* modulo the centre F^\times of G .

Jacquet-Langlands Correspondence. Let D, D' be central F -division algebras of dimension d^2, d'^2 respectively. Let m, m' be positive integers such that $md = m'd' = n$. Set $G = \mathrm{GL}_m(D), G' = \mathrm{GL}_{m'}(D')$. There is a unique bijection

$$(1.4.1) \quad \begin{aligned} \mathbf{j} : \mathcal{A}_m^\square(D) &\longrightarrow \mathcal{A}_{m'}^\square(D'), \\ \pi &\longmapsto \pi', \end{aligned}$$

with the following property. If $g \in G_{\mathrm{reg}}^{\mathrm{ell}}$ and if $g' \in G'_{\mathrm{reg}}^{\mathrm{ell}}$ is associate to g , then

$$(-1)^m \mathrm{tr} \pi(g) = (-1)^{m'} \mathrm{tr} \pi'(g').$$

We refer to the map \mathbf{j} as *the Jacquet-Langlands correspondence* between G and G' .

We note some obvious properties, using the same notation.

(1.4.2) If $\pi \in \mathcal{A}_m^\square(D)$ and $\pi' = \mathbf{j}\pi$, then the central characters $\omega_\pi, \omega_{\pi'}$ of π, π' are equal.

(1.4.3) If $\pi \in \mathcal{A}_m^\square(D)$ and $\pi' = \mathbf{j}\pi$, then $\mathbf{j}(\chi\pi) = \chi\pi'$, for any character χ of F^\times .

For an irreducible smooth representation π of (say) G , we let $t(\pi)$ denote the number of unramified characters χ of F^\times such that $\chi\pi \cong \pi$. From (1.4.3) we deduce that:

(1.4.4) If $\pi \in \mathcal{A}_m^\square(D)$ and $\pi' = \mathbf{j}\pi$, then $t(\pi) = t(\pi')$.

We shall also need the following fact, which is bound up in the proof of the existence and uniqueness of the Jacquet-Langlands correspondence in all cases. We treat it as a consequence from which we shall derive stronger results of the same kind.

(1.4.5) Let $\pi_1, \pi_2, \dots, \pi_r$ be distinct elements of $\mathcal{A}_m^\square(D)$, and set $G = \mathrm{GL}_m(D)$. The set of functions on $G_{\mathrm{reg}}^{\mathrm{ell}}$, defined by

$$\{\mathrm{tr} \pi_i \mid G_{\mathrm{reg}}^{\mathrm{ell}} : 1 \leq i \leq r\},$$

is linearly independent.

Proof. Let C be a central F -division algebra of dimension $n^2 = m^2 d^2$, and write $G' = \mathrm{GL}_1(C)$. The set $G_{\mathrm{reg}}^{\mathrm{ell}}$ is dense in G' and, for $\pi \in \mathcal{A}_1^\square(C)$, the function $\mathrm{tr} \pi$ is locally constant on G' . The result is therefore elementary for G' and holds in general as a consequence of (1.4.1). \square

1.5. Write $G = \mathrm{GL}_m(D)$ and let $(\pi, V) \in \mathcal{A}_m^\square(D)$ be square-integrable modulo centre. Let $(\check{\pi}, \check{V})$ be the contragredient of (π, V) and let $\dot{\mu}_G$ be a Haar measure on G/F^\times . For $v_i \in V$ and $\check{v}_i \in \check{V}$, we have the first Schur orthogonality relation

$$\int_{G/F^\times} \langle \check{\pi}(g)\check{v}_1, v_1 \rangle \langle \check{v}_2, \pi(g)v_2 \rangle d\dot{\mu}_G(g) = d(\pi, \dot{\mu}_G)^{-1} \langle \check{v}_1, v_2 \rangle \langle \check{v}_2, v_1 \rangle,$$

where $d(\pi, \dot{\mu}_G)$ is a positive real constant called the *formal degree* of π relative to $\dot{\mu}_G$. (See, for instance, [13] 10a.2 for an exposition of this topic.)

Clearly, if χ is a character of F^\times which is *unitary*, in that $|\chi| = 1$, then $\chi\pi$ is square-integrable mod. centre and $d(\chi\pi, \dot{\mu}_G) = d(\pi, \dot{\mu}_G)$. For an arbitrary $\pi \in \mathcal{A}_m^\square(D)$, there is an unramified character χ of F^\times such that $\pi_0 = \chi\pi$ is square-integrable mod. centre. We define $d(\pi, \dot{\mu}_G) = d(\pi_0, \dot{\mu}_G)$.

We discuss ways in which the Haar measure $\dot{\mu}_G$ may be conveniently normalized. In the case $m = 1$, the group D^\times/F^\times is compact, so one may choose $\dot{\mu}_{D^\times}$ to give D^\times/F^\times mass 1. For this choice of Haar measure, every $\pi \in \mathcal{A}_1^\square(D)$ has formal degree equal to $\dim \pi$.

In general, let $\pi \in \mathcal{A}_m^\square(D)$. The *cuspidal support* of π then consists of unramified twists of a cuspidal representation $\sigma \in \mathcal{A}_{m'}^\square(D)$, for some divisor m' of m . In the case where $m' = 1$ and σ is of dimension 1, the representation π is of the form $\chi \cdot \mathrm{St}_G$, where χ is a unitary character of F^\times and St_G is the *Steinberg representation* of G . (We recall more of this matter in Appendix 1 below.)

There is a unique Haar measure $\ddot{\mu}_G$ on G/F^\times such that

$$(1.5.1) \quad d(\mathrm{St}_G, \ddot{\mu}_G) = 1.$$

We set

$$(1.5.2) \quad \deg \pi = d(\pi, \ddot{\mu}_G), \quad \pi \in \mathcal{A}_m^\square(D),$$

and call $\deg \pi$ the *normalized formal degree* of π .

The following result may be found in [21] when F has characteristic zero, in [16] when F has characteristic p (but the scheme of proof in [16] applies equally in characteristic zero).

Theorem.

- (1) Let $\pi \in \mathcal{A}_m^\square(D)$; there is an open neighbourhood \mathcal{N} of 1 in G such that

$$\mathrm{tr} \pi(g) = (-1)^{m-1} \deg \pi, \quad g \in \mathcal{N} \cap G_{\mathrm{reg}}^{\mathrm{ell}}.$$

- (2) Using the notation of (1.4), we have

$$\deg \pi = \deg \pi',$$

for every $\pi \in \mathcal{A}_m^\square(D)$ and $\pi' = j\pi \in \mathcal{A}_{m'}^\square(D')$.

1.6. If G is any of the groups $\mathrm{GL}_m(D)$, an element $g \in G$ is called *pro-unipotent* if $g^{p^r} \rightarrow 1$ as $r \rightarrow \infty$. Let ${}_uG_{\mathrm{reg}}^{\mathrm{ell}}$ denote the set of pro-unipotent, elliptic regular elements of G .

Let \mathcal{N} be an open neighbourhood of 1 in $G = \mathrm{GL}_m(D)$. We say that \mathcal{N} is *u-small* if $\mathcal{N} \cap G_{\mathrm{reg}}^{\mathrm{ell}} \subset {}_uG_{\mathrm{reg}}^{\mathrm{ell}}$. Such neighbourhoods exist. For example, the group of elements $x \in \mathrm{GL}_m(\mathfrak{o}_D)$ such that $x \equiv 1 \pmod{\mathfrak{p}_D}$ is open in G and is a u-small neighbourhood of 1.

Lemma. Let $G = \mathrm{GL}_m(D)$, $G' = \mathrm{GL}_{m'}(D')$, for central F -division algebras D , D' such that $m^2 \dim D = m'^2 \dim D'$. Let \mathcal{N} be a u-small open neighbourhood of 1 in G . There exists a u-small open neighbourhood \mathcal{N}' of 1 in G' such that every element of $\mathcal{N}' \cap {}_uG'_{\mathrm{reg}}^{\mathrm{ell}}$ is associate to an element of \mathcal{N} .

Proof. See 2.9 below. \square

We use the lemma to prove the following result, needed in 7.9 at the end of the proof of the Second Comparison Theorem (7.1).

Proposition. *Let $(u_j)_{j \geq 1}$ be a sequence of elements of ${}_u G_{\text{reg}}^{\text{ell}}$ such that $u_j \rightarrow 1$ as $j \rightarrow \infty$. There is a sequence of elements $u'_j \in {}_u G'_{\text{reg}}{}^{\text{ell}}$ such that u'_j is associate to u_j , for each j , and $u'_j \rightarrow 1$.*

Proof. We have to show that, if \mathcal{N}' is an open neighbourhood of 1 in G' , then we may choose u'_j associate to u_j such that $u'_j \in \mathcal{N}'$ for all but finitely many values of j . We may assume that \mathcal{N}' is u -small. By the lemma, there is a u -small open neighbourhood \mathcal{N} of 1 in G such that every element of $\mathcal{N} \cap {}_u G_{\text{reg}}^{\text{ell}}$ is associate to an element of \mathcal{N}' . However, \mathcal{N} contains u_j for all but finitely many j . \square

2. Types and cuspidal representations

We recall material of a more algebraic nature. We work with a central simple F -algebra A of dimension n^2 , $n \geq 1$, and the group $G = A^\times$. We review the structure theory for the irreducible cuspidal representations of the locally profinite group G , as it is worked out in [38–41]. We give no detail at all, beyond establishing the vocabulary and standardizing the notation.

This analysis leads to a notion of *parametric degree* generalizing that of [9]. We connect it with the normalized formal degree of 1.5, to reach the starting point of the paper.

2.1. We recall standard facts concerning the algebra A . Most of this material can be found in, for example, [35].

Let V be a simple left A -module. As such, V is uniquely determined up to A -isomorphism. We define $\text{End}_A(V)$ so that V is a *right* $\text{End}_A(V)$ -module. The F -algebra $D = \text{End}_A(V)$ is a division algebra. The given action of A on V induces an isomorphism $A \cong \text{End}_D(V)$ of F -algebras¹.

Let V have dimension m as D -vector space. The choice of a D -basis of V induces an isomorphism $A \cong M_m(D)$. It follows that $\dim_F D = d^2$, where $md = n$, and $\dim_F V = nd$. The choice of a basis, and hence of an isomorphism $A \cong M_m(D)$, enables us to view D as embedded in A as the algebra of “scalar matrices” $\text{diag}(x, x, \dots, x)$, $x \in D$. This embedding does, of course, vary with the choice of basis.

¹The authors of [38–41] prefer to have $\text{End}_A(V)$ acting on the left, so D here corresponds to the *opposite* of D in those papers.

We mention some basic facts concerning subfields of A . Let $E \supset F$ be a subfield of A . The degree $[E:F]$ thus divides n . Conversely, any field extension E/F , of degree dividing n , is isomorphic to an F -subalgebra of A . Any two F -embeddings $E \rightarrow A$ are G -conjugate.

(2.1.1) *Let E/F be a subfield of A , of degree r , and let B denote the A -centralizer of E .*

- (1) *The algebra B is a central simple E -algebra of E -dimension n^2/r^2 .*
- (2) *There is an E -isomorphism $B \cong M_l(C)$, where C is a central E -division algebra of dimension c^2 , such that $c = d/\gcd(d, r)$ and $lc = n/r$.*

The assertion concerning the value of c follows from [35] Theorem 31.9 (or see [44]).

2.2. We make summary remarks (see [35] or [7]) concerning the class of hereditary \mathfrak{o}_F -orders in A .

We view V as a right D -vector space. An \mathfrak{o}_D -lattice in V is a finitely generated \mathfrak{o}_D -submodule of V which spans V over D (or F).

Let \mathcal{L} be an \mathfrak{o}_D -lattice chain in V . Thus \mathcal{L} is a non-empty set of right \mathfrak{o}_D -lattices in V , linearly ordered by inclusion and stable under translation by D^\times . It admits an enumeration $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$, with the following properties.

(2.2.1)

- (1) $L_i \supsetneq L_{i+1}$, $i \in \mathbb{Z}$, and
- (2) *there exists an integer $e \geq 1$ such that $L_{i+e} = L_i \mathfrak{p}_D$, $i \in \mathbb{Z}$.*

The integer $e = e_D(\mathcal{L})$ is the D -period of \mathcal{L} .

For $j \in \mathbb{Z}$, put

$$\mathfrak{A}_j(\mathcal{L}) = \{x \in A : xL_i \subset L_{i+j}, i \in \mathbb{Z}\}.$$

The set $\mathfrak{A} = \mathfrak{A}_0(\mathcal{L})$ is a ring, and an \mathfrak{o}_F -order in A . It is indeed a *hereditary \mathfrak{o}_F -order*, in the sense of [35]. All such orders arise, in this way, from lattice chains in V .

The set $\mathfrak{P} = \mathfrak{A}_1(\mathcal{L})$ is the Jacobson radical of \mathfrak{A} (we use the notation $\mathfrak{P} = \text{rad } \mathfrak{A}$) and $\mathfrak{A}_j(\mathcal{L}) = \mathfrak{P}^j$ (with a suitable interpretation in the case $j < 0$).

The lattice chain \mathcal{L} is the set of all \mathfrak{A} -lattices in V . We use the alternative notation $e(\mathfrak{A}|\mathfrak{o}_D) = e_D(\mathcal{L})$. We will also use the notation $e(\mathfrak{A}|\mathfrak{o}_F)$ for F -period of \mathcal{L} , defined by $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}^{e(\mathfrak{A}|\mathfrak{o}_F)}$. As $\mathfrak{p}_F\mathfrak{o}_D = \mathfrak{p}_D^d$, we have $e(\mathfrak{A}|\mathfrak{o}_F) = e(\mathfrak{A}|\mathfrak{o}_D)d$.

We set $U_{\mathfrak{A}} = U_{\mathfrak{A}}^0 = \mathfrak{A}^\times$ and $U_{\mathfrak{A}}^j = 1 + \mathfrak{P}^j$, $j \geq 1$. These groups $U_{\mathfrak{A}}^j$, $j \geq 0$, are compact open subgroups of G . We shall also need the group

$$\mathcal{K}_{\mathfrak{A}} = \{g \in G : g\mathfrak{A}g^{-1} = \mathfrak{A}\}.$$

This is an open subgroup of G , compact modulo the centre F^\times of G . It normalizes all of the groups $U_{\mathfrak{A}}^k$, $k \geq 0$. Writing $\mathfrak{A} = \mathfrak{A}_0(\mathcal{L})$ as above, we can also view $\mathcal{K}_{\mathfrak{A}}$ as the group of $g \in G$ for which $g\mathcal{L} = \mathcal{L}$.

The group $\mathcal{K}_{\mathfrak{A}}$ comes with a canonical homomorphism $v_{\mathfrak{A}} : \mathcal{K}_{\mathfrak{A}} \rightarrow \mathbb{Z}$ defined by the equivalent conditions $g\mathfrak{A} = \mathfrak{A}g = \mathfrak{P}^{v_{\mathfrak{A}}(g)}$ or $gL_i = L_{i+v_{\mathfrak{A}}(g)}$, for $g \in \mathcal{K}_{\mathfrak{A}}$ and $i \in \mathbb{Z}$. The homomorphism $v_{\mathfrak{A}} : \mathcal{K}_{\mathfrak{A}} \rightarrow \mathbb{Z}$ is surjective if and only if there exists $x \in \mathcal{K}_{\mathfrak{A}}$ such that $\mathfrak{P} = x\mathfrak{A} = \mathfrak{A}x$. Hereditary orders with this property are called *principal* orders [7].

Remark. Since we give no proofs in this area, we have no need of the notion of a “lattice sequence” in V , or the associated non-standard filtration of $U_{\mathfrak{A}}$, which plays such an important rôle in [38–41]. Indeed, we only ever deal with the standard filtration $(U_{\mathfrak{A}}^j)$ of $U_{\mathfrak{A}}$, and the hereditary order \mathfrak{A} will almost always be principal.

2.3. We combine the considerations of the preceding two paragraphs. Let \mathfrak{A} be a hereditary \mathfrak{o}_F -order in A and E/F a subfield of A . We say that \mathfrak{A} is *E-pure* if $E^\times \subset \mathcal{K}_{\mathfrak{A}}$.

We recall [41] 1.7:

(2.3.1) *Let E/F be a subfield of A and let B denote the A -centralizer of E .*

- (1) *Let \mathfrak{A} be an E -pure hereditary \mathfrak{o}_F -order in A . The set $\mathfrak{B} = \mathfrak{A} \cap B$ is a hereditary \mathfrak{o}_E -order in B and $\mathfrak{Q} = \mathfrak{P} \cap B$ is the Jacobson radical of \mathfrak{B} .*
- (2) *Let \mathfrak{B} be a hereditary \mathfrak{o}_E -order in B and put $\mathfrak{Q} = \text{rad } \mathfrak{B}$. There is a unique pair (ρ, \mathfrak{A}) , consisting of an integer $\rho \geq 1$ and an E -pure hereditary \mathfrak{o}_F -order in A , such that $\mathfrak{P}^k \cap B = \mathfrak{Q}^{k'}$, $k \in \mathbb{Z}$, where $\mathfrak{P} = \text{rad } \mathfrak{A}$ and k' is the least integer $\geq k/\rho$. Moreover, $\mathcal{K}_{\mathfrak{B}} = \mathcal{K}_{\mathfrak{A}} \cap B$.*

If \mathfrak{A} is any E -pure hereditary order in A , it has become common to refer to $\mathfrak{B} = \mathfrak{A} \cap B$ as the “trace” of \mathfrak{A} , and to \mathfrak{A} as a “continuation” of \mathfrak{B} to A . If we start from \mathfrak{B} , we refer to the order \mathfrak{A} , given by (2.3.1)(2), as the *canonical continuation* of \mathfrak{B} to A .

A special case is worthy of overt mention: see [5] for details.

(2.3.2) *Suppose, in the situation of (2.3.1)(2), that the \mathfrak{o}_E -order \mathfrak{B} is maximal. The canonical continuation \mathfrak{A} of \mathfrak{B} is then the unique E -pure hereditary \mathfrak{o}_F -order in A intersecting B in \mathfrak{B} . The order \mathfrak{A} is, moreover, maximal among E -pure hereditary orders in A .*

In the situation of (2.3.2), the orders \mathfrak{B} , \mathfrak{A} , are both principal.

2.4. We examine the irreducible cuspidal representations of G , starting with an extreme case. Let \mathfrak{A} be a *maximal* \mathfrak{o}_F -order in A . The order \mathfrak{A} is therefore given by an \mathfrak{o}_D -lattice chain of the form $\mathcal{L} = \{L\mathfrak{p}_D^i : i \in \mathbb{Z}\}$, for a fixed \mathfrak{o}_D -lattice L in V . Any two maximal orders in A are G -conjugate.

We may identify A with $M_m(D)$ via a basis of V , as in 2.1, chosen to be an \mathfrak{o}_D -basis of L . The order \mathfrak{A} then becomes identified with $M_m(\mathfrak{o}_D)$. Taking the case $\mathfrak{A} = M_m(\mathfrak{o}_D)$ and viewing D as the algebra of scalar matrices in A (as in 2.1), we get

$$\begin{aligned} \text{rad } \mathfrak{A} &= \mathfrak{p}_D \mathfrak{A} = \mathfrak{A} \mathfrak{p}_D = M_m(\mathfrak{p}_D), \\ \mathcal{K}_{\mathfrak{A}} &= D^\times U_{\mathfrak{A}}. \end{aligned}$$

We conclude that $U_{\mathfrak{A}}/U_{\mathfrak{A}}^1 \cong \text{GL}_m(\mathbb{k}_D)$. If ϖ_D is a prime element of D , viewed as an element of $\mathcal{K}_{\mathfrak{A}}$, then ϖ_D acts (by conjugation) on this group. The action is that of a generator of $\text{Gal}(\mathbb{k}_D/\mathbb{k}_F)$ acting on matrix coefficients.

Definition 1. *A maximal simple type of level zero in G is a pair (U, λ) , where*

- (1) $U = U_{\mathfrak{A}}$, for a maximal \mathfrak{o}_F -order \mathfrak{A} in A , and
- (2) λ is an irreducible representation of $U_{\mathfrak{A}}$, trivial on $U_{\mathfrak{A}}^1$ and inflated from an irreducible cuspidal representation of the finite group $U_{\mathfrak{A}}/U_{\mathfrak{A}}^1 \cong \text{GL}_m(\mathbb{k}_D)$.

Let (U, λ) , $U = U_{\mathfrak{A}}$, be a maximal simple type of level zero in G . The following lemma is a straightforward instance of the more general considerations of [24]. However, it can equally be proved by a direct calculation, essentially identical to that in the split case $G = \text{GL}_n(F)$.

Lemma.

