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Reshetnyak's Theorem and The Inner Distortion

Kai Rajala

Abstract: We prove that a quasilight mapping of finite distortion with locally n-integrable weak partials and locally integrable inner distortion is discrete and open.
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1. Introduction

We call $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, a mapping of finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $J_f \in L^1_{\mathrm{loc}}(\Omega)$, and if there exists a measurable function $K : \Omega \to [1, \infty)$ such that

$$|Df(x)|^n \leq K(x) J_f(x) \quad \text{a.e. } x \in \Omega.$$

Here $|Df(x)|$ and $J_f(x)$ are the operator norm and the Jacobian determinant of $Df(x)$, respectively. If $K \in L^\infty(\Omega)$, f is called quasiregular, or a mapping of bounded distortion.

For a mapping of finite distortion f, the outer and inner distortion functions K_O and K_I are defined as

$$K_O(x) = \frac{|Df(x)|^n}{J_f(x)} \quad \text{and} \quad K_I(x) = \frac{|D^\sharp f(x)|^n}{J_f(x)^{n-1}},$$

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respectively, when $0 < |Df(x)|$, $J_f(x) < \infty$, and $K_O(x) = K_I(x) = 1$ otherwise. Here $D^\sharp f(x)$ is the adjoint matrix of $Df(x)$. Then we have

$$K_I^{1/(n-1)}(x) \leq K_O(x) \leq K_I^{n-1}(x) \quad \text{a.e. } x \in \Omega.$$

In the late 1960s, Reshetnyak proved that a non-constant mapping of bounded distortion is always continuous, open and discrete. This theorem initiated the by now well-established theory of mappings of bounded distortion, see [13], [14], [6].

Recently, a lot of research has been done in order to find the sharp assumptions of Reshetnyak's theorem in the class of mappings of finite distortion, cf. [3], [4], [5], [7], [8], [11]. In this note we continue this line of research by giving a new partial result towards a conjecture of Iwaniec and Šverák [7].

Theorem 1.1. *Suppose that $f : \Omega \to \mathbb{R}^n$, $n \geq 2$, is a quasilight mapping of finite distortion satisfying $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $K_I \in L^1_{\mathrm{loc}}(\Omega)$. Then f is discrete and open.*

By definition, a mapping f is called quasilight if the components of every point-inverse $f^{-1}(y)$ are compact. The Iwaniec-Šverák conjecture is Theorem 1.1 without the quasilightness assumption. In [7] the conjecture is proved for $n = 2$. An example of Ball [2] shows that the integrability assumption on K_I cannot be relaxed in Theorem 1.1.

There are other partial results concerning the Iwaniec-Šverák conjecture, see [3], [4], [5] and [11]. The novelty in Theorem 1.1 lies in the fact that it only deals with the inner distortion; the previous results are proved under assumptions on the outer distortion function. In particular, Hencl and Malý [5] proved Theorem 1.1 assuming $K_O \in L^{n-1}_{\mathrm{loc}}(\Omega)$, and Manfredi and Villamor [11] without the quasilightness assumption when $K_O \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > n - 1$. It is clear that, when working with the inner distortion, one has to find methods different from those used in the above-mentioned works. We prove Theorem 1.1 by using the conformal modulus of $(n-1)$ -dimensional sets, the coarea formula, and elementary topological considerations. Also, we use several results concerning the theory of mappings of finite distortion. Another natural intermediate step towards the Iwaniec-Šverák conjecture would be the theorem of Manfredi and Villamor under the assumption $K_I \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 1$ (instead of the assumption on K_O), which we cannot prove. For closely related results on the global invertibility properties of Sobolev mappings, see [2, Theorem 2] and [15, Corollary 2].

