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Quasihyperbolic Geodesics in Convex Domains II

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Abstract: We consider the quasihyperbolic geometry of a convex domain in a uniformly convex Banach space. We show that quasihyperbolic geodesics are unique, that quasihyperbolic balls are convex and that in the finite-dimensional case, quasihyperbolic geodesics can be prolonged to geodesic rays.
Keywords: quasihyperbolic geodesic, quasihyperbolic ball.

1. Introduction

1.1. This paper is a continuation to [Vä4]. Throughout the paper, we assume that E is a real Banach space with $\dim E \geq 2$ and that $G \subsetneq E$ is a domain. We recall that the *quasihyperbolic length* of a rectifiable arc $\gamma \subset G$ or a path γ in G is the number

$$l_k(\gamma) = \int_\gamma \frac{|dx|}{\delta(x)},$$

where $\delta(x) = d(x, E \setminus G)$. For $a, b \in G$, the *quasihyperbolic distance* $k(a, b) = k_G(a, b)$ is defined by

$$k(a, b) = \inf l_k(\gamma)$$

over all rectifiable arcs γ joining a and b in G. An arc γ from a to b (written $\gamma \colon a \curvearrowright b$) is a *quasihyperbolic geodesic* or briefly a *geodesic* if $l_k(\gamma) = k(a, b)$.

The quasihyperbolic metric of a domain in R^n is due to F.W. Gehring, and it was first published in joint papers of him and his students B. Palka [GP] in 1976

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and B. Osgood [GO] in 1979. Important results on quasihyperbolic geodesics in domains $G \subset \mathsf{R}^n$ were obtained by G. Martin [Ma] in 1985. For example, he proved that the geodesics are C^1 smooth.

As in [Vä4], we shall consider the case where G is convex. This is considerably easier than the general case, mainly because the distance function δ is then concave in G. In [Vä4] it was proved that if E is a reflexive Banach space, then each pair of points $a, b \in G$ can be joined by a quasihyperbolic geodesic. We begin this paper by proving in Section 2 that if E is uniformly convex, then there is only one quasihyperbolic geodesic between given points of G. As far as we know, this is the first result on the uniqueness of quasihyperbolic geodesics (except for domains where geodesics are explicitly known). In nonconvex domains, for example, in $G = \mathsf{R}^n \setminus \{0\}$, the quasihyperbolic geodesics need not be unique.

We also show that quasihyperbolic balls are strictly convex in convex domains of uniformly convex spaces. In particular, these results hold in Hilbert spaces and therefore in euclidean spaces.

In Section 3 we show that if, moreover, $\dim E < \infty$, then each geodesic $\gamma\colon a \curvearrowright b$ in a convex domain G can be prolonged to a quasihyperbolic ray from a to a boundary point of G

We use the same notation as in [Vä4]. In particular, we write $s \wedge t = \min\{s, t\}$, $s \vee t = \max\{s, t\}$ for $s, t \in \mathsf{R}$.

2. Uniqueness and convexity

2.1. *Uniformly convex spaces.* A Banach space E is *uniformly convex* if for each $0 < t \le 2$ we have

$$\sup\{|x + y|/2\colon |x| = |y| = 1,\ |x - y| \ge t\} < 1.$$

For example, all Hilbert spaces and all L_p-spaces are uniformly convex for $1 < p < \infty$. For the basic properties of uniformly convex spaces, see [BL] or [FZ]. We need the following results; proofs can be found in [FZ, 8.11, 9.12] and [BL, 5.12, 5.21].

2.2. **Lemma.** *A uniformly convex space E has the following properties:*

(1) *If x and y are nonzero vectors in E with $|x + y| = |x| + |y|$, then $x = \lambda y$ for some $\lambda > 0$.*

(2) *E is reflexive.*

(3) *If $g\colon [a,b] \to E$ is absolutely continuous, then g is differentiable almost everywhere.*

2.3. *Conclusions.* From (3) it follows as usual (see e.g. [Vä1, 1.3, 4.1]) that if $g\colon [a,b] \to E$ is absolutely continuous, then the length of g is given by

$$l(g) = \int_a^b |g'(t)|dt. \tag{2.4}$$

More generally, if $\varrho\colon \operatorname{im} g \to \mathsf{R}$ is a continuous function, then

$$\int_g \varrho(x)|dx| = \int_a^b \varrho(g(t))|g'(t)|dt.$$

In particular, if $G \subset E$ is a domain and $g\colon [a,b] \to G$ is absolutely continuous, then

$$l_k(g) = \int_a^b \frac{|g'(t)|}{\delta(g(t))}dt. \tag{2.5}$$

It is often convenient to parametrize an arc or a path by quasihyperbolic length. We say that $g\colon [0,r] \to G$ is a *quasihyperbolic parametrization* if $l_k(g|[0,t]) = t$ for all $t \in [0,r]$. Then $r = l_k(g)$ and

$$|g'(t)| = \delta(g(t)) \tag{2.6}$$

almost everywhere. Every rectifiable arc $\gamma \subset G$ has a quasihyperbolic parametrization $g\colon [0,r] \to \gamma$, and g satisfies the Lipschitz condition $|g(s)-g(t)| \le M|s-t|$ where $M = \max\{\delta(x)\colon x \in \gamma\}$. If γ is a geodesic, then g is an isometry from $[0,r]$ into the metric space (G,k), and we say that g is a *geodesic path* from $g(0)$ to $g(r)$.