- (1) *Let τ be an irreducible representation of $U = U_{\mathfrak{A}}$, trivial on $U_{\mathfrak{A}}^1$. If τ intertwines with λ in G , there exists $x \in \mathcal{K}_{\mathfrak{A}}$ such that $\tau^x \cong \lambda$.*
- (2) *Let $\mathbf{J}(\lambda)$ denote the set of elements of G which intertwine the representation λ , let w_λ be the number of distinct $\mathcal{K}_{\mathfrak{A}}$ -conjugates of λ . The set $\mathbf{J}(\lambda)$ is then the unique subgroup of $\mathcal{K}_{\mathfrak{A}}$ of index w_λ and containing $F^\times U_{\mathfrak{A}}$. The group $\mathbf{J}(\lambda)$ normalizes the pair (U, λ) .*

The integer w_λ may also be described as follows. The representation λ is inflated from an irreducible cuspidal representation $\tilde{\lambda}$ of $\mathrm{GL}_m(\mathbb{k}_D)$, via an isomorphism $U_{\mathfrak{A}}/U_{\mathfrak{A}}^1 \cong \mathrm{GL}_m(\mathbb{k}_D)$. The number of distinct conjugates of $\tilde{\lambda}$, under the natural action of $\mathrm{Gal}(\mathbb{k}_D/\mathbb{k}_F)$, is then w_λ .

The group $\mathcal{K}_{\mathfrak{A}}/F^\times U_{\mathfrak{A}}$ is cyclic of order d , so w_λ divides d . In this situation, we define the *parametric degree* $\delta_0(\lambda)$ of λ by

$$(2.4.1) \quad \delta_0(\lambda) = mw_\lambda.$$

Remark 1. In practice, it is usually more convenient to think of $\tilde{\lambda}$ as being given, via the Green parametrization, by a character χ of \mathbf{l}^\times , where \mathbf{l}/\mathbb{k}_D is a field extension of degree m contained in $M_m(\mathbb{k}_D)$. Since $\tilde{\lambda}$ is cuspidal, the character χ is \mathbf{l}/\mathbb{k}_D -regular, in that its conjugates χ^σ , $\sigma \in \mathrm{Gal}(\mathbf{l}/\mathbb{k}_D)$ are distinct. Moreover,

$$(2.4.2) \quad \mathrm{tr} \tilde{\lambda}(x) = (-1)^{m-1} \sum_{\sigma \in \mathrm{Gal}(\mathbf{l}/\mathbb{k}_D)} \chi^\sigma(x),$$

for every \mathbf{l}/\mathbb{k}_D -regular element x of \mathbf{l}^\times . The number of distinct conjugates of χ under $\mathrm{Gal}(\mathbf{l}/\mathbb{k}_F)$ is then $\delta_0(\lambda) = mw_\lambda$. (For the Green parametrization, see [25] or [31], although the authors find the character tables in [34] particularly convenient. The summary in [14] §2 will be adequate for the purposes of this paper.)

We note the following straightforward consequence of the lemma.

Proposition. *Let (U, λ) be a maximal simple type in G of level zero.*

- (1) *The representation λ admits extension to a representation Λ of the group $\mathbf{J}(\lambda)$.*

- (2) The representation Λ is uniquely determined up to twisting with a character of the form $\chi_A = \chi \circ \text{Nrd}_A$, where χ is an unramified character of F^\times .
- (3) Let χ be an unramified character of F^\times . The representations Λ and $\chi_A \otimes \Lambda$ intertwine in G if and only if they are equivalent. Moreover, $\Lambda \cong \chi_A \otimes \Lambda$ if and only if $\chi^{\delta_0(\lambda)} = 1$.
- (4) The representation

$$\pi_\Lambda = c\text{-Ind}_{\mathbf{J}(\lambda)}^G \Lambda$$

of G is irreducible and cuspidal.

Representations $(\mathbf{J}(\lambda), \Lambda)$, of this form, constitute the class of *extended maximal simple types of level zero* in G .

Let π be an irreducible cuspidal representation of G . One says that π has *level zero* if it admits a non-zero vector fixed by $U_{\mathfrak{A}}^1$, for some maximal \mathfrak{o}_F -order \mathfrak{A} in A . The main result in this area is:

Theorem. *Let π be an irreducible cuspidal representation of G , of level zero.*

- (1) The representation π contains a maximal simple type (U, λ) of level zero.
- (2) There exists an extended maximal simple type $(\mathbf{J}(\lambda), \Lambda)$ such that $\Lambda \upharpoonright U \cong \lambda$ and $\pi \cong \pi_\Lambda$.
- (3) The representation π determines both (U, λ) and $(\mathbf{J}(\lambda), \Lambda)$ uniquely, up to G -conjugacy.
- (4) The number $t(\pi)$ of unramified characters χ of F^\times satisfying $\chi\pi \cong \pi$ is given by $t(\pi) = \delta_0(\lambda)$.

The proof is comparatively straightforward, mimicking the corresponding result in the split case. It can again be regarded as a simple instance of [24].

Definition 2. *Let π be an irreducible cuspidal representation of $G \cong \text{GL}_m(D)$, of level zero. The parametric degree $\delta(\pi)$ of π is $\delta_0(\lambda)$, where (U, λ) is a maximal simple type contained in π .*

Part (4) of the theorem implies that the definition of $\delta(\pi)$ is independent of the choice of type λ .

Remark 2. In the split case $G = \text{GL}_n(F)$, an irreducible cuspidal representation of G (of level zero) always satisfies $\delta(\pi) = n$. This, however, fails for general

groups $GL_m(D)$. If π is an irreducible cuspidal representation of $GL_m(D)$ of level zero, then m divides $\delta(\pi)$ and $\delta(\pi)$ divides n , but $\delta(\pi)$ is subject to some further combinatorial constraints. For, let \mathbf{l}/\mathbb{k}_D be a field extension of degree m as above, and write $\Gamma = \text{Gal}(\mathbf{l}/\mathbb{k}_F)$, $\Delta = \text{Gal}(\mathbf{l}/\mathbb{k}_D)$. Let χ be a character of \mathbf{l}^\times , and let Φ be the subgroup of Γ which fixes χ . The character χ gives rise to a maximal simple type if and only if $\Phi \cap \Delta$ is trivial. In other words, if we fix a positive integer δ , divisible by m and dividing n , there exists an irreducible cuspidal representation π of G , of level zero and with $\delta(\pi) = \delta$, if and only if n/δ is relatively prime to m .

2.5. We proceed to the general case. We follow [38–40] and, particularly, the summary in [41]. We abbreviate the treatment as much as possible, while keeping the terminology and notation strictly parallel to those of [17].

Let E/F be a finite field extension, and let $\beta \in E^\times$. We assume throughout that $v_E(\beta) < 0$ and that $E = F[\beta]$. Let A be a central simple F -algebra as before and let $\iota : E \rightarrow A$ be an F -embedding. Let \mathfrak{A} be an ιE -pure hereditary \mathfrak{o}_F -order in A . The period $e_{\mathfrak{A}} = e(\mathfrak{A}|\mathfrak{o}_F) = de(\mathfrak{A}|\mathfrak{o}_D)$ is divisible by $e(E|F)$ and $v_{\mathfrak{A}}(\iota\beta) = e_{\mathfrak{A}}v_E(\beta)/e(E|F)$. Any quadruple $[\mathfrak{A}, \ell_{\mathfrak{A}}, r, \iota\beta]$, with $\ell_{\mathfrak{A}} = -v_{\mathfrak{A}}(\iota\beta)$ and $r < \ell_{\mathfrak{A}}$, is then a *pure stratum in A* .

Attached to the pure stratum $[\mathfrak{A}, \ell_{\mathfrak{A}}, r, \iota\beta]$ is the *critical exponent* $k_0(\iota\beta, \mathfrak{A}) \in \mathbb{Z} \cup \{-\infty\}$. By definition, $[\mathfrak{A}, \ell_{\mathfrak{A}}, r, \iota\beta]$ is *simple* if $-k_0(\iota\beta, \mathfrak{A}) > r$. The critical exponent satisfies

$$e(E|F) k_0(\iota\beta, \mathfrak{A}) = e_{\mathfrak{A}} k_0(\beta, \mathfrak{A}(E)),$$

where $\mathfrak{A}(E)$ is the unique E -pure hereditary \mathfrak{o}_F -order in the algebra $\text{End}_F(E)$.

We henceforward assume that $[\mathfrak{A}, \ell_{\mathfrak{A}}, 0, \iota\beta]$ is a *simple* stratum. We identify E with its image in A and suppress the notation ι . The stratum gives rise to a pair

$$\mathfrak{H}(\beta, \mathfrak{A}) \subset \mathfrak{J}(\beta, \mathfrak{A}) \subset \mathfrak{A}$$

of \mathfrak{o}_F -orders in A . We take the standard filtration subgroups of unit groups

$$\begin{aligned} H^k(\beta, \mathfrak{A}) &= \mathfrak{H}(\beta, \mathfrak{A}) \cap U_{\mathfrak{A}}^k = \mathfrak{H}(\beta, \mathfrak{A})^\times \cap U_{\mathfrak{A}}^k, \\ J^k(\beta, \mathfrak{A}) &= \mathfrak{J}(\beta, \mathfrak{A}) \cap U_{\mathfrak{A}}^k = \mathfrak{J}(\beta, \mathfrak{A})^\times \cap U_{\mathfrak{A}}^k, \end{aligned}$$

for $k \in \mathbb{Z}$, $k \geq 0$.

We choose, once for all, a character ψ_F of F which is “of level one”. That is to say, $\mathfrak{p}_F \subset \text{Ker } \psi_F$ but $\psi_F |_{\mathfrak{o}_F} \neq 1$. We use ψ_F to construct the finite set $\mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ of so-called *simple characters* of the group $H^1(\beta, \mathfrak{A})$.

Comment. The definitions in [38] of the orders \mathfrak{H} , \mathfrak{J} , and the simple character set \mathcal{C} are necessarily indirect. However, as pointed out in [38] Propositions 3.42 and 3.45, one may see *ex post facto* that the obvious generalization of the approach of [17], via simple approximations to pure strata, applies without significant change. When we need to enter into more detail, this is the method we shall adopt. We therefore say no more at this stage concerning the constructions in [38].

We summarize some of the main points.

(2.5.1) *Let B denote the A -centralizer of E , let $\mathfrak{B} = \mathfrak{A} \cap B$, and let $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. Abbreviate $J^k = J^k(\beta, \mathfrak{A})$, and so on.*

- (1) *We have $J^0 = U_{\mathfrak{B}} J^1$ and $U_{\mathfrak{B}} \cap J^1 = U_{\mathfrak{B}}^1$.*
- (2) *There exists a unique irreducible representation η_θ of J^1 such that $\eta_\theta |_{H^1}$ contains θ .*
- (3) *For $g \in G$, the following are equivalent:*
 - (a) *g intertwines θ ;*
 - (b) *g intertwines η_θ ;*
 - (c) *$g \in J^1 B^\times J^1$.*
- (4) *If $g \in J^1 B^\times J^1$, the coset $J^1 g J^1$ supports a one-dimensional space of intertwining operators for the representation η_θ .*

We need to delineate more carefully a phenomenon underlying parts of (2.5.1). The quotient

$$(2.5.2) \quad \mathfrak{V}(\beta, \mathfrak{A}) = \mathfrak{J}^1(\beta, \mathfrak{A}) / \mathfrak{H}^1(\beta, \mathfrak{A}) = J^1 / H^1$$

is a vector space over \mathbb{k}_E and hence an elementary abelian p -group. It has a sequence of subspaces

$$\mathfrak{V}^k(\beta, \mathfrak{A}) = J^k / H^k, \quad k \geq 1.$$

For $x, y \in J^k(\beta, \mathfrak{A})$, the commutator $[x, y] = xyx^{-1}y^{-1}$ lies in H^{2k} , and we consider the pairing $h_\theta : (x, y) \mapsto \theta[x, y]$. The values of this pairing are p -th

roots of unity, and h_θ defines an alternating bilinear form

$$(2.5.3) \quad h_\theta : \mathfrak{Y}^k(\beta, \mathfrak{A}) \times \mathfrak{Y}^k(\beta, \mathfrak{A}) \longrightarrow \mathbb{F}_p, \quad k \geq 1,$$

over \mathbb{F}_p . We then have [38] Théorème 3.52:

(2.5.4) *The alternating form (2.5.3) is nondegenerate, for every $k \geq 1$. As alternating space, $\mathfrak{Y}(\beta, \mathfrak{A})$ is the orthogonal sum of nondegenerate alternating spaces isometric to $\mathfrak{Y}^k(\beta, \mathfrak{A})/\mathfrak{Y}^{k+1}(\beta, \mathfrak{A}) = J^k/J^{k+1}H^k$, $k \geq 1$.*

For the next step, we need a simplified version of the main result of [39]. As before, $[\mathfrak{A}, \ell, 0, \beta]$ is a simple stratum in A and $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. Again, B is the A -centralizer of $E = F[\beta]$ and $\mathfrak{B} = \mathfrak{A} \cap B$.

(2.5.5) *There exists an irreducible representation κ of $J = J^0(\beta, \mathfrak{A})$ such that*

- (1) $\kappa \upharpoonright J^1 \cong \eta_\theta$, and
- (2) κ is intertwined by every element of B^\times .

These conditions determine κ uniquely, up to tensoring with a character of $U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 = J/J^1$ of the form $\chi \circ \text{Nrd}_B$, where χ is a character of U_E trivial on U_E^1 .

We call κ a *wide extension*² of η_θ . Since $\dim \kappa$ has p -power dimension, two wide extensions κ_i of η_θ are equivalent if and only if the p -regular parts of the characters $\det \kappa_i$ are the same.

2.6. We use a special case of the foregoing to make a fundamental definition.

Definition. *A maximal simple type of positive level in G is a pair (J, λ) constructed as follows.*

- (1) *There is a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A , such that $J = J^0(\beta, \mathfrak{A})$, satisfying the following condition: if B is the A -centralizer of the field $E = F[\beta]$ and $\mathfrak{B} = \mathfrak{A} \cap B$, then \mathfrak{B} is a maximal \mathfrak{o}_E -order in B .*
- (2) *There is a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ and a maximal simple type $(U_{\mathfrak{B}}, \sigma_{\mathfrak{B}})$ of level zero in B^\times such that λ is a representation of J of the form*

$$\lambda = \sigma \otimes \kappa,$$

²This term is intended to replace the arcane expression “ β -extension” surviving from [17].

where

- (a) κ is a wide extension of η_θ , and
- (b) σ is the representation of J , trivial on J^1 , deduced from $\sigma_{\mathfrak{B}}$ via the isomorphism $U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 \cong J/J^1$.

Using the same notation, we note a particular property given by [40] Proposition 4.3.

(2.6.1) *Let $g \in G$. The following conditions are equivalent:*

- (1) g intertwines λ ;
- (2) g normalizes J and $\lambda^g \cong \lambda$;
- (3) $g \in \mathbf{J}_B(\sigma_{\mathfrak{B}})J^1$, where $\mathbf{J}_B(\sigma_{\mathfrak{B}})$ is the normalizer of the simple type $(U_{\mathfrak{B}}, \sigma_{\mathfrak{B}})$ in B^\times .

We write $\mathbf{J}(\lambda) = \mathbf{J}_B(\sigma_{\mathfrak{B}})J^1$.

We define the *parametric degree* $\delta_0(\lambda)$ of the maximal simple type (J, λ) by

$$\delta_0(\lambda) = \delta_0(\sigma_{\mathfrak{B}}) [E:F].$$

One may show directly that $\delta_0(\lambda)$ does indeed depend only on the isomorphism class of the pair (J, λ) , and not on the simple stratum and simple character we chose to describe it. There is no need to enter into the detail, since it is a consequence of 2.7 Proposition below.

When speaking of a *maximal simple type in G* , we mean either a maximal simple type of level zero or a maximal simple type of positive level.

We now extend the definition as before. We recall that, if χ is a character of F^\times , then χ_A denotes the character $\chi \circ \text{Nrd}_A$ of G .

(2.6.2) *Let (J, λ) be a maximal simple type in G , as in the Definition.*

- (1) *The representation λ admits extension to a representation Λ of $\mathbf{J}(\lambda)$.*
- (2) *The set of such extensions consists of the representations $\chi_A \otimes \Lambda$, where χ ranges over the unramified characters of F^\times .*
- (3) *The representation $\chi_A \otimes \Lambda$ intertwines with Λ if and only if it is equivalent to Λ . The group of unramified characters χ satisfying $\chi_A \otimes \Lambda \cong \Lambda$ has order $\delta_0(\sigma_{\mathfrak{B}})f(E|F)$.*

(4) The representation of G defined by

$$\pi_\Lambda = c\text{-Ind}_{\mathbf{J}(\lambda)}^G \Lambda$$

is irreducible and cuspidal. Moreover,

- (a) if χ is an unramified character of F^\times , then $\pi_{\chi_A \otimes \Lambda} \cong \chi \pi_\Lambda$, and
- (b) $t(\pi_\Lambda) = \delta_0(\sigma_{\mathfrak{B}})f(E|F)$.

A pair of the form $(\mathbf{J}(\lambda), \Lambda)$, obtained from a maximal simple type (J, λ) as in (2.6.2), is called an *extended maximal simple type of positive level* in G . We can now complete the picture with the following, which is the main result of [41].

Theorem. *Let π be an irreducible cuspidal representation of G , not of level zero. There is a maximal simple type (J, λ) in G , of positive level, such that π contains λ . Consequently, there exists an extended maximal simple type (\mathbf{J}, Λ) in G such that $\pi \cong c\text{-Ind}_{\mathbf{J}}^G \Lambda$.*

2.7. Let π be an irreducible cuspidal representation of $G = A^\times \cong \text{GL}_m(D)$. Attached to π we have the numerical invariant $t(\pi)$ (as in 1.4). Comparing central characters, we see that $t(\pi)$ divides n . The representation π contains a maximal simple type (J, λ) , the parametric degree $\delta_0(\lambda)$ of which also divides n , and $t(\pi)$ divides $\delta_0(\lambda)$.

Proposition. *Let π be an irreducible cuspidal representation of $G \cong \text{GL}_m(D)$. If (J_i, λ_i) , $i = 1, 2$, are maximal simple types occurring in π , then $\delta_0(\lambda_1) = \delta_0(\lambda_2)$.*

Proof. We need an auxiliary result. To state it, we need another notation. If a, b are positive rational numbers, we write $a \equiv b \pmod{p^{\mathbb{Z}}}$ to mean that ab^{-1} is a power (positive or negative) of p . We choose a Haar measure μ on G/F^\times , with the property that $\mu(PF^\times/F^\times)$ is an integral power of p , for any open pro- p subgroup P of G .

Lemma. *Let (J, λ) be a maximal simple type in G . If it has positive level and is based on a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ (as in 2.6 Definition), set $E = F[\beta]$. If it has level zero, set $E = F$.*

If τ is an irreducible representation of G containing λ , then τ is cuspidal and

$$(2.7.1) \quad d(\tau, \mu) \equiv \frac{t(\tau)}{n} \frac{q-1}{q^{n/e(E|F)}-1} \pmod{p^{\mathbb{Z}}}.$$

Proof. If τ is an irreducible representation of G containing λ , then

$$\tau \cong c\text{-Ind}_{\mathbf{J}(\lambda)}^G \Theta,$$

for an irreducible representation Θ of $\mathbf{J}(\lambda)$ extending λ . Indeed, $(\mathbf{J}(\lambda), \Theta)$ is an extended maximal simple type. Theorem A.14 of [8] shows that the formal degree $d(\tau, \dot{\mu})$ of π is given by

$$d(\tau, \dot{\mu}) = \frac{\dim \Theta}{\dot{\mu}(\mathbf{J}(\lambda))} = \frac{\dim \lambda}{\dot{\mu}(\mathbf{J}(\lambda))}.$$

(As remarked in §1, the proof of this result in [8] applies unchanged to our present situation.) We evaluate this expression to sufficient accuracy.

Suppose, for the moment, that we are in the positive level case and use the notation of the definition in 2.6. We have $\dim \Theta = \dim \lambda = \dim \sigma_{\mathfrak{B}} \cdot \dim \kappa$, and $\dim \kappa$ is a power of p . If $B \cong M_l(C)$, where C is a central E -division algebra such that $\dim_E C = c^2$, then $\sigma_{\mathfrak{B}}$ is the inflation of an irreducible cuspidal representation of $\text{GL}_l(\mathbb{k}_C)$. Therefore

$$\dim \sigma_{\mathfrak{B}} \equiv \frac{|\text{GL}_l(\mathbb{k}_C)|}{q_C^l - 1} \pmod{p^{\mathbb{Z}}},$$

where $q_C = |\mathbb{k}_C|$. The group $\mathbf{J}(\lambda)$ contains $F^\times J$, $J = J^0(\beta, \mathfrak{A})$, with index $e(E|F)lc/\delta_0(\sigma_{\mathfrak{B}})$. Thus

$$\begin{aligned} \dot{\mu}(\mathbf{J}(\lambda)/F^\times) &= \frac{e(E|F)lc}{\delta_0(\sigma_{\mathfrak{B}})} \dot{\mu}(J/U_F) \\ &= \frac{n}{f(E|F)\delta_0(\sigma_{\mathfrak{B}})} \dot{\mu}(J/U_F) \\ &= \frac{n}{t(\tau)} \dot{\mu}(J/U_F). \end{aligned}$$

The measure factor satisfies $\dot{\mu}(J/U_F) \equiv |\text{GL}_l(\mathbb{k}_C)|/(q-1) \pmod{p^{\mathbb{Z}}}$, giving

$$d(\tau, \dot{\mu}) \equiv \frac{t(\tau)}{n} \frac{q-1}{q_C^l - 1} \pmod{p^{\mathbb{Z}}},$$

while $q_C = q_E^c$, so

$$q_C^l = q_E^{lc} = q_E^{n/[E:F]} = q^{n/e(E|F)}.$$

Thus the lemma is proved when (J, λ) has positive level. When (J, λ) has level zero, one uses the same argument with $E = F$ and $\mathrm{GL}_l(\mathbb{k}_C) = \mathrm{GL}_m(\mathbb{k}_D)$. \square

We return to the statement of the proposition. If π contains the two maximal simple types (J_i, λ_i) , (2.7.1) implies (in the obvious notation) that $e(E_1|F) = e(E_2|F)$. However, $t(\pi) = f(E_i|F)\delta_0(\sigma_{\mathfrak{B}_i})$, and $\delta_0(\lambda_i) = [E_i:F]\delta_0(\sigma_{\mathfrak{B}_i})$, whence the result follows. \square

We may now make a formal definition.