2. Preliminaries

In this section we recall some known properties of mappings satisfying the assumptions of Theorem 1.1. First, let $f : \Omega \to \mathbb{R}^n$ be a continuous map, and $U \subset\subset \Omega$ open. Then the (local) topological degree $\mu(y, f, U)$ is well-defined for every $y \in \mathbb{R}^n \setminus f(\partial U)$, see [14, I.4]. We will use the following facts:

$$\mu(y, f, U) = 0 \text{ if } y \notin f(U), \tag{2.1}$$

$$\mu(y, f, U) = \mu(v, f, U) \tag{2.2}$$

whenever y and v lie in the same component of $\mathbb{R}^n \setminus f(\partial U)$, and

$$\mu(y, f, U) = \sum_{i=1}^{k} \mu(y, f, U_i) \tag{2.3}$$

if both sides are well-defined, and if $U_1, \ldots, U_k$ are disjoint open sets satisfying

$$U \cap f^{-1}(y) \subset \bigcup_{i=1}^{k} U_i \subset U.$$

We call f sense-preserving if $\mu(y, f, U) > 0$ whenever $y \in f(U) \setminus f(\partial U)$. Notice that if f is sense-preserving, then

$$\mu(y, f, U) \leq \mu(y, f, V)$$

whenever both sides are well-defined and $U \subset V$.

We say that f satisfies condition N if the n-measure $|f(E)| = 0$ whenever $|E| = 0$. For mappings of finite distortion with locally n-integrable partials, we have

Theorem 2.1 ([3, Theorem 1.3])**.** *Suppose that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a mapping of finite distortion. Then*

1. *f has a continuous representative,*
2. *f is sense-preserving,*
3. *f satisfies condition N,*
4. *f is differentiable almost everywhere in Ω.*

Part 3. implies that the change of variables formula holds for f. In fact, if $U \subset\subset \Omega$ is open, we have

$$N(y, f, U) = \mu(y, f, U) \tag{2.4}$$

for almost every $y \in \mathbb{R}^n \setminus f(\partial U)$, see [5, Proposition 2]. Here

$$N(y, f, U) = \operatorname{card}\{f^{-1}(y) \cap U\}.$$

Since $K_O^{1/(n-1)} \leq K_I$ almost everywhere, [9, Theorem 1.2] implies

Theorem 2.2. *Suppose that f is as in Theorem 1.1. Then $J_f(x) > 0$ for almost every $x \in \Omega$. In particular, if (A_i), $A_i \subset \Omega$, is a decreasing sequence of measurable sets so that $|A_1| < \infty$ and $\cap_i A_i \subset f^{-1}(y)$ for some $y \in \mathbb{R}^n$, then*

$$\int_{A_i} K_I \to 0 \quad \text{as } i \to \infty.$$

The following characterization of quasilightness will be useful in the sequel.

Theorem 2.3 ([16, Theorem 3.1])**.** *A mapping $f : \Omega \to \mathbb{R}^n$ is quasilight if and only if every point $x \in \Omega$ has a neighborhood $U \subset\subset \Omega$ such that $f(x) \notin f(\partial U)$.*

We call a mapping f light if every point-inverse $f^{-1}(y)$ is totally disconnected. Hence a light mapping is quasilight in particular.

Lemma 2.4 ([14, VI Lemma 5.6])**.** *If $f : \Omega \to \mathbb{R}^n$ is continuous, light and sense-preserving, then f is discrete and open.*

By combining Theorem 2.1 and Lemma 2.4, we see that in order to prove Theorem 1.1 it suffices to show that f is light. We conclude this section with a topological lemma.

Lemma 2.5. *Let f be as in Theorem 1.1. Suppose that $V \subset \mathbb{R}^n$ is homeomorphic to $B(0,1)$ and $\overline{V}$ to $\overline{B}(0,1)$, and that $\emptyset \neq U \subset\subset \Omega$ is a component of $f^{-1}(V)$. Then $f(\partial U) = \partial V$, and $f(U) = V$.*

Proof. First, $f(\partial U) \subset \partial V$ by the continuity of f. Hence, for every $a \in f(U)$, $\mu(a, f, U)$ is well-defined, and strictly positive by Theorem 2.1. By (2.1), there exists $b \in \mathbb{R}^n$ such that $\mu(b, f, U) = 0$. Hence, by (2.2), $f(\partial U)$ separates $f(U)$ and b, and so $f(\partial U) = \partial V$. Also, if there exists a point $p \in V \setminus f(U)$, then $\mu(p, f, U) = 0$. But p and $f(U)$ lie in the same component of $\mathbb{R}^n \setminus f(\partial U) = \mathbb{R}^n \setminus \partial V$.