2.7. *Correction.* In [Vä4], the treatment of derivatives was, unfortunately, somewhat careless. Since (2) implies (3) by [BL, 5.12], the argument of the proof of [Vä4, 2.1] is valid. However, in [Vä4, 4.1] one should assume that the space E is Hilbert, since the proof of [Vä4, 4.2] involves the derivative of a path.

The following lemma quickly implies both main results of this section.

2.8. **Lemma.** *Let G be a convex domain in a uniformly convex space. Let a, b_1, b_2 be points in G with $k(a,b_1) = k(a,b_2) = r$ and let $g_i\colon [0,r] \to G$ be geodesic paths from a to b_i. Let $\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$. Then for $g = \lambda g_1 + \mu g_2$ we have $l_k(g) \le r$. If $l_k(g) = r$, then $g_1 = g_2$.*

Proof. By (2.6) we have $|g_i'(t)| = \delta(g_i(t))$ for $i = 1, 2$ and for almost every $t \in [0, r]$. Hence

$$(2.9) \qquad |g'(t)| = |\lambda g_1'(t) + \mu g_2'(t)| \le \lambda |g_1'(t)| + \mu |g_2'(t)| = \lambda\delta(g_1(t)) + \mu\delta(g_2(t))$$

a.e. As δ is concave by [Vä4, 3.4], we get $|g'(t)| \le \delta(g(t))$ a.e. By (2.5) this yields $l_k(g) \le r$.

If $l_k(g) = r$, we have equality a.e. in (2.9). By 2.2(1), the vectors $g_1'(t)$ and $g_2'(t)$ have the same direction a.e. By (2.6) we can write

$$(2.10) \qquad g_i'(t) = \delta(g_i(t))F(t)$$

for almost every t and for $i = 1, 2$, where $F(t)$ is a unit vector independent of i. We can now apply a classical proof for the uniqueness of the solution of a differential equation to show that $g_1 = g_2$.

Write $h = g_1 - g_2$ and assume that there is $t_1 \in (0, r]$ with $h(t_1) \ne 0$. Set $t_2 = \max\{t \in [0, t_1) \colon h(t) = 0\}$. Replacing t_1 by a smaller number we may assume that $t_1 < t_2 + 1$. Set $M = \max\{|h(t)| \colon t_2 \le t \le t_1\}$ and choose $t_3 \in [t_2, t_1]$ with $|h(t_3)| = M$. Since δ is 1-Lipschitz, (2.10) gives

$$|h'(t)| = |\delta(g_1(t)) - \delta(g_2(t))||F(t)| \le |g_1(t) - g_2(t)| = |h(t)| \le M$$

a.e. on $[t_2, t_3]$. As h is absolutely continuous, this and (2.4) yield the desired contradiction

$$M = |h(t_3)| \le l(h|[t_2, t_3]) \le M(t_3 - t_2) < M. \ \square$$

2.11. **Theorem.** *Let G be a convex domain in a uniformly convex space E and let $a, b \in G$. Then there is a unique quasihyperbolic geodesic from a to b.*

Proof. As E is reflexive by 2.2(2), a geodesic exists by [Vä4, 2.1]. Assume that g_1 and g_2 are geodesic paths from a to b, and set $g = (g_1 + g_2)/2$. Applying Lemma 2.8 with $b_1 = b_2 = b$ we get $l_k(g) \le k(a, b)$. On the other hand, $k(a, b) \le l_k(g)$, and the last part of 2.8 gives $g_1 = g_2$. □

2.12. *Quasihyperbolic balls and spheres.* For open and closed quasihyperbolic balls and for quasihyperbolic spheres we use the notation $B_k(a, r), \bar{B}_k(a, r), S_k(a, r)$, where $a \in G$ and $r > 0$. We say that $B_k(a, r)$ is *strictly convex* if for each pair $x, y \in S_k(a, r)$, the open line segment between x and y is contained in $B_k(a, r)$. Equivalently, $B_k(a, r)$ is convex and $S_k(a, r)$ contains no line segments.

