Pure and Applied Mathematics Quarterly
Volume 7, Number 2
(*Special Issue: In honor of
Frederick W. Gehring, Part 2 of 2*)
319—343, 2011

An Invitation to n-Harmonic Hyperelasticity

Tadeusz Iwaniec and Jani Onninen*

Salutations to Fred W. Gehring

Abstract: We give an account of some recent developments in which quasiconformal theory and nonlinear elasticity share common problems of compelling mathematical interest. Geometric function theory is currently a field of enormous activity where the language and general framework of nonlinear elasticity is extremely fruitful and significant. As this interplay developed n-harmonic deformations became valid and well acknowledged as generalization of conformal mappings in $\mathbb{R}^n$. We have also found a place for n-harmonic deformations in the theory of hyperelasticity. J. Ball's fundamental paper [5] accounts for the principles of hyperelasticity and sets up mathematical problems. In presenting the recent advances we have relied on a few new existing articles [2, 3, 28, 29, 30, 31, 32, 33].
Keywords: n-Harmonics, Extremal problems, Quasiconformal mappings, Variational integrals, Free Lagrangians.

1. Introduction and Notation

Throughout this text $\mathbb{X}$ and $\mathbb{Y}$ will be nonempty bounded domains in $\mathbb{R}^n$, $n \geqslant 2$. We will be considering mappings $h : \mathbb{X} \to \mathbb{Y}$ and $f : \mathbb{Y} \to \mathbb{X}$ in various Sobolev

Received November 2, 2007.
AMS (2000) Classification. Primary 30C60, 35J15, 35J70.
*Iwaniec was supported by the NSF grants DMS-0301582 and DMS-0800416, and by the Academy of Finland grant 1128331. Onninen was supported by the NSF grant DMS-0701059 and by the Academy of Finland.

spaces. We always use the symbols $\mathscr{W}^{1,p}(\mathbb{X},\mathbb{Y})$ and $\mathscr{W}^{1,p}(\mathbb{Y},\mathbb{X})$, respectively, to denote such spaces. The basic concepts are:

1) The differential matrix, also referred to as deformation gradient;

$$Dh(x) = \left[\frac{\partial h^i}{\partial x_j}\right] \in \mathbb{R}^{n\times n}\,, \qquad h = (h^1, ..., h^n) \tag{1}$$

where $\mathbb{R}^{n\times n}$ is the space of $n\times n$-matrices. We reserve the notation $\mathbb{R}^{n\times n}_+$ for the space of matrices with positive determinant.

2) The Jacobian determinat

$$J(x,h) = \det Dh\,, \qquad J(x,h)\,\mathrm{d}x = \mathrm{d}h^1\wedge ...\wedge \mathrm{d}h^n \tag{2}$$

More generally, to all pairs of ordered ℓ-tuples $I=(i_1,...,i_\ell)$ and $J=(j_1,...,j_\ell)$, where $1\leqslant i_1<...<i_\ell\leqslant n$ and $1\leqslant j_1<...<j_\ell\leqslant n$, there correspond

3) the $\ell\times\ell$-subdeterminants and their matrix of size $\binom{n}{\ell}\times\binom{n}{\ell}$,

$$\frac{\partial h^I}{\partial x_J} = \frac{\partial\left(h^{i_1},...,h^{i_\ell}\right)}{\partial\left(x_{j_1},...,x_{j_\ell}\right)}\,, \qquad D^{\ell\times\ell}h = \left[\frac{\partial h^I}{\partial x_J}\right] \in \mathbb{R}^{\binom{n}{\ell}\times\binom{n}{\ell}} \tag{3}$$

For $\ell = n-1$, we obtain

4) Cramer's cofactor matrix

$$D^\sharp h = \left[(-1)^{i+j}\frac{\partial\left(h^1,...,h^{i-1},h^{i+1},...,h^n\right)}{\partial\left(x_1,...,x_{j-1},x_{j+1},...,x_n\right)}\right] \in \mathbb{R}^{n\times n} \tag{4}$$

By convention, $\frac{\partial h^I}{\partial x_J} = 1$ if $\ell = 0$. In this way the total number of all subdeterminants is:

$$\sum_{\ell=0}^{n}\binom{n}{\ell}^2 = \binom{2n}{n} \tag{5}$$

The $\ell\times\ell$-minors represent the infinitesimal deformations of ℓ-dimensional objects.

We call an orientation preserving homeomorphism $h:\mathbb{X}\xrightarrow{\text{onto}}\mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X},\mathbb{Y})$ a *deformation* of $\mathbb{X}$ onto $\mathbb{Y}$. Its domain $\mathbb{X}$ will be called the *reference configuration*, while its image the *deformed configuration*. We shall use these terms on frequent occasions in this text.

The general law of hyperelasticity tells us that there exists an energy integral

$$\mathcal{E}[h] = \int_{\mathbb{X}} E(x,h,Dh)\,\mathrm{d}x \tag{6}$$

where $E : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is a given *stored-energy* function characterizing mechanical properties of the material. The subject of study in hyperelasticity are deformations which minimize the energy, also called the stationary solutions. Except for a few problems in Section 3.2.2, we shall impose no particular boundary values on the mappings $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. Those with smallest energy will be called *absolute minimizers.* The question arises whether or not the absolute minimizer exists and is unique up to the energy preserving transformations. The main idea is the direct method of the calculus of variations which we shall apply to polyconvex energy functionals. The polyconvexity hypothesis, due to J. Ball [5], directly gives the lower seimicontinuity of the energy. Consequently, the existence of the minimizers subject to the given boundary data can be established without difficulties. A brief description of this method is as follows.

A stored-energy function is said to be *polyconvex* if it can be expressed (not necessarily uniquely) as a convex function of the minors of the deformation gradient; that is,

$$E(x, h, Dh) = \mathcal{W}\left(x, h, \frac{\partial h^I}{\partial x_J}\right) \tag{7}$$

where for each $(x, y) \in \mathbb{X}\times\mathbb{Y}$ the function $\mathcal{W}(x, y, \cdot) : \Omega \to \mathbb{R}$ is convex in a domain $\Omega \subset \mathbb{R}^{\binom{2n}{n}}$. The point to make here is that the nonlinear differential operators $\frac{\partial^I}{\partial x_J} : \mathscr{W}^{1,\ell}(\mathbb{X}, \mathbb{R}^n) \to \mathscr{L}^1(\mathbb{X}) \subset \mathscr{D}'(\mathbb{X})$ are continuous in the weak topology of $\mathscr{W}^{1,\ell}(\mathbb{X}, \mathbb{R}^n)$. Precisely, for every test function $\eta \in \mathscr{C}^\infty_\circ(\mathbb{X})$, we have

$$\int_{\mathbb{X}} \eta(x)\, \frac{\partial h_i^I}{\partial x_J}\, \mathrm{d}x \;\longrightarrow\; \int_{\mathbb{X}} \eta(x)\, \frac{\partial h^I}{\partial x_J}\, \mathrm{d}x \tag{8}$$

whenever h_i converges to h weakly in $\mathscr{W}^{1,\ell}(\mathbb{X}, \mathbb{R}^n)$. This is the key to the lower semicontinuity property of the energy; that is,

$$\mathcal{E}[h] \leqslant \liminf_{i\to\infty} \mathcal{E}[h_i] \tag{9}$$

Here the functions $h_i \in h_\circ + \mathscr{W}^{1,n}_\circ(\mathbb{X}, \mathbb{R}^n)$ have prescribed boundary data $h_\circ \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ and converge to h weakly in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$. For further details concerning this method see [12, 49]. The $\ell \times \ell$-minors of the differential and their linear combinations are called *null Lagrangians.* If the stored-energy function is a null Lagrangian, its integral depends only on the boundary values of $h : \mathbb{X} \to \mathbb{R}^n$.

In fact, we have

$$\int_{\mathbb{X}} \frac{\partial h^I}{\partial x_J} \, \mathrm{d}x = \int_{\mathbb{X}} \frac{\partial g^I}{\partial x_J} \, \mathrm{d}x \, , \qquad \text{whenever} \quad \begin{cases} g, h \in \mathscr{W}^{1,\ell}(\mathbb{X}, \mathbb{R}^n) \\ g - h \in \mathscr{W}_{\circ}^{1,\ell}(\mathbb{X}, \mathbb{R}^n) \end{cases} \tag{10}$$

Consequently, the Euler-Lagrange variational equation is satisfied by all mappings in $\mathscr{W}^{1,\ell}(\mathbb{X}, \mathbb{Y})$; for further reading see [6], [7], [38], [16], [15], [14] and [22]. However, we shall be largely concerned with *absolute minimizers*; that is, homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite energy without any restriction on the boundary. In most of this text all homeomorphisms will have nonnegative Jacobian determinant.