Hence, by (2.2), $\mu(p, f, U) = \mu(a, f, U) > 0$ whenever $a \in f(U)$. We conclude that $f(U) = V$. □

3. Preimages of radial segments

From now on we assume that f is as in Theorem 1.1. Recall from Section 2 that in order to prove Theorem 1.1 it suffices to show that f is light. We assume, in contrary, that there exists a point $a \in \mathbb{R}^n$ such that some component of $f^{-1}(a)$ has positive $\mathcal{H}^1$-measure. Without loss of generality, $a = 0 \in f(\Omega)$, and E is a component of $f^{-1}(0)$ so that $\mathcal{H}^1(E) > 0$. Then Theorem 1.1 is proved if we can show that $\mathcal{H}^1(E)$ has to be zero.

We denote the projection $(x_1, \ldots, x_n) \mapsto x_1$ by pr. By scaling and rotating, if necessary, we may assume that $\mathcal{H}^1(\mathrm{pr}(E)) = 1$. By Theorem 2.3, there exists a domain $G \subset\subset \Omega$ so that $E \subset G$, and a number $M > 0$ so that $|f(x)| \geq M$ for every $x \in \partial G$. Moreover, by Theorem 2.1, there exists $m \in \mathbb{N}$ so that

$$\mu(y, f, G) = m \quad \text{for every } y \in B(0, M). \tag{3.1}$$

For $0 < R < M$, we denote the E-component of $f^{-1}(B(0, R))$ by E_R. Then $E_R \subset G$. We define radial segments

$$I(R, \phi) = \{(t, \phi) : t \in (R/2, R)\}$$

in polar coordinates, and denote $A_R = B(0, R) \setminus \overline{B}(0, R/2)$, and $U_R = E_R \cap f^{-1}(A_R)$. The first main ingredient in the proof of Theorem 1.1 is the following.

Proposition 3.1. *There exists $0 < M_0 < M$, so that for each $R < M_0$ there exist $\phi_R \in S(0, 1)$ and $a_R \in \mathbb{R}$, so that if we denote*

$$L_R = \big(a_R - (4m)^{-1}, a_R + (4m)^{-1}\big),$$

then

$$L_R \subset \mathrm{pr}(E) \quad \textit{and} \quad \mathrm{pr}^{-1}(L_R) \cap E_R \cap f^{-1}(I(R, \phi_R)) = \emptyset.$$

Proof. For $R < M$, define

$$h_R : U_R \to S(0, 1), \quad h_R(x) = \frac{f(x)}{|f(x)|}.$$

Then $h_R^{-1}(\phi) = f^{-1}(I(R,\phi)) \cap E_R$ for every $\phi \in S(0,1)$. Also, we have

$$|J_{n-1} h_R(x)| \le \frac{|D^\sharp f(x)|}{|f(x)|^{n-1}} \quad \text{a.e. } x \in U_R,$$

for the $(n-1)$-dimensional Jacobian of h_R. Then, the coarea formula (cf. [10]), and Hölder's inequality yield

$$\begin{aligned}\int_{S(0,1)} \mathcal{H}^1(h_R^{-1}(\phi))\, d\mathcal{H}^{n-1}(\phi) &= \int_{U_R} |J_{n-1} h_R| \le \int_{U_R} \frac{|D^\sharp f|}{|f|^{n-1}} \\ &= \int_{U_R} \frac{K_I^{1/n} J_f^{(n-1)/n}}{|f|^{n-1}} \\ &\le \Big(\int_{U_R} K_I \Big)^{1/n} \Big(\int_{U_R} \frac{J_f}{|f|^n} \Big)^{(n-1)/n}.\end{aligned} \tag{3.2}$$

Since $\mu(y,f,E_R) \le m$ for every $y \in B(0,R)$, and $U_R \subset E_R$, the change of variables formula gives

$$\int_{U_R} \frac{J_f}{|f|^n} \le \int_{A_R} \frac{\mu(y,f,E_R)}{|y|^n}\, dy \le m\omega_{n-1} \log 2. \tag{3.3}$$

