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The $(6, p)$-arithmetic Hyperbolic Lattices in Dimension 3.

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To F.W. Gehring - friend and mentor

Abstract: We prove there are exactly 56 arithmetic Nielsen inequivalent lattices of hyperbolic 3-space which are generated by two elements of finite orders 6 and p with $p \geq 2$. In fact $p \in \{2, 3, 4, 6\}$ only. This continues our work to identify all arithmetic hyperbolic lattices generated by two elements of finite order.
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1 Introduction

There are infinitely many lattices in the group $\text{Isom}^+\mathbb{H}^3 \cong \text{PSL}(2, \mathbb{C})$, of orientation-preserving isometries of hyperbolic 3-space (equivalently Kleinian groups of finite covolume) which can be generated by two elements of finite orders p and q. For instance, all but finitely many $(p, 0) - (q, 0)$ Dehn surgeries on any of the infinitely many hyperbolic two-bridge links (or knots if $p = q$) will have (orbifold) fundamental groups which are such uniform (co-compact) lattices. In [6], we showed that, up to conjugacy, there are only finitely many arithmetic lattices which can

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be generated by two elements of finite order. This paper forms part of a programme to identify all these lattices. The genesis of this was a joint work with Gehring [3], where we identified the four arithmetic lattices which can be generated by a pair of parabolic elements, all of which were two-bridge knots or links. Subsequently, we identified the 20 non-uniform arithmetic lattices which can be generated by two elements of finite order [7]. More recently, we have established that there are only 17 arithmetic lattices with two generators of orders p and q where $p, q \geq 6$ [9]. It should be pointed out that Takeuchi [14] dealt with the two-dimensional case, showing that there were exactly 82 such lattices of which 33 had two generators of orders ≥ 6.

In this paper we determine all arithmetic lattices in $\mathrm{PSL}(2,\mathbb{C})$ generated by a pair of elements of finite order, one of which has order 6; we call these the $(6,p)$-arithmetic lattices. For $p \geq 6$, then from [9], we deduce that $p = 6$ and we list these groups obtained in [9] here for completeness. Furthermore, each such $(6,6)$-lattice gives rise to one or two $(6,2)$-lattices and all $(6,2)$-lattices arise in this way [4]. Thus the main thrust here is to determine the $(6,p)$ arithmetic lattices for $p = 3, 4, 5$. The techniques used in [6] and further developed in [9] can be applied to these cases to show that the degree of the defining field over $\mathbb{Q}$ cannot be any greater than 6. To further pinpoint the defining fields and parameters for these groups, the results of [9] are again employed but some new techniques have to be developed to handle the more complex computational problems that arise in these cases of small values of p. It should be noted that, in the most difficult case of $(3,2)$ arithmetic lattices, the first stage of determining candidate defining fields has also used new techniques dictated by the complexity of the computational number theory problems that arise [2].

2 The Results

For a two-generator group $\langle f, g\rangle \subset \mathrm{PSL}(2,\mathbb{C})$, the three complex numbers

$$(\gamma(f,g), \beta(f), \beta(g)) \tag{1}$$

where $\gamma(f,g) = \mathrm{tr}\,[f,g] - 2, \beta(f) = \mathrm{tr}^2 f - 4, \beta(g) = \mathrm{tr}^2 g - 4$, are well-defined (for any lift to $\mathrm{SL}(2,\mathbb{C})$) by f, g and form the *parameters* of the group $\langle f, g\rangle$. They define $\langle f,g\rangle$ uniquely up to conjugacy provided $\langle f,g\rangle$ is irreducible [4]. Recall that $\langle f,g\rangle$ is reducible if all elements have a common fixed point in their action on $\hat{\mathbb{C}}$ and this occurs precisely when $\gamma(f,g)=0$. Thus for our results we determine the parameters of all $(6,p)$ arithmetic lattices, $\langle f,g\rangle$ where $o(f)=6, o(g)=p$. When the orders are fixed, we abbreviate $\gamma(f,g)$ to γ. We further note that γ is determined by the Nielsen equivalence class of a pair of generators of the group. Thus a $(6,p)$ arithmetic lattice may have Nielsen inequivalent pairs of generators of the same order.