At this point we should bring on stage a special class of null Lagrangians. These are the ones whose integral depends only on the homotopy class of a deformation $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, regardless of its boundary values. Such special stored-energy functions will be called *free Lagrangians*. The first, and perhaps most important example, is the volume form

$$E(h, Dh) = G\big(\, h(x) \,\big) \, J(x, h) \, , \qquad \text{where} \;\; G \in \mathscr{L}^1(\mathbb{Y}) \tag{11}$$

subject to all orientation preserving homeomorphisms $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Indeed, by change of variables we obtain

$$\int_{\mathbb{X}} E(h, Dh) \, \mathrm{d}x = \int_{\mathbb{Y}} G(y) \, \mathrm{d}y \tag{12}$$

where the right hand side does not depend on h. Finding free Lagrangians for a specific pair of domains $\mathbb{X}$ and $\mathbb{Y}$ and applying them to a given energy integral is a work of art. We shall demonstrate these ideas in case of spherical ring domains in Section 6.

A further restriction, natural from the point of view of the hyperelasticity theory, is that of isotropic stored-energy functions. Singular values $\sigma_1, ..., \sigma_n$ of an $n \times n$ matrix $M \in \mathbb{R}^{n \times n}$ are the eigenvalues of the positive semidefinite symmetric matrix $\sqrt{M \, M^T}$. For $M = Dh$ these singular values are the *principal stretches* of the deformation $h : \mathbb{X} \to \mathbb{Y}$. A hyperelastic material is called isotropic if its stored-energy is a function of the principal stretches,

$$E(x, h, Dh) = \mathcal{W}(x, h, \sigma_1, ..., \sigma_n) \tag{13}$$

where $\mathcal{W} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^n_+ \to \mathbb{R}$ is symmetric with respect to the variables $\sigma_1, ..., \sigma_n$. Thus $E(x, h, Dh)$ is invariant under rotation of Dh. We shall encounter isotropic integrals in Section 4.

Let us now connect this discussion with mappings of finite distortion, see [23], [34], [35], [36] and [27]. A major role is played by the following definition.

Definition 1.1. An orientation preserving homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ is said to have finite distortion if $J(x, h) > 0$, except for the points where $Dh(x) = 0$.

As might be expected there cannot be too many exceptional points. For example such points cannot cover any open set almost everywhere. In what follows we will be working with various distortion functions of a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ and its inverse $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$. It will be understood, without saying it so all the time, that the mappings in question have finite distortion. The following function

$$K_O(x, h) = \begin{cases} \frac{|Dh(x)|^n}{n^{\frac{n}{2}}\, J(x,h)} & \text{if } J(x, h) \in \mathbb{R}^{n\times n}_+ \\ 1 & \text{if } J(x, h) = 0 \end{cases} \tag{14}$$

is called *outer distortion.*

Hereafter, we have used the Hilbert-Schmidt norm of a matrix $M \in \mathbb{R}^{n\times n}$, $|M| = \big(\operatorname{Tr} M\, M^T\big)^{\frac{1}{2}}$. The *inner distortion* of h is defined by the rule

$$K_I(x, h) = \begin{cases} \frac{|D^\sharp h(x)|^n}{n^{\frac{n}{2}}\, J(x,h)^{n-1}} & \text{if } J(x, h) \in \mathbb{R}^{n\times n}_+ \\ 1 & \text{if } J(x, h) = 0 \end{cases} \tag{15}$$

These two distortions are borderline cases of

$$K_\ell(x, h) = \begin{cases} \frac{\left|D^{\ell\times\ell} h(x)\right|^n}{\binom{n}{\ell}^{\frac{n}{2}} J(x,h)^\ell} & \text{if } J(x, h) \in \mathbb{R}^{n\times n}_+ \\ 1 & \text{if } J(x, h) = 0 \end{cases} \tag{16}$$

Here $\ell = 1, 2, ..., n - 1$. It is to be noted that all the above distortion functions are polyconvex. In fact, they are also isotropic.

2. n-Harmonic Integrals

We have now reached a stage where we may place all the above considerations in relation to the *n-harmonic integral*, also called *conformal energy*

$$\mathcal{E}_n[h] = \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x \tag{17}$$

There are several natural ways that the n-harmonic type integrals arise in Geometric Function Theory. If $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is a deformation of Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$, then

$$\mathcal{E}_n[h] = \int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x \geqslant n^{\frac{n}{2}} \int_{\mathbb{X}} J(x,h) \, \mathrm{d}x = n^{\frac{n}{2}} |\mathbb{Y}| \,, \tag{18}$$

by Hadamard's inequality. We have equality here if and only if h is conformal. We just see that conformal mappings are the absolute minimizers of the n-harmonic energy. More generally, every K-quasiconformal mapping $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is an absolute minimizer of its own energy integral defined by

$$\mathcal{E}_G[h] = \int_{\mathbb{X}} \langle G^{-1}(x) \, Dh \,, \, Dh \rangle^{\frac{n}{2}} \, \mathrm{d}x = n^{\frac{n}{2}} |\mathbb{Y}| \tag{19}$$

where $G = G(x)$, referred to as the *distortion tensor* of h, is a symmetric uniformly positive definite matrix of determinant 1. The map h solves the first order system of PDEs,

$$D^T h \, Dh = J(x,h)^{\frac{2}{n}} G(x) \qquad \text{the } \textit{Beltrami equation in } \mathbb{R}^n \tag{20}$$

For further reading on these topics see [47], [18], [8], [48] and [27]. An especially important and profound connection of the n-harmonic energy with mappings of finite distortion arises by considering to the inverse map $f = h^{-1} : \mathbb{Y} \to \mathbb{X}$.

THEOREM 2.1. *Suppose a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite distortion belongs to $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Then its inverse $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ has finite integrable distortion and we have*

$$\int_{\mathbb{X}} |Dh(x)|^n \, \mathrm{d}x = n^{\frac{n}{2}} \int_{\mathbb{Y}} K_I(y,f) \, \mathrm{d}y \tag{21}$$

A systematic treatment of this formula and related extremal problems have been undertaken in recent papers by the authors and their collaborators [3], [20], [19], [45], [21]. At this point in the discussion we shall also state without proofs the minimal regularity assumptions on f.

THEOREM 2.2. *Suppose a homeomorphism $f \in \mathscr{W}^{1,p}_{loc}(\mathbb{Y},\mathbb{X})$ has integrable inner distortion, where $p=1$ in the planar case and $p>n-1$ in higher dimensions. Then h has finite distortion and the identity (21) holds.* †

The natural question arises whether or not two topologically equivalent domains are n-harmonically equivalent. This is not always the case.

Example 2.1. Let $\mathbb{X} = \mathbb{U} \setminus \mathbb{E}$ and $\mathbb{Y} = \mathbb{V} \setminus \mathbb{F}$, where $\mathbb{U}$ and $\mathbb{V}$ are domains in $\mathbb{R}^n$, $n \geqslant 3$, while $\mathbb{E} \subset \mathbb{U}$ and $\mathbb{F} \subset \mathbb{V}$ are closed disks of dimension k and ℓ, respectively, which we refer to as cracks in $\mathbb{U}$ and $\mathbb{V}$. If $1 \leqslant k < \ell \leqslant n-1$, then there exists no homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Thus, the n-harmonic energy

$$\mathcal{E}[h] \;=\; \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x \tag{22}$$

is always infinite, even though there may exist a $\mathscr{C}^\infty$-smooth diffeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$.

Roughly speaking, an infinite conformal energy is required in order to increase the dimension of a crack $\mathbb{E} \subset \mathbb{U}$ if $\dim \mathbb{E} < n-1$. It is possible, however, to make a hole in $\mathbb{V}$ out of the crack $\mathbb{E} \subset \mathbb{U}$ of dimension $n-1$. It seems that flatness of the cracks in this example is redundant.

Question 2.1. Given a domain $\mathbb{U} \subset \mathbb{R}^n$ with a crack (a continuum $\mathbb{E} \subset \mathbb{U}$) of topological dimension k, does there exist a deformation $h : \mathbb{U} \setminus \mathbb{E} \xrightarrow{\text{onto}} \mathbb{V} \setminus \mathbb{F}$ of finite conformal energy such that $\mathbb{F}$ is a crack of topological dimension ℓ in a domain $\mathbb{V} \subset \mathbb{R}^n$, where $0 \leqslant k < \ell \leqslant n-1$.