Moreover, by Theorem 2.2,

$$\int_{U_R} K_I \to 0 \quad \text{as } R \to 0. \tag{3.4}$$

Now, by combining (3.2), (3.3) and (3.4), we have: for every $\epsilon > 0$ there exists $k < M$ so that

$$\int_{S(0,1)} \mathcal{H}^1(E_R \cap f^{-1}(I(R,\phi)))\, d\mathcal{H}^{n-1}(\phi) < \epsilon \tag{3.5}$$

for every $R \le k$. Moreover, by slightly changing the set A_R, we see that (3.5) also holds for $W_{R,\phi} = \overline{E_R} \cap f^{-1}(\overline{I(R,\phi)})$.

Let R be as above. Next, we claim that, for each $\phi \in S(0,1)$, $W_{R,\phi}$ consists of at most m components. Fix ϕ and let $\{J_i\}$, $i = 1,\dots,N$ be a finite set of preimage components of $\overline{I(R,\phi)}$ in $\overline{E_R}$. Denote by I_δ the closed δ-neighborhood of $I(R,\phi)$. Then I_δ has N_δ different preimage components $\tilde{I}_\delta^j$ containing some J_i. When δ is small enough, $\tilde{I}_\delta^j \subset G$ for every $j = 1,\dots,N_\delta$. Then, by Lemma 2.5, $f(\tilde{I}_\delta^j) = I_\delta$ for δ small enough. Moreover, for $\delta < \delta_0$ we have $N_\delta = N$. Then, if $y \in I(R,\phi)$, Theorem 2.1 and (2.3) yield

$$N \le \sum \mu(y,f,\operatorname{int} \tilde{I}_\delta^j) \le \mu(y,f,G) = m.$$

This proves the claim.

Suppose that $M_0 < M$ is small enough, so that (3.5) holds with $\epsilon = \omega_{n-1}(100m)^{-1}$. Then, in particular, for every $R \le M_0$ there exists $\phi_R \in S(0,1)$ such that

$$\mathcal{H}^1(\mathrm{pr}(W_{R,\phi_R})) < (100m)^{-1}.$$

Moreover, we showed that W_{R,ϕ_R} consists of at most m components. Now the proposition follows from our assumption $\mathcal{H}^1(\mathrm{pr}(E)) = 1$. □

4. Modulus estimates and the proof of Theorem 1.1

In this section we prove Theorem 1.1, except for an upper bound for the conformal modulus of certain $(n-1)$-dimensional sets (Proposition 4.2). For a measurable function $\omega \in L^1_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, and a family $\Lambda = \{V_i : i \in I\}$ of Borel sets, set

$$M_\omega \Lambda = \inf_{\rho \in X(\Lambda)} \int_\Omega \omega \rho^{n/(n-1)},$$

where $X(\Lambda)$ is the set of all Borel functions $\rho : \Omega \to [0,\infty]$ satisfying

$$\int_{V_i} \rho \, d\mathcal{H}^{n-1} \ge 1$$

for every $V_i \in \Lambda$ with $\mathcal{H}^{n-1}(V_i) > 0$. If $\omega = 1$ almost everywhere in Ω, we denote M_ω by M.

Now fix R and a_R as in Proposition 3.1. Denote $l = ((8m)^{-1}, (4m)^{-1})$,

$$V_t^+ = E_R \cap \mathrm{pr}^{-1}(\{a_R + t\}), \quad V_t^- = E_R \cap \mathrm{pr}^{-1}(\{a_R - t\}),$$
$$V_t = V_t^+ \cup V_t^-, \quad Q_R^+ = \{x \in V_t^+ : t \in l\}, \quad Q_R^- = \{x \in V_t^- : t \in l\},$$

and

$$\Lambda_R = \{V_t : t \in l\}.$$

Lemma 4.1. *We have*

$$(16m)^{-n/(n-1)} \Big(\int_{E_R} K_I \Big)^{-1/(n-1)} \le M_{K_I^{-1/(n-1)}} \Lambda_R \le m M f(\Lambda_R).$$

Proof. Since $f \in W^{1,n}(E_R, \mathbb{R}^n)$, the restrictions of f to the components G_t^j of V_t belong to $W^{1,n}(G_t^j, \mathbb{R}^n)$ for almost every $t \in l$. In particular, for those t the change of variables formula holds in V_t, see [12]. Also, Theorems 2.1 and 2.2 show that $\mathcal{H}^{n-1}(f(V_t)) > 0$ for almost every $t \in l$.