Definition. *Let π be an irreducible cuspidal representation of $G \cong \mathrm{GL}_m(D)$. The parametric degree $\delta(\pi)$ of π is $\delta(\pi) = \delta_0(\lambda)$, for any maximal simple type (J, λ) contained in π .*

We have already remarked, at the beginning of the paragraph:

(2.7.3) *Let π be an irreducible cuspidal representation of G . The integer $\delta(\pi)$ divides n and is divisible by $t(\pi)$.*

2.8. We now state a result connecting the preceding material with that of §1. The proof, however, requires a distinctive family of ideas, so we have given it separate treatment in Appendix 1.

Let $G = \mathrm{GL}_m(D)$ as before, with $\dim D = d^2$ and $n = md$. Take $\pi \in \mathcal{A}_m^\square(D)$. If π is cuspidal, its parametric degree $\delta(\pi)$ has just been defined in 2.7. Otherwise, there is a positive divisor r of m and an irreducible cuspidal representation σ of $\mathrm{GL}_r(D)$ such that the cuspidal support of π consists of unramified twists of σ (1.5). We set

$$(2.8.1) \quad \delta(\pi) = \delta(\sigma).$$

In this situation, we also have the relation

$$(2.8.2) \quad t(\pi) = t(\sigma).$$

Theorem. *Let $\pi \in \mathcal{A}_m^\square(D)$. The normalized formal degree $\deg \pi$ satisfies*

$$\deg \pi \equiv t(\pi) (q^n - 1) / (q^{t(\pi)n/\delta(\pi)} - 1) \pmod{p^{\mathbb{Z}}}.$$

Proof. See Appendix 1 below. \square

Corollary 1. *Let $j : \mathcal{A}_n^\square(F) \rightarrow \mathcal{A}_m^\square(D)$ denote the Jacquet-Langlands correspondence. If $\pi \in \mathcal{A}_n^\square(F)$, then $\delta(j\pi) = \delta(\pi)$.*

Proof. Write $\tau = j\pi$. We have $t(\tau) = t(\pi)$ (1.4.4) and $\deg \pi = \deg j\pi$ (1.5 Theorem). Consequently,

$$(q^{t(\pi)n/\delta(\pi)} - 1) \equiv (q^{t(\pi)n/\delta(\tau)} - 1) \pmod{p^{\mathbb{Z}}},$$

whence the assertion follows. \square

Let $\pi \in \mathcal{A}_m^\square(D)$. The integer $t(\pi)$ divides $\delta(\pi)$ (2.7.3). We say that π is *essentially tame* if p does not divide $\delta(\pi)/t(\pi)$.

Lemma. *Let $\pi \in \mathcal{A}_m^\square(D)$ be cuspidal. The representation π is essentially tame if and only if either*

- (a) π has level zero, or
- (b) π has positive level and contains a maximal simple type based on a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ such that the field extension $F[\beta]/F$ is tamely ramified.

A general element $\pi \in \mathcal{A}_m^\square(D)$ is essentially tame if and only if some element of its cuspidal support is essentially tame.

Proof. If π has level zero, then $t(\pi) = \delta(\pi)$ (2.4 Theorem), and so π is essentially tame. If π has positive level, it contains a maximal simple type based on a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$, and $\delta(\pi)/t(\pi) = e(F[\beta]|F)$. The final assertion follows from (2.8.1), (2.8.2). \square

The proof of (b) shows that, if π is essentially tame, then $F[\gamma]/F$ is tamely ramified for any simple stratum $[\mathfrak{A}', \ell', 0, \gamma]$ underlying a maximal simple type occurring in π .

Notation. *Let $\mathcal{A}_m^{\text{et}}(D)$ denote the set of $\pi \in \mathcal{A}_m^\square(D)$ which are essentially tame and satisfy $\delta(\pi) = n$.*

All elements of $\mathcal{A}_m^{\text{et}}(D)$ are cuspidal and, in particular, $\mathcal{A}_n^{\text{et}}(F)$ is the set of equivalence classes of irreducible, essentially tame, cuspidal representations of $\text{GL}_n(F)$. (This notation agrees with that of [11].) From Corollary 1 and (1.4.4), we obtain:

Corollary 2. *The Jacquet-Langlands correspondence $j : \mathcal{A}_n^\square(F) \rightarrow \mathcal{A}_m^\square(D)$ induces a bijection $\mathcal{A}_n^{\text{et}}(F) \rightarrow \mathcal{A}_m^{\text{et}}(D)$.*

Corollary 2 provides the true starting point of the paper. It is our task to render the map $\mathcal{A}_n^{\text{et}}(F) \rightarrow \mathcal{A}_m^{\text{et}}(D)$ explicit.

2.9. We prove 1.6 Lemma. For the first step, let C be a central F -division algebra of dimension $n^2 = m^2d^2$, $d^2 = \dim D$, and suppose that $G' = \text{GL}_1(C)$. In the algebra $A = M_m(D)$, we choose a maximal order \mathfrak{M} . Shrinking the given neighbourhood \mathcal{N} , we may assume that it takes the form $U_{\mathfrak{M}}^k$, for some integer $k \geq 1$. Set $\mathcal{N}' = U_C^{mk} = 1 + \mathfrak{p}_C^{mk}$, and let $x \in \mathcal{N}'$ be elliptic regular. Let $E = F[x]$, let $e' = e(F[x]|F)$ and let e be the least common multiple of e' and d . Since d divides e and e divides n , the order \mathfrak{M} contains a principal order \mathfrak{A} of F -period e . There is an F -embedding $\phi : E \rightarrow A$ such that \mathfrak{A} is $\phi(E)$ -pure. We write $x = 1+y$ and compare valuations:

$$v_{\mathfrak{A}}(\phi y) = ev_E(y)/e' = ev_C(y)/n \geq emk/n = ek/d.$$

However, $U_{\mathfrak{A}}^{ek/d} \subset U_{\mathfrak{M}}^k$, so x is associate to an element of \mathcal{N} , as required.

We next treat the case where $G = \text{GL}_1(C)$ and $G' = \text{GL}_m(D)$. We take \mathcal{N} of the form U_C^{mk} , for some $k \geq 1$. We fix a maximal order \mathfrak{M} in $M_m(D)$ as before, and set $\mathcal{N}' = U_{\mathfrak{M}}^k$. If $x = 1+y \in \mathcal{N}'$ is elliptic regular, we put $E = F[x]$ and define $e' = e(E|F)$ in the same way. We have

$$v_E(y) \geq e'v_{\mathfrak{M}}(y)/d \geq e'k/d.$$

If we choose an F -embedding $\phi : E \rightarrow C$, we get

$$v_C(\phi y) = nv_E(y)/e' \geq kn/d = mk.$$

It follows that x is associate to an element of \mathcal{N} , as required.

The general case of the proposition now follows from a simple transitivity argument. \square

3. Tame ramification

Let A be a central simple F -algebra, exactly as in §2. We set out, in an explicit form, some special properties of simple strata $[\mathfrak{A}, \ell, r, \beta]$ in A for which the field extension $F[\beta]/F$ is *tamely ramified*.

3.1. We start by recalling a fundamental property of tamely ramified simple strata.

Let E/F be a finite, tamely ramified field extension, let $\beta \in E^\times$ satisfy $E = F[\beta]$ and $v_E(\beta) < 0$. Let $A(E)$ denote the split central simple F -algebra $\text{End}_F(E)$ and let $\mathfrak{A}(E)$ denote the unique E -pure hereditary \mathfrak{o}_F -order in $A(E)$. Thus $\mathfrak{A}(E)$ is associated to the lattice chain $\{\mathfrak{p}_E^k : k \in \mathbb{Z}\}$ in the F -vector space E . Write $k_F(\beta) = k_0(\beta, \mathfrak{A}(E))$. We recall that $k_F(\beta) \leq v_E(\beta)$ if and only if β is minimal over F .

Proposition. *Suppose that $k_F(\beta) > v_E(\beta)$. There exists a simple stratum $[\mathfrak{A}(E), -v_E(\beta), -k_F(\beta), \gamma]$, equivalent to $[\mathfrak{A}(E), -v_E(\beta), -k_F(\beta), \beta]$ and with $\gamma \in E$.*

Proof. See 8.5 Lemma of [14]. \square

Let A be a central simple F -algebra of dimension n^2 . Suppose that E has been embedded in A as an F -subalgebra. Let \mathfrak{A} be an E -pure hereditary \mathfrak{o}_F -order in A .

Corollary. *Let $\ell = -v_{\mathfrak{A}}(\beta)$ and $r = -k_0(\beta, \mathfrak{A})$. The stratum $[\mathfrak{A}, \ell, r, \gamma]$ is then simple and equivalent to $[\mathfrak{A}, \ell, r, \beta]$.*

Proof. We have $\ell = -v_E(\beta)e(\mathfrak{A}|\mathfrak{o}_F)e(E|F)^{-1}$ and $r = -k(\beta)e(\mathfrak{A}|\mathfrak{o}_F)e(E|F)^{-1}$ (2.5). Applying these identities to γ , in place of β , the result follows. \square

3.2. We take a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A , such that $E = F[\beta]/F$ is *tamely ramified*. We define the orders $\mathfrak{H}(\beta, \mathfrak{A})$, $\mathfrak{J}(\beta, \mathfrak{A})$ and the simple character set $\mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ following the recipes in Propositions 3.42, 3.45 of [38]: these are strictly parallel to the treatment of the split case in [17]. It is convenient, in the present general discussion to impose the following simplifying assumption.

Hypothesis. *There is a subfield $\tilde{E} \supset E$ of A such that \mathfrak{A} is \tilde{E} -pure, \tilde{E}/E is unramified and $[\tilde{E}:F] = n$.*

The field \tilde{E} is its own centralizer in A and $\mathfrak{o}_{\tilde{E}}$ is the unique $\mathfrak{o}_{\tilde{E}}$ -order in \tilde{E} . It follows from (2.3.2) that \mathfrak{A} is the unique \tilde{E} -pure hereditary order in A . In particular, the hypothesis holds when the stratum $[\mathfrak{A}, \ell, 0, \beta]$ underlies a maximal simple type (as in 2.6 Definition). From the start of §4, we will always be in that situation.

Let K/F be a subfield of \tilde{E} : in particular, K/F is tamely ramified. We let A_K denote the A -centralizer of K and $\mathfrak{A}_K = \mathfrak{A} \cap A_K$. Thus (2.3.2) \mathfrak{A}_K is the unique \tilde{E} -pure hereditary \mathfrak{o}_K -order in A_K . Let $G_K = A_K^\times$.

Proposition.

(1) *The quadruple $[\mathfrak{A}_K, \ell, 0, \beta]$ is a simple stratum in A_K and*

$$\begin{aligned} \mathfrak{H}(\beta, \mathfrak{A}_K) &= \mathfrak{H}(\beta, \mathfrak{A}) \cap A_K, & \mathfrak{J}(\beta, \mathfrak{A}_K) &= \mathfrak{J}(\beta, \mathfrak{A}) \cap A_K, \\ H^1(\beta, \mathfrak{A}_K) &= H^1(\beta, \mathfrak{A}) \cap G_K, & J^1(\beta, \mathfrak{A}_K) &= J^1(\beta, \mathfrak{A}) \cap G_K. \end{aligned}$$

(2) *Let $\psi_K = \psi_F \circ \text{Tr}_{K/F}$ and let $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. The restriction $\theta_K = \theta \mid H^1(\beta, \mathfrak{A}_K)$ then lies in $\mathcal{C}(\mathfrak{A}_K, \beta, \psi_K)$.*

Proof. Regarding the simplicity of $[\mathfrak{A}_K, \ell, 0, \beta]$: this reduces to an assertion concerning the relation between $k_F(\beta)$ and $k_K(\beta)$. In other words, we need only consider the split case, where the result is given by [8] 2.4.

For the other assertions, we proceed by “induction along β ” in the manner familiar from [17]. Suppose first that β is minimal over F . Thus either $\beta \in K$ or β is minimal over K (cf. [8] 2.9). Let B denote the A -centralizer of β and set $\mathfrak{B} = \mathfrak{A} \cap B$. We have $\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{P}^{[\ell/2]+1}$, where $\mathfrak{P} = \text{rad } \mathfrak{A}$. Define an integer ℓ' by $\beta \mathfrak{A}_K = \mathfrak{P}_K^{-\ell'}$, where $\mathfrak{P}_K = \text{rad } \mathfrak{A}_K$. We then have $\mathfrak{P}^{[\ell/2]+1} \cap A_K \supset \mathfrak{P}_K^{[\ell'/2]+1}$ whence $\mathfrak{H}(\beta, \mathfrak{A}) \cap A_K \supset \mathfrak{H}(\beta, \mathfrak{A}_K)$. Similarly for \mathfrak{J} and, in multiplicative terms,

$$H^1(\beta, \mathfrak{A}_K) \subset H^1(\beta, \mathfrak{A}) \cap G_K, \quad J^1(\beta, \mathfrak{A}_K) \subset J^1(\beta, \mathfrak{A}) \cap G_K.$$

Take $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ and consider the character $\vartheta = \theta \mid H^1(\beta, \mathfrak{A}) \cap G_K$. Certainly, ϑ is normalized by $J^1(\beta, \mathfrak{A}) \cap G_K$. Using the explicit form for simple characters, one sees that the restriction $\theta_K = \vartheta \mid H^1(\beta, \mathfrak{A}_K)$ lies in $\mathcal{C}(\mathfrak{A}_K, \beta, \psi_K)$. In particular, θ_K is intertwined by the subgroup $J^1(\beta, \mathfrak{A}) \cap G_K$ of $U_{\mathfrak{A}_K}^1$ and stable under conjugation by $J^1(\beta, \mathfrak{A}_K)$. It follows that $J^1(\beta, \mathfrak{A}) \cap G_K$ is contained in, and hence equal to, $J^1(\beta, \mathfrak{A}_K)$. The nondegeneracy property (2.5.4) implies that the character θ_K cannot be extended to a $J^1(\beta, \mathfrak{A}_K)$ -stable character of a subgroup of $J^1(\beta, \mathfrak{A}_K)$ containing $H^1(\beta, \mathfrak{A}_K)$ strictly. All assertions follow in this case.

For the general step of the induction, let $r = -k_0(\beta, \mathfrak{A}) < \ell$, and choose $\gamma \in E$ so that $[\mathfrak{A}, \ell, r, \gamma]$ is simple and equivalent to $[\mathfrak{A}, \ell, 0, \beta]$. Define the integer r' by

$(\beta-\gamma)\mathfrak{A}_K = \mathfrak{P}_K^{-r'}$. This gives $\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{H}^{[r/2]+1}(\gamma, \mathfrak{A})$, where \mathfrak{B} is the \mathfrak{A} -centralizer of β . We have $\mathfrak{P}^{[r/2]+1} \cap A_K = \mathfrak{P}_K^{[r'/2]+1}$, so by inductive hypothesis we get $\mathfrak{H}(\beta, \mathfrak{A}) \cap A_K \supset \mathfrak{H}(\beta, \mathfrak{A}_K)$. Similarly for \mathfrak{J} . The argument concludes exactly as before. \square

We remark that some of the hypotheses can be weakened here, but we are not currently concerned with achieving generality.

3.3. By the term *1-pair* we mean a pair $(E/F, \xi)$ in which E/F is a finite, tamely ramified field extension and ξ is a character of U_E^1 . Two such pairs $(E_i/F, \xi_i)$, $i = 1, 2$, are said to be F -isomorphic if there is an F -isomorphism $\alpha : E_1 \rightarrow E_2$ such that $\xi_1 = \xi_2 \circ \alpha$.

The 1-pair $(E/F, \xi)$ is called *admissible* if ξ does not factor through $N_{E/K}$, for any field K satisfying $F \subset K \subsetneq E$.

We return to the central simple algebra A as above, and the simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A , with $E = F[\beta]/F$ tamely ramified and satisfying 3.2 Hypothesis. Let B denote the A -centralizer of β and set $\mathfrak{B} = \mathfrak{A} \cap B$. Let $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. The restriction of θ to $H^1(\beta, \mathfrak{A}) \cap B = U_{\mathfrak{B}}^1$ factors through Nrd_B , [38] Proposition 3.47. That is, there exists a unique character ξ_θ of $U_E^1 = \text{Nrd}_B(U_{\mathfrak{B}}^1)$ such that $\theta|_{U_{\mathfrak{B}}^1} = \xi_\theta \circ \text{Nrd}_B$.

Proposition. *The 1-pair $(E/F, \xi_\theta)$ is admissible: if $r_0 = -k_F(\beta)$ then $\xi_\theta|_{U_E^{r_0}}$ does not factor through $N_{E/K}$ when $F \subset K \subsetneq E$.*

Proof. We first recall [11] A.1 Lemma:

Lemma. *Let E/F be a finite, tamely ramified field extension, and let ψ be a character of U_E^i , for some $i \geq 1$. There exists a unique sub-extension E'/F of E/F , such that ψ factors through $N_{E'/E'}$, and which is minimal for this property.*

We return to the proposition. If $E = F$, there is nothing to say, so we assume the contrary. We set $\ell_0 = -v_E(\beta)$ and proceed by induction along β .

Assume first that β is minimal over F . In this case, $r_0 = \ell_0$. The character ξ_θ has level ℓ_0 , and $\xi_\theta(1+x) = \psi_F(\text{Tr}_{E/F} \beta x)$, $x \in \mathfrak{p}_E^{\ell_0}$. Therefore $\xi_\theta|_{U_E^{\ell_0}}$ does not factor through any non-trivial norm map over F .

In general, we choose a simple stratum $[\mathfrak{A}(E), \ell_0, r_0, \gamma]$ equivalent to $[\mathfrak{A}(E), \ell_0, r_0, \beta]$, with $\gamma \in E$ (3.1). The element $\beta-\gamma \in E$ is then minimal over $F[\gamma]$. The

character ξ_θ is of the form $\xi_1\xi_2$, for factors ξ_i as follows. First, ξ_1 has level r and $\xi_1(1+y) = \psi_F(\text{Tr}_{E/F}(\beta-\gamma)y)$, $y \in \mathfrak{p}_E^{r_0}$. As in the first step, $\xi_1 \mid U_E^{r_0}$ does not factor through $N_{E/K}$ if $F[\gamma] \subset K \subsetneq E$. The other factor is of the form $\xi_2 = \xi_0 \circ N_{E/F[\gamma]}$, for a character ξ_0 of $U_{F[\gamma]}^1$. By inductive hypothesis, $\xi_2 \mid U_E^{r_0+1}$ does not factor through $N_{E/K}$ if $F \subset K \subsetneq F[\gamma]$. Consider the set of fields L , with $F \subset L \subset E$, such that $\xi_2 \mid U_E^{r_0+1}$ factors through $N_{E/L}$. By the lemma, this set has a unique minimal element, which is clearly $F[\gamma]$. We conclude that if $\xi_\theta \mid U_E^{r_0}$ factors through $N_{E/K}$, then $K \supset F[\gamma]$. It follows that if $\xi_\theta \mid U_E^{r_0}$ factors through $N_{E/K}$, then $K = E$, as required. \square

Comment. Théorème 3.53 of [38] gives a bijection

$$\mathcal{C}(\mathfrak{A}(E), \beta, \psi_F) \cong \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$$

that is canonical, for a fixed choice of β . Characters corresponding under this bijection give rise to the same admissible 1-pair. In the split case, elements of $\mathcal{C}(\mathfrak{A}(E), \beta, \psi_F)$ give rise to isomorphic 1-pairs if and only if they are conjugate in $A(E)^\times = \text{Aut}_F(E)$. This level of precision is not presently available in the general case.

4. Attached pairs

In this section, we give an approximate description of the elements of $\mathcal{A}_m^{\text{et}}(D)$, in terms of *admissible pairs*. Initially, this process involves arbitrary choices and has no particular invariance properties. The main results of later sections will show that it provides a *canonical* parametrization of the representations in $\mathcal{A}_m^{\text{et}}(D)$, generalizing that for $\mathcal{A}_n^{\text{et}}(F)$ in [11].