No	p	γ	$p(z)$	Δ_k	$\mathrm{Ram}_f(A)$	Covol.	Min.
1	3	-1	z^2+z+1	-3	$\emptyset$	0.0846	0.0846
2	3	$-1+\sqrt{-3}$	z^2+2z+4	-3	$\emptyset$	1.692	0.0846
3	3	$2+\sqrt{-3}$	z^2-4z+7	-3	$\emptyset$	2.707	0.0846
4	3	$1.190+2.547i$	z^3-2z^2+7z+3	-231	$\mathcal{P}_3=zR_k$	6.211	1.552
5	3	$-1.696+1.436i$	z^3+4z^2+7z+3	-87	$\mathcal{P}_3=zR_k$	1.418	0.354
6	3	$-2.630+1.091i$	$z^3+6z^2+12z+6$	-108	$\mathcal{P}_2$	2.308	0.289
7	3	$-0.5+2.131i$	$z^4+2z^3+6z^2+5z+1$	-1323	$\emptyset$	1.946	0.243
8	4	-1	z^2+z+1	-3	$\emptyset$	0.211	0.0846
9	4	$\sqrt{-6}$	z^2+6	-24	$\mathcal{P}_2,\mathcal{P}_3$	2.591	0.648
10	4	$-1.289+1.807i$	z^3+3z^2+6z+2	-216	$\mathcal{P}_2,\mathcal{P}_2',\mathcal{P}_3$	2.086	0.522
11	4	$-2.287+1.350i$	z^3+5z^2+9z+3	-204	$\mathcal{P}_3=(z+2)R_k$	3.277	1.639
12	4	$0.362+2.876i$	z^4+8z^2+6z+1	-2412	$\mathcal{P}_2,\mathcal{P}_2'$	5.287	0.441
13	6	-1	z^2+z+1	-3	$\emptyset$	0.507	0.0846
14	6	$\sqrt{-3}$	z^2+z+1	-3	$\emptyset$	1.016	0.0846
15	6	$-1+i$	z^2+2z+2	-4	$\mathcal{P}_2,\mathcal{P}_3$	1.221	0.305
16	6	$1+3i$	$z^2-2z+10$	-4	$\mathcal{P}_2,\mathcal{P}_5=1/3(5+z)R_k$	6.106	0.153
17	6	$-1+\sqrt{-7}$	z^2+2z+8	-7	$\mathcal{P}_2=(z/2+1)R_k,\mathcal{P}_3$	7.111	0.889
18	6	$-2+\sqrt{-2}$	z^2+4z+6	-8	$\mathcal{P}_3,\mathcal{P}_3'$	4.015	0.502
19	6	$4.110+2.432i$	$z^3-8z^2+21z+5$	-23	$\mathcal{P}_5=zR_k$	8.799	0.0786
20	6	$3.067+2.328i$	* $z^3-6z^2+14z+2$	-44	$\mathcal{P}_2$	3.707	0.0662
21	6	$2.124+2.747i$	$z^3-4z^2+11z+3$	-31	$\mathcal{P}_3'=(z^2-4z+11)R_k$	4.222	0.264
22	6	$1.092+2.052i$	* z^3-2z^2+5z+1	-23	$\mathcal{P}_3$	2.043	0.511
23	6	$0.124+2.837i$	** z^3+8z+2	-44	$\mathcal{P}_2$	3.707	0.0662
24	6	$-0.892+1.954i$	z^3+2z^2+5z+1	-31	$\mathcal{P}_3=(z+1)R_k$	2.639	0.0660
25	6	$-1.877+0.745i$	** z^3+4z^2+5z+1	-23	$\mathcal{P}_3$	2.043	0.511
26	6	$-2.884+0.590i$	$z^3+6z^2+10z+2$	-76	$\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_3'$	5.293	0.662
27	6	$-1.876+2.133i$	z^3+4z^2+9z+2	-59	$\mathcal{P}_2=zR_k$	11.989	0.107

Table 1. The cases $p=3,4$ and 6.

The defining field of the arithmetic lattice is obtained from γ and the commensurability class of the lattice is given by a quaternion algebra over the defining field. The isomorphism class of a quaternion algebra is determined by its ramification set which is also obtainable from γ (see [10] and §3) so that the arithmetic data surrounding these groups is deducible and is given in Table 1.

Notes on Table 1.

- If $\gamma \notin \mathbb{R}$, then $p(z)$ is the minimum polynomial of γ and it determines the defining field k. For $\gamma \in \mathbb{R}$, $p(z)$ determines the defining field.
- If a prime ideal $\mathcal{P}_p$ or $\mathcal{P}'_p$ in $\mathrm{Ram}_f(A)$ is not uniquely determined by the rational prime p, we give an explicit description.
- "Min." is the covolume of the smallest orbifold in the commensurability class determined by γ.
- The asterixed pairs (20,23) and (22,25) define two groups, each member of the pair having a distinct Nielsen equivalence class of generators.

Theorem 2.1 *Let $\Gamma = \langle f, g\rangle$ be an arithmetic lattice in $\mathrm{Isom}^+(\mathbb{H}^3)$ with $o(f) = 6, o(g) = p$. Then $p = 2, 3, 4$ or 6. There are 53 conjugacy classes of such groups and 56 Nielsen equivalence classes of pairs of generators. These groups are described in Tables 1 to 3. The geometric descriptions in Tables 2 and 3 are clarified in the remarks preceding these tables.*

We also give a geometric description of these groups. Since this relates to the final process of determining which candidate values γ actually give rise to arithmetic lattices, we describe that now. The methods to be described in §3, 4 and 5 produce a list of polynomials with integer coefficients with a root γ which defines a group $\langle f, g\rangle$ with $o(f) = 6, o(g) = p$ which is a subgroup of an arithmetic Kleinian group. It remains to decide if this group has finite covolume and so is a $(6, p)$ lattice.

A computer program has been developed, initially by J. McKenzie and named JSnap to study subgroups Γ of $\mathrm{PSL}(2, \mathbb{C})$ which have two generators of finite order. This is effectively an implementation of the Dirichlet routine in the J. Week's program Snappea [15]. This program aims to find a Dirichlet region for Γ. It is important to note that in our cases we know *a priori* that our candidates Γ are discrete. JSnap runs and either produces a fundamental domain - either of finite or infinite volume - or produces an error message if it cannot put together a fundamental domain after looking at words in the generators of a given bounded length. For all the groups obtained from the values of γ on our candidate list, JSnap always produces a fundamental domain which is either compact or meets the sphere at infinity in an open set. These latter cases can be eliminated and in the former cases, JSnap also produces an approximate volume.

On the other hand, many examples of $(6, p)$ lattices are obtained via $(6, 0) - (p, 0)$ surgery on hyperbolic link complements. Implementing this on Snappea yields an approximate volume for the resulting orbifold. Comparing these with the volumes produced by JSnap yields likely candidates. The matrix representation then given by Snappea is used to verify that the commutator traces and hence γ agree. As both come as the roots of a monic polynomial with integer coefficients, this comparison is exact. In this way we can attempt to identify the orbit space corresponding to each γ.

The $(6, 6)$ arithmetic lattices which have Nielsen inequivalent pairs of generators were identified as follows. Each is an index two subgroup of a group obtained by $(2, 0)$ surgery on one component C_1 and $(6, 0)$ surgery on the other component C_2 of a two-bridge link complement, in particular 7^2_1 and 9^2_1. If this yields $\langle f, g\rangle$ with $o(f) = 6, o(g) = 2$ then $\langle f, gfg\rangle$ is a $(6, 6)$ arithmetic lattice. However, carrying out $(6, 0) - (2, 0)$ surgery on components C_1, C_2 respectively, to obtain $\langle f', g'\rangle$, the subgroup $\langle f', g'f'g'\rangle$ is the same $(6, 6)$ lattice since these links admit automorphisms which interchange the components but the generators are not in the same Nielsen equivalence class (as the γ parameters are different). This can be verified starting from a $(6, 6)$ lattice and using the retriangulation procedure of Snappea.