Now fix $\rho \in X(f(\Lambda_R))$. Then, for almost every $t \in l$, the change of variables formula yields

$$\int_{V_t} (\rho \circ f)|D^\sharp f| \, d\mathcal{H}^{n-1} \geq \int_{fV_t} \rho \, d\mathcal{H}^{n-1} \geq 1, \tag{4.1}$$

i.e. the function $\rho' : E_R \to [0,\infty]$, defined as $\rho'(x) = (\rho \circ f)(x)|D^\sharp f(x)|$ for $x \in V_t$, $t \in l$, when (4.1) holds, $\rho'(x) = \infty$ when $x \in V_t$, $t \in l$, and (4.1) does not hold, and $\rho'(x) = 0$ otherwise, belongs to $X(\Lambda_R)$. Now, by using the change of variables formula in E_R, with the fact that $\mu(y, f, E_R) \leq m$ for every $y \in B(0,R)$, we have

$$\begin{aligned} \int_{E_R} (\rho')^{n/(n-1)} K_I^{-1/(n-1)} &= \int_{E_R} (\rho \circ f)^{n/(n-1)} |D^\sharp f|^{n/(n-1)} K_I^{-1/(n-1)} \\ &= \int_{E_R} (\rho \circ f)^{n/(n-1)} J_f \\ &\leq \int_{\mathbb{R}^n} \rho(y)^{n/(n-1)} \mu(y, f, E_R) \, dy \leq m \int_{\mathbb{R}^n} \rho^{n/(n-1)}. \end{aligned}$$

Since $\rho \in X(f(\Lambda_R))$ is arbitrary, the second inequality in the lemma follows.

To prove the first inequality, fix $g \in X(\Lambda_R)$. Then, for every $t \in l$,

$$1 \leq \int_{V_t^+} g \, d\mathcal{H}^{n-1} + \int_{V_t^-} g \, d\mathcal{H}^{n-1}.$$

By Fubini's theorem,

$$(8m)^{-1} \leq \int_{Q_R^+} g + \int_{Q_R^-} g,$$

so that one of the integrals, say the one over Q_R^+, is greater than $(16m)^{-1}$. Then, Hölder's inequality yields

$$(16m)^{-1} \leq \int_{Q_R^+} g K_I^{-1/n} K_I^{1/n} \leq \Big(\int_{Q_R^+} g^{n/(n-1)} K_I^{-1/(n-1)} \Big)^{(n-1)/n} \Big(\int_{Q_R^+} K_I \Big)^{1/n}. \tag{4.2}$$

Since g is arbitrary, (4.2) proves the first inequality in the lemma. □

In order to complete the proof of Theorem 1.1, we need an upper bound for $Mf(\Lambda_R)$.

Proposition 4.2.

$$Mf(\Lambda_R) \leq C,$$

where $C > 0$ only depends on n.

We will prove Proposition 4.2 in Section 5. Assuming the proposition, Theorem 1.1 now follows: combining Lemma 4.1 with the proposition yields

$$(16m)^{-n/(n-1)}\Big(\int_{E_R} K_I\Big)^{-1/(n-1)} \le mC,$$

where C does not depend on R. Thus,

$$\int_{E_R} K_I \ge T > 0,$$

with T independent of R. This contradicts Theorem 2.2, since

$$\bigcap_{R>0} E_R = E.$$

We conclude that $\mathcal{H}^1(E) = 0$, as desired.

5. Proof of Proposition 4.2

We assume that $n \ge 3$. For $n = 2$ the proposition is trivial. The idea for the proof is to show, using Proposition 3.1, that the sets $f(V_t)$ separate $I(R, \phi_R)$ and another "large" set in A_R. There are some technicalities, though, that slightly complicate matters.