2.13. **Theorem.** *Let G be a convex domain in a uniformly convex space E and let $a \in G$. Then each quasihyperbolic ball $B_k(a,r)$ is strictly convex.*

Proof. Let $b_1 \neq b_2$ be points in G with $k(a,b_1) = k(a,b_2) = r$ and let $\lambda, \mu > 0,\ \lambda + \mu = 1$. Choose geodesic paths g_i from a to b_i. Then 2.8 gives $l_k(g) < r$ for $g = \lambda g_1 + \mu g_2$, whence $\lambda b_1 + \mu b_2 = g(r) \in B_k(a,r)$. $\square$

2.14. *Question.* Is every quasihyperbolic ball $B_k(a,r)$ in a convex domain *quasihyperbolically convex*? This means that if $\gamma\colon x \curvearrowright y$ is a quasihyperbolic geodesic with $x, y \in B_k(a,r)$, then $\gamma \subset B_k(a,r)$.

2.15. *Digression: Uniform domains of higher order.* Let $p \geq 0$ be an integer and let $c \geq 1$. A domain G is said to be *homotopically* (p,c)*-uniform* if each continuous map $f\colon S^p \to G$ has a continuous extension $g\colon \bar{B}^{p+1} \to G$ satisfying the conditions

$$d(x, |f|) \leq c\delta(x) \text{ for all } x \in |g|, \tag{2.16}$$

$$d(|g|) \leq cd(|f|), \tag{2.17}$$

where $|f| = \operatorname{im} f$.

This definition is due to P. Alestalo [Al]. By [Al, 6.9], it is quantitatively equivalent to the definition of J. Heinonen and S. Yang [HY]. For $p = 0,\ E = \mathsf{R}^n$, it is n-quantitatively equivalent to the classical definition [MS] of c-uniform domains.

In [HY, 2.1] it was proved that if $G \subset \mathsf{R}^n$ is c-uniform and if quasihyperbolic geodesics in G are unique, then G is homotopically (p,c')-uniform for every p with $c' = c'(c,n)$. By Theorem 2.11 we see that this holds in every convex domain of R^n. However, we show that an improved dimension-free version of this result is easily proved directly.

2.18. **Theorem.** *Suppose that $G \subset E$ is a convex $(0,c)$-uniform domain. Then G is homotopically (p,c')-uniform for all $p \geq 0$ with $c' = 2c(c+1)$.*

Proof. Let $f\colon S^p \to G$ be continuous. Choose points $a, b \in |f|$ with $|a - b| = d(|f|)$. Define $f_0\colon S^0 \to G$ by $f_0(-1) = a,\ f_0(1) = b$. As G is $(0,c)$-uniform, f_0 extends to a path $g_0\colon [-1,1] \to G$ such that (2.16) and (2.17) hold with the substitution $(f, g) \mapsto (f_0, g_0)$. Choose a point $z \in |g_0|$ such that $|z-a| = |z-b| =: r$. Then

$$r \le d(|g_0|) \le cd(|f_0|) = cd(|f|). \tag{2.19}$$

We may assume that $z = 0$. Define $g\colon \bar{B}^{p+1} \to G$ by $g(tx) = tfx$ for $x \in S^p$, $0 \le t \le 1$. Clearly g is continuous and $g|S^p = f$. We show that g satisfies (2.16) and (2.17) with $c \mapsto c'$.

For $x, y \in S^p$ and $s, t \in [0, 1]$ we have

$$|g(sx) - g(ty)| = |sfx - tfy| \le |fx| + |fy|.$$

Here

$$|fx| \le |fx - a| + |a| \le d(|f|) + r \le (c+1)d(|f|), \tag{2.20}$$

whence $d(|g|) \le 2(c+1)d(|f|)$, which implies (2.17).

Since $d(|f|) = |a-b| \le 2r$, the inequality (2.20) yields $d(tfx, |f|) \le |tfx - fx| = (1-t)|fx| \le (1-t)(c+1)d(|f|) \le 2(1-t)(c+1)r$. As δ is concave by [Vä4, 3.4], condition (2.16) for (f_0, g_0) gives

$$\delta(tfx) \ge (1-t)\delta(0) + t\delta(fx) \ge (1-t)\delta(0) \ge (1-t)r/c.$$

Combining the estimates we obtain (2.16). □

3. Prolongation of geodesics

In this section we consider a convex domain G in a *finite-dimensional* uniformly convex space E. We show in 3.12 that each quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$ can be prolonged to an arbitrarily long geodesic $\gamma_1\colon a \curvearrowright b_1$ and in fact to a *geodesic ray* (Th. 3.18) from a to a point in ∂G.