A negative answer to this question would rule out quasiconformal equivalence of the domains $\mathbb{U}\setminus\mathbb{E}$ and $\mathbb{V}\setminus\mathbb{F}$. Considering Example 2.1 in relation with Theorem 2.1 we find that there is no homeomorphism $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ in $\mathscr{W}^{1,p}(\mathbb{Y}, \mathbb{X})$ having integrable distortion. Various nonexistence results concerning quasiconformal deformations are well known. For example, no quasiconformal mapping of a ball can produce a spike which points into the ball [17]. It would be interesting to describe all pairs of the domains $\mathbb{X}$ and $\mathbb{Y}$ for which there exist deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite conformal energy, or deformations $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ with integrable inner distortion.

†Even more, after the submission of this article, Theorem 2.2 has been proved as well [10].

2.1. **The Nitsche Phenomenon.** Computing the infimum of energy integrals by the direct method of variation we must pass to the limit of a minimizing sequence $h_j : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. In this procedure the injectivity of the weak limit $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is most likely to be lost. Nevertheless, with a little geometric assumption on $\mathbb{Y}$, h has right inverse $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$, see [31]. Precisely, we require that $\partial\mathbb{Y}$ consists of at least two finitely many boundary components. Moreover, f has bounded variation, meaning that Df exists as a Borel measure. Although h need not be injective, it is still a legitimate deformation, for no cracks are produced by h, see [42] for further questions. We shall illustrate the lack of injectivity in case of mappings $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ between annuli

$$\mathbb{X} = \mathbb{A} = \mathbb{A}\,(r,R) = \big\{x \in \mathbb{R}^n\,; \quad r < |x| < R\big\} \tag{23}$$

$$\mathbb{Y} = \mathbb{A}^* = \mathbb{A}^*\,(r,R) = \big\{y \in \mathbb{R}^n\,; \quad r_* < |y| < R_*\big\} \tag{24}$$

We shall only consider orientation preserving homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ which take the inner boundary of $\mathbb{A}$ onto the inner boundary of $\mathbb{A}^*$ in the sense of cluster sets. Let us commence with the planar annuli for which the energy $\mathcal{E}[h] = \int_{\mathbb{A}} |Dh(x)|^2\,\mathrm{d}x$ assumes its minimum value on injective maps.

THEOREM 2.3. (The Nitsche bound) *Suppose the reference annulus $\mathbb{A}$ is conformally not too fat relative to the deformed configuration $\mathbb{A}^*$. Precisely, we assume the following Nitsche bound*

$$\frac{R}{r} \leqslant \frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1}\,, \qquad \textit{or equivalently} \quad \frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right) \tag{25}$$

Then the complex harmonic function

$$h(z) = \lambda\left(z + \frac{\omega}{\overline{z}}\right)\,, \qquad \begin{cases} \omega = \frac{rR(Rr_* - rR_*)}{RR_* - rr_*} \\ \lambda = \frac{RR_* - rr_*}{R^2 - r^2} \end{cases} \tag{26}$$

is a diffeomorphism of $\mathbb{A}$ onto $\mathbb{A}^*$. This map is a unique minimizer of the conformal energy up to the rotation of $\mathbb{A}$ and $\mathbb{A}^*$.

The proof is based on a point-wise estimate of $|Dh|^2$ in terms of a free Lagrangian,

$$|Dh(x)|^2 \geqslant L(x, h, Dh) \tag{27}$$

Integrating over $\mathbb{A}$, we see that for every deformations $h, h_\circ : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of finite energy

$$\mathcal{E}[h] = \int_{\mathbb{A}} |Dh|^2 \geqslant \int_{\mathbb{A}} L(x,h,Dh)\,\mathrm{d}x = \int_{\mathbb{A}} L(x,h,Dh_\circ)\,\mathrm{d}x \tag{28}$$

Now the task is to find a free Lagrangian $E : \mathbb{A}\times\mathbb{A}^*\times\mathbb{R}^{2\times 2} \to \mathbb{R}$ and a deformation $h_\circ : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ such that

$$\int_{\mathbb{A}} L(x,h,Dh_\circ)\,\mathrm{d}x = \int_{\mathbb{A}} |Dh_\circ|^2 \tag{29}$$

Indeed, such a null Lagrangian has been found in [32], see also [2], and [28] for related results. It turns out that the Nitsche map in (26) gives the desired identity (29). Although the above general idea is natural and simple, the execution of this plan is far from being easy. The difficulty lies chiefly in controlling the chain of point-wise inequalities that we encounter on the way to (27). The Nitsche map must give us equality at every link of this chain. Luckily, it was possible. Although, it is far from obvious, one can show that homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with smallest conformal energy are harmonic maps; the components are harmonic functions. To this effect one must perform a variation of the conformal energy within homeomorphisms that leads to the Laplace equation. Such a variation is furnished by the Radó-Kneser-Choquet theorem [46], [37], [9] and [13]. It asserts that a harmonic extension of a homeomorphism $h : \partial\mathbb{U} \xrightarrow{\text{onto}} \partial\mathbb{D}$ ($\mathbb{D}$ is a bounded convex domain in $\mathbb{R}^2$) is a homeomorphism of $\mathbb{U}$ onto $\mathbb{D}$. Furthermore, by the theorem of Lewy [39] the Jacobian of an injective harmonic map $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ does not vanish. Thus h is a $\mathscr{C}^\infty$-diffeomorphism. Analogous questions concerning n-harmonic extensions of a boundary map, $n \geqslant 3$, remain open. However, a complete analogy of the Choquet-Kneser-Lewy-Radó theory for n-harmonic maps, at least in dimensions $n \geqslant 7$ looks very doubtful, see the famous example of Milnor [40]. We omit further details.

Let us take a closer look at the critical case in the Nitsche bound; that is,

$$\frac{R_*}{r_*} = \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right), \quad \text{or equivalently} \quad \frac{R}{r} = \frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1} \tag{30}$$

The Nitsche map takes the form

$$h(z) = \frac{r_*}{2\,r}\left(z + \frac{r^2}{\overline{z}}\right) \tag{31}$$

Note that the Jacobian

$$J(z,h) = \frac{r_*^2}{4\,r^2}\left(1 - \frac{r^4}{|z|^4}\right) \tag{32}$$

vanishes on the inner boundary circle $|z| = r$. Crossing this circle we loose injectivity of h and there is a reason for this:

Conjecture 2.1. (Nitsche, [43], 1962) There is no harmonic deformation (homeomorphism) $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ if $\frac{R}{r} > \frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1}$. ‡

The Nitsche bound (25) has originated from a study of minimal surfaces [44]. It tells us that the conformal modulus of a minimal graph over a given annulus $\mathbb{A}^*$ does not exceed $\log\left(\frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1}\right)$. The greatest modulus is attained on a half slab of a catenoid. The Nitsche bound should not be regarded as answering the general question of existence of the extremals $h : \mathbb{A} \to \mathbb{A}^*$. It says, however, that when $\mathbb{A}$ is too fat relative to $\mathbb{A}^*$, then one must accept noninjective weak limits of homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ as legitimate deformations. One of many possible candidates for a legitimate deformation is facilitated by squeezing a portion of $\mathbb{A}$ (hammering flat) onto the inner boundary of $\mathbb{A}^*$. We shall do it in a way so that the rest of $\mathbb{A}$ remains in the Nitsche critical configuration with $\mathbb{A}^*$.

$$h(z) = \begin{cases} r_* \frac{z}{|z|}\,, & r < |z| < \rho \qquad \text{hammering map} \\ \frac{r_*}{2\rho}\left(z + \frac{\rho^2}{\bar z}\right)\,, & \rho < |z| < R \qquad \text{critical Nitsche map} \end{cases} \tag{33}$$

where the radius $\rho \in (r, R)$ is determined by the equation

$$\frac{R_*}{r_*} = \frac{1}{2}\left(\frac{R}{\rho} + \frac{\rho}{R}\right) \tag{34}$$

It is true, though somewhat unexpected, that this deformation of $\mathbb{A}$ onto $\mathbb{A}^*$ indeed requires smallest conformal energy among all legitimate deformations. Moreover, such extremal map is unique up to the rotation of the annuli. Let us emphasize that not all harmonic homeomorphisms minimize the energy; they are only equilibrium solutions. In fact there exist many univalent harmonic maps from $\mathbb{A}$ onto $\mathbb{A}^*$ when $\frac{R_*}{r_*} > \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$. In view of [25] we now know that only one (up to a rotation) in the borderline case, and none otherwise.

‡The fascinating and engaging Nitsche conjecture remained open for almost a half of a century. Most recently L. V. Kovalev and the authors of this survey provided, among further generalizations, an affirmative answer to Nitsche's conjecture [24, 25, 26].