Fix a point $\xi \in \mathrm{pr}^{-1}(a_R) \cap E$, and denote by W the ξ-component of $\mathbb{R}^n \setminus (V_{(8m)^{-1}} \cup \partial E_R)$. Notice that, by the definition of V_t,

$$\mathrm{pr}(W) \subset (a_R - (8m)^{-1}, a_R + (8m)^{-1}). \tag{5.1}$$

Lemma 5.1. *For almost every $r \in (R/2, R)$ there exist $q_r \in W$ and a neighborhood $U_r \subset W$ of q_r so that $|p_r| = |f(q_r)| = r$ and*

$$f^{-1}(p_r) \cap U_r = \{q_r\}. \tag{5.2}$$

Proof. First, by Proposition 3.1, there exists a segment α joining ∂E_R and ξ in $W \cap \mathrm{pr}^{-1}(a_R)$. Fix a small $\epsilon > 0$. Then, for any $x \in B^{n-1}(0, \epsilon)$, we can choose a segment α_x as follows: if $\tilde\alpha$ is the line spanned by α, then $\tilde\alpha_x = \tilde\alpha + x$, $x \in B^{n-1}(0, \epsilon) \subset H$, where $H \ni 0$ is the hyperplane orthogonal to $\tilde\alpha$. Moreover, α_x is a segment in $\tilde\alpha_x$ joining ∂E_R and $B(\xi, \epsilon)$ in W. Choose ϵ to be small enough, so that $f(\alpha_x)$ connects $S(0, R)$ and $S(0, R/2)$ for every $x \in B^{n-1}(0, \epsilon)$.

By the definition of a mapping of finite distortion, and Theorems 2.1 and 2.2, there exists $x_0 \in B^{n-1}(0, \epsilon)$ so that

(1) f is absolutely continuous on α_{x_0},
(2) f is differentiable $\mathcal{H}^1$-almost everywhere on α_{x_0},
(3) $J_f > 0$ $\mathcal{H}^1$-almost everywhere on α_{x_0}.

If f is differentiable at $z \in \alpha_{x_0}$, and $J_f(z) > 0$, then, for every $\nu > 0$ small enough,

$$f(z) \notin f(S(z,\nu)).$$

Because this is true for almost every $z \in \alpha_{x_0}$, the absolute continuity of f on α_{x_0} completes the proof. □

Denote by D the exceptional set in Lemma 5.1. For a radius $r \in (R/2, R) \setminus D$, denote $\{\beta_r\} = S(0,r) \cap I(R,\phi_R)$. By (5.1), Lemma 5.1 and (5.3) below, $\beta_r \neq p_r$ for every r.

Lemma 5.2. *Let $\kappa : [0,1] \to S(0,r)$ be a one-to-one C^∞-path such that $\kappa(0) = p_r$ and $\kappa(1) = \beta_r$. Then, for every $t \in ((8m)^{-1}, (4m)^{-1})$,*

$$\kappa((0,1)) \cap f(V_t) \neq \emptyset.$$

Proof. Recall that

$$\text{pr}^{-1}((a_R - (4m)^{-1}, a_R + (4m)^{-1})) \cap E_R \cap f^{-1}(\beta_r) = \emptyset \tag{5.3}$$

by Proposition 3.1. For q_r and U_r as in Lemma 5.1, denote by $\tilde{\kappa}$ the q_r-component of $f^{-1}(\kappa([0,1]))$. By using Lemma 2.5 as below, we see that $\tilde{\kappa} \neq \{q_r\}$. Then, by (5.2), we find $s \in (0,1)$, and a component κ' of $f^{-1}(\kappa([s,1]))$ so that $\kappa' \cap U_r \neq \emptyset$ and $\kappa' \subset \tilde{\kappa}$.