We remark that the prolongation is not in general unique. For example, let $G \subset \mathsf{R}^2$ be the strip $\{(x,y)\colon |y| < 1\}$. The horizontal segment $[0, e_1]$ is a geodesic and it has an infinite number of prolongations to geodesic rays, namely any line segment $[0, re_1]$, $r \ge 1$, extended either by a horizontal half line or by a quarter of one of the circles $S(re_1 + e_2, 1)$, $S(re_1 - e_2, 1)$. This bifurcation was observed already in [Ma, 4.11].

We first consider sequences of paths and recall that the length cannot increase in a limiting process:

3.1. **Lemma.** *Suppose that* (X, d) *is a metric space and that* $g_m \colon [a, b] \to X$ *is a sequence of rectifiable paths converging pointwise to a path* $g \colon [a, b] \to X$*. Then* $l(g) \le \liminf_{m\to\infty} l(g_m)$*.*

Proof. For each subdivision $a = t_0 < \cdots < t_n = b$ we have

$$\sum_{j=1}^{n} d(g(t_{j-1}), g(t_j)) = \lim_{m\to\infty} \sum_{j=1}^{n} d(g_m(t_{j-1}), g_m(t_j)) \le \liminf_{m\to\infty} l(g_m),$$

and the lemma follows. □

3.2. *Convention.* In 3.3–3.12, E is a uniformly convex space with $\dim E < \infty$ and $G \subset E$ is a convex domain.

The next lemma shows that geodesics depend continuously on the end points.

3.3. **Lemma.** *Let* $\gamma_j \colon a_j \curvearrowright b_j$ *be a sequence of geodesics in* G *such that* $k(a_j, b_j) = r$ *for all* j *and such that* $a_j \to a \in G$, $b_j \to b \in G$*. Let* $g_j \colon [0, r] \to \gamma_j$ *be quasihyperbolic parametrizations. Then the sequence* (g_j) *converges uniformly to a geodesic path* g *from* a *to* b*.*

Proof. Since geodesics are unique by 2.11, it suffices to show that (g_j) has a subsequence converging uniformly to a geodesic path g from a to b. Since $a_j \to a$, there is a quasihyperbolic ball $\bar{B} = \bar{B}_k(a, R)$ containing every γ_j. As $\bar{B}$ is compact, we may apply Ascoli's theorem and find a subsequence, still written as (g_j), converging uniformly to a path $g \colon [0, r] \to G$ with $g(0) = a$, $g(r) = b$. By 3.1 we have $l_k(g) \le \liminf_{j\to\infty} l_k(g_j) = r$, and the lemma is proved. □

3.4. *Quasihyperbolic projection.* Let $a \in G$, $0 < r < s$. For each $x \in S_k(a, s)$ there is a unique geodesic $\gamma \colon a \curvearrowright x$, which meets $S_k(a, r)$ at a unique point y. Setting $y = fx$ we obtain a map

$$f \colon S_k(a, s) \to S_k(a, r),$$

called the *quasihyperbolic projection.* Because of the aforementioned bifurcation of geodesics, the quasihyperbolic projection need not be injective. However, we show that it is a continuous surjection.

3.5. **Lemma.** *The quasihyperbolic projection is continuous.*

Proof. Let (x_j) be a sequence in $S_k(a,s)$ converging to a point x. Let $g_j\colon [0,s] \to \bar{B}_k(a,s)$ be the geodesic path from a to x_j. By 3.3 the sequence (g_j) converges to a geodesic path g from a to x, and $fx_j = g_j(r) \to g(r) = fx$. □

3.6. *The map* Φ. To simplify notation we assume that $0 \in G$ and write $B_k(r) = B_k(0,r)$, $S_k(r) = S_k(0,r)$. For $r > 0$ define a map $\Phi\colon \bar{B}_k(r) \to \bar{B}(r)$ by

$$\Phi x = \begin{cases} k(x,0)x/|x| & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

From the convexity of the quasihyperbolic balls $B_k(t)$ (Theorem 2.13) it follows that for each ray l from the origin, the function $x \mapsto k(x,0)$ is strictly increasing on $l \cap G$. Hence Φ maps $l \cap \bar{B}_k(r)$ bijectively onto $l \cap \bar{B}(r)$. Since Φ is clearly continuous and since $\bar{B}_k(r)$ is compact, we obtain:

3.7. **Lemma.** *The map* $\Phi\colon \bar{B}_k(r) \to \bar{B}(r)$ *is a homeomorphism.* □

3.8. **Theorem.** *Suppose that G is a convex domain in a finite-dimensional uniformly convex space E. Let $a \in G$ and let $0 < r < s$. Then the quasihyperbolic projection $f\colon S_k(a,s) \to S_k(a,r)$ is surjective.*

Proof. We may assume that $a = 0$. For $0 < u \le s$ we let f_u denote the quasihyperbolic projection of $S_k(s)$ into $S_k(u)$.

Assume that the theorem is false. Let u_0 denote the supremum of all $u \le s$ such that f_u is not surjective. Then $0 < u_0 \le s$. We first show that f_{u_0} is surjective.