A presentation can be obtained for those groups for which a surgery description has been identified. This is true in particular for all $(6, 6)$ cocompact arithmetic lattices. The non-cocompact $(6, p)$ arithmetic lattices were obtained in [7].

Group No. 4 on Table 2 We were unable to identify this group beyond the matrix representation (see (7)). In particular, this group does not arise as $(6, 0) - (4, 0)$ surgery on any of the two-bridge links which appear in Rolfsen's tables [12]. We also tried a number of other two component links to match volumes, without success. On the arithmetic side, this field does not appear in any examples that we know so we were unable to get a handle on any group in the commensurability class. In principle, we could obtain a presentation of this group from the face pairing of the Dirichlet domain, but our present implementation of JSnap does not yield enough information to allow this.

No	p	γ	Description
1	3	-1	Non-cocompact $3-3-6$ Coxeter group $\langle a,b \vert a^3=b^6=[a,b]^3=([a,b]a)^2=(b^{-1}[a,b])^2=1\rangle$
2	3	$-1+\sqrt{-3}$	Non-cocompact generalised triangle group $\langle a,b \vert a^3=b^6=(aba^{-1}bab^{-1})^2=1\rangle$
3	3	$2+\sqrt{-3}$	Non-cocompact generalised triangle group $\langle a,b \vert a^3=b^6=((ab)^2ab^{-1}(a^{-1}b^{-1})^2)^2=1\rangle$
4	3	$1.190+2.547i$	?
5	3	$-1.696+1.436i$	$(6,0)-(3,0)$ surgery on $(8/3)$
6	3	$-2.630+1.901i$	$(6,0)-(3,0)$ surgery on $(12/5)$
7	3	$-0.5+2.131i$	$(6,0)-(3,0)$ surgery on $(10/3)$
8	4	-1	Non-cocompact $4-3-6$ Coxeter group $\langle a,b \vert a^4=b^6=[a,b]^3=([a,b]a)^2=(b^{-1}[a,b])^2=1\rangle$
9	4	$\sqrt{-6}$	$(6,0)-(4,0)$ surgery on $(10/3)$
10	4	$-1.298+1.807i$	$(6,0)-(4,0)$ surgery on $(8/3)$
11	4	$-2.287+1.350i$	$(6,0)-(4,0)$ surgery on $(12/5)$
12	4	$0.362+2.876i$	$(6,0)-(4,0)$ surgery on 8^2_4
13	6	-1	Non-cocompact $6-3-6$ Coxeter group $\langle a,b, \vert a^6=b^6=[a,b]^3=([a,b]a)^2=(b^{-1}[a,b])^2=1\rangle$
14	6	$\sqrt{-3}$	Non-cocompact group $\langle a,b \vert a^6=b^6=(b^{-1}a)^2b[a^{-1},b][a,b][a,b^{-1}]a^{-1}=([b^{-1},a]ba^2)^2=1\rangle$
15	6	$-1+i$	$(6,0)$ surgery on $(5/3)$
16	6	$1+3i$	$(6,0)-(6,0)$ surgery on $(24/7)$
17	6	$-1+\sqrt{-7}$	$(6,0)-(6,0)$ surgery on $(30/11)$
18	6	$-2+\sqrt{-2}$	$(6,0)-(6,0$ surgery on $(12/5)$
19	6	$4.110+2.432i$	$(6,0)$ surgery on $(65/11)$
20	6	$3.067+2.328i$	* $(6,0)$ surgery on $(13/3)$
21	6	$2.124+2.747i$	$(6,0)$ surgery on $(15/11)$
22	6	$1.092+2.052i$	* $(6,0)$ surgery on $(7/3)$
23	6	$0.124+2.837i$	** $(6,0)$ surgery on $(13/3)$
24	6	$-0.892+1.954i$	$(6,0)-(6,0)$ surgery on $(8/3)$
25	6	$-1.877+0.745i$	** $(6,0)$ surgery on $(7/3)$
26	6	$-2.884+0.590i$	$(6,0)-(6,0)$ surgery on $(20/9)$
27	6	$-1.876+2.133i$	$(6,0)-(6,0)$ surgery on $(130/51)$

Table 2. Geometric Description of Groups on Table 1.

In Table 3 below we describe all $(6,2)$ arithmetic lattices. In all cases except

No. 1 on Table 3, each $(6,2)$ lattice is an extension of order 2 of a $(6,6)$ lattice, and, for No. 1, the $(6,2)$ lattice is also $(6,6)$ lattice [4, 7]. We indicate the related $(6,6)$ lattice from Tables 1 and 2. As commensurable groups have the same defining field and quaternion algebra, that arithmetic data is not repeated on Table 3.