We assume that $\kappa' \cap V_t = \emptyset$. Since $f(\partial E_R) = S(0,R)$, we conclude that κ' is compact. On the other hand, $\beta_r = \kappa(1) \notin f(\kappa')$ by (5.3). Thus there exists $t \in (s,1)$ so that

$$t = \max\{\tau : \kappa(\tau) \in f(\kappa')\}. \tag{5.4}$$

Choose a point $x_t \in f^{-1}(\kappa(t)) \cap \kappa'$. By our assumption on κ', the x_t-component of $f^{-1}(\kappa(t))$ does not intersect V_t. Then there exists a ball $B = B(\kappa(t), \epsilon)$ so that the x_t-component U_t of $f^{-1}(B)$ does not intersect V_t. By Lemma 2.5 $f(U_t) = B$, and since κ is C^∞, applying Lemma 2.5 to the ϵ-neighborhoods of $\kappa((t-\delta, t+\delta))$ for small enough δ, and the x_t-components of their preimages, shows that actually $\kappa([t, t+\delta)) \subset f(\kappa')$, contradicting (5.4). The proof is complete. □

Lemma 5.3. *For every $r \in (R/2, R) \setminus D$, there exists a Borel function $\rho_r : S(0,r) \to [0,\infty]$ so that, whenever $t \in ((8m)^{-1}, (4m)^{-1})$,*

$$\int_{S(0,r)\cap f(V_t)} \rho_r \, d\mathcal{H}^{n-2} \geq C_1/r, \tag{5.5}$$

and

$$\int_{S(0,r)} \rho_r^{n/(n-1)} \, d\mathcal{H}^{n-1} \leq C_2/r, \tag{5.6}$$

where the constants $C_1, C_2 > 0$ only depend on n.

Proof. We first map $S(0,r)$ onto $S(e_n/2, 1/2)$ by a map T which is a composition of scaling, translation and rotation, so that $T(\beta_r) = e_n$. Then, if $\rho : S(e_n/2, 1/2) \to [0,\infty]$ satisfies

$$\int_{(T\circ f)(V_t)} \rho \, d\mathcal{H}^{n-2} \geq C_1(n) \tag{5.7}$$

for all $t \in ((8m)^{-1}, (4m)^{-1})$, and

$$\int_{S(e_n/2,1/2)} \rho^{n/(n-1)} \, d\mathcal{H}^{n-1} \leq C_2(n), \tag{5.8}$$

then the function $\rho_r = r^{1-n}(\rho \circ T)$ satisfies (5.5) and (5.6). Hence it suffices to show (5.7) and (5.8).

If we map $S(e_n/2, 1/2)$ onto $\overline{\mathbb{R}}^{n-1}$ by the stereographic projection h,

$$h(x) = e_n + (x - e_n)/|x - e_n|^2,$$

then $e_n = T(\beta_r)$ gets mapped to ∞. We denote

$$a = (h \circ T)(p_r) \in \mathbb{R}^{n-1}.$$

We define $\rho : \mathbb{R}^{n-1} \to [0,\infty]$,

$$\rho(x) = |x - a|^{2-n}(1 + |x|^2)^{n-2},$$

and denote $Y_t = (h \circ T \circ f)(V_t)$. Then we have to show that

$$\int_{Y_t} \frac{\rho(x)}{(1+|x|^2)^{n-2}} \, d\mathcal{H}^{n-2}(x) = \int_{Y_t} |x - a|^{2-n} \, d\mathcal{H}^{n-2}(x) \geq C_1(n) \tag{5.9}$$

for all $t \in ((8m)^{-1}, (4m)^{-1})$, and

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \frac{\rho^{n/(n-1)}(x)}{(1+|x|^2)^{n-1}} \, d\mathcal{H}^{n-1}(x) &= \int_{\mathbb{R}^{n-1}} \frac{|x-a|^{1-n+1/(n-1)}}{(1+|x|^2)^{1/(n-1)}} \, d\mathcal{H}^{n-1}(x) \\ &\leq C_2(n).\end{aligned} \tag{5.10}$$

By Lemma 5.2, for every $\alpha \in S^{n-2}(0,1)$, the half-line

$$I_\alpha = \{a + \alpha t : t > 0\}$$

intersects Y_t. For $i \in \mathbb{Z}$, denote $A_i = B(a, 2^i) \setminus B(a, 2^{i-1})$, and

$$\Phi_i = \{\alpha \in S^{n-2}(0,1) : I_\alpha \cap A_i \cap Y_t \neq \emptyset\}.$$

Then a projection argument shows that

$$\int_{Y_t \cap A_i} |x-a|^{2-n} \, d\mathcal{H}^{n-2}(x) \geq C(n)\mathcal{H}^{n-2}(\Phi_i). \tag{5.11}$$

Since $\sum_i \mathcal{H}^{n-2}(\Phi_i) = \omega_{n-2}$, (5.9) follows by summing over i.