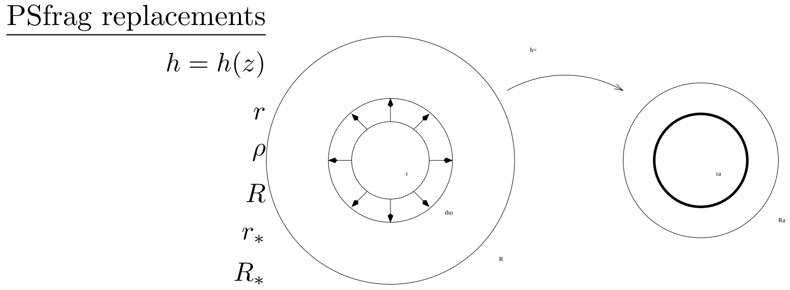

FIGURE 1. Hammering a portion of an annulus flat.

An exactly similar Nitsche phenomenon pertains to higher dimensions [28]. The counterparts of the Nitsche map are n-harmonics; that is, radial mappings $h(x) = H\big(|x|\big)\frac{x}{|x|}$ that satisfy the n-harmonic system

$$\text{(35)} \qquad \operatorname{div} |Dh|^{n-2} Dh = 0 \qquad \text{in the punctured space } \mathbb{R}^n_\circ = \mathbb{R}^n \setminus \{0\}$$

This second order Lagrange equation for h can be nicely reduced to the first order ODE for H with one free parameter $c \in \mathbb{R}$, namely

$$\text{(36)} \qquad \Big[(n-1)H^2 + t^2\dot{H}^2\Big]^{\frac{n-2}{2}} \Big(H^2 - t^2\dot{H}^2\Big) \equiv c$$

In n-dimensions, the Nitsche condition takes the form

$$\text{(37)} \qquad N\left(\frac{R}{r}\right) \leqslant \frac{R_*}{r_*}$$

A precise formula for the function $N : [1,\infty) \to [1,\infty)$ has been given in [28]. For example, $N(t) = \frac{1}{2}\left(t + \frac{1}{t}\right)$ in dimension 2. In all dimensions, we always have $N\left(\frac{R}{r}\right) < \frac{R}{r}$. Under the Nitsche condition (37) there is one and only one radial n-harmonic homeomorphism which takes $\mathbb{A}$ onto $\mathbb{A}^*$.

With these remarks in mind, Theorem 2.3 extends to dimension 3 as follows.

THEOREM 2.4. *Suppose that the annuli $\mathbb{A}$ and $\mathbb{A}^*$ satisfy the Nitsche condition (37), $n = 3$. Then the smallest conformal energy among all homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is attained on the radial n-harmonic map. This extremal map is unique up to the rotation of the annuli.*

Now, consider the case below the Nitsche bound, $\frac{R_*}{r_*} < N\left(\frac{R}{r}\right)$ and $n \geqslant 3$. As in the planar case, among all the legitimate deformations (weak limits of homeomorphisms) the n-harmonic energy assumes its smallest value on

$$h(z) = \begin{cases} r_* \frac{z}{|z|}\,, & r < |z| < \rho \qquad \text{hammering map} \\ H\big(|x|\big)\,\frac{x}{|x|}\,, & \rho < |z| < R \qquad \text{critical Nitsche map} \end{cases} \tag{38}$$

where $\rho \in (r, R)$ is determined by the equation $\frac{R_*}{r_*} = N\left(\frac{R}{\rho}\right)$.

Because of rotational symmetry of the annuli and conformal invariance of the n-harmonic integral one might naturally expect that the extremal deformations are radial mappings. Proving this in dimension 3 requires a rather sophisticated use of free Lagrangians. However, in spite of our geometric intuition the case $n \geqslant 4$ is dramatically different. The extremal mappings remain radially symmetric only when the target annulus is not too fat. In [28] we gave an example of the annuli for which no radial homeomorphism $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ minimizes the n-harmonic energy. In addition to (37) an upper bound on $\frac{R_*}{r_*}$ came to the rescue,

$$N\left(\frac{R}{r}\right) \leqslant \frac{R_*}{r_*} < M\left(\frac{R}{r}\right) \tag{39}$$

where the function $M : [1,\infty) \to [1,\infty)$ (call it upper Nitsche bound) has been identified in [28]. Note that $N\left(\frac{R}{r}\right) < \frac{R}{r} < M\left(\frac{R}{r}\right)$.

THEOREM 2.5. *Let the annuli $\mathbb{A}, \mathbb{A}^* \subset \mathbb{R}^n$, $n \geqslant 4$, satisfy the lower and upper bounds in (39). Then among all homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ the smallest conformal energy is attained on radial n-harmonics. Such extremal maps are unique up to rotation of the annuli.*

Theorems 2.3 and 2.4 show in particular that $M(t) \equiv \infty$ in dimensions $n = 2, 3$. This means that no upper bound for the ratio $\frac{R_*}{R}$ is required in dimensions $n = 2, 3$.

2.2. **Elasticity Function.** There is an interesting common feature of all radial extremals for spherical annuli in $\mathbb{R}^n$ in the isotropic case. It concerns the elasticity function of $h = H\big(|x|\big)\frac{x}{|x|}$, defined by the rule

$$\eta(t) = \frac{t\dot{H}(t)}{H(t)} \qquad \text{for} \quad r < t < R \tag{40}$$

It follows directly from the equation (36) that there are only three possibilities for the n-harmonics, depending whether $c > 0$, $c = 0$, or $c < 0$.

$$\eta(t) = \frac{t\dot{H}}{H} < 1\,, \qquad \text{for all } r < t < R \quad \textit{(inelastic case)} \tag{41}$$

$$\eta(t) = \frac{t\,\dot{H}}{H} = 1\,, \qquad \text{for all } r < t < R \quad \textit{(conformal case)} \tag{42}$$

$$\eta(t) = \frac{t\,\dot{H}}{H} > 1\,, \qquad \text{for all } r < t < R \quad \textit{(elastic case)} \tag{43}$$

As a matter of fact each of these cases pertains to the annuli $\mathbb{X}$ and $\mathbb{Y}$; it can be recognized without knowing the extremal deformation $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. To this effect we observe that

$$\frac{\int_r^R \eta(t)\,\frac{\mathrm{d}t}{t}}{\int_r^R \frac{\mathrm{d}t}{t}} = \frac{\log \frac{R_*}{r_*}}{\log \frac{R}{r}} = \frac{\operatorname{Mod}\mathbb{Y}}{\operatorname{Mod}\mathbb{X}} \tag{44}$$

Hence

$$\eta(t) < 1 \qquad \text{iff} \qquad \operatorname{Mod}\mathbb{Y} < \operatorname{Mod}\mathbb{X} \tag{45}$$

$$\eta(t) = 1 \qquad \text{iff} \qquad \operatorname{Mod}\mathbb{Y} = \operatorname{Mod}\mathbb{X} \tag{46}$$

$$\eta(t) > 1 \qquad \text{iff} \qquad \operatorname{Mod}\mathbb{Y} > \operatorname{Mod}\mathbb{X} \tag{47}$$

The above three cases occur here not by chance see Section 4 for a glimpse of more general isotropic energy integrals.

3. The Total Energy

In general, the inverse of a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,\infty}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ need not belong to $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{Y}, \mathbb{X})$. From now on we consider only orientation preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$ between bounded domains in $\mathbb{R}^n$ whose inverse $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ also lies in $\mathscr{W}^{1,n}(\mathbb{Y}, \mathbb{X})$. Thus we may speak of n-harmonic energy integrals for both h and f,

$$\mathcal{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x \qquad \text{and} \qquad \mathcal{E}_{\mathbb{Y}}[f] = \int_{\mathbb{Y}} |Df(y)|^n \,\mathrm{d}y \tag{48}$$

Let us combine these two integrals into so-called total n-harmonic energy

$$\mathcal{E}[h, f] = \frac{\alpha}{|\mathbb{Y}|} \int_{\mathbb{X}} |Dh(x)|^n \,\mathrm{d}x + \frac{\beta}{|\mathbb{X}|} \int_{\mathbb{Y}} |Df(y)|^n \,\mathrm{d}y\,, \qquad \alpha + \beta = 1 \tag{49}$$

where α and β are given positive numbers. In view of Theorems 2.1 and 2.2 this translates into the average of the *mean total distortion*; that is,

$$n^{\frac{n}{2}}\,\mathcal{E}[f,h] = \alpha \fint_{\mathbb{Y}} K_{I}(y,f)\,\mathrm{d}y + \beta \fint_{\mathbb{X}} K_{I}(x,h)\,\mathrm{d}x \geqslant 1 \tag{50}$$

It measures, in the average sense, the deviation of f and h from conformal mappings. Indeed, the mean total distortion equals 1 if and only if h and f are conformal. Let us emphasize in this connection that classical Teichmüller theory deals with mappings of smallest $\mathscr{L}^\infty$-norms of the distortion of h and f. In case of planar simply connected domains, we infer via the *Riemann Mapping Theorem* [27] that the minimizers of the mean total energy are conformal mappings, in which case $K_I(x,h) \equiv 1 = K_I(y,f)$. However, in higher dimensions or even for planar multiply connected domains, conformal mappings between $\mathbb{X}$ and $\mathbb{Y}$ may not exist. This divergence from the classical conformal theory is best illustrated in case of annuli, see Section 6.