4.1. We recall a standard definition, cf. [11], [26].

Definition. Let E/F be a finite, tamely ramified field extension and let ξ be a character of E^\times . The pair $(E/F, \xi)$ is admissible if

- (1) ξ does not factor through $N_{E/K}$, for any field K such that $F \subset K \subsetneq E$,
and
- (2) if $\xi \mid U_E^1$ does factor through $N_{E/K}$, $F \subset K \subset E$, then E/K is unramified.

We define F -isomorphism of admissible pairs in the same way as for 1-pairs (3.3). The degree of $(E/F, \xi)$ is $[E:F]$.

From 3.3 Lemma, we have:

Lemma. *Let $(E/F, \xi)$ be an admissible pair. There is a unique sub-extension E_0/F of E/F such that $\xi \mid U_E^1$ factors through N_{E/E_0} , and which is minimal for this property.*

The extension E/E_0 is, by definition, unramified. Putting the lemma another way, there is a unique admissible 1-pair $(E_0/F, \xi_0)$ with $E_0 \subset E$ and $\xi \mid U_E^1 = \xi_0 \circ N_{E/E_0}$.

4.2. Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ have level zero. By 2.4 Theorem (1), π contains a representation λ of $\text{GL}_m(\mathfrak{o}_D)$, inflated from an irreducible cuspidal representation $\tilde{\lambda}$ of $\text{GL}_m(\mathbb{k}_D)$. The representation $\tilde{\lambda}$ is given, via the Green parametrization, by a character $\tilde{\xi}$ of \mathfrak{l}^\times , where $\mathfrak{l}/\mathbb{k}_D$ is a field extension of degree m (cf. (2.4.2)). Since (by definition) $\delta(\pi) = n$, the character $\tilde{\xi}$ is $\mathfrak{l}/\mathbb{k}_F$ -regular.

Let E/F be unramified of degree n . We identify \mathbb{k}_E with \mathfrak{l} . We define a tamely ramified character ξ of E^\times by deeming that $\xi \mid U_E$ shall be the inflation of $\tilde{\xi}$ and that $\xi \mid F^\times$ shall be the central character of π .

Proposition. *The pair $(E/F, \xi)$ is admissible. The process $\pi \mapsto (E/F, \xi)$ induces a canonical bijection between the set of $\pi \in \mathcal{A}_m^{\text{et}}(D)$ of level zero and the set of F -isomorphism classes of admissible pairs $(E/F, \xi)$, of degree n , in which ξ is tamely ramified.*

Proof. The conjugates $\xi^\sigma \mid U_E$, $\sigma \in \text{Gal}(E/F)$, are distinct, so $(E/F, \xi)$ is admissible. The second assertion restates 2.4 Theorem (3). \square

4.3. Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ be of positive level. Thus π contains an extended maximal simple type (\mathbf{J}, λ) , as in 2.6 Theorem. The extended type (\mathbf{J}, λ) is based on a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in $A = \text{M}_m(D)$ such that the field extension $E_0 = F[\beta]/F$ is tamely ramified (2.8 Lemma). The order \mathfrak{A} is a maximal among E_0 -pure hereditary orders in A . We let B denote the A -centralizer of E_0 and $\mathfrak{B} = \mathfrak{A} \cap B$. Thus \mathfrak{B} is a maximal \mathfrak{o}_{E_0} -order in B . As in the general discussion of 2.5, there exists an unramified field extension E/E_0 inside B , such that \mathfrak{B} is E -pure and $[E:F] = n$. In particular, 3.2 Hypothesis applies.

The restriction $\lambda \mid H^1(\beta, \mathfrak{A})$ is a multiple of some $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. There is a unique character ξ_0 of $U_{E_0}^1$ satisfying $\theta \mid U_{\mathfrak{B}}^1 = \xi_0 \circ \text{Nrd}_B$ [38] Proposition 3.47. The 1-pair $(E_0/F, \xi_0)$ is admissible (3.3 Proposition).

For the next step of the construction, we need to temporarily fix a prime element ϖ_F of F .

Lemma 1. *There exists a unique character ξ_w of E^\times such that*

- (a) $\xi_w | U_E^1 = \xi_0 \circ N_{E/E_0}$,
- (b) $\xi_w(\varpi_F) = 1$, and
- (c) $\xi_w^{p^a} = 1$, for some integer a .

The proof is immediate. For the next step, let η be the unique irreducible representation of $J^1 = J^1(\beta, \mathfrak{A})$ containing θ . Abbreviate also $J = J(\beta, \mathfrak{A})$.

Lemma 2. *There exists a unique irreducible representation Λ_w of \mathbf{J} such that*

- (a) $\Lambda_w | J$ is a wide extension of η ,
- (b) $\varpi_F \in \text{Ker } \Lambda_w$, and
- (c) $(\det \Lambda_w)^{p^b} = 1$, for some integer b .

Proof. Conditions (a) and (c) determine $\Lambda_w | J$ uniquely (cf. (2.5.5)), and are certainly achievable. Taken together with (b), they determine the restriction of Λ to $F^\times J$. The hypothesis $\delta(\pi) = n$ implies that the group \mathbf{J} is $E^\times J = E_0^\times J$, which contains $F^\times J$ with index $e(E_0|F)$. Since p does not divide $e(E_0|F)$ and $\dim \Lambda_w = \dim \eta$ is a power of p , the lemma follows. \square

As in the discussion of 2.6, we have a decomposition $\Lambda = \Lambda_t \otimes \Lambda_w$, for a uniquely determined representation Λ_t of \mathbf{J} , trivial on J^1 . Writing $\mathbf{J}_B = \mathbf{J} \cap B$, we have $\mathbf{J} = \mathbf{J}_B J^1$ and the pair $(\mathbf{J}_B, \Lambda_t | \mathbf{J}_B)$ is an extended maximal simple type in B^\times of level zero. Following 2.4 and 4.2, it gives rise to an admissible pair $(E/E_0, \xi_t)$ in which ξ_t is tamely ramified. We define $\xi = \xi_t \cdot \xi_w$.

Proposition. *The pair $(E/F, \xi)$ is admissible of degree n , and satisfies $t(\pi) = f(E|F)$. The definition of ξ is independent of the choice of ϖ_F in its construction.*

The proof is immediate. An admissible pair obtained from π , in this manner, will be said to be *attached to* π .

Remark. Take E , E_0 and ξ_w as above. Let $(E/E_0, \vartheta)$ be an admissible pair with ϑ tamely ramified. The pair $(E/F, \vartheta \xi_w)$ is then admissible and, following through the constructions of 2.6, there exists a representation $\pi' \in \mathcal{A}_m^{\text{et}}(D)$ to which this admissible pair is attached.

4.4. Let $P_n(F)$ denote the set of F -isomorphism classes of admissible pairs over F , of degree n . In [11] 2.3, we produced a *canonical bijection*

$$(4.4.1) \quad \begin{aligned} P_n(F) &\longrightarrow \mathcal{A}_n^{\text{et}}(F), \\ (E/F, \xi) &\longmapsto {}_F\Pi_\xi. \end{aligned}$$

In this case, the construction of 4.3 is the inverse map ${}_F\Pi_\xi \mapsto (E/F, \xi)$. In the split case therefore, the construction is canonical, and yields a bijection $\mathcal{A}_n^{\text{et}}(F) \rightarrow P_n(F)$.

5. Totally ramified representations

In this section, we prove a special case of our main result. Throughout, D denotes a central F -division algebra of dimension d^2 , $d \geq 1$, and $A = M_m(D)$, $G = \text{GL}_m(D)$, where $md = n$.

5.1. We establish the notation and hypotheses for the rest of the section. We take $\pi \in \mathcal{A}_m^{\text{et}}(D)$ and assume it is *totally ramified*, that is, $t(\pi) = 1$. If we exclude the trivial case $n = 1$, the representation π then cannot be of level zero (*cf.* 2.4 Theorem (4)).

We attach to π an admissible pair $(E/F, \xi)$ as in 4.3, using an extended maximal simple type (\mathbf{J}, A) in π , which contains a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$, for some simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A . In this case, the field $F[\beta]$ is E . By 4.3 Proposition, the extension E/F is totally ramified of degree n (and Hypothesis 3.2 is trivially verified). It follows that the hereditary order \mathfrak{A} is *minimal*, and $\text{rad } \mathfrak{A} = \varpi \mathfrak{A}$, for any prime element ϖ of E . The group \mathbf{J} is $E^\times J^1$, where $J^1 = J^1(\beta, \mathfrak{A})$. Therefore $\Lambda \mid J^1$ is equivalent to the unique irreducible representation η of J^1 which contains θ .

5.2. We need a property of the character of the representation Λ .

Lemma 1. *There is a constant $\epsilon_A = \pm 1$ such that*

$$(5.2.1) \quad \text{tr } \Lambda(x) = \epsilon_A \xi(x),$$

for every $x \in E$ with $v_E(x)$ relatively prime to n .

Proof. This is a standard application of the Glauberman correspondence, strictly parallel to the corresponding result 4.1 Lemma in [11]. \square

As recalled in 2.5, the space $\mathfrak{V} = \mathfrak{V}(\beta, \mathfrak{A})$ carries a nondegenerate alternating form h_θ . The group E^\times acts on \mathfrak{V} via its natural conjugation action on $J^1(\beta, \mathfrak{A})$, and this action fixes the form h_θ (as follows from (2.5.1)(3)). The pair (\mathfrak{V}, h_θ) thus provides a symplectic representation over \mathbb{F}_p of the cyclic group $\Psi = \Psi_{E/F} = E^\times / F^\times U_E^1$. The *symplectic sign* $t_\Psi(\mathfrak{V}(\beta, \mathfrak{A}))$ is thus defined, as in [14] 3.4.

Lemma 2. *The sign ϵ_A of (5.2.1) is given by*

$$(5.2.2) \quad \epsilon_A = t_\Psi(\mathfrak{V}(\beta, \mathfrak{A})).$$

Proof. This is an instance of [6] (8.6.1). \square

5.3. We state the main result of the section. Let $j_A : \mathcal{A}_n^{\text{et}}(F) \rightarrow \mathcal{A}_m^{\text{et}}(D)$ denote the bijection induced by the Jacquet-Langlands correspondence $\mathcal{A}_n^\square(F) \rightarrow \mathcal{A}_m^\square(D)$. We use the notation ${}_F\Pi_\xi, (E/F, \xi) \in P_n(F)$, as in 4.4.

Theorem. *Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ be totally ramified, and let $(E/F, \xi)$ be an admissible pair attached to π . If ν denotes the unramified character of E^\times given by*

$$\nu(x) = (-1)^{m(d-1)v_E(x)}, \quad x \in E^\times,$$

then

$$\pi = j_A({}_F\Pi_{\nu\xi}).$$

Before proving the theorem, we note some consequences.

Corollary. *Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ be totally ramified.*

- (1) *Any two admissible pairs attached to π are F -isomorphic.*
- (2) *The attachment process induces a canonical bijection between the set of totally ramified elements of $\mathcal{A}_m^{\text{et}}(D)$ and the set of F -isomorphism classes of admissible pairs $(E/F, \xi)$ in which E/F is totally ramified of degree n .*

Proof. The assertions are valid in the case $\pi \in \mathcal{A}_n^{\text{et}}(F)$, by [11] 2.3 Theorem. The general case follows from the theorem. \square

5.4. We start the proof of the theorem. We use the Mackey formula (1.2.2) to follow the corresponding passage of [11].

Proposition 1.

- (1) *There exists $g \in G_{\text{reg}}^{\text{ell}}$ such that $\gcd(v_F(\text{Nrd}_A g), n) = 1$ and $\text{tr } \pi(g) \neq 0$.*
- (2) *Let $g \in G_{\text{reg}}^{\text{ell}}$ satisfy $\gcd(v_F(\text{Nrd}_A g), n) = 1$. If $\text{tr } \pi(g) \neq 0$, then g is G -conjugate to an element of E^\times .*

Proof. Let X_n denote the group of unramified characters χ of F^\times satisfying $\chi^n = 1$. Since π is totally ramified, the representations $\chi\pi, \chi \in X_n$, are distinct, whence the character functions $\text{tr } \chi\pi : g \mapsto \chi(\text{Nrd}_A g) \text{tr } \pi(g)$ are linearly independent on $G_{\text{reg}}^{\text{ell}}$ (1.4.5). The first assertion follows directly. The proof of the second is identical to that of part (1) of 4.1 Proposition of [11]. \square

Let $\text{Aut}(E|F)$ denote the group of F -automorphisms of the field E . Following the proof of part (2) of 4.1 Proposition of [11], we get:

Proposition 2. *If $g \in E^\times$ satisfies $\gcd(v_E(g), n) = 1$ then*

$$\text{tr } \pi(g) = \epsilon_A \sum_{\alpha \in \text{Aut}(E|F)} \xi^\alpha(g),$$

where $\epsilon_A = \pm 1$ is given by (5.2.2).

5.5. Define $\tau \in \mathcal{A}_n^{\text{et}}(F)$ by the condition $\pi = \mathbf{j}_A(\tau)$. There exists an admissible pair $(K/F, \zeta)$, uniquely determined up to F -isomorphism, such that $\tau = {}_F\Pi_\zeta$ [11] 2.3. We have $t(\tau) = t(\pi) = 1$ (1.4.4), so τ is totally ramified. It follows that K/F is totally ramified of degree n .

Applying 5.4 Proposition 1 to τ , the character relation $\text{tr } \pi = (-1)^{m(d-1)} \text{tr } \tau$ implies that K is F -isomorphic to E . We henceforth take $K = E$.

We apply 5.4 Proposition 2 to the representations π, τ . If ϖ is a prime element of E , we get

$$(5.5.1) \quad \epsilon_A \sum_{\alpha \in \text{Aut}(E|F)} \xi^\alpha(\varpi^a u) = (-1)^{m(d-1)} \epsilon_M \sum_{\alpha \in \text{Aut}(E|F)} \zeta^\alpha(\varpi^a u),$$

for $u \in U_E$, any integer a relatively prime to n , and a constant $\epsilon_M = \pm 1$ obtained as follows. The representation τ contains a simple character $\theta_M \in \mathcal{C}(\mathfrak{M}, \gamma, \psi_F)$, for some simple stratum $[\mathfrak{M}, \ell_M, 0, \gamma]$ in $M = M_n(F)$, with $F[\gamma] = E$ and $\theta_M | U_E^1 = \zeta | U_E^1$. The sign ϵ_M is then $t_\Psi(\mathfrak{V}(\gamma, \mathfrak{M}))$, in the notation of (5.2).

We fix a and view the two sides of (5.5.1) as functions of u . Since E/F is totally ramified and $(E/F, \xi)$ is admissible, the characters $\xi^\alpha \mid U_E$, $\alpha \in \text{Aut}(E|F)$, are distinct and hence linearly independent as functions on U_E . The same applies with ζ in place of ξ . We deduce that, for some $\alpha \in \text{Aut}(E|F)$, we have $\xi \mid U_E = \zeta^\alpha \mid U_E$. Replacing $(E/F, \zeta)$ by the isomorphic pair $(E/F, \zeta^\alpha)$ leaves τ unchanged, so we can assume that ζ agrees with ξ on U_E .

Comparing coefficients of $\xi \mid U_E$ in (5.5.1), we obtain

$$\zeta(\varpi^a u) = (-1)^{m(d-1)} \epsilon_A \epsilon_M \xi(\varpi^a u).$$

To complete the proof, it remains only to show that

$$(5.5.2) \quad \epsilon_A = \epsilon_M.$$

5.6. We prove (5.5.2). As before, let \mathfrak{A} , \mathfrak{M} be the unique E -pure hereditary orders in $A = M_m(D)$, $M = M_n(F)$ respectively. Each of these has F -period $n = e(E|F)$ and so is a minimal hereditary order. We have simple strata $[\mathfrak{A}, l, 0, \beta]$, $[\mathfrak{M}, l, 0, \gamma]$ in A , M , with $E = F[\beta] = F[\gamma]$. The ϵ -invariants are given by

$$\epsilon_A = t_\Psi(\mathfrak{V}(\mathfrak{A}, \beta)), \quad \epsilon_M = t_\Psi(\mathfrak{V}(\mathfrak{M}, \gamma)),$$

where $\Psi = \Psi_{E/F}$. This group is cyclic, generated by the image of ϖ . The desired relation (5.5.2) thus follows from:

Proposition. *The symplectic $\mathbb{F}_p\Psi$ -modules $\mathfrak{V}(\mathfrak{A}, \beta)$, $\mathfrak{V}(\mathfrak{M}, \gamma)$ are isometric.*

Proof. Indeed, it is enough to show that $\mathfrak{V}(\mathfrak{A}, \beta)$, $\mathfrak{V}(\mathfrak{M}, \gamma)$ are isomorphic as $\mathbb{F}_p\Psi$ -modules [14] 3.3.

Let $\mathfrak{P} = \text{rad } \mathfrak{A}$. If ϖ is a prime element of E , we have $\mathfrak{P} = \varpi\mathfrak{A}$, so all the quotients $\mathfrak{P}^j/\mathfrak{P}^{j+1}$, $j \in \mathbb{Z}$, are Ψ -isomorphic to $\mathfrak{A}/\mathfrak{P}$. Similarly for \mathfrak{M} and $\mathfrak{P}_{\mathfrak{M}} = \text{rad } \mathfrak{M}$.

Lemma. *Each of the $\mathbb{F}_p\Psi$ -modules $\mathfrak{A}/\mathfrak{P}$, $\mathfrak{M}/\mathfrak{P}_{\mathfrak{M}}$ is isomorphic to the regular representation $\mathbb{k}_F\Psi$.*

Proof. In the case of \mathfrak{M} , the quotient $\mathfrak{M}/\mathfrak{P}_{\mathfrak{M}}$ is a sum of n copies of \mathbb{k}_F , permuted cyclically by ϖ , with ϖ^n acting trivially: it is therefore of the desired form.

In the case of \mathfrak{A} , the quotient $\mathfrak{A}/\mathfrak{P}$ is the direct sum of m copies of \mathbb{k}_D . The element ϖ permutes these cyclically, and ϖ^m acts on each factor \mathbb{k}_D as a generator of $\text{Gal}(\mathbb{k}_D/\mathbb{k}_F)$. By the Normal Basis Theorem, the $\mathbb{k}_F\Psi^m$ -module \mathbb{k}_D is free of rank one, so $\mathfrak{A}/\mathfrak{P}$ is $\mathbb{k}_F\Psi$ -free of rank one, as required. \square

It is now convenient to choose the prime element ϖ , as we may, so that $\varpi^n \in F$. Thus, if E' is a field, $F \subset E' \subset E$, then $E' = F[\varpi^{[E:E']}]$.

Let $i \geq 0$ be an integer, and let E_i/F be the least sub-extension of E/F such that $\xi \mid U_E^{i+1}$ factors through N_{E/E_i} (cf. 3.3 Lemma). We recall that $i \geq 1$ is a *jump* of ξ if $E_i \neq E_{i-1}$. Denote by \mathfrak{W}_M the Ψ -module $\mathfrak{M}/\mathfrak{P}_M$, and let $\mathfrak{W}_M^{(i)}$ denote the submodule of E_i^\times fixed points: these are the same as the ϖ^{n_i} -fixed points, $n_i = [E:E_i]$.

Exactly as in [14] 8.5 (see also (2.5.4) above), the semisimple Ψ -module $\mathfrak{V}(\mathfrak{M}, \beta)$ is the direct sum of the modules $\mathfrak{W}_M^{(j)}/\mathfrak{W}_M^{(j+1)}$, where $2j$ ranges over the even jumps of ξ . The same observation applies with $\mathfrak{A}, \mathfrak{W} = \mathfrak{A}/\mathfrak{P}$ in place of $\mathfrak{M}, \mathfrak{W}_M$. The proposition now follows from the lemma. \square

This completes the proof of 5.3 Theorem. \square

6. Comparison and parametrization

In this section, we extend the scope of 5.3 Corollary to obtain a canonical parametrization of the representations $\pi \in \mathcal{A}_m^{\text{et}}(D)$ in terms of admissible pairs. Simultaneously, we give the first step to describing the relation between this parametrization and the Jacquet-Langlands correspondence.

6.1. Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ have attached simple pair $(E/F, \xi)$. We use all the associated notations from the definition of $(E/F, \xi)$ in §4. In particular, π contains an extended maximal simple type (\mathbf{J}, A) . If π has positive level, (\mathbf{J}, A) is based on a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A , satisfying Hypothesis 3.2. The representation A contains a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. We set $E_0 = F[\beta]$ and let B denote the A -centralizer of E_0 . We also set $\mathfrak{B} = \mathfrak{A} \cap B, \mathfrak{Q} = \mathfrak{P} \cap B = \text{rad } \mathfrak{B}$.

We prove:

Parametrization Theorem.

- (1) *Let $\pi \in \mathcal{A}_m^{\text{et}}(D)$ have attached simple pair $(E/F, \xi)$. If $(E'/F, \xi')$ is another simple pair attached to π , then $(E'/F, \xi')$ is F -isomorphic to*

$(E/F, \xi)$.

