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Modeling Oscillatory Components with The Homogeneous Spaces $\dot{BMO}^{-\alpha}$ and $\dot{W}^{-\alpha,p*}$

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Abstract: This paper is devoted to the decomposition of an image f into $u+v$, with u a piecewise-smooth or "cartoon" component, and v an oscillatory component (texture or noise), in a variational approach. The cartoon component u is modeled by a function of bounded variation, while v, usually represented by a square integrable function, is now being modeled by a more refined and weaker texture norm, as a distribution. Generalizing the idea of Y. Meyer [34], where $v \in F = \mathrm{div}(BMO) = \dot{BMO}^{-1}$, we model here the texture component by the action of the Riesz potentials on v that belongs to BMO or to L^p. In an earlier work [28], the authors proposed energy minimization models to approximate (BV, F) decompositions explicitely expressing the texture as divergence of vector fields in BMO. In this paper, we consider an equivalent more isotropic norm of the space F in terms of the Riesz potentials, and study models where the Riesz potentials of oscillatory components belong to BMO or to L^p, $1 \leq p < \infty$ (thus we consider oscillatory components in $\dot{BMO}^{\alpha}$ or in $\dot{W}^{\alpha,p}$, with $\alpha < 0$). Theoretical, experimental results and comparisons to validate the proposed methods are

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1 Introduction and motivations

We assume that a given grayscale image f is defined on $\mathbb{R}^n$ or on $\overline{\Omega} = [0,1]^n \subset \mathbb{R}^n$, with $\Omega = (0,1)^n$. When f is defined on Ω, we assume that f is periodic and Ω is the fundamental domain (or we can also assume that f is extended by zero outside $\overline{\Omega}$).

An important problem in image analysis is the decomposition of f into $u + v$, where u is piecewise-smooth containing the geometric components of f and v is oscillatory, typically texture or noise. A general variational method for decomposing $f \in X_1 + X_2$ into $u + v$, with $u \in X_1$ and $v \in X_2$, can be defined by the minimization problem

$$\inf_{(u,v)\in X_1\times X_2} \left\{ F_1(u) + \lambda F_2(v) : f = u + v \right\}, \tag{1}$$

where $F_1, F_2 \geq 0$ are functionals and X_1, X_2 are spaces of functions or distributions such that $F_1(u) < \infty$ and $F_2(v) < \infty$, if and only if $(u,v) \in X_1 \times X_2$. The constant $\lambda > 0$ is a tuning parameter. A good model for (1) is given by a choice of X_1 and X_2 so that with the given desired properties of u and v, we obtain $F_1(u) << F_1(v)$ and $F_2(v) << F_2(u)$.

In standard approaches, the space L^2 is used to model v when f denotes the image of a real scene, u is a piecewise-smooth approximation of f (made up of homogeneous regions with sharp boundaries), and v is a residual (additive Gaussian noise or small details). For example, in the Mumford and Shah model [37] for two dimensional image segmentation, $f \in L^\infty(\Omega) \subset L^2(\Omega)$ is split into $u \in SBV(\Omega)$ [4], a piecewise-smooth function with its discontinuity set J_u composed of a union of curves of total finite length, and $v = f - u \in L^2(\Omega)$ representing noise or texture. The (non-convex) model in the weak formulation is [35]

$$\inf_{(u,v)\in SBV(\Omega)\times L^2(\Omega)} \Big\{ \int_{\Omega\setminus J_u} |\nabla u|^2 dx + \alpha \mathcal{H}^1(J_u) + \beta \|v\|^2_{L^2(\Omega)},\ f = u + v \Big\}, \tag{2}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure, and $\alpha, \beta > 0$ are tuning parameters. With the above notations, the first two terms in the energy from (2) compose $F_1(u)$, while the third term makes $F_2(v)$. A related decomposition is obtained by the total variation minimization model of Rudin, Osher, and Fatemi [41] for image denoising. The (convex) decomposition model is

$$\inf_{(u,v)\in BV(\Omega)\times L^2(\Omega)} \Big\{ |u|_{BV(\Omega)} + \lambda \|v\|^2_{L^2(\Omega)}, \quad f = u + v \Big\}, \tag{3}$$

where $|u|_{BV(\Omega)} = \int_\Omega |Du|$ (the semi-norm on the space BV) [16], and $\lambda > 0$ is a tuning parameter. This model is strictly convex and is easily solved in practice. However, it has some limitations pointed out by several authors ([46], [47], [34] among others). If $f = \alpha\chi_D$ is a multiple of the characteristic function of a disk D centered at the origin and of radius R, we would like the minimizer u to be f if R is not too small. However, for any $R \geq \frac{1}{\lambda\alpha}$ and any finite $\lambda > 0$, we have [34]