In order to prove (5.10), we first consider the case $|a| > 1$. We divide $\mathbb{R}^{n-1}$ to $N_1 = B^{n-1}(a, |a|/2)$, $N_2 = B^{n-1}(0, |a|/2)$ and $N_3 = \mathbb{R}^{n-1} \setminus (N_1 \cup N_2)$. Then

$$\begin{aligned}\int_{N_1} \frac{|x-a|^{1-n+1/(n-1)}}{(1+|x|^2)^{1/(n-1)}} \, d\mathcal{H}^{n-1}(x) &\leq C|a|^{\frac{-2}{n-1}} \int_{N_1} |x-a|^{1-n+1/(n-1)} \, d\mathcal{H}^{n-1}(x) \\ &\leq C|a|^{\frac{-1}{n-1}},\end{aligned}$$

$$\begin{aligned}\int_{N_2} \frac{|x-a|^{1-n+1/(n-1)}}{(1+|x|^2)^{1/(n-1)}} \, d\mathcal{H}^{n-1}(x) &\leq C|a|^{-2+\frac{1}{n-1}} \int_0^{|a|/2} \frac{r}{(1+r^2)^{1/(n-1)}} \, dr \\ &\leq C|a|^{\frac{-1}{n-1}},\end{aligned}$$

and, since $10|x| \geq |x-a|$ for $x \in N_3$,

$$\begin{aligned}\int_{N_3} \frac{|x-a|^{1-n+1/(n-1)}}{(1+|x|^2)^{1/(n-1)}} \, d\mathcal{H}^{n-1}(x) &\leq C \int_{N_3} |x-a|^{1-n-1/(n-1)} \, d\mathcal{H}^{n-1}(x) \\ &\leq C|a|^{\frac{-1}{n-1}}.\end{aligned}$$

Combining the integrals proves (5.10) in the case $|a| > 1$. The case $|a| \leq 1$ is similar, but now it suffices to consider the division $\tilde{N}_1 = B^{n-1}(a, 3)$, $\tilde{N}_2 = \mathbb{R}^{n-1} \setminus \tilde{N}_1$.

□

Define $\rho : A_R \to [0, \infty]$, $\rho(x) = \rho_{|x|}(x)$, where ρ_r is as in Lemma 5.3 for $r \notin D$, and $\rho_r = 0$ otherwise. Since the restrictions of f to the components G_t^j of V_t belong to $W^{1,n}(G_t^j, \mathbb{R}^n)$ for almost every t, $f(V_t)$ is countably $(n-1)$ -rectifiable for $t \in ((8m)^{-1}, (4m)^{-1}) \setminus Q$, where $\mathcal{H}^1(Q) = 0$. Then, Lemma 5.3, and the coarea formula for rectifiable sets, cf. [1, Theorem 2.93 and Remark 2.94], yield

$$\int_{f(V_t)} \rho \, d\mathcal{H}^{n-1} \geq C(n) \int_{R/2}^{R} \int_{f(V_t)\cap S(0,r)} \rho \, d\mathcal{H}^{n-2} \, dr \geq C(n)$$

for every $t \in ((8m)^{-1}, (4m)^{-1}) \setminus Q$. Also, by Lemma 5.3,

$$\int_{A_R} \rho^{n/(n-1)} = \int_{R/2}^{R} \int_{S(0,r)} \rho^{n/(n-1)} \, d\mathcal{H}^{n-1} \, dr \leq C(n) \int_{R/2}^{R} \frac{dr}{r} \leq C(n).$$

By Theorem 2.1, $M\{f(V_t) : t \in Q\} = 0$. The proof of Proposition 4.2 is complete.

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Kai Rajala
University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35
FI-40014 University of Jyväskylä
Finland
E-mail: kirajala@maths.jyu.fi