- (2) Let $(E/F, \xi)$ be an admissible pair of degree n . There exists a unique element ${}_D\Pi_\xi \in \mathcal{A}_m^{\text{et}}(D)$ with attached admissible pair F -isomorphic to $(E/F, \xi)$. The map

$$\begin{aligned} P_n(F) &\longrightarrow \mathcal{A}_m^{\text{et}}(D), \\ (E/F, \xi) &\longmapsto {}_D\Pi_\xi, \end{aligned}$$

is a bijection.

Remark. For representations π of level zero and admissible pairs $(E/F, \xi)$ with ξ tamely ramified, the theorem reduces to 4.2 Proposition.

The simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ gives rise to the group $\mathfrak{V} = \mathfrak{V}(\beta, \mathfrak{A})$, which furnishes a symplectic representation of the finite abelian group $\Psi = E^\times / U_E^1 F^\times$ over \mathbb{F}_p . In particular, it gives a symplectic representation of the cyclic group μ_E , so the fine symplectic signs $t_{\mu_E}^k(\mathfrak{V})$, $k = 0, 1$, are defined as in [14] §3. We recall that $t_{\mu_E}^0(\mathfrak{V}) = \pm 1$ is a constant, while $t_{\mu_E}^1(\mathfrak{V})$ is a character of μ_E with values in $\{\pm 1\}$.

The symplectic representation of Ψ over \mathbb{F}_p afforded by $\mathfrak{V}(\beta, \mathfrak{A})$ depends, up to isometry, only on the algebra A and the restriction of ξ to U_E^1 . Indeed, as in [14], it depends only on the sequence of “jump fields” E_i/F defined by this character. We therefore use a notation parallel to that of [14],

$$(6.1.1) \quad {}_A\epsilon_{(E/F, \xi)}^k(\mu_E) = t_{\mu_E}^k(\mathfrak{V}), \quad k = 0, 1,$$

or, more commonly, the abbreviated version

$$(6.1.2) \quad \epsilon_A^k(\mu_E) = {}_A\epsilon_{(E/F, \xi)}^k(\mu_E).$$

We remark that, in the case where π has level zero, the ϵ -invariants are all trivial.

We shall also prove:

First Comparison Theorem. Let $(E/F, \xi)$ be an admissible pair of degree n . There exists a unique tamely ramified character $\nu = {}_D\nu_\xi$ of E^\times with the following properties.

- (1) The restriction of ν to μ_E is the character

$$\epsilon_A^1(\mu_E) \epsilon_M^1(\mu_E) = {}_A\epsilon_{(E/F, \xi)}^1(\mu_E) \cdot {}_M\epsilon_{(E/F, \xi)}^1(\mu_E).$$

(2) *The pair $(E/F, \nu\xi)$ is admissible and*

$${}_D\Pi_\xi = \mathbf{j}_A({}_F\Pi_{\nu\xi}),$$

where $\mathbf{j}_A : \mathcal{A}_n^{\text{et}}(F) \rightarrow \mathcal{A}_m^{\text{et}}(D)$ is the Jacquet-Langlands correspondence.

The proofs of the two theorems are entwined, and occupy the rest of the section.

6.2. We take $\pi \in \mathcal{A}_m^{\text{et}}(D)$ with attached simple pair $(E/F, \xi)$, and use the notation of 6.1. Until declared otherwise (at the end of 6.8), we assume that π does not have level zero or, equivalently, that ξ is not tamely ramified.

As in 2.1, the E_0 -algebra B takes the form $B \cong M_l(C)$, for an integer l and a central E_0 -division algebra C of dimension c^2 , $lc = n/[E_0:F]$.

Let K/F be the maximal unramified sub-extension of E/F . Set

$$\Sigma = \text{Gal}(K/F), \quad \Sigma_0 = \text{Gal}(E/E_0).$$

If $K_0 = K \cap E_0$, restriction of operators identifies Σ_0 with $\text{Gal}(K/K_0)$.

Let A_K denote the A -centralizer of K , $G_K = A_K^\times$. Thus $A_K \cong M_{m_K}(D_K)$, for a central K -division algebra D_K of K -dimension d_K^2 , say, and $m_K d_K = n/[K:F]$.

Set $\mathfrak{A}_K = \mathfrak{A} \cap A_K$. Thus \mathfrak{A}_K is a hereditary \mathfrak{o}_K -order in A_K . It is E -pure, and is the unique E -pure hereditary order in A_K (cf. 2.3). From 3.2 Proposition, we have:

(6.2.1)

- (1) *The stratum $[\mathfrak{A}_K, \ell, 0, \beta]$ is simple.*
- (2) *We have the relations*

$$H^1(\beta, \mathfrak{A}_K) = H^1(\beta, \mathfrak{A}) \cap G_K, \quad J^1(\beta, \mathfrak{A}_K) = J^1(\beta, \mathfrak{A}) \cap G_K.$$

- (3) *The restriction $\theta \mid H^1(\beta, \mathfrak{A}_K) = \theta_K$ lies in $\mathcal{C}(\mathfrak{A}_K, \beta, \psi_K)$, where $\psi_K = \psi_F \circ \text{Tr}_{K/F}$.*

The discussion in 3.2 likewise gives the relation

$$(6.2.2) \quad H^1(\beta, \mathfrak{A}) \cap E = U_E^1, \quad \text{and} \quad \theta \mid U_E^1 = \xi \mid U_E^1.$$

The same holds with $(\mathfrak{A}_K, \theta_K)$ replacing (\mathfrak{A}, θ) .

6.3. We fix a root of unity $\zeta \in \mu_K = \mu_E$ such that $K = F[\zeta]$. We let u range over the set ${}_u(G_K)_{\text{reg}}^{\text{ell}}$ of pro-unipotent, elliptic regular elements of G_K . (Regularity and ellipticity are measured relative to the base field K in this set.) The element $h = \zeta u$ then lies in $G_{\text{reg}}^{\text{ell}}$, so we may use (1.2.2) to evaluate the character of π at h ,

$$(6.3.1) \quad \text{tr } \pi(h) = \sum_{x \in G/\mathbf{J}} \text{tr } \Lambda(x^{-1}hx).$$

A coset $x\mathbf{J}$ can only contribute to the sum if $x^{-1}hx \in \mathbf{J}$ or, equivalently, $x^{-1}hx \in J^0$.

Lemma. *Let $x \in G$ and suppose $x^{-1}hx \in J^0$. There then exists $y \in N_G(K^\times)$ such that $x\mathbf{J} = y\mathbf{J}$. For any such y , we have $y^{-1}uy \in J^1$ (and hence $y^{-1}uy \in J^1 \cap G_K$).*

Proof. This is identical to the proof of 5.4 Lemma 1 in [14]: as noted there, the argument applies to pro-unipotent, elliptic regular elements u of G_K . \square

The sum in (6.3.1) is therefore effectively taken over the space $N_G(K^\times)\mathbf{J}/\mathbf{J}$, which we now describe more conveniently.

Let Σ_t denote the unique subgroup of $\Sigma_0 = \text{Gal}(E/E_0)$ of index c . The residue field \mathbb{k}_E is a maximal subfield of the algebra $\mathfrak{B}/\mathfrak{Q} \cong M_l(\mathbb{k}_C)$. The group Σ is canonically identified with $\text{Gal}(\mathbb{k}_E/\mathbb{k}_F)$ via its actions on roots of unity. The subgroups Σ_0, Σ_t are then $\text{Gal}(\mathbb{k}_E/\mathbb{k}_{E_0})$ and $\text{Gal}(\mathbb{k}_E/\mathbb{k}_C)$ respectively.

For each $\sigma \in \Sigma$, we choose $t_\sigma \in N_G(K^\times)$ with image σ under the canonical isomorphism $N_G(K^\times)/G_K \cong \Sigma$. Observe that the subgroup Σ_t of Σ is the image of $\mathbf{J} \cap N_G(K^\times)$ under this map.

Proposition. *We have*

$$N_G(K^\times)\mathbf{J} = \bigcup_{\sigma \in \Sigma/\Sigma_t} G_K t_\sigma \mathbf{J},$$

the union being disjoint, and hence a bijection

$$\begin{aligned} G_K \backslash N_G(K^\times)\mathbf{J}/\mathbf{J} &\longrightarrow \Sigma/\Sigma_t, \\ G_K t_\sigma \mathbf{J} &\longmapsto \sigma \Sigma_t. \end{aligned}$$

Proof. The space $N_G(K^\times)\mathbf{J}$ surely decomposes as the union of double cosets $G_K t_\sigma \mathbf{J}$, $\sigma \in \Sigma$. We have to show that, if $\sigma, \tau \in \Sigma$, then $G_K t_\sigma \mathbf{J} = G_K t_\tau \mathbf{J}$ if and only if $\sigma^{-1}\tau \in \Sigma_t$.

The elements t_σ, t_τ normalize G_K , whence $G_K t_\sigma \mathbf{J} = G_K t_\tau \mathbf{J}$ if and only if $t_\sigma^{-1}t_\tau \in G_K \mathbf{J}$. This, in turn, is equivalent to the automorphism $\alpha \mapsto \alpha^{\sigma^{-1}\tau}$ being realized as conjugation by an element of \mathbf{J} . That is, $\sigma^{-1}\tau \in \Sigma_t$, as required. \square

The formula (6.3.1) has become

$$(6.3.2) \quad \text{tr } \pi(\zeta u) = \sum_{\alpha \in \Sigma/\Sigma_t} \sum_{y \in G_K/\mathbf{J}_K} \text{tr } \Lambda(y^{-1}\zeta^\alpha u^\alpha y).$$

This expression is valid for any K/F -regular root of unity $\zeta \in \mu_K$ and any $u \in {}_u(G_K)_{\text{reg}}^{\text{ell}}$. Only finitely many of the terms $\text{tr } \Lambda(y^{-1}\zeta^\alpha u^\alpha y)$ are non-zero (1.2 Lemma).

6.4. Let η_K be the unique irreducible representation of $J_K^1 = J^1(\beta, \mathfrak{A}_K)$ which contains θ_K . The field extension $E = K[\beta]/K$ is totally ramified and maximal in A_K . It follows that the group $J_K = J(\beta, \mathfrak{A}_K) = J \cap G_K$ is $U_E J_K^1 = \mu_E J_K^1$.

We take a prime element ϖ_F of F , and use it to define a factorization $\xi = \xi_t \cdot \xi_w$ of ξ , reversing the procedure of 4.3. Explicitly, ξ_w agrees with ξ on U_E^1 , it has finite p -power order, and $\xi_w(\varpi_F) = 1$. The character ξ_t is $\xi_w^{-1}\xi$.

Since $E = E_0 K$, the pair $(E/K, \xi_w)$ is admissible and totally ramified. The character ξ_t is tamely ramified, and the pair $(E/E_0, \xi_t)$ is admissible.

We extend η_K to a representation ${}_K \lambda_w$ of $J_K = \mu_E J_K^1$ by deeming that ${}_K \lambda_w \mid \mu_E$ be trivial. The pair $(J_K, {}_K \lambda_w)$ is then a maximal simple type in G_K . The group $\mathbf{J}({}_K \lambda_w)$ is $\mathbf{J}_K = \mathbf{J} \cap G_K = E^\times J_K^1$. We extend ${}_K \lambda_w$ to a representation ${}_K \Lambda_w$ of \mathbf{J}_K so that $\varpi_F \in \text{Ker } {}_K \Lambda_w$ and $\det {}_K \Lambda_w$ has finite, p -power order. Thus $(\mathbf{J}_K, {}_K \Lambda_w)$ is an extended maximal simple type in G_K , with the property $\mu_K \subset \text{Ker } {}_K \Lambda_w$. We define

$$\rho_w = c\text{-Ind}_{\mathbf{J}_K}^{G_K} {}_K \Lambda_w.$$

This is an irreducible, totally ramified, essentially tame, cuspidal representation of G_K , with maximal parametric degree

$$\delta(\rho_w) = [E:K] = (\dim_K A_K)^{1/2},$$

and attached admissible pair $(E/K, \xi_w)$.

In the formula (6.3.2), ζ^α commutes with y and, by 6.3 Lemma, the element $y^{-1}u^\alpha y$ lies in J_K^1 . We decompose $\Lambda = \Lambda_t \otimes \Lambda_w$, as in the definition of 4.3. Since Λ_t is trivial on J^1 , we are reduced to

$$(6.4.1) \quad \mathrm{tr} \pi(\zeta u) = \sum_{\alpha \in \Sigma/\Sigma_t} \mathrm{tr} \Lambda_t(\zeta^\alpha) \sum_{y \in G_K/\mathbf{J}_K} \mathrm{tr} \Lambda_w(\zeta^\alpha y^{-1}u^\alpha y).$$

The definitions of Λ_w , ${}_K\Lambda_w$ yield

$$\begin{aligned} \mathrm{tr} \Lambda_w(\zeta^\alpha y^{-1}u^\alpha y) &= \epsilon_A \mathrm{tr} {}_K\Lambda_w(y^{-1}u^\alpha y) \\ &= \epsilon_A \mathrm{tr} \eta_K(y^{-1}u^\alpha y), \end{aligned}$$

where

$$\epsilon_A = t_{(\zeta)}(\mathfrak{B}) = t_{\mu_E}^0(\mathfrak{B}) t_{\mu_E}^1(\mathfrak{B}; \zeta^\alpha) = t_{\mu_E}^0(\mathfrak{B}) t_{\mu_E}^1(\mathfrak{B}; \zeta),$$

and $\mathfrak{B} = \mathfrak{B}(\beta, \mathfrak{A})$ (cf. [14] (3.4.3), (5.6.2)). The inner sum in (6.4.1) is therefore reduced to

$$\sum_{y \in G_K/\mathbf{J}_K} \mathrm{tr} \Lambda_w(\zeta^\alpha y^{-1}u^\alpha y) = \epsilon_A \mathrm{tr} \rho_w^{\alpha^{-1}}(u).$$

Overall, we have

$$(6.4.2) \quad \mathrm{tr} \pi(\zeta u) = \epsilon_A \sum_{\alpha \in \Sigma/\Sigma_t} \mathrm{tr} \Lambda_t(\zeta^\alpha) \mathrm{tr} \rho_w^{\alpha^{-1}}(u).$$

This relation is valid for all Σ -regular roots of unity $\zeta \in \mu_K = \mu_E$ and all pro-unipotent, elliptic regular elements u of G_K .

6.5. We examine the representations $\rho_w^\alpha = \rho_w \circ \alpha$, $\alpha \in \Sigma$, of the group G_K .

Lemma. *Let $\alpha \in \Sigma$. The following conditions are equivalent.*

- (1) $\rho_w^\alpha \cong \rho_w$;
- (2) *there is an unramified character χ of K^\times such that $\rho_w^\alpha \cong \chi \rho_w$;*
- (3) $\alpha \in \Sigma_0$.

Proof. The representation ρ_w is, by construction, attached to the admissible pair $(E/K, \xi_w)$ and E/K is totally ramified. If $\alpha \in \Sigma_0$, then ρ_w^α is attached to the admissible pair $(E/K, \xi_w^\alpha)$, while $\xi_w^\alpha = \xi_w$. By 5.3 Corollary, we have $\rho_w^\alpha \cong \rho_w$.

Thus (3) \Rightarrow (1) and surely (1) \Rightarrow (2). We assume, therefore, that (2) holds. Let $(E'/K, \xi')$ be the admissible pair attached to ρ_w^α : since ρ_w^α is totally ramified, the pair $(E'/K, \xi')$ is uniquely determined up to K -isomorphism (5.3 Corollary). It satisfies the following condition: there is a field isomorphism $\alpha' : E \rightarrow E'$, such that $\alpha' | K = \alpha$, which carries ξ_w to ξ' . However, the hypothesis $\chi\rho_w \cong \rho_w^\alpha$ implies that the admissible 1-pairs $(E/K, \xi_w | U_E^1), (E'/K, \xi' | U_{E'}^1)$ are K -isomorphic. If this isomorphism is realized by a K -isomorphism $\alpha'' : E \rightarrow E'$, then $\alpha''^{-1}\alpha'$ is an automorphism of E , extending the automorphism α of K , which fixes $\xi_w | U_E^1 = \xi | U_E^1$. It follows that $\alpha''^{-1}\alpha' \in \Sigma_0$. We may therefore take $E' = E$ and the same argument gives $\alpha \in \Sigma_0$, as required. \square

6.6. In order to apply 6.5 Lemma, we need a general result on linear independence of cuspidal characters.

Linear Independence Lemma. *Let $\pi_1, \pi_2, \dots, \pi_r$ be irreducible, totally ramified, cuspidal representations of G , all of parametric degree n . Suppose*

- (a) *the central characters of the π_i all agree on some prime element of F and on μ_F ;*
- (b) *the representation π_i is not equivalent to an unramified twist of π_j whenever $i \neq j$.*

The characters $\text{tr } \pi_i, 1 \leq i \leq r$, then form a linearly independent set of functions on ${}_u G_{\text{reg}}^{\text{cell}}$.

Proof. The Jacquet-Langlands correspondence commutes with twisting by characters of F^\times (1.4.3), and preserves central characters (1.4.2). It also preserves parametric degree 2.8 Corollary 1. It is therefore enough to treat the case $G = \text{GL}_1(D)$, for some central F -division algebra D of dimension n^2 .

We use the classification theory of [4] for representations of G . Since π_j is totally ramified, there is a simple stratum $[\mathfrak{o}_D, l_j, 0, \beta_j]$ in D , and a simple character $\theta_j \in \mathcal{C}(\mathfrak{o}_D, \beta_j, \psi)$ which occurs in π_j . Since π_j is totally ramified of parametric degree n , the field extension $F[\beta_j]/F$ is totally ramified of degree n . The hypotheses on the representations π_j are equivalent to θ_i not intertwining with θ_j when $i \neq j$. The proof now follows the same course as that of 5.7 Lemma (3) in [14]. \square

6.7. In (6.4.2), the definition of A_t gives

$$\mathrm{tr} A_t(\zeta^\alpha) = (-1)^{l-1} \sum_{\gamma \in \Sigma_t} \xi_t(\zeta^{\alpha\gamma}),$$

where, we recall, $B \cong M_l(C)$ and $U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 \cong \mathrm{GL}_l(\mathbb{k}_C)$. We therefore re-write (6.4.2) in the form

$$(6.7.1) \quad \mathrm{tr} \pi(\zeta u) = (-1)^{l-1} \epsilon_A \sum_{\alpha \in \Sigma/\Sigma_0} \mathrm{tr} \rho_w^\alpha(u) \sum_{\gamma \in \Sigma_0} \xi_t(\zeta^{\alpha\gamma}).$$

We turn to the corresponding situation in $\mathrm{GL}_n(F)$. We take $\tau \in \mathcal{A}_n^{\mathrm{et}}(F)$ to satisfy $\mathbf{j}\tau = \pi$. The representation τ is of the form ${}_F\Pi_\phi$, for an admissible pair $(E'/F, \phi)$, uniquely determined up to isomorphism. The relation $t(\tau) = t(\pi)$ implies $f(E|F) = f(E'|F)$, so we may identify the maximal unramified sub-extension of E'/F with K/F and continue to use the notation $\Sigma = \mathrm{Gal}(K/F)$ on either side. Let E'_0/F be the minimal sub-extension of E'/F such that $\phi \mid U_{E'}^1$ factors through N_{E'/E'_0} and set $\Sigma'_0 = \mathrm{Gal}(E'/E'_0)$. We use the same prime element ϖ_F of F to achieve a factorization $\phi = \phi_t \cdot \phi_w$, as was applied to ξ in 6.4.

Applying (6.7.1) to this case, we get an expression

$$(6.7.2) \quad \mathrm{tr} \tau(\zeta v) = (-1)^{k-1} \epsilon_M \sum_{\alpha \in \Sigma/\Sigma'_0} \mathrm{tr} \sigma_w^\alpha(v) \sum_{\gamma \in \Sigma'_0} \phi_t(\zeta^{\alpha\gamma}).$$

Here, $k = [E':E'_0]$. The simple stratum in $M = M_n(F)$ attached to $(E'/F, \phi)$ is $[\mathfrak{M}, \ell_M, 0, \delta]$, say, and

$$\epsilon_M = t_{\langle \zeta \rangle}(\mathfrak{B}(\delta, \mathfrak{M})) = \epsilon_M^0(\boldsymbol{\mu}_E) \epsilon_M^1(\boldsymbol{\mu}_E; \zeta).$$

The representation σ_w is ${}_K\Pi_{\phi_w}$. The equation (6.7.2) is valid for all K/F -regular roots of unity $\zeta \in \boldsymbol{\mu}_K$ and all pro-unipotent, elliptic regular elements v of the $\mathrm{GL}_n(F)$ -centralizer of K^\times .

We now use the lemmas of 6.5, 6.6 to conclude that, for a suitable choice of $(E'/F, \phi)$, we get

$$\rho_w = \mathbf{j}_K(\chi\sigma_w),$$

for an unramified character χ of K^\times , where \mathbf{j}_K is a Jacquet-Langlands correspondence with base field K . Therefore (5.3 Corollary) $(E'/K, \phi_w)$ is K -isomorphic

to an unramified twist of $(E/K, \xi_w)$. We may henceforth take $E' = E$, $E'_0 = E_0$, and $\Sigma = \Sigma'$. In particular, $k = lc$, $c^2 = \dim_{E_0} C$.