3.1. **Polyconvex and Conformally Coerced Integrals.** More general mathematical models of hyperelastic deformations [1, 5, 11] are concerned with mappings $h : \mathbb{X} \to \mathbb{Y}$ which minimize a given stored-energy of the form

$$\mathcal{E}[h] \;=\; \int_{\mathbb{X}} \mathcal{H}(x,h,Dh)\,\mathrm{d}x\,, \qquad \text{where } \; Dh \in \mathbb{R}^{n\times n}_{+} \tag{51}$$

This energy integral can usually be turned around so as to yield, at least formally, the energy of the inverse map. By change of variables we obtain

$$\mathcal{E}[f] = \int_{\mathbb{Y}} \mathcal{F}(y,f,Df)\,\mathrm{d}y \tag{52}$$

where

$$\mathcal{F}(y,x,F) = \mathcal{H}(x,y,F^{-1})\det F \tag{53}$$

for matrices $F \in \mathbb{R}^{n\times n}_{+}$. Here are such examples. In dimension $n = 2$, the $\mathscr{L}^1$-norm of Dh is the same as the $\mathscr{L}^1$-norm of Df

$$\mathcal{E}_1[h] = \int_{\mathbb{X}} |Dh(x)|\,\mathrm{d}x = \int_{\mathbb{Y}} |Df(y)|\,\mathrm{d}y = \mathcal{E}_1[f] \tag{54}$$

In general, however, convexity of a stored-energy function is lost upon passing to the inverse map. For instance, when $n > 2$, the $\mathscr{L}^1$-norm of the deformation gradient translates into the $\mathscr{L}^1$-norm of the cofactors for the inverse map

$$\mathcal{E}_1[h] = \int_{\mathbb{X}} |Dh(x)|\,\mathrm{d}x = \int_{\mathbb{Y}} |D^\sharp f(y)|\,\mathrm{d}y = \mathcal{E}_\sharp[f] \tag{55}$$

It is our conviction that a viable theory of elastic deformations should make no distinction between $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ and $f : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$, like in Geometric Function Theory. Therefore, the notion of convexity is not well suited to such theory. But the concept of polyconvexity, invented and studied by J. Ball [5], [4] suits well.

Proposition 3.1. *Polyconvexity of $\mathcal{H} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ for $h : \mathbb{X} \to \mathbb{Y}$ is equivalent to polyconvexity of $\mathcal{F} : \mathbb{Y} \times \mathbb{X} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ for $f : \mathbb{Y} \to \mathbb{X}$.*

It is interesting to note that Proposition 3.1 actually extends to other familiar concepts of convexity in the Calculus of Variations, such as Morrey's notion of quasiconvexity or rank-one convexity, [41]. But we shall not enter into this territory here.

We say that the integrand $\mathcal{H} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ or, equivalently, $\mathcal{F} : \mathbb{Y} \times \mathbb{X} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ is *conformally coerced* if

$$\begin{cases} c_1\, |H|^n \leqslant \mathcal{H}(x,y,H)\,, & \text{for all } H \in \mathbb{R}^{n\times n}_+ \\ c_2\, |F|^n \leqslant \mathcal{F}(y,x,F)\,, & \text{for all } F \in \mathbb{R}^{n\times n}_+ \end{cases} \tag{56}$$

where c_1 and c_2 are positive constants. Hereafter, we assume that $\mathcal{H}$ depends continuously on $(x,y) \in \mathbb{X} \times \mathbb{Y}$, though weaker regularity (Lusin's measurability) suffices, see [32].

3.2. **Existence Theorems.** We are now in a position to formulate the most general existence results.

3.2.1. *Free Boundary Values.*

THEOREM 3.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded domains with at least two but finitely many boundary components. Suppose $\mathcal{H} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ is continuous polyconvex and conformally coerced. Then the energy integral*

$$\mathcal{E}[h] \;=\; \int_{\mathbb{X}} \mathcal{H}(x,\, h,\, Dh)\, \mathrm{d}x\;, \tag{57}$$

subject to all orientation preserving homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,n}(\mathbb{X},\mathbb{Y})$, assumes its minimum value.

Here it is tacitly assumed that the domain of definition of $\mathcal{E}$ is not empty. The reader may wish to recall Example 2.1 to see that it may be empty.

3.2.2. *The Dirichlet Boundary Value Problem.* One might consider the Dirichlet problem for the energy functional

$$\mathcal{E}_{\mathbb{X}}[h] = \int_{\mathbb{X}} \left[\alpha \, |Dh(x)|^2 \; + \; \beta \, \frac{|Dh(x)|^2}{J(x,h)} \right] \mathrm{d}x \qquad \alpha, \beta > 0 \tag{58}$$

for mappings $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{R}^2)$ with nonnegative Jacobian determinant. Suppose we are given the Dirichlet data $h_\circ \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{R}^2)$, $h_\circ : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, which is an orientation preserving homeomorphism of finite energy. We wish to find $h \in h_\circ + \mathscr{W}_\circ^{1,2}(\mathbb{X}, \mathbb{R}^2)$ with $J(x,h) \geqslant 0$ having smallest energy. A priori, h need not be a homeomorphisms. Before making a statement let us recall that the inverse of $h_\circ$, denoted by $f_\circ : \mathbb{Y} \to \mathbb{X}$, has its own finite energy

$$\begin{aligned} \mathcal{E}_{\mathbb{Y}}[f_\circ] &= \int_{\mathbb{Y}} \left[\beta \, |Df_\circ(y)|^2 \; + \; \alpha \, \frac{|Df_\circ(y)|^2}{J(y, f_\circ)} \right] \mathrm{d}y \\ &= \int_{\mathbb{X}} \left[\alpha \, |Dh_\circ(x)|^2 \; + \; \beta \, \frac{|Dh_\circ(x)|^2}{J(x, h_\circ)} \right] \mathrm{d}x = \mathcal{E}_{\mathbb{X}}[h_\circ] \end{aligned} \tag{59}$$

It is not obvious, though true, that every $h \in h_\circ + \mathscr{W}_\circ^{1,2}(\mathbb{X}, \mathbb{R}^2)$, $J(x,h) \geqslant 0$, with $\mathcal{E}_{\mathbb{X}}[h] < \infty$ is a homeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$. [§] Thus in particular the above properties of the Dirichlet data $h_\circ$ and $f_\circ$ also hold for every contender h and its inverse $f : \mathbb{Y} \to \mathbb{X}$. Finally, using lower semicontinuity of polyconvex functionals we conclude

THEOREM 3.2. *The energy $\mathcal{E}_{\mathbb{X}}[h]$ defined for mappings $h \in h_\circ + \mathscr{W}_\circ^{1,2}(\mathbb{X}, \mathbb{R}^2)$ with $J(x,h) \geqslant 0$ attains its minimum. Every minimizer is a homeomorphism of $\mathbb{X}$ onto $\mathbb{Y}$.*

4. Isotropic Integrals

We consider now isotropic materials whose stored-energy function takes the form

$$\mathcal{H}(Dh) = W(\rho_1, \rho_2, ..., \rho_n) \tag{60}$$

[§] As $h \in \mathscr{W}^{1,2}(\mathbb{X}, \mathbb{R}^2)$ has integrable distortion it must be open and discrete [34]. Then the degree theory tells us that h is indeed a homeomorphism, see [32] and [6].

where $W : \mathbb{R}_+ \times ... \times \mathbb{R}_+ \to \mathbb{R}$ is a symmetric function in the principal stretchings $\rho_1, ..., \rho_n$. Thus $\mathcal{H}$ depends solely on the norms of the cofactor matrices $D^{\ell\times\ell}h$.

$$\mathcal{H}(Dh) = \Gamma\left(..., \left|D^{\ell\times\ell}h\right|, ...\right), \quad \text{where } \Gamma : \mathbb{R}_+ \times ... \times \mathbb{R}_+ \to [0, \infty)$$

Note that $D^{n\times n}h = J(x, h)$. Suppose now that $\Gamma \in \mathscr{C}^\infty(\mathbb{R}_+ \times \cdots \times \mathbb{R}_+)$ and denote by $\Gamma_1, \Gamma_2, ..., \Gamma_n$ and $\Gamma_{k\,\ell}$, $k, \ell = 1, 2, ..., n$, the first and second order partial derivatives of Γ. To ensure polyconvexity of $\mathcal{H}$ we assume further that:

i) The Hessian matrix $[\Gamma_{k\,\ell}]$ is nonnegative definite.
ii) The first $n - 1$ terms $\Gamma_1, ..., \Gamma_{n-1}$ are strictly positive.