No	p	γ	Description
1.	2	$(-3+\sqrt{-3})/2$	Coincides with group No. 14 on Tables 1 and 2. $\langle a,b \mid a^2=b^6=(aba^{-1}bab^{-1})^3=1\rangle$
2.	2	$(1+\sqrt{-3})/2$	Doubly-covered by No. 14 $\langle a,b \mid a^2=b^6=(b^{-1}a)^4b[aba,b][ab^{-1}a,b][ab^{-1}a,b^{-1}]aba$ $=([b^{-1},ab^{-1}a]bab^{-2}b)^2=1\rangle$
3.	2	i	Doubly-covered by No. 15
4.	2	$-1+i$	Doubly-covered by No. 15, $(2,0)-(6,0)$ on $(10/3)$
5.	2	$(-1+\sqrt{-3})/2$	Doubly-covered by No. 13 $\langle a,b \mid a^2=b^6=[aba,b]^3=([aba,b]aba)^2=(b^{-1}[aba,b])^2$ $=(b^{-1}[aba,b]aba)^2=1\rangle$
6.	2	$-2+i$	Doubly-covered by No. 16, $(2,0)-(6,0)$ on $(48/17)$
7.	2	$1+i$	Doubly-covered by No. 16.
8.	2	$(-3+\sqrt{-7})/2$	Doubly-covered by No. 17, $(2,0)-(6,0)$ on $(60/19)$
9.	2	$(1+\sqrt{-7})/2$	Doubly-covered by No. 17.
10.	2	$\sqrt{-2}$	Doubly-covered by No. 18.
11.	2	$-1+\sqrt{-2}$	Doubly-covered by No. 18.
12.	2	$-2.662+0.562i$	Doubly-covered by No.19.
13.	2	$1.662+0.562i$	Doubly-coverd by No. 19.
14.	2	$-2.420+0.606i$	Doubly-covered by No. 20.
15.	2	$1.420+0.606i$	Doubly-covered by No. 20.
16.	2	$-2.233+0.793i$	Doubly-covered by No. 21.
17.	2	$1.233+0.793i$	Doubly-covered by No. 21.
18.	2	$-1.877+0.745i$	Doubly-covered by No. 22.
19.	2	$0.877+0.745i$	Doubly-covered by No. 22.
20.	2	$0.772+1.115i$	Doubly-covered by No. 23.
21.	2	$-1.772+1.115i$	Doubly-covered by No. 23.
22.	2	$0.341+1.161i$	Doubly-covered by No.24.
23.	2	$-1.341+1.161i$	Doubly-covered by No.24.
24.	2	$-0.215+1.307i$	Doubly-covered by No. 25.
25.	2	$-0.785+1.307i$	Doubly-covered by No. 25.
26.	2	$-0.319+1.633i$	Doubly-covered by No. 26.
27.	2	$-0.681+1.633i$	Doubly-covered by No. 26.
28.	2	$0.227+1.468i$	Doubly-covered by No. 27.
29.	2	$-1.227+1.468i$	Doubly-covered by No. 27.

Table 3. The cases where $p=2$.

Up to complex conjugation, the γ values are related by the equation

$$\gamma(6,2)(\gamma(6,2)+1)=\gamma(6,6).$$

We now discuss how one can determine the geometric data, orbifold graph and presentation from the cases above of groups generated by two elements of order 6. If $\langle f,g\rangle$ is generated by an element f of order 6 and g of order 2, then $\langle f,gfg\rangle$ is generated by two elements of order 6 and has index at most two in $\langle f,g\rangle$. These groups are therefore simultaneously lattices - or otherwise. This process can be reversed. If $\langle f,h\rangle$ is a group generated by f and h of order 6, then there are two elements of order two, say g_1 and g_2, whose axes bisect the common perpendicular between the axes of f and h such that $h^{\pm 1}=g_i f g_i$. Again $\langle f,g_i\rangle$ and $\langle f,h\rangle$ are simultaneously lattices or not. Should the group arise from surgery on a knot or link, these two elements of order 2 can often be identified as symmetries of the underlying knot or link. As an index two subgroup, one can find a presentation for $\langle f,g_i\rangle$ from that of $\langle f,h\rangle$ by adding in the generator g_i and the relator $f^{-1}=g_i h g_i$. Let us work through the example of the figure eight knot illustrated below with these two symmetries.

The orbifold fundamental group of $(6,0)$ surgery on the figure eight knot complement is $\langle a,b : a^6=b^6=ba^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1}a\rangle$(which is No. 15 on Tables 1 and 2). If we put $c^2=1$ and $cac=b$ or $cac=b^{-1}$ we arrive at the two groups

$$\langle a,c : a^6=c^2=(caca^{-1}ca^{-1})^2cacaca^{-1}ca=1\rangle$$

$$\langle a,c : a^6=c^2=((ca^{-1})^2(ca)^3)^2=1\rangle.$$

The first is a presentation for $(2,0)-(6,0)$ Dehn surgery on the two bridge link with slope $(10/3)$ (No. 4 on Table 3) and the second is a generalised triangle group of the type discussed in [5] (No. 3 on Table 3). It is typical that with the addition of these two symmetries the number of crossings of the quotient link (if it is a link) is quite large - outside of any common tables.

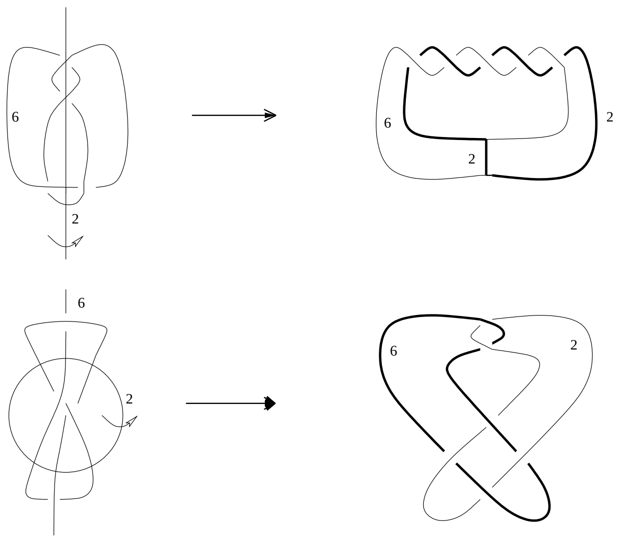

Figure 8 knot and symmetries.