We use 6.6 Lemma again to compare coefficients and obtain

$$(-1)^{m(d-1)+(lc-1)+m_K(d_K-1)} \epsilon_M \sum_{\gamma \in \Sigma_0} \phi_t(\zeta^\gamma) = (-1)^{l-1} \epsilon_A \sum_{\gamma \in \Sigma_0} \xi_t(\zeta^\gamma).$$

Re-writing in terms of the finer symplectic invariants $\epsilon_M^j = \epsilon_M^j(\mu_E)$ etc., this equation becomes

$$(-1)^{m(d-1)+(lc-1)+m_K(d_K-1)} \epsilon_M^0 \sum_{\gamma \in \Sigma_0} \epsilon_M^1 \phi_t(\zeta^\gamma) = (-1)^{l-1} \epsilon_A^0 \sum_{\gamma \in \Sigma_0} \epsilon_A^1 \xi_t(\zeta^\gamma).$$

We can vary the character ξ_t over all the tamely ramified characters of E^\times , subject to the pair $(E/E_0, \xi_t)$ being admissible (cf. 4.3 Remark 2). There will always exist a tamely ramified admissible pair $(E/E_0, \phi_t)$ for which this last relation holds, with ζ ranging over all Σ -regular elements of μ_K . We deduce from 2.3 Corollary of [14] that:

Lemma. *The characters $\epsilon_M^1 \phi_t$, $\epsilon_A^1 \xi_t$ of \mathbb{k}_E^\times lie in the same Σ_0 -orbit, and*

$$(6.7.3) \quad (-1)^{m(d-1)+(lc-1)+m_K(d_K-1)} \epsilon_M^0 = (-1)^{l-1} \epsilon_A^0.$$

The formula (6.7.3) can be re-written as

$$(6.7.4) \quad \epsilon_M^0(\mu_E) \epsilon_A^0(\mu_E) = (-1)^{m(d-1)+m_K(d_K-1)+l(c-1)}.$$

We have

$$(6.7.5) \quad m_K d_K = n/[K:F], \quad lc = n/[E_0:F],$$

while (as in 2.1)

$$(6.7.6) \quad d_K = d/(d, [K:F]), \quad c = d/(d, [E_0:F]).$$

6.8. Adjusting our choices within Galois orbits, we can assume that the character ϕ satisfies

$$(6.8.1) \quad \phi \mid U_E = \epsilon_M^1 \epsilon_A^1 \xi \mid U_E.$$

Let $(E'/F, \xi')$ be an admissible pair attached to π . The F -isomorphism class of $(E/F, \phi)$ is uniquely determined by τ [11] 2.3, and hence also by π . In other words, $(E'/F, \xi')$ must give rise to the same pair $(E/F, \phi)$ as $(E/F, \xi)$. Thus E' is F -isomorphic to E , and we may as well take $E = E'$. The relation (6.8.1) implies that the 1-pairs $(E/F, \xi \mid U_E^1)$, $(E/F, \phi \mid U_E^1)$ and $(E/F, \xi' \mid U_E^1)$ are all F -isomorphic. We may assume that they are the same. It is the restriction $\xi \mid U_E^1$ which determines the symplectic sign character ϵ_A^1 , so (6.8.1) now implies that the pair $(E/F, \xi' \mid U_E)$ is F -isomorphic to $(E/F, \xi \mid U_E)$ in the obvious sense. Again, we may assume they are the same. It follows that $\xi' = \chi \xi$, for some unramified character of E^\times . This implies $\tau = {}_F\Pi_{\chi\phi} = {}_F\Pi_\phi$, whence $(E/F, \chi\phi)$ is isomorphic to $(E/F, \phi)$. The admissibility of $(E/F, \phi)$ implies $\chi = 1$, and we have proved part (1) of the Parametrization Theorem.

We prove part (2) of the Parametrization Theorem. We have bijections

$$\begin{aligned} P_n(F) &\longrightarrow \mathcal{A}_n^{\text{et}}(F), & \mathcal{A}_n^{\text{et}}(F) &\longrightarrow \mathcal{A}_m^{\text{et}}(D), \\ (E/F, \xi) &\longmapsto {}_F\Pi_\xi, & \pi &\longmapsto \mathbf{j}\pi, \end{aligned}$$

given by 2.3 Theorem of [11] and Corollary 2 of 2.8 respectively. We also have an injective map $\mathcal{A}_m^{\text{et}}(D) \rightarrow P_n(F)$ given by mapping $\pi \in \mathcal{A}_m^{\text{et}}(D)$ to its attached admissible pair. Composing, we get a map

$$P_n(F) \longrightarrow \mathcal{A}_n^{\text{et}}(F) \longrightarrow \mathcal{A}_m^{\text{et}}(D) \longrightarrow P_n(F),$$

which preserves restrictions to 1-units and to F^\times . It is injective, by part (1) of the Parametrization Theorem. There are only finitely many admissible pairs $(E/F, \xi)$ with specified values for $\xi \mid F^\times$, $\xi \mid U_E^1$. Consequently the map is bijective, and the result follows.

The preceding calculations also yield the First Comparison Theorem, although we have so far excluded representations of level zero. There, the result follows from 4.2 Proposition since, in this case, the symplectic sign characters are trivial. Comparing central characters, we see that the character ν is trivial as well. \square

6.9. We comment further on the formula (6.7.4) and the character $\epsilon_M^1 \epsilon_A^1$ of \mathbb{k}_E^\times . It is shown in [14] 8.4 that the constant $\epsilon_M^0(\boldsymbol{\mu}_E)$ and the character $\epsilon_M^1(\boldsymbol{\mu}_E)$ determine each other. The same applies in general.

Theorem. *The following conditions are equivalent:*

- (1) $p \neq 2$ and $\epsilon_A^0(\boldsymbol{\mu}_E) = -1$;
- (2) the character $\epsilon_A^1(\boldsymbol{\mu}_E)$ is non-trivial.

If $f(E|F)$ is odd, then $\epsilon_A^0(\boldsymbol{\mu}_E) = +1$ and $\epsilon_A^1(\boldsymbol{\mu}_E)$ is trivial.

Proof. The case $p = 2$ is uninteresting, since \mathbb{k}_E^\times has no character of order 2. We henceforward exclude it.

We write $\ell = \mathbb{k}_E$ and $\Omega = \text{Gal}(\ell/\mathbb{k}_F)$. The space $\mathfrak{V} = \mathfrak{V}(\beta, \mathfrak{A})$ in our standard notation is a ℓ -vector space on which the group ℓ^\times acts via the natural conjugation action of $\boldsymbol{\mu}_E$. It is therefore a direct sum of spaces of the form Λ_ω , $\omega \in \Omega$, where Λ_ω is a one-dimensional ℓ -vector space on which ℓ^\times acts by

$$x : \lambda \longmapsto x^\omega x^{-1} \lambda, \quad x \in \ell^\times, \lambda \in \Lambda_\omega.$$

Let us write

$$\mathfrak{V} = \bigoplus_{\omega \in \Omega} \Lambda_\omega^{v(\omega)} = \bigoplus_{\omega \in \Omega} \mathfrak{V}_\omega,$$

for various integers $v(\omega) \geq 0$. Since \mathfrak{V} is symplectic, we have $v(\omega^{-1}) = v(\omega)$ and $v(1) \equiv 0 \pmod{2}$.

The component \mathfrak{V}_1 is hyperbolic and ℓ^\times acts trivially. Therefore $t_{\ell^\times}^0(\mathfrak{V}_1) = +1$ and the character $t_{\ell^\times}^1(\mathfrak{V}_1)$ is trivial. If $\omega^2 \neq 1$, the hyperbolic space $\mathfrak{V}_\omega \oplus \mathfrak{V}_{\omega^{-1}}$ has trivial t -invariants [14] 7.3. If Ω has odd order, there is nothing more to be said. We therefore assume the contrary.

If $\omega^2 = 1 \neq \omega$, the space Λ_ω is anisotropic symplectic and the same result gives

$$t_{\ell^\times}^0(\Lambda_\omega) = -1, \quad t_{\ell^\times}^1(\Lambda_\omega) \neq 1.$$

Therefore

$$\epsilon_A^0 = t_{\ell^\times}^0(\mathfrak{V}) = (-1)^{v(\omega_2)}, \quad \epsilon_A^1 = t_{\ell^\times}^1(\mathfrak{V}) = \psi_2^{v(\omega_2)},$$

where ω_2 is the element of Ω of order 2 and ψ_2 is the character of ℓ^\times of order 2.

All assertions have now been proved. \square

The theorem applies equally with M in place of A . Therefore:

Corollary. *The character $\epsilon_A^1(\boldsymbol{\mu}_E)\epsilon_M^1(\boldsymbol{\mu}_E)$ of $\boldsymbol{\mu}_E$ is non-trivial if and only if $p \neq 2$ and $\epsilon_A^0(\boldsymbol{\mu}_E)\epsilon_M^0(\boldsymbol{\mu}_E) = -1$.*

We recall that the value of $\epsilon_A^0(\boldsymbol{\mu}_E)\epsilon_M^0(\boldsymbol{\mu}_E)$ is given explicitly by the relations (6.7.4–6).

We recall also that the invariants $\epsilon_M^j(\boldsymbol{\mu}_E)$ are calculated, in terms of the admissible pair $(E/F, \xi)$, in the Ramification Theorem of [14] 8.3. These calculations and the corollary together yield explicit formulæ for $\epsilon_A^j(\boldsymbol{\mu}_E)$.

7. Completion of the comparison

We continue with the admissible pair $(E/F, \xi)$ and the central simple F -algebra $A \cong M_m(D)$, where D is of dimension d^2 and $[E:F] = n = md$. We complete our determination of the tamely ramified character $\nu = {}_D\nu_\xi$ encountered in the Comparison Theorem of 6.1. To do this, we have to calculate its value at one prime element ϖ of E . We observe that there is nothing to do in the case where E/F is unramified: if ϖ_F is a prime element of F , then $\nu(\varpi_F) = 1$ because the Jacquet-Langlands correspondence preserves central characters.

7.1. We continue with the notation of §6. In particular, we use the symplectic $\Psi_{E/F}$ -modules $\mathfrak{A} = \mathfrak{A}(\beta, \mathfrak{A})$ and $\mathfrak{A}_M = \mathfrak{A}(\delta, \mathfrak{M})$ of 6.7. As before, E_0/F is the minimal sub-extension of E/F such that $\xi \mid U_E^1$ factors through N_{E/E_0} . We prove:

Second Comparison Theorem. *Let ϖ be any prime element of E_0 such that $\varpi^r \in F$, for some integer r not divisible by p . Let $L = F[\varpi]$, let A_L denote the A -centralizer of L , and write $A_L \cong M_{m_L}(D_L)$, for a central L -division algebra D_L of dimension d_L^2 .*

The character $\nu = {}_D\nu_\xi$ of E^\times then satisfies

$$(-1)^{m(d-1)} \nu(\varpi) = (-1)^{m_L(d_L-1)} \epsilon_M(\varpi) \epsilon_A(\varpi),$$

where

$$\epsilon_M(\varpi) = t_{\langle \varpi \rangle}(\mathfrak{A}_M), \quad \epsilon_A(\varpi) = t_{\langle \varpi \rangle}(\mathfrak{A}).$$

The proof will occupy the remainder of the section.

Observe that the condition on ϖ can be rephrased as follows: the group $\langle \varpi, \boldsymbol{\mu}_E \rangle$, generated by ϖ and $\boldsymbol{\mu}_E$, contains a prime element ϖ_F of F .

7.2. Before proceeding to the proof, we point out a couple of consequences. By definition, the ϵ -invariants appearing in our formulæ for ${}_D\nu_\xi$ depend only on the simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ determined by the admissible pair $(E/F, \xi)$. This, in turn, is constructed from the restriction $\xi | U_E^1$. Consequently:

Corollary 1. *For $i = 1, 2$, let $(E/F, \xi_i)$ be an admissible pair with $[E:F] = n$. If $\xi_1^{-1}\xi_2$ is tamely ramified, then ${}_D\nu_{\xi_1} = {}_D\nu_{\xi_2}$.*

We remark that the quantities $\epsilon_A(\varpi)$, $\epsilon_M(\varpi)$ are amenable to calculation, in terms of the structure of the admissible pair $(E/F, \xi)$, using the methods of [14] §7, §8. We shall not pursue the matter here beyond pointing out that, using a more elaborate version of the argument of 5.6 along with the analysis in [14] §7 and 8.5, one obtains:

Corollary 2. *Let $m \geq 1$ be an integer. Let D, D' be central F -division algebras such that $\dim_F D = \dim_F D' = d^2$, $d \geq 1$. If $(E/F, \xi)$ is an admissible pair in which $[E:F] = md$, then*

- (1) ${}_D\nu_\xi = {}_{D'}\nu_\xi$, and
- (2) if $j : \mathcal{A}_m^{\text{et}}(D) \rightarrow \mathcal{A}_m^{\text{et}}(D')$ is the Jacquet-Langlands correspondence, then

$$j({}_D\Pi_\xi) = {}_{D'}\Pi_\xi.$$

7.3. We start the proof of the Second Comparison Theorem of 7.1.

We take $\pi = {}_D\Pi_\xi$ and $\tau = {}_F\Pi_{\nu\xi}$ as before, so that $\pi = j_A(\tau)$. In particular, π contains the extended maximal simple type (\mathbf{J}, Λ) , giving rise to the admissible pair $(E/F, \xi)$. We compare the character values $\text{tr } \pi(\varpi u)$, $\text{tr } \tau(\varpi u)$, for an element u of ${}_u(G_L)_{\text{reg}}^{\text{ell}}$.

Proposition. *Let $x \in G$ and suppose that $\text{tr } \Lambda(x^{-1}\varpi ux) \neq 0$. There then exists $y \in x\mathbf{J}$ such that $y^{-1}\varpi y \in E^\times$. For any such y , we have $y^{-1}uy \in J^0$.*

Proof. The proof is exactly parallel to that of 6.1 Proposition of [14], so we omit the details. \square

Let $L = F[\varpi]$, let $\Sigma_L = \text{Gal}(E/L)$ and let G_L be the G -centralizer of L . The extension E/L is unramified, so any F -embedding $L \rightarrow E$ extends to

an F -automorphism of E . The element y of the proposition therefore lies in $G_L N_G(E^\times)$ and we have

$$(7.3.1) \quad \mathrm{tr} \pi(\varpi u) = \sum_{x \in G_L N_G(E^\times) \mathbf{J} / \mathbf{J}} \mathrm{tr} \Lambda(x^{-1} \varpi u x).$$

We need to clarify the nature of the coset space $G_L N_G(E^\times) \mathbf{J} / \mathbf{J}$. We set $\mathcal{Y} = N_G(E^\times) / E^\times = \mathrm{Aut}(E|F)$. Take the subgroup Σ_t of Σ_0 as in 6.3. We choose representatives $t_\alpha \in N_G(E^\times)$, $\alpha \in \mathcal{Y}$, as in 6.3.

Lemma. *There is a canonical bijection*

$$(7.3.2) \quad \begin{aligned} \Sigma_L \backslash \mathcal{Y} / \Sigma_t &\longrightarrow G_L \backslash G_L N_G(E^\times) \mathbf{J} / \mathbf{J}, \\ \Sigma_L \alpha \Sigma_t &\longmapsto G_L t_\alpha \mathbf{J}. \end{aligned}$$

Proof. We remarked in the proof of 6.3 Lemma that the image of $\mathbf{J} \cap N_G(K^\times)$ in $\Sigma = \mathrm{Gal}(K/F)$ is the subgroup Σ_t . It follows that the image of $\mathbf{J} \cap N_G(E^\times)$ in \mathcal{Y} is the subgroup Σ_t again, viewed as a subgroup of $\Sigma_0 = \mathrm{Gal}(E/E_0)$. Thus we have a bijection $\mathcal{Y} / \Sigma_t \rightarrow N_G(E^\times) \mathbf{J} / \mathbf{J}$ given by $\alpha \Sigma_t \mapsto t_\alpha \mathbf{J}$, $\alpha \in \mathcal{Y}$.

For $\alpha \in \mathcal{Y}$, the double coset $\Sigma_L \alpha \Sigma_t$ is the disjoint union of the cosets $\sigma \alpha \Sigma_t$, where σ ranges over a set of representatives for $\Sigma_L / (\alpha \Sigma_t \alpha^{-1} \cap \Sigma_L)$. Under the map $\gamma \mapsto t_\gamma$, all of these cosets have image contained in $G_L t_\alpha \mathbf{J}$. The map (7.3.2) is therefore well-defined and surjective.

Let $\alpha, \beta \in \mathcal{Y}$, and suppose that $t_\alpha = g t_\beta j$, with $g \in G_L$, and $j \in \mathbf{J}$. Since $\mathbf{J} = E^\times J^0$, we may as well assume $j \in J^0$. The element j conjugates ϖ^β to ϖ^α . We write $\varpi^\alpha = \zeta_\alpha \varpi$, $\varpi^\beta = \zeta_\beta \varpi$, for roots of unity $\zeta_\alpha, \zeta_\beta \in \mu_K$. We have $j^{-1} \varpi j \equiv \varpi \pmod{J^1}$, so $j^{-1} \zeta_\alpha j \equiv \zeta_\beta \pmod{J^1}$. Any such conjugation, realized by an element of J^0 , is given by an element σ of Σ_t . Therefore $\alpha \sigma \beta^{-1}$ fixes ϖ , so $\alpha \sigma \beta^{-1} \in \Sigma_L$. In other words, $\alpha \in \Sigma_L \beta \Sigma_t$; this implies that the map (7.3.2) is injective and completes the proof. \square

We therefore decompose the character formula (7.3.1) as

$$(7.3.3) \quad \mathrm{tr} \pi(\varpi u) = \sum_{\alpha \in \Sigma_L \backslash \mathcal{Y} / \Sigma_t} \mathrm{tr} \pi(\varpi u; \alpha),$$

where

$$\begin{aligned}
 \text{tr } \pi(\varpi u; \alpha) &= \sum_{y \in G_L t_\alpha \mathbf{J}/\mathbf{J}} \text{tr } \Lambda(y^{-1} \varpi u y) \\
 (7.3.4) \qquad \qquad &= \sum_{y \in G_L^\alpha \mathbf{J}/\mathbf{J}} \text{tr } \Lambda(y^{-1} \varpi^\alpha u^\alpha y).
 \end{aligned}$$

7.4. We examine the function $\text{tr } \pi(\varpi u; \alpha)$ of (7.3.4). Write $Z = L^\alpha$, so that $G_L^\alpha = G_Z = A_Z^\times$ is the G -centralizer of Z^\times .

Set $\mathfrak{A}_Z = \mathfrak{A} \cap A_Z$. We recall from 3.2 Proposition that the stratum $[\mathfrak{A}_Z, \ell, 0, \beta]$ is simple, $H^1(\beta, \mathfrak{A}_Z) = H^1(\beta, \mathfrak{A}) \cap G_Z$ and $J^k(\beta, \mathfrak{A}_Z) = J^k(\beta, \mathfrak{A}) \cap G_Z$, $k = 0, 1$. We use the obvious abbreviations

$$H_Z^1 = H^1(\beta, \mathfrak{A}_Z), \quad J_Z^k = J^k(\beta, \mathfrak{A}_Z),$$

for $k = 0, 1$.

The character $\theta_Z = \theta \mid H^1(\beta, \mathfrak{A}_Z)$ lies in $\mathcal{C}(\mathfrak{A}_Z, \beta, \psi_Z)$. We consider a maximal simple type (J_Z^0, λ_Z) in G_Z containing θ_Z . We recall (2.6) that $\mathbf{J}(\lambda_Z)$ denotes the G_Z -normalizer of λ_Z and that, in the same notation, the group we have been calling \mathbf{J} is $\mathbf{J}(\lambda)$.

Lemma. *Using the preceding notation, we have $\mathbf{J}(\lambda_Z) \supset \mathbf{J}(\lambda) \cap G_Z$. Moreover, $\mathbf{J}(\lambda_Z) = \mathbf{J}(\lambda) \cap G_Z$ if and only if $\delta_0(\lambda_Z) = [E:Z]$.*

Proof. Since, by hypothesis, $\delta_0(\lambda) = [E:F] = n$, we have $\mathbf{J}(\lambda) = E^\times J^0$, giving $\mathbf{J}(\lambda) \cap G_Z = E^\times J_Z^0$. The lemma thus follows from (2.6.1). \square

7.5. Bearing 7.4 Lemma in mind, we write $\mathbf{J}_Z = \mathbf{J} \cap G_Z$. Thus $G_Z \mathbf{J}/\mathbf{J} = G_Z/\mathbf{J}_Z$ and we obtain

$$\text{tr } \pi(\varpi u; \alpha) = \sum_{y \in G_Z/\mathbf{J}_Z} \text{tr } \Lambda(\varpi^\alpha y^{-1} u^\alpha y).$$

The element $y^{-1} u^\alpha y$ lies in J^0 , it commutes with ϖ^α , and it is pro-unipotent (7.3 Proposition).