Note that we do not assume here as it is in many interesting cases, that the dependence of Γ on the Jacobian determinant is monotone.

There is an interesting common feature of all such isotropic integrals, at least when $\mathbb{X}$ and $\mathbb{Y}$ are annuli. It turns out that the elasticity function η of the extremal radial map $h = H\big(|x|\big)\frac{x}{|x|}$ always satisfies the same conditions as n-harmonics do; that is, (41), (42) or (43). Moreover H is either convex, linear or concave, see Figure 2.

5. Free Lagrangians

For $\mathbb{X}$ and $\mathbb{Y}$ bounded domains in $\mathbb{R}^n$ we consider the class $\mathcal{F}^p(\mathbb{X}, \mathbb{Y})$ of all homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leqslant p < \infty$. This class splits into two subclasses $\mathcal{F}^p_+(\mathbb{X}, \mathbb{Y})$ and $\mathcal{F}^p_-(\mathbb{X}, \mathbb{Y})$, according to whether h preserves of reverses orientation.

Definition 5.1. A *free Lagrangian* in the space $\mathcal{F}^p(\mathbb{X}, \mathbb{Y})$ is a continuous function $L : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ such that the energy integral

$$\mathcal{E}[h] = \int_{\mathbb{X}} L(x, h, Dh)\,\mathrm{d}x\,, \qquad \text{for } h \in \mathcal{F}^p(\mathbb{X}, \mathbb{Y}) \tag{61}$$

is well defined (converges) and its value depends only on the homotopy class of $h \in \mathcal{F}^p(\mathbb{X}, \mathbb{Y})$.

In the mathematical models of elasticity one prevents impenetrability of matter by assuming that $J(x, h) > 0$, almost everywhere. Actually, in our subsequent examples this condition will always be satisfied for mappings of finite energy. The most elementary example of a free Lagrangian is

$$L(x, h, Dh) = F(x)\,, \qquad \text{where } \; F \in \mathscr{L}^1(\mathbb{X}) \tag{62}$$

Thus $\mathcal{E}[h] = \int_{\mathbb{X}} F(x)\, \mathrm{d}x$ for all $h \in \mathscr{F}^p(\mathbb{X}, \mathbb{Y})$. Less trivial though still simple is the dual example obtained by pulling back a volume form $\omega = G(y)\, \mathrm{d}y$, with $G \in \mathscr{L}^1(\mathbb{Y})$, via a homeomorphism $h \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$.

$$L(x, h, Dh) = G(h)\, J(x, h) \tag{63}$$

Thus $\mathcal{E}[h] = \pm \int_{\mathbb{Y}} G(y)\, \mathrm{d}y$ for $h \in \mathscr{F}^n_+(\mathbb{X}, \mathbb{Y})$ and $h \in \mathscr{F}^n_-(\mathbb{X}, \mathbb{Y})$, respectively.

The idea of pulling back a differential ℓ-form on $\mathbb{Y}$ by a homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is effective in constructing other free Lagrangians, see [28] and [32]. Here we confine ourselves to annuli $\mathbb{A} = \{x \in \mathbb{R}^n ; \quad r < |x| < R\}$ and $\mathbb{A}^* = \{x \in \mathbb{R}^n ; \quad r_* < |x| < R_*\}$ and a subfamily $\mathscr{F}^p_\circ(\mathbb{A}, \mathbb{A}^*) \subset \mathscr{F}^p_+(\mathbb{A}, \mathbb{A}^*)$ of homeomorphisms $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ preserving the order of boundary components; that is, $|h(x)| = r_*$ for $|x| = r$ and $|h(x)| = R_*$ for $|x| = R$.

In the class $\mathscr{F}^{n-1}_\circ(\mathbb{A}, \mathbb{A}^*)$ we can define the *tangential energy* of $h = (h^1, ..., h^n)$ by

$$\begin{aligned} \mathcal{E}_T[h] &\stackrel{\text{def}}{=\!=} \int_{\mathbb{A}} \sum_{i=1}^n \frac{h^i\, dh^1 \wedge ... \wedge dh^{i-1} \wedge d|x| \wedge dh^{i+1} \wedge ... \wedge dh^n}{|x|\, |h|^n} \\ &= \omega_{n-1} \log \frac{R}{r} = \operatorname{Mod} \mathbb{A} \end{aligned} \tag{64}$$

where ω_{n-1} stands for the $(n-1)$-surface area of $\mathbb{S}^{n-1}$.

A dual example is furnished by the *normal energy* of h

$$\begin{aligned} \mathcal{E}_N[h] &\stackrel{\text{def}}{=\!=} \int_{\mathbb{A}} \sum_{i=1}^n \frac{x_i\, dx_1 \wedge ... \wedge dx_{i-1} \wedge d|h| \wedge dx_{i+1} \wedge ... \wedge dx_n}{|h|\, |x|^n} \\ &= \omega_{n-1} \log \frac{R_*}{r_*} = \operatorname{Mod} \mathbb{A}^* \end{aligned} \tag{65}$$

provided $h \in \mathscr{F}^1_\circ(\mathbb{A}, \mathbb{A}^*)$. The duality between these two free Lagrangians is empasized by the following identity $\mathcal{E}_T[h] = \mathcal{E}_N[f]$, where $f : \mathbb{A}^* \to \mathbb{A}$ is the inverse of h. In studying deformations of annuli it is convenient to use polar coordinates. Accordingly, the tangential and normal components of the deformation gradient can be introduced by the identity

$$|Dh(x)|^2 = \sum_{i,j=1}^n \left| \frac{\partial h^i}{\partial x_j} \right|^2 = |h_N|^2 + (n-1)\, |h_T|^2 \tag{66}$$

see [28].

Now the integral for the normal energy of h takes the form

$$\mathcal{E}_N[h] = \int_{\mathbb{A}} \frac{|h|_N}{|h|} \frac{\mathrm{d}x}{|x|^{n-1}} = \omega_{n-1} \log \frac{R_*}{r_*} = \operatorname{Mod} \mathbb{A}^* \tag{67}$$

Thus the differential expression $E(x,h,Dh) = \frac{|h|_N}{|x|^{n-1}|h|}$ is a free Lagrangian. The same observation pertains to a slightly more general case. Given $\Psi \in \mathscr{C}^1[r_*, R_*]$ we find that

$$\mathcal{E}_\Psi[h] \overset{\text{def}}{=\!=} \int_{\mathbb{A}} \frac{\Psi'\big(|h|\big)\,|h|_N}{|x|^{n-1}}\,\mathrm{d}x = \omega_{n-1}\,[\Psi(R_*) - \Psi(r_*)] \tag{68}$$

As a generalization of this idea we now consider the following free Lagrangian

$$E(x,h,Dh) = \frac{\Phi_t\big(|x|,|h|\big) + \Phi_\tau\big(|x|,|h|\big)\,|h|_N}{|x|^{n-1}} \tag{69}$$

where $\Phi = \Phi(t,\tau)$ is any $\mathscr{C}^1$-function of two variables $(t,\tau) \in [r,R] \times [r_*,R_*]$ and Φ_t and Φ_τ denote its partial derivatives. Indeed, using the polar coordinates $x = t \cdot \omega$, where $\omega \in \mathbb{S}^{n-1}$, we compute the energy independently of h,

$$\begin{aligned}
\mathcal{E}_\Phi[h] &\overset{\text{def}}{=\!=} \int_{\mathbb{A}} E(x,h,Dh)\,\mathrm{d}x \\
&= \int_{\mathbb{S}^{n-1}} \left(\int_r^R \left[\Phi_t\big(t, |h(t\omega)|\big) + \Phi_\tau\big(t,|h(t\omega)|\big) \frac{\mathrm{d}|h(t\omega)|}{\mathrm{d}t} \right] \mathrm{d}t \right) \mathrm{d}\omega \\
&= \int_{\mathbb{S}^{n-1}} \left(\int_r^R \frac{\mathrm{d}}{\mathrm{d}t} \Phi\big(t,|h(t\omega)|\big)\,\mathrm{d}t \right) \mathrm{d}\omega \\
&= \int_{\mathbb{S}^{n-1}} [\Phi(R,R_*) - \Phi(r,r_*)]\,\mathrm{d}\omega \\
&= \omega_{n-1}\,[\Phi(R,R_*) - \Phi(r,r_*)] \qquad (70)
\end{aligned}$$

as desired.