Let $y \in S_k(u_0)$. Choose a sequence of numbers $u_j \in (u_0, s)$ and points $y_j \in S_k(u_j)$ such that $u_j \to u_0$ and $y_j \to y$. Since the quasihyperbolic projections f_{u_j} are surjective, there are points $x_j \in S_k(s)$ with $f_{u_j}x_j = y_j$. Let $g_j\colon [0,s] \to \bar{B}_k(s)$ be the (unique) geodesic path from 0 to x_j. Then $g_j(u_j) = y_j$. Passing to a subsequence we may assume that the sequence (x_j) converges to a point $x \in S_k(s)$. By Lemma 3.3, the maps g_j converge to a geodesic path g from 0 to x. Now

$$k(y_j, g(u_0)) \le k(g_j(u_j), g_j(u_0)) + k(g_j(u_0), g(u_0)) = u_j - u_0 + k(g_j(u_0), g(u_0)) \to 0$$

as $j \to \infty$, whence $y = g(u_0) = f_{u_0}x$, which implies that f_{u_0} is surjective.

For $0 < u < u_0$ let φ_u denote the quasihyperbolic projection of $S_k(u_0)$ into $S_k(u)$. We show that there is $u_1 < u_0$ such that φ_u is surjective for all $u \in (u_1, u_0)$. For these u, also $f_u = \varphi_u \circ f_{u_0}$ is surjective contradicting the definition of u_0 and proving the theorem.

Set $m = d(0, S_k(u_0/2))$. As the map $\Phi^{-1}\colon \bar{B}(u_0) \to \bar{B}_k(u_0)$ is uniformly continuous, there is $\eta > 0$ such that

$$|\Phi^{-1}x - \Phi^{-1}y| \le m/2 \tag{3.9}$$

for all $x, y \in \bar{B}(u_0)$ with $|x-y| \le \eta$. Since the quasihyperbolic metric is bilipschitz equivalent to the norm metric in compact sets, there is $K \ge 1$ such that

$$|x - y| \le Kk(x, y) \tag{3.10}$$

for all $x, y \in \bar{B}_k(u_0)$. We show that one can choose

$$u_1 = \max\{u_0 - \eta, u_0 - m/2K, u_0/2\}.$$

Let $u_1 < u < u_0$. Let $p\colon E \setminus \{0\} \to S_k(u)$ be the radial projection, which sends all points of a ray l from the origin to the unique point of $l \cap S_k(u)$, and set $p_u = p|S_k(u_0)$. As $S_k(t)$ is the boundary of the bounded convex open set $B_k(t)$ for all $t > 0$, the map $p_u\colon S_k(u_0) \to S_k(u)$ is a homeomorphism between topological spheres. Hence the topological degree of p_u is ± 1 depending on orientation. Consequently, it suffices to show that the maps φ_u and p_u are homotopic. For this it suffices to show that

$$|\varphi_u x - x| \le m/2, \quad |p_u x - x| \le m/2 \tag{3.11}$$

for all $x \in S_k(u_0)$, because then $0 \notin [\varphi_u x, p_u x]$, and the desired homotopy is given by

$$h_t(x) = p((1-t)\varphi_u x + t p_u x).$$

Let $x \in S_k(u_0)$. Then $k(\varphi_u x, x) = u_0 - u \le m/2K$, and the first part of (3.11) follows from (3.10). Furthermore,

$$|\Phi p_u x - \Phi x| = |k(p_u x, 0) p_u x/|p_u x| - k(x, 0) x/|x|| = k(x, 0) - k(p_u x, 0) = u_0 - u \le \eta,$$

whence $|p_u x - x| \le m/2$ by (3.9), and the theorem is proved. □

3.12. **Corollary.** *Let $G \subset E$ be as in 3.8, let $\gamma\colon a \curvearrowright b$ be a geodesic and let $s > k(a, b)$. Then there is a geodesic $\gamma_1\colon a \curvearrowright b_1$ with $\gamma \subset \gamma_1$ and $k(a, b_1) = s$.* □

3.13. *Nonconvex domains.* None of the theorems 2.11, 2.13 and 3.12 is true without the condition that G be convex. For example, in the domain $G = \mathsf{R}^2 \setminus \{0\}$ there are two quasihyperbolic geodesics (semicircles) between the points e_1 and $-e_1$, and $\bar{B}_k(e_1, \pi)$ is not convex, not even a topological disk. Moreover, these geodesics have no prolongation. However, the following *conjectures* look natural:

Let G be a domain in a reasonable space, say in R^n.

(1) There is a universal constant $r_0 > 0$ such that for $r < r_0$, each quasihyperbolic ball $B_k(a, r)$ is strictly convex.