We factorize Λ in the form $\Lambda = \Lambda_t \otimes \Lambda_w$, relative to a prime element ϖ_F of F lying in $\langle \mu_E, \varpi \rangle$ (cf. 7.1). We so obtain

$$\text{tr } \Lambda(\varpi^\alpha y^{-1} u^\alpha y) = \text{tr } \Lambda_t(\varpi^\alpha y^{-1} u^\alpha y) \cdot \text{tr } \Lambda_w(\varpi^\alpha y^{-1} u^\alpha y).$$

We first control the contribution from the “wild” factor $\mathrm{tr} \Lambda_w$. The element $y^{-1}u^\alpha y$ lies in a subgroup \mathcal{J} of J^0 , containing J^1 and normalized by ϖ^α , such that \mathcal{J}/J^1 is unipotent. In particular, \mathcal{J} is a pro- p group. Using the Glauberman correspondence directly, we conclude that there is a unique irreducible representation ϑ of $\mathcal{J}_Z = \mathcal{J} \cap J_Z^0$ and a sign ϵ such that

$$(7.5.1) \quad \mathrm{tr} \Lambda_w(\varpi^\alpha t) = \epsilon \mathrm{tr} \vartheta(t),$$

for all $t \in \mathcal{J}_Z$. The relation (7.5.1) holds, in particular, for $t \in J_Z^1$. Therefore $\vartheta \mid J_Z^1 \cong {}_Z\eta_\xi$, and so

$$(7.5.2) \quad \epsilon = \epsilon_A(\varpi^\alpha) = t_{\langle \varpi^\alpha \rangle}(\mathfrak{V}), \quad \mathfrak{V} = \mathfrak{V}(\beta, \mathfrak{A}).$$

Wide Extension Lemma. *There is a wide extension κ_Z of ${}_Z\eta_\xi$ such that $\vartheta \cong \kappa_Z \mid \mathcal{J}_Z$.*

The proof is given in Appendix 2.

7.6. We first apply the Wide Extension Lemma of 7.5 to the case $\alpha = 1$. We obtain

$$(7.6.1) \quad \mathrm{tr} \Lambda(\varpi y^{-1}uy) = \epsilon_A(\varpi) \xi(\varpi) \mathrm{tr} \Lambda_t(y^{-1}uy) \cdot \mathrm{tr} \kappa_L(y^{-1}uy),$$

where κ_L is a wide extension of ${}_L\eta_\xi$: the choice of κ_L is immaterial (2.5.5).

Proposition. *The pair $(E/L, \xi)$ is admissible. Write $A_L \cong M_{m_L}(D_L)$, for a central L -division algebra D_L , and set $\pi_L = {}_{D_L}\Pi_\xi$. We then have*

$$(7.6.2) \quad \mathrm{tr} \pi(\varpi u; 1) = \epsilon_A(\varpi) \xi(\varpi) \mathrm{tr} \pi_L(u).$$

Proof. The first assertion is clear. There is an extended maximal simple type $(\mathbf{J}(L), \Lambda_L)$ in G_L , containing the simple character θ_L and with attached admissible pair $(E/L, \xi)$. The group $\mathbf{J}(L)$ is $\mathbf{J}_L = \mathbf{J} \cap G_L$ by 7.4 Lemma. The result then follows from (7.6.1). \square

7.7. We consider the general term

$$\mathrm{tr} \pi(\varpi u; \alpha) = \sum_{y \in G_Z/J_Z} \mathrm{tr} \Lambda(\varpi^\alpha y^{-1}u^\alpha y),$$

where $\alpha \in \mathcal{Y}$ and $Z = L^\alpha = F[\varpi^\alpha]$. We have

$$\mathrm{tr} \Lambda((\varpi^\alpha y^{-1}u^\alpha y) = \epsilon_A(\varpi^\alpha) \mathrm{tr} \kappa_Z(y^{-1}u^\alpha y) \mathrm{tr} \Lambda_t(\varpi^\alpha y^{-1}u^\alpha y).$$

Lemma 1. *There is a collection $\Lambda_{t,i}$, $1 \leq i \leq r$, of extended maximal simple types of level zero in G_Z such that*

$$\mathrm{tr} \Lambda_t(\varpi^\alpha j) = (-1)^{m-m_L} \sum_{i=1}^r \mathrm{tr} \Lambda_{t,i}(\varpi^\alpha j),$$

for every pro-unipotent element j of J_Z^0 . Moreover, if $\lambda_{t,i} = \Lambda_{t,i} \mid J_Z^0$, then $\delta_0(\lambda_{t,i}) = [E:E_0Z]$, for all i .

Proof. The result can be read off from the discussion in paragraph 6.10 of [14]. \square

The type $\Lambda_{t,i}$ is thus determined by a tamely ramified admissible pair of the form $(E/ZE_0, \psi_i)$. We set $\chi_i = \psi_i \cdot \xi_\omega$; this gives a family of admissible pairs $(E/Z, \chi_i)$, $1 \leq i \leq r$. Applying Lemma 1 and 7.4 Lemma, we may argue as in 7.6 Proposition to get

$$\mathrm{tr} \pi(\varpi u; \alpha) = (-1)^{m-m_L} \epsilon_A(\varpi^\alpha) \sum_{i=1}^r \mathrm{tr}_{D_Z} \Pi_{\chi_i}(\varpi^\alpha u^\alpha).$$

It is more convenient to re-write this relation in the form

$$\begin{aligned} \mathrm{tr} \pi(\varpi u; \alpha^{-1}) &= (-1)^{m-m_L} \epsilon_A(\varpi^{\alpha^{-1}}) \sum_{i=1}^r \mathrm{tr}_{D_L} \Pi_{\chi_i^\alpha}(\varpi u) \\ &= (-1)^{m-m_L} \epsilon_A(\varpi^{\alpha^{-1}}) \xi^\alpha(\varpi) \sum_{i=1}^r \mathrm{tr}_{D_L} \Pi_{\chi_i^\alpha}(u), \end{aligned}$$

for $\alpha \in \Sigma_t \backslash \Upsilon / \Sigma_L$.

Consider the admissible pair $(E/L, \chi_i^\alpha)$. We recall that $\xi \mid U_E^1 = \xi_0 \circ N_{E/E_0}$, the 1-pair $(E_0/F, \xi_0)$ being admissible.

Lemma 2.

- (1) *The character $\chi_i^\alpha \mid U_E^1$ factors through N_{E/LE_0^α} , and LE_0^α/L is the minimal sub-extension of E/L with this property.*
- (2) *The restriction $\chi_i^\alpha \mid U_E^1$ is of the form $\xi_0^\alpha \circ N_{LE_0^\alpha/E_0^\alpha}$.*
- (3) *The 1-pair $(LE_0^\alpha/L, \xi_0^\alpha \circ N_{LE_0^\alpha/E_0^\alpha})$ is L -isomorphic to $(E_0/L, \xi_0)$ if and only if $\alpha \in \Sigma_L$.*

Proof. By definition, $\chi_i \mid U_E^1 = \xi \mid U_E^1$, so we need only treat the character ξ . Looking back at the proof of [11] A.1 Lemma (stated above as 3.3 Lemma), the field E_0 arises as follows. We choose a finite extension \tilde{E}/E such that \tilde{E}/F is Galois and set $\Gamma = \text{Gal}(\tilde{E}/F)$. If Δ is the subgroup of Γ which fixes $\xi \mid U_E^1$, the field E_0 is \tilde{E}^Δ . The character ξ_0 is then uniquely determined by $\xi \mid U_E^1$.

Parts (1) and (2) of the lemma follow immediately. In part (3), one implication is obvious. In the opposite direction, the extension E/L is unramified, so if LE_0^α is L -isomorphic to E_0 then $LE_0^\alpha = E_0$ and the isomorphism is realized by an L -automorphism of E_0 , that is, by an element σ of Σ_L . If the pairs are to be isomorphic, this automorphism must carry ξ_0^α to ξ_0 . This implies that $\sigma^{-1}\alpha$ fixes ξ_0 , whence $\sigma^{-1}\alpha \in \Sigma_0$ and $\alpha \in \Sigma_L$, as required. \square

Remark. There is the following more precise version of part (3) of the lemma. If $\alpha, \beta \in \Upsilon$, the pairs

$$(LE_0^\alpha/L, \xi_0^\alpha \circ N_{LE_0^\alpha/E_0^\alpha}), \quad (LE_0^\beta/L, \xi_0^\beta \circ N_{LE_0^\beta/E_0^\beta}),$$

are L -isomorphic if and only if $\Sigma_0\alpha\Sigma_L = \Sigma_0\beta\Sigma_L$. We do not need this fact, so we omit the proof.

7.8. We return, for the moment, to a general situation. If $\Phi \in \mathcal{A}_m^{\text{et}}(D)$, we may write $\Phi = {}_D\Pi_\vartheta$, for an admissible pair $(H/F, \vartheta)$. This pair determines an admissible 1-pair $(H_0/F, \vartheta_0)$, the F -isomorphism class of which is determined by Φ . We extend our terminology and say that $(H_0/F, \vartheta_0)$ is the 1-pair attached to Φ .

Linear Independence Lemma. *Let $\pi_1, \pi_2, \dots, \pi_r \in \mathcal{A}_m^{\text{et}}(D)$. Let $(E_i/F, \gamma_i)$ be an admissible 1-pair attached to π_i , $1 \leq i \leq r$. Suppose that $(E_i/F, \gamma_i)$ is not F -isomorphic to $(E_j/F, \gamma_j)$ when $i \neq j$. The functions $u \mapsto \text{tr } \pi_i(u)$, $1 \leq i \leq r$, on ${}_uG_{\text{reg}}^{\text{ell}}$ then form a linearly independent set.*

Proof. Let H be a central F -division algebra of dimension n^2 and let \mathbf{j}_H denote the Jacquet-Langlands correspondence $\mathcal{A}_m^{\text{et}}(D) \rightarrow \mathcal{A}_1^{\text{et}}(H)$. The admissible 1-pair attached to $\mathbf{j}_H(\pi_i)$ is $(E_i/F, \gamma_i)$, $1 \leq i \leq r$, by the First Comparison Theorem of 6.1. We may therefore assume we are in the case $m = 1$. So, until the end of the proof of the Linear Independence Lemma, A is a division algebra.

Lemma. *Suppose A is a division algebra, and let θ_i be a simple character occurring in π_i , $1 \leq i \leq r$. If the characters θ_i, θ_j intertwine in G , then $i = j$.*

Proof. If θ_i intertwines with θ_j , then θ_i is G -conjugate to θ_j [4] (11.2.9). The 1-pairs $(E_i/F, \gamma_i), (E_j/F, \gamma_j)$ are then F -isomorphic, as follows from the Parametrization Theorem 6.1 and the construction of the attached pair. \square

Let $\mathcal{H}(G)$ be the space of locally constant, compactly supported functions $G \rightarrow \mathbb{C}$, viewed as convolution algebra relative to a choice of Haar measure on G . Let $e_i \in \mathcal{H}(G)$ be the idempotent corresponding to the simple character θ_i . Thus e_i has support contained in $U_A^1 = U_{\mathfrak{o}_A}^1$ and, if (π, V) is an admissible representation of G , we have

$$\int_G \text{tr } \pi(g) e_i(g) dg = \dim V^{\theta_i},$$

where V^{θ_i} is the θ_i -isotypic subspace of V .

Since A is a division algebra, the set ${}_{\mathfrak{u}}G_{\text{reg}}^{\text{ell}}$ is open dense in U_A^1 . Consequently, if we have a linear dependence relation

$$\sum_{i=1}^r a_i \text{tr } \pi_i(u) = 0, \quad u \in {}_{\mathfrak{u}}G_{\text{reg}}^{\text{ell}},$$

we may integrate it against e_j to get

$$\begin{aligned} 0 &= \sum_{i=1}^r a_i \int_{U_A^1} \text{tr } \pi_i(u) e_j(u) du \\ &= \sum_{i=1}^r a_i \int_G \text{tr } \pi_i(g) e_j(g) dg = c_j a_j, \end{aligned}$$

where $c_j > 0$ is the dimension of the θ_j -isotypic subspace of π_j . Thus $a_j = 0$, as required. \square

7.9. We return to the context of 7.7 and consider the representation $\tau = {}_F\Pi_{\nu\xi}$. As a special case of (7.6.1), we have

$$\text{tr } \tau(\varpi u; 1) = \epsilon_M(\varpi) \nu\xi(\varpi) \text{tr } \tau_L(u),$$

where $\tau_L = {}_L\Pi_{\nu\xi}$ and u ranges over the pro-unipotent, elliptic regular elements of the $\text{GL}_n(F)$ -centralizer $\text{GL}_n(F)_L$ of some copy of L embedded in $M_n(F)$. From

7.7 Lemma, the Linear Independence Lemma of 7.8 and the defining relation $\pi = j_A \tau$, we obtain

$$\text{tr } \pi(\varpi u; 1) = (-1)^{m(d-1)} \text{tr } \tau(\varpi u; 1),$$

and hence

$$(7.9.1) \quad (-1)^{m(d-1)} \epsilon_M(\varpi) \nu \xi(\varpi) \text{tr } \tau_L(u) = \epsilon_A(\varpi) \xi(\varpi) \text{tr } \pi_L(u),$$

this relation being valid for all $u \in {}_u(G_L)_{\text{reg}}^{\text{ell}}$. (Note that, in the left hand side, u denotes an element of $\text{GL}_n(F)_L$ associate to u .)

We recall that $A_L \cong M_{m_L}(D_L)$, for an L -division algebra D_L of dimension d_L^2 . We let $u \rightarrow 1$ in the right hand side of (7.9.1). The associate element u in the left hand side then also tends to 1 (1.6 Proposition). From 1.5 Theorem (1), we get

$$(-1)^{m(d-1)} \epsilon_M(\varpi) \nu \xi(\varpi) (-1)^{m_L d_L - 1} c_{\tau_L} = \epsilon_A(\varpi) \xi(\varpi) (-1)^{m_L - 1} c_{\pi_L},$$

for certain positive constants c_{τ_L}, c_{π_L} . These cancel, since every other factor is a root of unity. Therefore

$$(-1)^{m(d-1)} \nu(\varpi) = (-1)^{m_L(d_L-1)} \epsilon_M(\varpi) \epsilon_A(\varpi),$$

as required to prove the theorem. \square

APPENDIX 1.

Parametric degree and formal degree

Here we prove Theorem 2.8, which connects the normalized formal degree $\text{deg } \pi$ and the parametric degree $\delta(\pi)$ of a representation $\pi \in \mathcal{A}_m^\square(D)$. This is an exercise, parallel to [9] and generalizing §7.7 of [17]. The novelty derives from the Hecke algebra calculations of [41], [24].

A1.1. Let $G = \text{GL}_m(D)$, $A = M_m(D)$, where D is a central F -division algebra of dimension d^2 , $d \geq 1$. We set $n = md$. We recall a well-known property of the essentially square-integrable representations of G — see for instance [21] Théorème B.2.b — but we give a rather different proof.

Proposition. *Let $\pi \in \mathcal{A}_m^\square(D)$. There is then a positive divisor r of m and an irreducible cuspidal representation π_0 of $\mathrm{GL}_{m/r}(D)$ such that the cuspidal support of π consists of unramified twists of π_0 .*

Proof. Replacing π by $\chi\pi$, where χ is a character of F^\times , changes nothing. We may therefore assume π to be square-integrable. It follows, from an argument identical to the proof Proposition 8.5.10 of [17], that π cannot be of the form $\mathrm{Ind}_P^G \tau$, where P is a proper parabolic subgroup of G and τ is an irreducible representation of P trivial on the unipotent radical of P . However, if the cuspidal support of π is not of the stated form, then Theorem 3.1 of [36] asserts that π is an irreducible, parabolically induced representation of just this kind. \square

A1.2. It follows from Théorème 5.23 of [41] that any $\pi \in \mathcal{A}_m^\square(D)$ contains a *simple type*, in the sense of [40].

In order to review the definition of a simple type, it will be simpler to introduce a new scheme of notation. Let π be an irreducible *cuspidal* representation of $\mathrm{GL}_m(D)$. Thus π contains a maximal simple type (J, λ) . We assume initially that this type is of positive level, and use the notational scheme of 2.6 above. Thus there is a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in $A = M_m(D)$ so that $J = J^0(\beta, \mathfrak{A})$. The hereditary order \mathfrak{A} is maximal among E_0 -pure hereditary orders in A , where E_0 denotes the field $F[\beta]$. We denote by B the A -centralizer of E_0 , so that $B \cong M_l(C)$, where C is a central E_0 -division algebra of dimension c^2 , and $lc = n/[E_0:F]$. The order $\mathfrak{B} = \mathfrak{A} \cap B$ is a maximal \mathfrak{o}_{E_0} -order in B .

The restriction of λ to $H^1(\beta, \mathfrak{A})$ is a multiple of a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. Let η be the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ containing θ , and let κ be some wide extension of η . The representation λ is then of the form $\lambda = \sigma \otimes \kappa$, for an irreducible representation σ of $J(\beta, \mathfrak{A})$ trivial on $J^1(\beta, \mathfrak{A})$. The restriction $\sigma_{\mathfrak{B}}$ of σ to $U_{\mathfrak{B}} = J(\beta, \mathfrak{A}) \cap B$ is trivial on $U_{\mathfrak{B}}^1 = J^1(\beta, \mathfrak{A}) \cap B$. The pair $(U_{\mathfrak{B}}, \sigma_{\mathfrak{B}})$ is a maximal simple type of level zero in the group B^\times . The representation $\sigma_{\mathfrak{B}}$ is the inflation of an irreducible cuspidal representation $\tilde{\sigma}$ of $U_{\mathfrak{B}}/U_{\mathfrak{B}}^1 \cong \mathrm{GL}_l(\mathbb{k}_C)$.

In this scheme, we have

$$(A1.2.1) \quad \delta(\pi) = \delta_0(\sigma_{\mathfrak{B}})[E_0:F], \quad t(\pi) = \delta(\pi)/e(E_0|F).$$

We add a further datum to this list. We fix an unramified field extension

E/E_0 , inside B , so that $[E:F] = n$ and \mathfrak{B} is E -pure. Thus \mathfrak{A} is E -pure. These conditions determine \mathfrak{A} and \mathfrak{B} uniquely in terms of E .

The modifications required for the level zero case are both clear and straightforward.

A1.3. Continuing with the notation of A1.2, let $r \geq 1$ be an integer and let $\bar{A} = M_{mr}(D)$, $\bar{G} = \mathrm{GL}_{mr}(D)$. We embed the field E in \bar{A} , in some way, as an F -subalgebra. We choose an E -pure hereditary \mathfrak{o}_F -order $\bar{\mathfrak{A}}$ in \bar{A} , which is *minimal* relative to this property. The F -period of $\bar{\mathfrak{A}}$ is thus r times the F -period of \mathfrak{A} , and we get a simple stratum $[\bar{\mathfrak{A}}, r\ell, 0, \beta]$ in \bar{A} .

Let \bar{B} denote the centralizer of E_0 in \bar{A} : thus $\bar{B} \cong M_{lr}(C)$. We set $\bar{\mathfrak{B}} = \bar{\mathfrak{A}} \cap \bar{B}$.

The transfer process of [38] Théorème 3.53 determines, from θ and β , a simple character $\bar{\theta} \in \mathcal{C}(\bar{\mathfrak{A}}, \beta, \psi_F)$. Let $\bar{\eta}$ be the unique irreducible representation of $J^1(\beta, \bar{\mathfrak{A}})$ which contains $\bar{\theta}$. Following the procedures of [39], starting from Théorème 2.18, we obtain from κ a wide extension $\bar{\kappa}$ of $\bar{\eta}$.

We have $J(\beta, \bar{\mathfrak{A}}) = U_{\bar{\mathfrak{B}}} J^1(\beta, \bar{\mathfrak{A}})$ and $U_{\bar{\mathfrak{B}}} \cap J^1(\beta, \bar{\mathfrak{A}}) = U_{\bar{\mathfrak{B}}}^1$. The group $U_{\bar{\mathfrak{B}}}/U_{\bar{\mathfrak{B}}}^1$ is the direct product of r copies of $\mathrm{GL}_l(\mathbb{k}_C)$. We let $\bar{\sigma}_{\bar{\mathfrak{B}}}$ be the inflation to $U_{\bar{\mathfrak{B}}}$ of $\tilde{\sigma} \otimes \tilde{\sigma} \otimes \cdots \otimes \tilde{\sigma}$ (r factors). We extend $\bar{\sigma}_{\bar{\mathfrak{B}}}$ to a representation $\bar{\sigma}$ of $J(\beta, \bar{\mathfrak{A}})$ trivial on $J^1(\beta, \bar{\mathfrak{A}})$. We set

$$\bar{\lambda} = \bar{\sigma} \otimes \bar{\kappa}, \quad \bar{J} = J(\beta, \bar{\mathfrak{A}}).$$

The representation $\bar{\lambda}$ is irreducible. The pair $(\bar{J}, \bar{\lambda})$ is an instance of a *simple type in \bar{G}* , in the sense of [40]. The property we require is [41] Théorème 5.23:

(A1.3.1) *Let Π be an irreducible smooth representation of \bar{G} . The following conditions are equivalent:*

- (1) *Π contains the simple type $(\bar{J}, \bar{\lambda})$;*
- (2) *the cuspidal support of Π consists of unramified twists of π .*

A1.4. We recall and extend some standard material from, for example, [17] 7.7 and [18] §8. We fix an integer $r \geq 1$, and consider those representations $\tau \in \mathcal{A}_r^\square(F)$ which have, as cuspidal support, unramified characters of F^\times . Among these is the *Steinberg representation*, which we denote St_r^F . Indeed, any other representation of this kind is an unramified twist of St_r^F , and has the same formal degree. A calculation originating in [3] yields:

(A1.4.1) Let $\dot{\mu}_r^F$ denote a Haar measure on $\mathrm{GL}_r(F)/F^\times$, and let $\mathcal{I}_r(F)$ be an Iwahori subgroup of $\mathrm{GL}_r(F)$. The formal degree of the Steinberg representation St_r^F satisfies

$$\dot{\mu}_r^F(\mathcal{I}_r(F)F^\times) d(\mathrm{St}_r^F, \dot{\mu}_r^F) = \frac{1}{r} \frac{(q-1)^r}{q^r-1},$$

where $q = q_F = |\mathbb{k}_F|$.