6. Extremal Deformations of Annuli, $n=2$

We shall now demonstrate how to minimize the total harmonic energy of $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ and its inverse $f : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$

$$\begin{aligned}
\mathcal{E}[h,f] &= \alpha \int_{\mathbb{A}} |Dh(x)|^2\,\mathrm{d}x \;+\; \beta \int_{\mathbb{A}^*} |Df(y)|^2\,\mathrm{d}y \\
&= \int_{\mathbb{A}} \left[\alpha\,|Dh(x)|^2 \;+\; \beta\,\frac{|Dh(x)|^2}{J(x,h)} \right] \mathrm{d}x\,, \qquad \alpha,\beta > 0 \qquad (71)
\end{aligned}$$

We shall consider deformations $h \in \mathcal{F}^2_\circ(\mathbb{A}, \mathbb{A}^*)$. It involves no loss of generality in assuming that the deformed annulus is conformally fatter than the reference annulus, since otherwise we could reverse the situation by considering the inverse map $f : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$. Thus we assume that $\frac{R}{r} < \frac{R_*}{r_*}$. By virtue of Theorem 3.1 we already know that a minimizer in $\mathcal{F}^2_\circ(\mathbb{A}, \mathbb{A}^*)$ exists. But we shall not rely on this fact though. Geometric intuition tells us that we have to examine radial mappings as candidates for the minimum energy,

$$h(x) = H\big(|x|\big)\,\frac{x}{|x|}\,, \qquad r < |x| < R \tag{72}$$

where $H : [r, R] \to [r_*, R_*]$ is continuously increasing from $H(r) = r_*$ to $H(R) = R_*$. Presupposing that this is the case we first derive the equilibrium equation for H

$$\ddot{H} = \Big(H - t\dot{H}\Big)\,\frac{\Big(\alpha\, H\, \dot{H} \;+\; \beta\, t\Big)\; \dot{H}^2}{\Big(\alpha\, t\, \dot{H}^3 \;+\; \beta\, H\Big)\; t\, H} \tag{73}$$

where dots stand for the derivatives of $H = H(t)$. Rather routine analysis of this ODE reveals that the above boundary value problem has a unique solution. A similar equation holds for the inverse mapping $F = F(\tau)$, $r_* < \tau < R_*$,

$$\ddot{F} = \Big(F - \tau\dot{F}\Big)\,\frac{\Big(\beta\, F\, \dot{F} \;+\; \alpha\, \tau\Big)\; \dot{F}^2}{\Big(\beta\, \tau\, \dot{F}^3 \;+\; \alpha\, F\Big)\; \tau\, F} \tag{74}$$

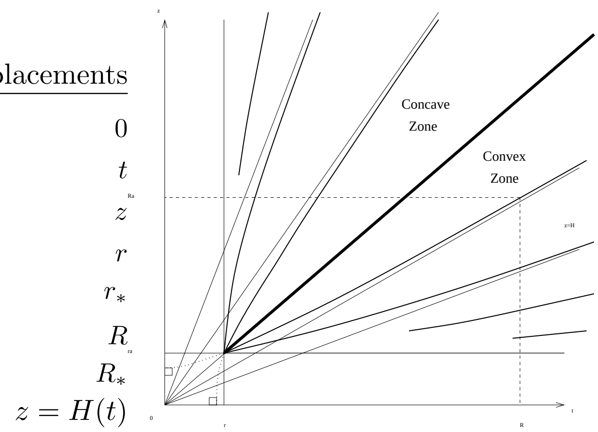

FIGURE 2. Radial extremals, $\alpha > 0$ and $\beta > 0$.

We shall make use of the functions $H\big(|x|\big)$, $\dot{H}\big(|x|\big)$, $F\big(|h|\big)$ and $\dot{F}\big(|h|\big)$ to construct free Lagrangians. It is by no means obvious at this point which free Lagrangians should be selected and combined for our purpose. After a deep analysis of the problem, too involved to be discussed in detail, we came out with the following point-wise estimate

$$E(x,h,Dh) = \alpha\,|Dh|^2 \;+\; \beta\,\frac{|Dh|^2}{J(x,h)} \geqslant A\big(|x|\big) + B\big(|h|\big)\,J(x,h) + \frac{\Phi_t\big(|x|,|h|\big) + \Phi_\tau\big(|x|,|h|\big)|h|_N}{|x|} \tag{75}$$

where

$$A(t) = 2\,\beta\,\frac{H(t)}{t\,\dot{H}(t)} \;+\; \frac{\alpha}{t^2}\Big[H^2(t) \;-\; t^2\dot{H}^2(t)\Big] \tag{76}$$

$$B(\tau) = 2\,\alpha\,\frac{\tau\dot{F}(\tau)}{F(\tau)} \;+\; \frac{\beta}{\tau^2}\Big[F^2(t) \;-\; \tau^2\dot{F}^2(\tau)\Big] \tag{77}$$

The function $\Phi : [r,R]\times[r_*,R_*] \to \mathbb{R}$ can also be expressed in terms of $H(t)$, $\dot{H}(t)$, $F(\tau)$ and $\dot{F}(\tau)$. But we shall not include the explicit description of Φ here; the reader may wish to consult [32] for the derivation of inequality (75). It is important to emphasize that (75) turns into equality if and only if h is a rotation of the radial map $H\big(|x|\big)\frac{x}{|x|}$, where H satisfies the equilibrium equation (73). We are now in a position to compute the minimum energy

$$\begin{aligned}\int_{\mathbb{A}} E(x,h,Dh)\,\mathrm{d}x &\geqslant \int_{\mathbb{A}} A\big(|x|\big)\,\mathrm{d}x + \int_{\mathbb{A}} B\big(|h|\big)\,J(x,h)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{A}} \frac{\Phi_t\big(|x|,|h|\big) \;+\; \Phi_\tau\big(|x|,|h|\big)\,|h|_N}{|x|} \\ &= 2\pi\int_r^R t\,A(t)\,\mathrm{d}t + 2\pi\int_{r_*}^{R_*}\tau\,B(\tau)\,\mathrm{d}\tau \\ &\quad + 2\pi\left[\Phi(R,R_*) - \Phi(r,r_*)\right]\end{aligned} \tag{78}$$

for all $h \in \mathcal{F}^2_\circ(\mathbb{A},\mathbb{A}^*)$. We have equality for $h = H\big(|x|\big)\frac{x}{|x|}$.

7. The Failure of Radial Symmetry, $n \geqslant 4$

It seems natural to speculate that the rotational symmetry of the annuli $\mathbb{A},\mathbb{A}^* \subset \mathbb{R}^n$ and the rotational invariance of $E(x,h,Dh)$ imply radial symmetry of the extremal deformations. This is not always the case. Let us recall the mean total distortion of $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ and its inverse $f : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$,

$$n^{-\frac{n}{2}}\mathcal{E}[f,h] = \beta \fint_{\mathbb{A}} K_I(x,h)\,\mathrm{d}x + \alpha \fint_{\mathbb{A}^*} K_I(y,f)\,\mathrm{d}y\,, \quad \alpha,\beta > 0 \tag{79}$$

or, equivalently

$$\mathcal{E}[f,h] = \frac{\alpha}{|\mathbb{A}^*|}\int_{\mathbb{A}} |Dh(x)|^n\ \mathrm{d}x + \frac{\beta}{|\mathbb{A}|}\int_{\mathbb{A}^*} |Df(y)|^n\ \mathrm{d}y\,, \quad \alpha,\beta > 0 \tag{80}$$

The existence of the extremal deformation has been established in Theorem 3.1. For $n = 2$ this extremal map is indeed radially symmetric. Although we did not study (80) for $n = 3$ it seems that radial symmetry of the extremal map for (80) still remains valid. This is certainly true when $\beta = 0$, see [28]. Ending, we have some surprise for the reader.

THEOREM 7.1. *For each $n \geqslant 4$ there are annuli $\mathbb{A}, \mathbb{A}^* \subset \mathbb{R}^n$ such that $\mathcal{E}[f,h]$ does not assume its minimum value on any radial map. Here in dimension $n = 4$ we assume that $\alpha > 3\beta$ or $\beta > 3\alpha$.*

References

[1] Antman, S. S. *Fundamental mathematical problems in the theory of nonlinear elasticity.* Univ. Maryland, College Park, Md., (1975) 35–54.