3 Two-generator Arithmetic Lattices

As noted in §1, we wish to find all parameters as described at (1) for a group $\Gamma = \langle f, g\rangle$ where $o(f) = 6, o(g) = p$ and Γ is an arithmetic Kleinian group. We can assume that f, g are primitive so that $\mathrm{tr}^2 f = 3, \mathrm{tr}^2 g = 4\cos^2 \pi/p$ and $\beta(f) = -1, \beta(g) = -4\sin^2 \pi/p$. Thus, for each fixed p, we will determine $\gamma = \gamma(6, p) \in \mathbb{C}$. When γ is real, all such groups have been obtained in [8]. These are given by the parameters $(-1, -1, -3), (-1, -1, -2), (-1, -1, -1)$ and have corresponding values of $p = 3, 4$ and 6 respectively. We assume henceforth that $\gamma \in \mathbb{C} \setminus \mathbb{R}$ and, as remarked in §1, we need only consider $p = 3, 4$ and 5.

Let $\Gamma = \langle f, g\rangle$ be a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$. There is an associated invariant trace field $k\Gamma$ and an invariant quaternion algebra $A\Gamma$ over $k\Gamma$ which are invariants of the commensurability class of Γ [10, §3.3]. For a two-generator group as above

$$k\Gamma = \mathbb{Q}(\mathrm{tr}^2 f, \mathrm{tr}^2 g, \mathrm{tr}\, f \mathrm{tr}\, g \mathrm{tr}\, fg) \tag{2}$$

([10, Lemma 3.5.7]) and $A\Gamma$ has the Hilbert symbol ([10, Theorem 3.6.2])

$$A\Gamma = \left(\frac{\mathrm{tr}^2 f(\mathrm{tr}^2 f - 4), \mathrm{tr}^2 f \mathrm{tr}^2 g(\mathrm{tr}\,[f,g] - 2)}{k\Gamma}\right). \tag{3}$$

Furthermore, if Γ is a finite covolume Kleinian group, then Γ is arithmetic if and only if the following three conditions hold:

(A) $k\Gamma$ is a number field with exactly one complex place,

(B) $\mathrm{tr}\, h$ is an algebraic integer for each $h \in \Gamma$,

(C) $A\Gamma$ is ramified at all real places of $k\Gamma$.

([10, Theorem 8.3.2]). From this, extending similar results in [6, 9], we obtain the following theorem on which our identification of the $(6,p)$ arithmetic lattices will be based.

Theorem 3.1 *Let $\Gamma = \langle f, g\rangle$ be a non-elementary subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *with f of order 6 and g of order p, $p \geq 3$. Let $\gamma(f,g) = \gamma \in \mathbb{C} \setminus \mathbb{R}$. Then Γ is an arithmetic Kleinian group if and only if*

1. *γ is an algebraic integer,*
2. *$\mathbb{Q}(\gamma) \supset L = \mathbb{Q}(\cos 2\pi/p)$ and $\mathbb{Q}(\gamma)$ is a number field with exactly one complex place,*
3. *if $\tau : \mathbb{Q}(\gamma) \to \mathbb{R}$ is a real embedding such that $\tau|_L = \sigma$, then*

$$-\sigma(\sin^2 \pi/p) < \tau(\gamma) < 0, \tag{4}$$

4. *$\mathbb{Q}(\lambda) = \mathbb{Q}(\gamma)$ where*

$$12\cos^2 \pi/p\, \gamma = \lambda^2 - 12\cos^2 \pi/p\, \lambda + 12\cos^2 \pi/p\, (4\cos^2 \pi/p - 1) \tag{5}$$

5. Γ has finite co-volume.

Sketch of proof: The traces of all elements in Γ are integer polynomials in $\mathrm{tr}\, f, \mathrm{tr}\, g, \mathrm{tr}\, fg$ and Fricke's identity shows that

$$\mathrm{tr}\,[f,g] = \mathrm{tr}^2 f + \mathrm{tr}^2 g + \mathrm{tr}^2 fg - \mathrm{tr}\, f \mathrm{tr}\, g \mathrm{tr}\, fg - 2. \tag{6}$$

In this case, $\mathrm{tr}^2 f = 3, \mathrm{tr}^2 g = 4\cos^2 \pi/p$, so that $\mathrm{tr}\, f, \mathrm{tr}\, g$ are algebraic integers. Thus from (6) traces of all elements of Γ will be algebraic integers if $\gamma (= \mathrm{tr}^2[f,g] - 2)$ is an algebraic integer.

Let $\lambda = \mathrm{tr}\, f \mathrm{tr}\, g \mathrm{tr}\, fg$ so that $k\Gamma = \mathbb{Q}(\cos 2\pi/p, \lambda)$. Multiplying (6) by $\mathrm{tr}^2 f \mathrm{tr}^2 g$ yields the equation (5), so that, since $\gamma \in \mathbb{C} \setminus \mathbb{R}$, then $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Recall that every proper subfield of a field with one complex place is real. Thus if $k\Gamma = \mathbb{Q}(\cos 2\pi/p, \lambda)$ has exactly one complex place, then $k\Gamma = \mathbb{Q}(\gamma) = \mathbb{Q}(\lambda)$ and $\mathbb{Q}(\gamma) \supset \mathbb{Q}(\cos 2\pi/p)$. Conversely, if $\mathbb{Q}(\gamma)$ is a field with one complex place with $\mathbb{Q}(\gamma) \supset \mathbb{Q}(\cos 2\pi/p)$ and $\mathbb{Q}(\gamma) = \mathbb{Q}(\lambda)$ where λ is defined by (5), then $k\Gamma = \mathbb{Q}(4\cos^2 \pi/p, \lambda)$ has exactly one complex place.

Let $\tau : \mathbb{Q}(\gamma) \to \mathbb{R}$ be a real embedding such that $\tau\, |_L = \sigma$. Since $\mathbb{Q}(\gamma) = \mathbb{Q}(\lambda)$ has just one complex place, the discriminant of the quadratic in λ at (5) must be positive at real embeddings. This forces $-\sigma(\sin^2 \pi/p) < \tau(\gamma)$.