$$u = (\alpha - \frac{1}{\lambda R})\chi_D, \ v = \frac{1}{\lambda R}\chi_D.$$

Model (3) is of the form $|u|_{BV(\Omega)} + \lambda\|f-u\|^q_{L^p(\Omega)}$, $p \geq 1$, $q \geq 1$, and the loss of intensity property is always present when we have $q > 1$ while keeping the total variation. In particular, we no longer have an intensity loss if we substitute $\|\cdot\|^2_{L^2}$ in (3) with $\|\cdot\|_{L^2}$ or $\|\cdot\|_{L^1}$, which was proposed in the continuous case by Cheon, Paranjpye, Vese and Osher [11], and further analysis in the L^1 case was made by Chan and Esedoglu [10], among others (see also earlier works of S. Alliney [1] and [2] in the discrete one dimensional case).

We are interested in function spaces that give small penalties to oscillations. As noted in [34], oscillatory components do not have small norms in L^2 or L^1. Moreover, Alvarez, Gousseau and Morel [21], [3] argue that BV is not a good choice to model natural images. To overcome these drawbacks, we can relax the condition on $F_1(u) = |u|_{BV}$ or on $F_2(v) = \|v\|_{L^p}$, for $p = 1$ or $p = 2$. One way is to use a non-convex regularization in u (like in (2), [19], [9], [50], [31], etc.), that is weaker than $|\cdot|_{BV}$. Another way is to use weaker norms than the L^p norm. Here we keep a convex BV regularization, and consider weaker norms than the L^p norm, following [34]. Mumford and Gidas [36] also show that, under some assumptions, natural images are drawn from probability distributions supported by generalized functions, and not by functions.

Y. Meyer [34] questions the model (3) and proposes more refined versions, using weaker norms of generalized functions to model v, instead of the $\|\cdot\|_{L^2}^2$. Among the spaces proposed in [34] to better model the texture component, is the space F and the minimization model

$$\inf_{(u,v)\in BV\times F}\Big\{|u|_{BV}+\lambda\|v\|_F,\quad f=u+v\Big\}, \tag{4}$$

where F is defined below.

Definition 1. In two dimensions, the space F consists of distributions v which can be written as

$$v=\operatorname{div}(\vec{g}) \text{ in } \mathcal{D}',\quad \vec{g}=(g_1,g_2)\in BMO^2, \text{ with}$$

$$\|v\|_F=\inf\Big\{\|g_1\|_{BMO}+\|g_2\|_{BMO}:\ v=\operatorname{div}(\vec{g}) \text{ in } \mathcal{D}',\ \vec{g}\in BMO^2\Big\}.$$

The space BMO is defined below.

Definition 2. We say that $f\in L^1_{loc}$ belongs to BMO [24], [43], if

$$\|f\|_{BMO}=\sup_Q\frac{1}{|Q|}\int_Q|f-f_Q|dx<\infty,$$

where Q is a square (it is sufficient to consider squares with sides parallel with the axes), and $f_Q=|Q|^{-1}\int_Q f(x)dx$ denotes the mean value of f over the square Q.

An equivalent norm of BMO can be obtained by taking the supremum over dyadic squares and their 1/3 translations, as in the work [17] by the first two authors. For $f\in BMO(\Omega)$, the supremum is over squares $Q\subset\Omega$.

In [34], Y. Meyer also proposed two other function spaces to model the oscillatory component v, denoted by G and E, with $u\in BV\subset L^2\subset G\subset F\subset E$. The space G is defined like F but having $\vec{g}$ in $(L^\infty)^2$ instead of $(BMO)^2$, while $E=\dot{B}^{-1}_{\infty,\infty}=\triangle(\dot{B}^1_{\infty,\infty})$ is a homogeneous Besov space of regularity index -1.

Meyer's G-model is approximated and studied in [51]-[52], [39], [6], [5], [8], [38], [22], [53], [12], [29], [27], among others. Meyer's E model was studied and discussed in [7], [18] and [26].