[2] Astala, K., Iwaniec, T., and Martin, G. J. *Deformations of smallest mean distortion.* Arch. Rational Mech. Anal., to appear.

[3] Astala, K., Iwaniec, T., Martin, G. J., and Onninen, J. *Extremal mappings of finite distortion.* Proc. London Math. Soc. (3) 91 (2005) no. 3, 655–702.

[4] Ball, J. M. *Constitutive inequalities and existence theorems in nonlinear elastostatics.* Nonlinear analysis and mechanics: Heriot-Watt Symposium (Edinburgh, 1976), Vol. I, pp. 187–241. Res. Notes in Math., No. 17, Pitman, London, 1977.

[5] Ball, J.M. *Convexity conditions and existence theorems in nonlinear elasticity.* Arch. Rational Mech. Anal. 63 (1978) 337–403.

[6] Ball, J.M. *Global invertibility of Sobolev functions and the interpenetration of matter.* Proc. Roy. Soc. Edinburgh Sect. A 88, no. 3-4, (1981) 315–328.

[7] Ball, J. M., Currie, J. C., and Olver, P. J. *Null Lagrangians, weak continuity, and variational problems of arbitrary order.* J. Funct. Anal. 41 (1981) no. 2, 135–174.

[8] Bojarski, B., and Iwaniec, T. *Analytical foundations of the theory of quasiconformal mappings in R^n.* Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983) no. 2, 257–324.

[9] Choquet, G., *Sur un type de transformation analytique généralisant la représentation conforme et définie au moyen de fonctions harmoniques,* Bull. Sci. Math., 69, (1945) 156-165.

[10] Csörnyei, M, Hencl, S., and Malý J. *Homeomorphisms in the Sobolev space $W^{1,n-1}$.* J. Reine Angew. Math., to appear.

[11] Ciarlet, P. G. *Mathematical elasticity.* Vol. I. Three-dimensional elasticity. Studies in Mathematics and its Applications, 20. North-Holland Publishing Co., Amsterdam 1988.

[12] Dacorogna, B. *Direct methods in the calculus of variations.* Applied Mathematical Sciences, 78. Springer-Verlag, Berlin, 1989.

[13] Duren, P. *Harmonic mappings between planar domains*, Cambridge Tracts in Mathematics, 156. Cambridge University Press, Cambridge, 2004.

[14] Franchis, M. De. *La piú generale funzione d'invarianze per criteri sufficient di minimo con condizione di Dirichlet per integrali pluridimensionali del primo ordine dipendenti da un vettore a piú componenti,* Atti Accad. Naz. Lincei Rend. C;. Sci. Fis. Mat. Natur. (8) 37 (1964) 129–140.

[15] Edelen, D. G. B. *The null set of the Euler-Lagrange operator.* Arch. Rational Mech. Anal. 11 (1962) 117–121.

[16] Ericksen, J. L. *Nilpotent energies in liquid crystal theory.* Arch. Rational Mech. Anal. 10 (1962) 189–196.

[17] Gehring, F. W.; Väisälä, J. *The coefficients of quasiconformality of domains in space.* Acta Math. 114, (1965) 1–70.

[18] Heinonen, J., Kilpeläinen, T., and Martio, O. *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs, (1993).

[19] Hencl, S., and Koskela, P. *Regularity of the inverse of a planar Sobolev homeomorphism.* Arch. Ration. Mech. Anal. 1 80 (2006) no. 1, 75–95.

[20] Hencl, S. Koskela, P and Onninen, J. *A note on extremal mappings of finite distortion.* A note on extremal mappings o f finite distortion. Math. Res. Lett. 12 (2005) no. 2-3, 231–237.

[21] Hencl, S. Koskela, P., and Onninen, J. *Homeomorphisms of bounded variation.* Arch. Ration. Mech. Anal. **186** (2007), no. 3, 351–360.

[22] Iwaniec, T. *On the concept of the weak Jacobian and Hessian.* Rep. Univ. Jyväskylä Dep. Math. Stat., 83, Univ. Jyväskylä, Jyväskylä, (2001) 181–205.

[23] Iwaniec, T., Koskela, P., and Onninen, J. *Mappings of finite distortion: monotonicity and continuity.* Invent. Math. 144 (2001) no. 3, 507–531.

[24] Iwaniec, T., Kovalev, L. V., and Onninen, J. *Harmonic mappings of an annulus, Nitsche conjecture and its generalizations.*, arXiv:0903.2665.

[25] Iwaniec, T., Kovalev, L. V., and Onninen, J. *The Nitsche conjecture.*, arXiv:0908.1253.

[26] Iwaniec, T., Kovalev, L. V., and Onninen, J. *Doubly connected minimal surfaces and extremal harmonic mappings.* submitted, available at arXiv: 0912.3542.

[27] Iwaniec, T. and Martin, G. J. *Geometric Function Theory and Non-linear Analysis.* Oxford Mathematical Monographs, 2001.

[28] Iwaniec, T., and Onninen, J. *n-Harmonic mappings between annuli.* Submitted for publication.

[29] Iwaniec, T., and Onninen, J. *p-harmonic energy of deformations between punctured balls.* Adv. Calc. Var. **2** (2009), no. 1, 93–107.

[30] Iwaniec, T., and Onninen, J. *Deformations of finite conformal energy: existence and removability of singularities.* Proc. London Math. Soc., to appear.

[31] Iwaniec, T., and Onninen, J. *Deformations of finite conformal energy: boundary behavior and limit theorems.* Trans. Amer. Math. Soc., to appear.

[32] Iwaniec, T., and Onninen, J. *Hyperelastic deformations of smallest total energy.* Arch. Ration. Mech. 194 (2009) 927–986

[33] Iwaniec, T., and Onninen, J. *Neohookean deformations of annuli, existence, uniqueness and radial symmetry.* Math Ann., to appear.

[34] Iwaniec, T. and Šverák, V. *On mappings with integrable dilatation.* Proc. Amer. Math. Soc. 118 (1993) no. 1, 181–188.

[35] Kauhanen, J., Koskela, P., and Malý, J. *Mappings of finite distortion: condition N.* Michigan Math. J. 49 (2001) no. 1, 169–181.

[36] Kauhanen, J., Koskela, P., and Malý, J. *Mappings of finite distortion: discreteness and openness.* Arch. Ration. Mech. Anal. 160 (2001) no. 2, 135–151.

[37] Kneser, H., *Lösung der Aufgabe 41,* Jahresber. Deutsch. Math.-Verein., 35, (1926) 123-124.

[38] Landers, A. W. *Invariant multiple integrals in the calculus of variations.* Contributions to the Calculus of Variations, 1938–1941, University of Chicago Press, Chicago, Ill. (1942) 175–207.

[39] Lewy, H. *On the non-vanishing of the Jacobian in certain one-to-one mappings.* Bulletin Amer. Math. Soc., 42, (1936) 689–692.

[40] Milnor, J.*On manifolds homeomorphic to the* 7*-sphere.* Ann. of Math. (2) 64 (1956) 399–405.

[41] Morrey, C.B. *Quasi-convexity and the lower semicontinuity of multiple integrals.* Pacific J. Math. 2 (1952) 25–53.

[42] Müller, S., and Spector, S. J. *An existence theory for nonlinear elasticity that allows for cavitation.* Arch. Rational Mech. Anal. 131 (1995) no. 1, 1–66.

[43] Nitsche, J.C.C. *On the modulus of doubly connected regions under harmonic mappings*, Amer. Math. Monthly, 69, (1962) 781–782.

[44] J. C. C. Nitsche, *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin-New York, 1975.

[45] Onninen, J. *Regularity of the inverse of spatial mappings with finite distortion*, Calc. Var. Partial Differential Equations 26 (2006) no. 3, 331–341.

[46] Radó, T., *Aufgabe 41.*, Jahresber. Deutsch. Math.-Verein., 35, (1926) 49.

[47] Reshetnyak, Yu. G. *Space mappings with bounded distortion.* Translations of Mathematical Monographs, 73. American Mathematical Society, Providence, RI, 1989.

[48] Rickman, S. *Quasiregular mappings.* Springer-Verlag, Berlin, 1993.

[49] Struwe, M. *Variational methods. Applications to nonlinear differential equations and Hamiltonian systems.* Springer-Verlag (1990).

Tadeusz Iwaniec
Department of Mathematics
Syracuse University
Syracuse NY 13244, USA

E-mail: tiwaniec@mailbox.syr.edu

Department of Mathematics and Statistics,
University of Helsinki
FI-00014 Helsinki
Finland

Jani Onninen
Department of Mathematics
Syracuse University
Syracuse NY 13244, USA
E-mail: jkonnine@syr.edu