We recall that an Iwahori subgroup of $\mathrm{GL}_r(F)$ is the same as the unit group of a *minimal* hereditary \mathfrak{o}_F -order in $M_r(F)$.

We extend this notation by letting $\mathcal{I}_r(D)$ denote the unit group of some minimal hereditary order in $M_r(D)$, when D is a central F -division algebra of dimension d^2 . Again we consider the elements of $\mathcal{A}_r^\square(D)$ having unramified characters of $D^\times = \mathrm{GL}_1(D)$ as cuspidal support. Among these is the Steinberg representation St_r^D .

Proposition. Let $\dot{\mu}_r^D$ be a Haar measure on $\mathrm{GL}_r(D)/F^\times$. The formal degree of the Steinberg representation St_r^D satisfies

$$\dot{\mu}_r^D(\mathcal{I}_r(D)F^\times) d(\mathrm{St}_r^D, \dot{\mu}_r^D) = \frac{1}{dr} \frac{(q^d-1)^r}{q^{dr}-1}.$$

Proof. Set $G = \mathrm{GL}_r(D)$ and let $\mathcal{H}(G, \mathcal{I}_r(D))$ be the convolution algebra of compactly supported functions $G \rightarrow \mathbb{C}$ which are left- and right-invariant under translation by $\mathcal{I}_r(D)$. As a normalized Hilbert algebra (in the sense of [15] §3 or [22]), this is isomorphic to the affine Hecke algebra $\mathcal{H}(r, q_D)$ (in the notation of [17] (5.4.6)). This, in turn, is the same as $\mathcal{H}(\mathrm{GL}_r(K), \mathcal{I}_r(K))$, where K/F is an unramified field extension of degree d . Under the composite isomorphism $\mathcal{H}(G, \mathcal{I}_r(D)) \cong \mathcal{H}(\mathrm{GL}_r(K), \mathcal{I}_r(K))$, functions with support contained in $F^\times \mathcal{I}_r(D)$ are mapped to the functions with support contained in $(K^\times)^d \mathcal{I}_r(K)$, where $(K^\times)^d$ denotes the group of d -th powers in K^\times . As in [17] 7.7 (or following the more general analysis in [15]), we get the relation

$$\begin{aligned} \dot{\mu}_r^D(\mathcal{I}_r(D)F^\times) d(\mathrm{St}_r^D, \dot{\mu}_r^D) &= d^{-1} \dot{\mu}_r^K(\mathcal{I}_r(K)K^\times) d(\mathrm{St}_r^K, \dot{\mu}_r^K) \\ &= \frac{1}{dr} \frac{(q^d-1)^r}{q^{dr}-1}, \end{aligned}$$

as required. \square

A1.5. We return to the situation of A1.2–3. Let $\Pi \in \mathcal{A}_{mr}^\square(D)$ have cuspidal support consisting of unramified twists of $\pi \in \mathcal{A}_m^\square(D)$. By (A1.3.1), the representation Π contains the simple type $(\bar{J}, \bar{\lambda})$.

We consider the Hecke algebra $\mathcal{H}(\bar{G}, \bar{\lambda})$. Appealing to [40] Théorème 4.6, $\mathcal{H}(\bar{G}, \bar{\lambda})$ is isomorphic, as Hilbert algebra, to the affine Hecke algebra

$$\mathcal{H}(r, q_E) = \mathcal{H}(\mathrm{GL}_r(E), \mathcal{I}_r(E)).$$

However, under the isomorphism $\mathcal{H}(\bar{G}, \bar{\lambda}) \rightarrow \mathcal{H}(\mathrm{GL}_r(E), \mathcal{I}_r(E))$, functions supported in $F^\times \bar{J}$ become functions supported in $(E^\times)^k \mathcal{I}_r(E)$, where k is the integer $e(E_0|F)lc/\delta_0(\sigma_{\mathfrak{B}})$. This gives us the relation

$$\dot{\mu}_{mr}^D(\bar{J}F^\times) d(\Pi, \dot{\mu}_{mr}^D) = \frac{\delta_0(\sigma_{\mathfrak{B}})}{e(E_0|F)lc} \dot{\mu}_r^E(\mathcal{I}_r(E)E^\times) d(\mathrm{St}_r^E, \dot{\mu}_r^E) \dim \bar{\lambda}.$$

From the definitions, $\delta(\pi) = \delta(\Pi) = [E_0:F] \delta_0(\sigma_{\mathfrak{B}})$, so (2.1.1)

$$\frac{\delta_0(\sigma_{\mathfrak{B}})}{e(E_0|F)lc} = t(\Pi)/n.$$

Since (A1.4.1)

$$\dot{\mu}_r^E(\mathcal{I}_r(E)E^\times) d(\mathrm{St}_r^E, \dot{\mu}_r^E) = r^{-1} \frac{(q_E - 1)^r}{q_E^r - 1},$$

we have

$$\dot{\mu}_{mr}^D(\bar{J}F^\times) d(\Pi, \dot{\mu}_{mr}^D) = \frac{t(\Pi)}{nr} \frac{(q_E - 1)^r}{q_E^r - 1} \dim \bar{\lambda}.$$

We now choose $\dot{\mu}_{mr}^D$ so that $d(\mathrm{St}_{mr}^D, \dot{\mu}_{mr}^D) = 1$. That is,

$$\dot{\mu}_{mr}^D(\mathcal{I}_{mr}(D)F^\times) = \frac{1}{dmr} \frac{(q^d - 1)^{mr}}{q^{dmr} - 1}.$$

This yields

$$(A1.5.1) \quad \deg \Pi = (\mathcal{I}_{mr}(D)F^\times : \bar{J}F^\times) t(\Pi) \frac{q^{dmr} - 1}{(q^d - 1)^{mr}} \frac{(q_E - 1)^r}{q_E^r - 1} \dim \bar{\lambda}.$$

A1.6. We now prove 2.8 Theorem. In our present scheme of notation A1.2, A1.3, it reads:

Theorem. *Let $\Pi \in \mathcal{A}_{mr}^\square(D)$ have cuspidal support consisting of unramified twists of a cuspidal representation π of $\mathrm{GL}_m(D)$. We then have*

$$(A1.6.1) \quad \deg \Pi \equiv t(\Pi) \frac{(q^{nr} - 1)}{(q^{t(\Pi)nr/\delta(\Pi)} - 1)} \pmod{p^{\mathbb{Z}}}.$$

Proof. We recall that $t(\Pi) = t(\pi)$ and $\delta(\Pi) = \delta(\pi)$ in our present scheme.

We proceed by simplifying the expression (A1.5.1). The generalized group index term there contributes

$$(\mathcal{I}_{mr}(D)F^\times : \bar{J}F^\times) \equiv (q^d - 1)^{mr} |\mathrm{GL}_l(\mathbb{k}_C)|^{-r} \pmod{p^{\mathbb{Z}}}.$$

On the other hand,

$$\dim \bar{\lambda} \equiv |\mathrm{GL}_l(\mathbb{k}_C)|^r (q_C^l - 1)^{-r} \pmod{p^{\mathbb{Z}}},$$

so

$$\deg \Pi \equiv t(\Pi) (q^{nr} - 1) (q_E - 1)^r (q_E^r - 1)^{-1} (q_C^l - 1)^{-r} \pmod{p^{\mathbb{Z}}}.$$

Now we recall that $q_E = q_C^l = q_{E_0}^{lc} = q^{n/e(E_0|F)}$, resulting in

$$\deg \Pi \equiv t(\Pi) (q^{nr} - 1) (q_E^r - 1)^{-1} \pmod{p^{\mathbb{Z}}}.$$

In the final term $(q_E^r - 1)^{-1}$, we have $q_E^r = q^{nr/e(E_0|F)}$. However,

$$t(\Pi) = t(\pi) = \delta(\pi)/e(E_0|F) = \delta(\Pi)/e(E_0|F),$$

yielding $nr/e(E_0|F) = nrt(\Pi)/\delta(\Pi)$. The relation (A1.6.1) has been proved. \square

Remark. Using (A1.3.1) and the Hecke algebra isomorphisms cited in the preceding proof, it is a straightforward matter to refine A1.1 Proposition to give a complete description of the non-cuspidal elements of $\mathcal{A}_m^\square(D)$: the argument is identical to that of Theorem 8.3 of [18].

APPENDIX 2.

Wide extensions and the Glauberman correspondence

We prove the Wide Extension Lemma of 7.5.

A2.1. We start in an abstract setting, recalling a fundamental result from [23].

We are given a finite group T and a cyclic group of automorphisms Z of T , such that $|Z|$ is relatively prime to $|T|$. We think of Z as acting on T by conjugation, so that we may form the semi-direct product group $Z \ltimes T = ZT$. The group Z acts on the set $\text{Irr } T$ of equivalence classes of irreducible representations of T by $\zeta : \rho \mapsto \rho^\zeta$, $\zeta \in Z$, $\rho \in \text{Irr } T$, where ρ^ζ denotes the representation $t \mapsto \rho(\zeta t \zeta^{-1})$, $t \in T$.

We denote by T^Z the group of Z -fixed points in T , and by $\text{Irr}(T^Z)$ the set of equivalence classes of irreducible representations of T^Z . We summarize the main points needed from [23].

(A2.1.1) *Let ζ generate the group Z .*

- (1) *If τ is an irreducible representation of ZT such that $\tau \upharpoonright T$ is not irreducible, then*

$$\text{tr } \tau(\zeta t) = 0, \quad t \in T.$$

- (2) *Let $\rho \in (\text{Irr } T)^Z$.*

- (a) *There exists a unique irreducible representation $\tilde{\rho}$ of ZT such that $\tilde{\rho} \upharpoonright T \cong \rho$ and $\det \tilde{\rho} \upharpoonright Z = 1$.*
 (b) *There is a unique irreducible representation ρ_Z of T^Z , and a constant $\epsilon = \epsilon_\zeta(\rho) = \pm 1$ such that*

$$\text{tr } \tilde{\rho}(\zeta t) = \epsilon \text{tr } \rho_Z(t), \quad t \in T^Z.$$

- (c) *The map $\rho \mapsto \rho_Z$ is a bijection $(\text{Irr } T)^Z \rightarrow \text{Irr}(T^Z)$, independent of the choice of ζ .*

The sign $\epsilon_\zeta(\rho)$ may indeed vary with both ζ and ρ . The bijection $(\text{Irr } T)^Z \rightarrow \text{Irr}(T^Z)$ is what we call the *Glauberman correspondence*.

There is a more general version of (b).

(A2.1.2) *Let $\rho \in (\text{Irr } T)^Z$, and let $\boldsymbol{\rho}$ be a representation of ZT such that $\boldsymbol{\rho} \upharpoonright T \cong \rho$. There is a constant root of unity α , of order dividing $2|Z|$, such that*

$$\text{tr } \boldsymbol{\rho}(\zeta t) = \alpha \text{tr } \rho_Z(t), \quad t \in T^Z.$$

We denote by $\langle \cdot, \cdot \rangle$ the standard inner product of class functions on a finite group. We call the elements of $\text{Irr } T \setminus (\text{Irr } T)^Z$ *Z-irrelevant*.

Proposition. *Let θ be a finite-dimensional representation of T such that $\theta^\zeta \cong \theta$. Let $\boldsymbol{\theta}$ be a representation of ZT such that $\boldsymbol{\theta} \upharpoonright T \cong \theta$, let $\text{tr}_\zeta(\boldsymbol{\theta})$ denote the class function $j \mapsto \text{tr } \boldsymbol{\theta}(\zeta j)$ on T^Z , and let $\rho \in (\text{Irr } T)^Z$.*

- (1) *If $\langle \text{tr}(\rho_Z), \text{tr}_\zeta(\boldsymbol{\theta}) \rangle = a \in \mathbb{C}$, then ρ occurs in θ with multiplicity $\geq |a|$.*
- (2) *If ρ occurs in θ with multiplicity one, then $\langle \text{tr}(\rho_Z), \text{tr}_\zeta(\boldsymbol{\theta}) \rangle \neq 0$.*

Proof. Let $\rho_i, 1 \leq i \leq r$, be the Z -stable irreducible components of θ , repeated according to multiplicity. Thus

$$\theta = \bigoplus_{i=1}^r \rho_i \oplus \vartheta,$$

where ϑ is a sum of Z -irrelevant irreducible representations of T . It follows that

$$\boldsymbol{\theta} = \vartheta \oplus \bigoplus_{i=1}^r \rho_i,$$

for representations ϑ, ρ_i of ZT such that $\rho_i \upharpoonright T \cong \rho_i$ and $\vartheta \upharpoonright T \cong \vartheta$. By (A2.1.1), (A2.1.2),

$$\text{tr}_\zeta(\boldsymbol{\theta}) = \sum_i \alpha_i \text{tr}(\rho_{i,Z}),$$

for various roots of unity α_i . The class functions $\text{tr}(\rho_{i,Z})$, for distinct ρ_i , are linearly independent and both assertions follow. \square

We will not actually use part (1) of the proposition; it is included to add perspective.

We observe that all of the foregoing applies equally to the case where T is a *pro p -group* and Z is a cyclic group of continuous automorphisms of T of finite order relatively prime to p . Of course, in this more general context, we have to consider only *smooth* representations of T and T^Z .

A2.2. We translate to a broader context. We are given a locally profinite group G and a compact open subgroup T of G which is a pro p -group. We are also given a finite cyclic group $Z = \langle \zeta \rangle$ of continuous automorphisms of G , which has order relatively prime to p and which stabilizes T . The group T^Z of Z -fixed points in T is then closed in G and is a pro p -group. We can form the semi-direct product $Z \ltimes G$, topologized so that the obvious map $G \rightarrow Z \ltimes G$ is a homeomorphism of G with an open subgroup of $Z \ltimes G$.

We prove the main technical result of this appendix.

Proposition. *Let ρ be an irreducible smooth representation of T and $g \in G$. Suppose that*

- (a) $\rho^\zeta \cong \rho$,
- (b) g commutes with ζ , and
- (c) $\dim \text{Hom}_{T \cap T^g}(\rho, \rho^g) = 1$.

The element g then intertwines the representation ρ_Z .

Proof. Let τ be the unique irreducible representation of $T \cap T^g$ occurring in both $\rho \upharpoonright T \cap T^g$ and $\rho^g \upharpoonright T \cap T^g$. The group T^g is normalized by Z and the representations ρ, ρ^g are both Z -stable, whence it follows that τ is Z -stable.

Let $\rho_* = \text{Ind}_T^{ZT} \rho = \bigoplus_\chi \chi \otimes \tilde{\rho}$, where χ runs over the characters of Z , viewed by inflation as one-dimensional representations of ZT . The representations $\chi \otimes \tilde{\rho}$ are pairwise distinct, being distinguished by their determinant characters. Surely $(\rho^g)_* \cong (\rho_*)^g$. Frobenius Reciprocity, in the guise of [17] 4.1.5, gives

$$\dim \text{Hom}_{Z(T \cap T^g)}(\rho_*, \rho_*^g) = |Z|,$$

since $|Z|$ is the number of T, T -double cosets contained in $ZTgZT$. Consequently:

Lemma. *Let $\tilde{\rho}$ be some representation of ZT extending ρ . There is a unique representation $\tilde{\rho}_1$ of ZT , extending ρ , such that g intertwines $\tilde{\rho}$ with $\tilde{\rho}_1$. Moreover,*

$$(A2.2.1) \quad \dim \text{Hom}_{Z(T \cap T^g)}(\tilde{\rho}, \tilde{\rho}_1^g) = 1.$$

That is, there is a unique irreducible representation $\hat{\tau}$ of $Z(T \cap T^g)$ occurring in both $\tilde{\rho} \upharpoonright Z(T \cap T^g)$ and $\tilde{\rho}_1^g \upharpoonright Z(T \cap T^g)$. It occurs in both with multiplicity one. On the other hand, $\tilde{\rho} \upharpoonright Z(T \cap T^g)$ contains some irreducible representation $\tilde{\tau}$ extending τ . We may choose an extension $\tilde{\rho}_2$ of ρ so that $\tilde{\tau}$ occurs in $\tilde{\rho}_2^g \upharpoonright Z(T \cap T^g)$: thus $\text{Hom}_{Z(T \cap T^g)}(\tilde{\rho}, \tilde{\rho}_2) \neq 0$, implying $\tilde{\rho}_2 = \tilde{\rho}_1$ and $\hat{\tau} = \tilde{\tau}$. Put another way, $\hat{\tau} \upharpoonright T \cap T^g = \tau$.

We apply A2.1 Proposition (2) to the representation τ of $T \cap T^g$, the rôle of θ being taken by $\tilde{\rho}_1^g \upharpoonright Z(T \cap T^g)$. We deduce that $\langle \tau_Z, \text{tr}_\zeta(\tilde{\rho}_1^g) \rangle \neq 0$, where the inner product is that on class functions on the group $T^Z \cap (T^Z)^g$.

By (A2.1.2), the function $\text{tr}_\zeta(\tilde{\rho}_1^g)$ is a linear combination of the characters tr_{φ_Z} , where φ ranges over the Z -stable irreducible components of $\tilde{\rho}_1^g \upharpoonright T \cap$

T^g . However, the function $\text{tr}_\zeta(\tilde{\rho}_1^g)$ on $T^Z \cap (T^Z)^g$ is the restriction of $\text{tr}(\rho^g)_Z$, multiplied by a root of unity. Thus τ_Z occurs in $(\rho^g)_Z \upharpoonright T^Z \cap (T^Z)^g$ with multiplicity one. Surely $(\rho^g)_Z = (\rho_Z)^g$, so g intertwines ρ_Z , as required. \square

A2.3. We apply A2.2 Proposition to proving the Wide Extension Lemma of 7.5. We return to the notation of that paragraph.

We work in the group $G = \text{GL}_m(D)$; the automorphism ζ is conjugation by the element ϖ^α of 7.5. The place of the group T is taken by the group \mathcal{J} . It will be convenient to take \mathcal{J} as large as possible. This is achieved as follows. The group J/J^1 is of the form $\text{GL}_l(\mathbb{k}_C)$. The element ϖ^α acts here, its group of fixed points being of the form $J_Z/J_Z^1 \cong \text{GL}_r(\mathbb{k}')$, for an integer r and a finite extension \mathbb{k}'/\mathbb{k}_C . We may take for \mathcal{J} the inverse image in J of the unipotent radical of a Borel subgroup of J_Z/J_Z^1 .

The restriction of Λ_w to J is a wide extension κ of η . Because of (2.5.1)(4), we may apply A2.2 Proposition to κ to show that the representation ϑ of (7.5.1) is intertwined by every element of the G_Z -centralizer of β . The desired result is then given by the following general property of wide extensions, for which we revert to the base field F and the notation of §2.

Lemma. *Let $[\mathfrak{A}, \ell, 0, \beta]$ be a simple stratum in $A = \text{M}_m(D)$, and let B denote the A -centralizer of β . Let X be a subgroup of $J = J(\beta, \mathfrak{A})$, containing $J^1 = J^1(\beta, \mathfrak{A})$, such that X/J^1 is the unipotent radical of the finite reductive group J/J^1 .*

Let ρ be an irreducible representation of X such that $\rho \upharpoonright J^1$ is equivalent to the unique irreducible representation η_θ of J^1 containing a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$. Suppose that ρ is intertwined by every element of B^\times . We then have $\rho \cong \kappa \upharpoonright X$, for any wide extension κ of η .

Proof. Let κ be some wide extension of η , and write $\kappa_X = \kappa \upharpoonright X$. By (2.5.5), the representation κ_X does not depend on the choice of κ . Since $\rho \upharpoonright J^1 \cong \eta \cong \kappa_X \upharpoonright J^1$, there is a character ϕ of X , trivial on J^1 , such that $\rho \cong \phi \otimes \kappa_X$. The intertwining property of ρ implies that ϕ is intertwined by every element of B^\times . This implies that ϕ is trivial, as required. \square

Remark. This argument applies in considerably greater generality, but we only need the restricted case treated above.

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