(2) There is $r_0 > 0$ such that if $k(a, b) < r_0$, then there is one and only one geodesic joining a and b.

(3) There is $r_0 > 0$ such that if $k(a, b) < r < r_0$, then a geodesic $\gamma\colon a \curvearrowright b$ can be prolonged to a geodesic $\gamma_1\colon a \curvearrowright b_1$ such that $\gamma \subset \gamma_1$ and $k(a, b_1) = r$.

It is easy to see that (1) implies (2) in uniformly convex spaces with the constant $2r_0$. The proof of Theorem 3.12 shows that (2) implies (3) in finite-dimensional spaces with the same constant r_0. The results of the present paper show that for convex domains, (1) and (2) are true in uniformly convex spaces and that (3) holds in finite-dimensional uniformly convex spaces, all with $r_0 = \infty$.

We believe that (1) is true with $r_0 = 1$.

3.14. *Geodesic rays.* Suppose that G is a (not necessarily convex) domain in a Banach space E. A *geodesic ray path* in G is an isometry g of the half line $[0, \infty)$ into the metric space (G, k_G), and the image set $\operatorname{im} g \subset G$ is a *geodesic ray.* Equivalently, a set $\gamma^* \subset G$ is a geodesic ray if

(1) γ^* is homeomorphic to $[0, \infty)$,

(2) each compact subarc of γ^* is a geodesic,

(3) $l_k(\gamma^*) = \infty$.

The following result is from [Vä4, 3.2].

3.15. **Lemma.** *Let $\gamma\colon a \curvearrowright b$ be a quasihyperbolic geodesic in a convex domain $G \subset E$. Then $l(\gamma) \le c_0|a - b|$ where c_0 is a universal constant.* □

3.16. *The point at infinity.* In what follows, the boundary ∂G of a domain $G \subset E$ is always considered in the one-point extension $\dot{E} = E \cup \{\infty\}$ of E. This is the Hausdorff space where the neighborhoods of the point ∞ are the complements of all closed bounded sets. If $\dim E < \infty$, this space is the one-point compactification of E. We have $\infty \in \partial G$ if and only if G is unbounded. A map $g\colon [0, \infty) \to G$ converges to ∞ as $t \to \infty$ if and only if $|g(t)| \to \infty$ as $t \to \infty$.

3.17. **Theorem.** *Let G be a convex domain in a Banach space E and let $g\colon [0,\infty) \to G$ be a geodesic ray path. Then either $g(t) \to \infty$ as $t \to \infty$ or $l(g) < \infty$ and $g(t)$ converges to a finite point $b \in \partial G$.*

Proof. Set $a = g(0)$. Assume that $g(t)$ does not converge to ∞. Then there is $M > 0$ and a sequence (t_j) such that $t_j \to \infty$ and $|g(t_j) - a| \le M$. By 3.15 this implies that

$$l(g|[0,t_j]) \le c_0|a - g(t_j)| \le c_0 M.$$

for all j. Hence g is rectifiable. Since E is complete, $g(t) \to b \in \bar{G}$ as $t \to \infty$. On the other hand, $k(a, g(t)) = t \to \infty$, and thus $b \in \partial G$. □

We say that the geodesic ray path g of Theorem 3.17 (or the geodesic ray $\operatorname{im} g$) *joins* the point $a = g(0)$ and $b \in \partial G$, possibly $b = \infty$.

We next give a ray version of the prolongation theorem 3.12.

3.18. **Theorem.** *Let G be a convex domain in a finite-dimensional uniformly convex space, and let $\gamma\colon a \curvearrowright b$ be a geodesic in G. Then γ has a prolongation to a geodesic ray γ^* from a to a point $b^* \in \partial G$ with $\gamma \subset \gamma^*$.*

Proof. Applying 3.12 we choose a sequence of geodesics $\gamma \subset \gamma_1 \subset \gamma_2 \subset \dots$, $\gamma_j\colon a \curvearrowright b_j$ such that $l_k(\gamma_j) \to \infty$. Then $\gamma^* = \bigcup\{\gamma_j : j \in \mathsf{N}\}$ is the desired geodesic ray. □

We next show that if G is a convex domain in a finite-dimensional Banach space, then each point $a \in G$ can be joined to each point $b \in \partial G$ by a quasihyperbolic geodesic ray. We shall make use of the theory of uniform domains.

3.19. *Uniform domains.* We recall that an arc $\gamma\colon a \curvearrowright b$ in a domain G is *c-uniform* in G, $c \ge 1$, if it satisfies the conditions

(1) $l(\gamma[a,x]) \wedge l(\gamma[x,b]) \le c\delta(x)$ for all $x \in \gamma$,

(2) $l(\gamma) \le c|a - b|$.

A domain G is *c-uniform* if each pair of points in G can be joined by a c-uniform arc.