Finally, $A\Gamma$ is ramified at all real embeddings of $k\Gamma$ if and only if the Hilbert symbol entries are negative at all such τ. Thus substituting in (3), $A\Gamma = \left(\frac{-3, 12\cos^2 \pi/p\, \gamma}{k\Gamma}\right)$ and so $\tau(\gamma) < 0$. $\square$

The theorem is stated in the above form, as we first use conditions 1., 2. and 3. together with a geometric condition (see Theorem 3.2 below) which is a necessary consequence of condition 5, to obtain restrictions on the parameter γ. This is carried out in §4. Further refinements are carried out in §5 before we invoke condition 4 and the resulting groups Γ are then subgroups of arithmetic Kleinian groups [10, Corollary 8.3.7]. Finally these are tested using JSnap as described in §2 to determine which have finite covolume.

If $\Gamma = \langle f, g\rangle$ has finite covolume, then it cannot be isomorphic to the free product $\langle f\rangle * \langle g\rangle$. Let Γ be normalised so that f, g are represented by the matrices

$$\begin{pmatrix} \sqrt{3}/2 & i/2 \\ i/2 & \sqrt{3}/2 \end{pmatrix}, \quad \begin{pmatrix} \cos\pi/p & iw\sin\pi/p \\ iw^{-1}\sin\pi/p & \cos\pi/p \end{pmatrix} \tag{7}$$

for w which can be chosen to have $|w| \leq 1$ and $\Re(w) \geq 0$ [6]. If the isometric circles of g lie inside the intersection of the isometric circles of f, then Γ is a free product. This manifests itself as bounds on w which can be translated into bounds on γ since $\gamma = \frac{1}{4}\sin^2\pi/p\,(w - 1/w)^2$. We deduce the following bounds proved in [9, §3].

Theorem 3.2 *Let $\langle f, g\rangle$ be a Kleinian group where $o(f) = 6, o(g) = p$ and $p = 3, 4$ or 5. If Γ is not a free product of $\langle f\rangle$ and $\langle g\rangle$ then*

$$|\Im(\gamma)| \leq \frac{5}{2}\sqrt{(25 + 11\sqrt{5})/2}. \tag{8}$$

$$-\frac{11 + 5\sqrt{5}}{4} \leq R_\ell \leq \Re(\gamma) \leq R_u = (\sqrt{3} + 2\cos\pi/p)^2. \tag{9}$$

(For each value of p, a lower bound R_ℓ for $\Re(\gamma)$ can be computed which satisfies the above inequality for all values of p).

$$|\gamma| \leq (\sqrt{3} + 2\cos\pi/p)^2 \tag{10}$$

Furthermore, if $x \in (-\sin^2\pi/p, 0)$ then

$$|\gamma - x| \leq (2 + \sqrt{3}\cos\pi/p)^2. \tag{11}$$

4 Degree Bounds

From Theorem 3.1, we note that γ is an algebraic integer, such that $\mathbb{Q}(\gamma)$ has exactly one complex place and that $\mathbb{Q}(\gamma)$ must contain $L = \mathbb{Q}(\cos 2\pi/p)$. Let $[\mathbb{Q}(\gamma) : L] = r$. Note that, for $p = 3, 4$, $L = \mathbb{Q}$ and for $p = 5$, $L = \mathbb{Q}(\sqrt{5})$. In this section we obtain bounds on r.

Let N denote the absolute norm $N : \mathbb{Q}(\gamma) \to \mathbb{Q}$. Since γ is a non-zero algebraic integer, $N(\gamma) \geq 1$. The inequality (10) gives a bound for $|\gamma|$ and $|\bar{\gamma}|$. For $p = 3$ and 4, the remaining $r - 2$ conjugates of γ are bounded in absolute value by $\sin^2\pi/p$ by (4). For $p = 5$, $r - 2$ conjugates over $\mathbb{Q}$ are likewise bounded by $\sin^2\pi/5$ while the remaining r conjugates over $\mathbb{Q}$ are bounded by the $L \mid \mathbb{Q}$-conjugate $\sin^2 2\pi/5$ by (4). These yield the following bounds:

$$\begin{array}{ll} p=3 & 1 \le |N(\gamma)| \le (1+\sqrt{3})^4(\sin^2\pi/3)^{r-2} \\ p=4 & 1 \le |N(\gamma)| \le (\sqrt{3}+\sqrt{2})^4\,(\sin^2\pi/4)^{r-2} \\ p=5 & 1 \le |N(\gamma)| \le (\sqrt{3}+2\cos\pi/5)^4\,(\sin^2\pi/5)^{r-2}\,(\sin^2 2\pi/5)^r. \end{array}$$

As r increases, the right hand sides of these inequalities decrease to 0 so by direct calculation we obtain $r \le 15, 8, 5$ respectively. We can improve upon these initial bounds by considering the relative discriminant $\delta_{\mathbb{Q}(\gamma)|L}$ (which is the absolute discriminant for $p = 3, 4$). Recall Schur's bound [13] that if $-1 \le y_1 < y_2 < \cdots < y_n \le 1$, with $n \ge 3$ then

$$\prod_{1\le i<j\le n}(y_i-y_j)^2 \le M_n = \frac{2^2 3^3\cdots n^n 2^2\cdots(n-2)^{n-2}}{3^3 5^5\cdots(2n-3)^{2n-3}}.$$

Let $x_3, x_4, \ldots, x_r$ denote the real roots of the minimum polynomial of γ over L. Then, $\mathcal{B} = \{1, \gamma, \gamma^2, \ldots, \gamma^{r-1}\}$ is a power basis of integers of $\mathbb{Q}(\gamma)$ over L. Thus for $p = 3, 4$

$$|\Delta_{\mathbb{Q}(\gamma)}| \le |\text{discr }\mathcal{B}| = |\gamma-\bar\gamma|^2\left[\prod_{i=3}^{r}|\gamma-x_i|^2|\bar\gamma-x_i|^2\right]\prod_{3\le i<j\le r}(x_i-x_j)^2.$$

We noted above that $x_i \in (-\sin^2\pi/p, 0)$. Thus scaling Schur's bound and using (8) and (11) we obtain

$$|\Delta_{\mathbb{Q}(\gamma)}| \le 620(2+\sqrt{3}/2)^{8(r-2)}(3/8)^{(r-2)(r-3)}M_{r-2} \quad \text{for } p=3.$$

$$|\Delta_{\mathbb{Q}(\gamma)}| \le 620(2+\sqrt{3}/\sqrt{2})^{8(r-2)}(1/4)^{(r-2)(r-3)}M_{r-2} \quad \text{for } p=4$$

For $p = 5$, let $\tau : \mathbb{Q}(\gamma) \to \mathbb{R}$ be such that $\tau|_L = \sigma$, the non-trivial Galois automorphism of L. Then $\{1, \tau(\gamma), \tau(\gamma^2), \ldots, \tau(\gamma^{r-1})\}$ is a power basis of integers of $\tau(\mathbb{Q}(\gamma))$ over L, and the roots of the minimum polynomial of $\tau(\gamma)$ over L all lie in the interval $(-\sin^2 2\pi/5, 0)$. Thus

$$|N_{L|\mathbb{Q}}(\delta_{\mathbb{Q}(\gamma)|L})| \le |\text{discr }\mathcal{B}||\sigma(\text{discr }\mathcal{B})|$$

$$\le 620(2+\sqrt{3}\cos\pi/5)^{8(r-2)}\left(\frac{\sin^2\pi/5}{2}\right)^{(r-2)(r-3)}M_{r-2}\left(\frac{\sin^2 2\pi/5}{2}\right)^{r(r-1)}M_r.$$

For the small values of r thrown up by the first set of bounds, the right-hand sides of these inequalities can be evaluated. For a lower bound for the left hand sides, we note that $\mathbb{Q}(\gamma)$ is a field with exactly one complex place. Let D_n denote the minimum absolute value of the discriminant of any field of degree n over $\mathbb{Q}$ with exactly one complex place. For $p = 3, 4$, $|\Delta_{\mathbb{Q}(\gamma)}| \geq D_r$. For $p = 5$, since $|\Delta_{\mathbb{Q}(\gamma)}| = |\Delta_L^r N_{L|\mathbb{Q}}(\delta_{\mathbb{Q}(\gamma)|L})|$ there will be a lower bound of $D_{2r}/5^r$. For $n \leq 8$, D_n is known explicitly and for $n \geq 9$, lower bounds for D_n due to Odlyzko [11] can be used (see Table 2 in [9]). In this way we obtain further bounds on r. There are further slight refinements to these methods. Note that the upper bound for $|N(\gamma)|$ occurs when the real x_i cluster at one end of the interval given by (4). But in these circumstances, the discriminant will be small. This is quantified as a "balancing argument" in [9, §4]. All these yield the following bounds:

Theorem 4.1 *Let $\Gamma = \langle f, g\rangle$ be an arithmetic Kleinian group generated by f of order* 6 *and g of order $p \in \{3, 4, 5\}$. Let $r = [\mathbb{Q}(\gamma) : \mathbb{Q}(\cos 2\pi/p)]$.*

- $r \leq 3$ *if* $p = 5$
- $r \leq 6$ *if* $p = 4$
- $r \leq 6$ *if* $p = 3$.

5 The Remaining Cases

We first remark on the case $p = 5$. Here γ satisfies a quadratic or cubic polynomial with coefficients $c_i \in R_{\mathbb{Q}(\sqrt{5})}$. Thus $c_i = (a + b\sqrt{5})/2$ where $a, b \in \mathbb{Z}$ with $a \equiv b(\mathrm{mod}\ 2)$. From the bounds on $|\gamma|, \Re(\gamma)$ at (9) and (10) and on the conjugates of γ at (4) we obtain upper and lower bounds u_1, ℓ_1 for c_i and u_2, ℓ_2 for $\sigma(c_i) = (a - b\sqrt{5})/2$. From these we obtain candidate values for the integers a, b for each c_i. This results in a not-too-large collection of polynomials which we can further examine to determine if their roots, and the roots of the conjugate polynomial, satisfy the bounds dictated by the inequalities (4), (9) and (10). The resulting values of γ then satisfy 1,2 and 3 of Theorem 3.1 and we invoke condition 4. In the polynomial of a candidate γ, we use (5) to obtain a polynomial of double the degree for λ which, by Condition 4 of Theorem 3.1, must factorise over $\mathbb{Q}(\gamma)$. We find that none of the candidate polynomials for γ satisfy this factorisation condition. This method was employed in [9, §6] and for more on this crucial factorisation condition, see below.

For the cases where $p = 3, 4$, γ satisfies a polynomial of degree ≤ 6 with rational integer coefficients.The bounds on the coefficients obtained from the inequalities (4), (9) and (10) give, in general, too large a collection of polynomials to solve and apply the conditions of Theorem 3.1, particularly 4. So we use various methods to reduce the size of the problem. We illustrate this with one case, which is neither the most difficult nor the easiest but contains all the ideas necessary to handle the other cases. This is the case $p = 3$ and $r = 4$ which does eventually produce one example (No 7 on Table 1).

Let γ have minimum polynomial $p(z) = z^4 + c_3z^3 + c_2z^2 + c_1z + c_0$, so that it has roots $\gamma, \bar{\gamma}, x_1, x_2$ where $x_i \in (-3/4, 0)$, and the bounds on $\gamma, \bar{\gamma}$ are given at (9) and (10). Expressing the c_i in terms of these roots, e.g. $c_3 = -2\Re(\gamma) - (x_1 + x_2)$, we quickly obtain the bounds $-14 \leq c_3 \leq 12$, $-22 \leq c_2 \leq 72$, $-6 \leq c_1 \leq 91$, $1 \leq c_0 \leq 31$. For implementing and programming our search, particularly for higher degrees, it is useful to cut down the size of this search space as much as possible and we give two methods of proceeding from here.