A bounded convex domain is uniform; see, for example, [Vä2, 2.19]. The following useful result is from [GO, Th. 2]; see also [Vä3, 10.9].

3.20. **Lemma.** *Let γ be a quasihyperbolic geodesic in a c-uniform domain $G \subset E$. Then γ is a c'-uniform arc in G with $c' = c'(c)$.* $\square$

3.21. **Theorem.** *Let E be a finite-dimensional Banach space, let the domain $G \subset E$ be uniform or convex, and let $a \in G$, $b \in \partial G$. Then there is a geodesic ray from a to b.*

Proof. Choose geodesic paths g_j from a to points $x_j \in G$ such that $x_j \to b$. Then $g_j\colon [0, r_j] \to G$ with $r_j = k(a, x_j) \to \infty$. For each $t \geq 0$, the points $g_j(t)$ are defined for large j, and they lie in the compact set $\bar{B}_k(a,t)$. Since the maps g_j are isometries into the space (G, k), we may apply Ascoli's theorem to get a subsequence, still written as (g_j), which converges to a 1-Lipschitz map $g\colon [0, \infty) \to (G, k)$. By 3.1 we see that g is a geodesic ray path. By 3.19(2) and by 3.20, the proof of 3.17 is valid for uniform domains. Hence $g(t)$ converges to a point $b_1 \in \partial G$ as $t \to \infty$. It remains to prove that $b_1 = b$. Assume that $b_1 \neq b$.

Suppose first that G is c-uniform and that $b \neq \infty$. Let c' be the constant given by 3.20. Since the arcs $\gamma_j = \operatorname{im} g_j$ are c'-uniform, we have $l(\gamma_j) \leq c'|a - x_j|$, which implies that $b_1 \neq \infty$. Set

$$s = |a - b_1| \wedge |b - b_1|, \quad \varepsilon = s/6c'.$$

Choose $t \geq 0$ such that $|g(t) - b_1| \leq \varepsilon$ and then an integer j such that $r_j \geq t$, $|g_j(t) - g(t)| \leq \varepsilon$ and $|x_j - b| \leq \varepsilon$. Setting $y_j = g_j(t)$ we have

$$\delta(y_j) \leq |y_j - b_1| \leq 2\varepsilon.$$

Since γ_j is a c'-uniform arc, we get

$$c'\delta(y_j) \geq |y_j - a| \wedge |y_j - x_j| \geq s - 3\varepsilon \geq 3c'\varepsilon \geq 3c'\delta(y_j)/2,$$

a contradiction.

The proofs for the other cases are variations of the argument above. If G is c-uniform and $b = \infty$, we set $s = |a - b_1|$, $\varepsilon = s/6c'$, choose t as above and j such that $|y_j - g(t)| \leq \varepsilon$ and $|x_j - b_1| \geq s$. We again obtain a contradiction $c'\delta(y_j) \geq s - 2\varepsilon \geq 2c'\delta(y_j)$.

If G is convex and bounded, then G is uniform, and the assertion follows from the first part of the proof. Suppose that G is convex and unbounded. Assume first that $b \neq \infty$. Set $R = 2c_0|a - b|$, where c_0 is given by 3.15. The domain $G_0 = G \cap B(a, R)$ is convex and bounded. By the previous case, there is a geodesic ray γ

from a to b in G_0. For each $x \in \gamma$ we have by 3.15 $|x-a| \le l(\gamma) \le c_0|a-b| = R/2$ and similarly $\delta(x) \le |x-b| \le R/2 \le d(x, S(a,R))$. Hence γ is a geodesic ray in G.

Finally assume that $b = \infty$, $b_1 \ne \infty$. Set $R_1 = 6c_0|a-b_1|$. Then $G_1 = G \cap B(a, R_1)$ is a c-uniform domain for some c. Set $s = |a-b_1|$, $\varepsilon = s/6c'$. Choose $t \ge 0$ as before and an integer j such that $|y_j - b_1| \le 2\varepsilon$ and such that $|x_j - b_1| \ge s$. Next choose $t_j \in [t, r_j]$ such that $|z_j - b_1| = s$ for $z_j = g_j(t_j)$. Since $|a - z_j| \le 2s$, the arc $\beta = \gamma_j[a, z_j]$ lies in $\bar{B}(a, 2c_0 s) = \bar{B}(a, R_1/3)$. Hence β is a geodesic in G_1 and hence c'-uniform. As before we obtain the contradiction $c'\delta(y_j) \ge s - 2\varepsilon \ge 2c'\delta(y_j)$. □

3.22. *Quasihyperbolic lines.* A *geodesic line path* in a domain $G \subset E$ is an isometry $g\colon \mathsf{R} \to (G, k)$, and the set $\gamma^{**} = \operatorname{im} g$ is a *geodesic line.* Assume that G is convex or uniform. Since γ^{**} is the union of two quasihyperbolic rays, Theorem 3.17 and its uniform version imply that the limits $b = \lim_{t\to-\infty} g(t)$ and $b' = \lim_{t\to\infty} g(t)$ exist and that $b, b' \in \partial G$. We say that g and γ^{**} *join* b to b'.