Method 1. First of all, we can bound one coefficient in terms of another. For instance, we have

$$-c_3 \leq 2\Re(\gamma) \leq -c_3 + 3/2$$

which can then be used to bound $c_2 = |\gamma|^2 + 2(x_1 + x_2)\Re(\gamma) + x_1x_2$. This gives

$$\text{Min}(0, \frac{3}{2}(c_3 - \frac{3}{2})) \leq c_2 \leq (1 + \sqrt{3})^4 + \frac{9}{16} + \text{Max}(\frac{3}{2}c_3, 0).$$

Also, as was used in [9, §6], $c_1 = -c_0(1/x_1 + 1/x_2) - 2\Re(\gamma)x_1x_2$, yielding

$$\frac{8}{3}c_0 - \frac{9}{8}(1 + \sqrt{3})^2 < c_1 < c_0 \left(\frac{0.75(1 + \sqrt{3})^2}{c_0} + \left(\frac{(1 + \sqrt{3})^4}{c_0} \right)^{1/2} \right) - \frac{9}{8}R_\ell,$$

where R_ℓ is defined in Theorem 3.2 Since the quartic $p(z)$ has two real roots in the interval $(-0.75, 0)$, $p'(z)$ has at least one root in that interval. Since $p'(z)$

is a cubic involving only c_3, c_2, c_1 we can use this to limit the possibilities of c_1 depending on c_3, c_2.

Using these sorts of refinements to reduce the search space produces a list of candidate polynomials with one complex pair of roots satisfying (9) and (10) and two real roots in the interval $(-0.75, 0)$.

We now implement condition 4 of Theorem 3.1. From (5), $\gamma = (\lambda^2 - 3\lambda)/3$. Substituting into a candidate polynomial $p(z)$ gives $q(\lambda) = 3^4 p((\lambda^2 - 3\lambda)/3)$, a monic integral polynomial of degree 8, $q(\lambda) = \lambda^8 - 12\lambda^7 + (54 + 3c_3)\lambda^6 + \cdots$. This must factor as the product $(\lambda^4 + a_3\lambda^3 + \cdots + a_0)(\lambda^4 + b_3\lambda^3 + \cdots + b_0)$ where $a_i, b_i \in \mathbb{Z}$, one factor being the minimum polynomial of λ, the other of $3-\lambda$. This leads to a non-linear system of eight equations over the integers e.g $a_3 + b_3 = -12, a_2 + b_2 + a_3 b_3 = 54 - 3c_3$ etc. Such a system can be solved numerically yielding many solutions from which we must select those that are approximately integral. The implementation of this on a machine becomes impractical if there are, as in this case, a large number of candidate polynomials to be considered. To shortcut this, solve each $q(\lambda) = 0$ numerically. For each set of 8 roots, select one root, and test with all possible triples of the other roots, for those for which the product of all 4 roots and the sum of all 4 roots is approximately integral as these must correspond to two of the coefficients of the polynomial satisfied by λ (or $3 - \lambda$). This drastically reduces the number of polynomials to be considered and hence the number of non-linear systems to be solved.

Method 2. In this variation, we use equation (5) at the outset in the following way.

$$12\gamma + 9 = (2\lambda - 3)^2 \tag{12}$$

so that $\mu = -(2\lambda - 3)$ is an algebraic integer such that $\mathbb{Q}(\gamma) = \mathbb{Q}(\mu)$. If $p(z)$ is the minimum polynomial of γ, then $q(v) := 12^4 p((v-9)/12) = a_0 + a_1 v + \cdots + v^4$, with $a_i \in \mathbb{Z}$ linear combinations of the c_i e.g. $a_3 = 12(c_3 - 3)$. Now if $b_0 + b_1 x + \cdots + x^4$ is the minimum polynomial of μ, then μ and $-\mu$ satisfy $q(x^2) = 0$ so that $q(x^2) = (b_0 + b_1 x + \cdots + x^4)(b_0 - b_1 x + \cdots + x^4)$ and we have four non-linear equations relating the b_i to the a_i and hence to the c_i e.g. $2b_2 - b_3^2 = a_3$. The divisibility of the coefficients a_i by powers of 2 and 3 impose necessary conditions on the coefficients b_i. Precisely, $b_0 = 9(2b_0' + 1), b_1 = 18b_1', b_2 = 6b_2', b_3 = 6b_3'$ where

$b_i' \in \mathbb{Z}$. Just as bounds were obtained for γ, we can in the same way obtain bounds on μ. Thus, since the quaternion algebra is to be ramified at the real places, (12) and (3) show that $\sigma(\mu) \in (-3,3)$ for σ a real embedding. Also, from the matrix representation at (7), we have that $\mu = -(2\lambda - 3) = 3(w + w^{-1})/2$ and the isometric circle conditions given in §3, then yield $|\mu| \leq 3(4/\sqrt{3}+1)$ and $0 \leq \Re(\mu) < 3(4/\sqrt{3}+1)$. Using these we obtain bounds on the b_i'. If necessary, e.g. in the analogous case when $r = 6$, we can also use the extensions as described in Method 1 to obtain improved bounds. The divisibility conditions on the b_i as stated above are only necessary conditions and having obtained candidate values for the b_i we can then determine the a_i and solve for c_i which must also be integers (and must also satisfy the bounds as described in Method 1). All this produces a small number of polynomials for μ and hence for γ.

The outcome of either Method 1 or Method 2 (and the authors independently used these two methods to cross-check their calculations) is a set of polynomials which satisfy the first four conditions of Theorem 3.1 and the bounds of Theorem 3.2. Further tests for the related groups to be free products $\langle f\rangle * \langle g\rangle$ using isometric circles are carried out as in [7] before the final list of candidate polynomials is handled by JSnap as described in §2.

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