One can easily prove line versions of the results on geodesic rays. Applying the prolongation theorem 3.12 to both endpoints of a geodesic we obtain

3.23. **Theorem.** *Let G be a convex domain in a finite-dimensional uniformly convex space, and let $\gamma\colon a \curvearrowright b$ be a geodesic in G. Then γ has a prolongation to a geodesic line γ^{**} containing γ.* □

We finally give the line version of 3.21.

3.24. **Theorem.** *Let E be a finite-dimensional Banach space, let the domain $G \subset E$ be uniform or convex, and let $b, b' \in \partial G$, $b \ne b'$. Then there is a geodesic line from b to b'.*

Proof. We sketch the proof in the harder case where G is convex and unbounded. We may assume that $b \ne \infty$. Choose a number $R > 0$ such that $b' \notin \bar{B}(b, R)$. Next choose sequences (x_j) and (x'_j) in G such that $x_j \to b$, $x'_j \to b'$ and such that $|x_j - b| \le R/3$ and $|x'_j - b| \ge R$ for all j. Let $\gamma_j\colon x_j \curvearrowright x'_j$ be a geodesic, let y_j be the first point of γ_j with $|y_j - b| = R$ and let $z_j \in \beta_j = \gamma_j[x_j, y_j]$ be a point with $|z_j - b| = 2R/3$. Since β_j is a geodesic also in the domain $G_0 = G \cap B(b, 2R)$ and since G_0 is c-uniform for some c, we have $\delta(z_j) \ge R/3c'$ for all j, where c' is given by 3.20. We may therefore assume that $z_j \to z_0 \in G$.

Let $g_j\colon [r_j, r'_j] \to G$ be the quasihyperbolic parametrization of γ_j such that $g_j(0) = z_j$. For each $z \in \mathsf{R}$, the points $g_j(t)$ are defined for large j and $k(g_j(t), z_0) \le |t| + k(z_j, z_0)$ is bounded. Hence we may again apply Ascoli's theorem to get a subsequence of (g_j) converging to a geodesic line path $g\colon \mathsf{R} \to G$. As in 3.21 we can show that g joins b to b'. □

Added in proof. November 2009. The second author has proved in [Vä5] that the three conjectures of 3.13 are true in the plane R^2.

References

[Al] P. Alestalo, Uniform domains of higher order, Ann. Acad. Sci. Fenn. Math. Diss. 94, 1994, 1–48.

[BL] Y. Benyamini, J. Lindenstrauss, Geometric nonlinear functional analysis I, Amer. Math. Soc. Colloquium Publications 48, 2000.

[FZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, Functional analysis and infinite-dimensional geometry, Springer, 2001.

[GO] F.W. Gehring and B.G. Osgood, Uniform domains and the quasi-hyperbolic metric, J. Analyse Math. 36, 1979, 50–74.

[GP] F.W. Gehring and B.P. Palka, Quasiconformally homogeneous domains, J. Analyse Math 30, 1976, 172–199.

[HY] J. Heinonen and S. Yang, Strongly uniform domains and periodic quasiconformal maps, Ann. Acad. Sci. Fenn. Math. 20, 1995, 123–148.

[Ma] G.J. Martin, Quasiconformal and bi-Lipschitz homeomorphisms, uniform domains and the quasihyperbolic metric, Trans. Amer. Math. Soc. 292, 1985, 169–191.

[MO] G.J. Martin and B.G. Osgood, The quasihyperbolic metric and associated estimates on the hyperbolic metric, J. Anal. Math. 47, 1986, 37–53.

[MS] O. Martio and J. Sarvas, Injectivity theorems in plane and space, Ann. Acad. Sci. Fenn. Math. 4, 1978/1979, 383–401.

[Vä1] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Mathematics 229, Springer, 1971.

[Vä2] J. Väisälä, Relatively and inner uniform domains, Conformal Geom. Dyn. 2, 1998, 56–88.

[Vä3] J. Väisälä, The free quasiworld. Freely quasiconformal and related maps in Banach spaces, Quasiconformal geometry and dynamics (Lublin 1996), Banach Center Publications, vol. 48, Polish Academy of Sciences, Warsaw, 1999, 55-118.

[Vä4] J. Väisälä, Quasihyperbolic geodesics in convex domains, Results Math. 48, 2005, 184–195.

[Vä5] J. Väisälä, Quasihyperbolic geometry of planar domains, Ann. Acad. Sci. Fenn. Math. 34, 2009, 447–473.

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