In [28], the third and fourth authors proposed several methods to numerically compute the BMO-norm of a function defined on a bounded domain Ω, and approximate Meyer's F-model (4) by the convex variational relaxed problem,

$$\inf_{u\in BV(\Omega),\ \vec{g}\in BMO(\Omega)^2}\Big\{|u|_{BV(\Omega)}+\mu\|f-u-\mathrm{div}\vec{g}\|^2_{L^2(\Omega)}+ \\ \lambda\left[\|g_1\|_{BMO(\Omega)}+\|g_2\|_{BMO(\Omega)}\right]\Big\}. \quad (5)$$

As $\mu\to\infty$, this model approximates model (4). An equivalent model was also proposed in [28], by setting $\vec{g}=\nabla g$, i.e. $v=\Delta g$, and minimizing

$$\inf\Big\{|u|_{BV(\Omega)}+\mu\|f-u-\Delta g\|^2_{L^2(\Omega)}+\lambda\left[\|g_x\|_{BMO(\Omega)}+\|g_y\|_{BMO(\Omega)}\right] \\ : u\in BV(\Omega), g, \triangle g\in L^2(\Omega), \nabla g\in BMO(\Omega)^2\Big\}. \quad (6)$$

Formulations (5) and (6) are still approximations to Meyer's F-model. In these models, a given image f is decomposed into $u+v+r$, where $u\in BV(\Omega)$ is piecewise smooth, $v=\mathrm{div}(\vec{g})\in F$ or $v=\Delta g=\mathrm{div}(\nabla g)\in F$ consists of oscillatory components, and $r=f-u-v\in L^2(\Omega)$ is a residual. Numerically, r is negligible. The significance of r is also discussed in [18].

Other related decomposition models using wavelets are by I. Daubechies and G. Teschke [14], R. Coifman and D. Donoho [13], J. L. Starck, M. Elad, and D. Donoho [42], F. Malgouyres [33], S. Lintner and F. Malgouyres [32], A. Haddad and Y. Meyer [22], A. Haddad, [23], or J. Gilles [20].

In this paper, we consider an equivalent norm for the space F in terms of the Riesz potentials, and study models where the action of the Riesz potentials on the oscillatory components belong to BMO. In other words, we model the oscillatory component v by imposing that $(-\Delta)^{\alpha/2}v$ belongs to BMO, for some $\alpha<0$, i.e. $v\in \dot{BMO}^\alpha$. If $\alpha=-1$, we recover the space F, but now the equivalent norm is defined in an isotropic way and we can obtain exact decompositions (4), and equivalent decompositions as in (5) and (6).

As a byproduct and for comparison, we also consider models when $(-\Delta)^{\alpha/2}v\in L^p$, $1\le p<\infty$, i.e. v belongs to the homogeneous potential Sobolev space $\dot{W}^{\alpha,p}$, for some $\alpha<0$. The case $1\le p<\infty$ and $\alpha=-1$ reduces to the case from [51], [52]. The case $p=2$ and $\alpha=-1$ reduces to the model from [39] in an equivalent

PDE formulation, while the more general case with $\alpha < 0$ and $p = 2$ reduces to the models proposed by L. Lieu [29], [30], and also related with the proposal from [36].

As noted in [34] in more details, the space $F = B\dot{M}O^{-1}$ has also been used in an analysis of the Navier-Stokes equations by Koch and Tataru [25], where $B\dot{M}O^{-1}$ is defined through another isotropic equivalent norm, also recalled in the next section.

We conclude this section by an example, motivating that oscillatory functions (not captured in the $BV-$cartoon component, having large total variations), have small $B\dot{M}O^{\alpha}$ norms ($\alpha < 0$) and thus are rather captured in the texture component (therefore such spaces do not penalize oscillations). Consider for simplicity the case $\alpha = -1$ (similar discussion for general $\alpha < 0$). Let $v_m(x) = \cos(mx)$, then $v_m(x) = \left(\frac{1}{m}\sin(mx)\right)'$ and $\|v_m\|_G = \frac{1}{m}$. Since we have the embedding $G \subset F$, this implies that $\|v_m\|_F \leq C\|v_m\|_G \to 0$ as $m \to \infty$. Thus, more oscillations give smaller $B\dot{M}O^{-1}$ norm, or in other words, oscillations are encouraged in the texture component v in a minimization model such as (4). By comparison, if $v \in L^p$, $p \geq 1$, then $\|v_m\|_{L^p} \to \mathit{constant} > 0$, as $m \to \infty$.

2 The homogeneous spaces $B\dot{M}O^{\alpha}$ and $\dot{W}^{\alpha,p}$

In this section, we consider a general form of function spaces, and in the definitions we make no distinction between periodic functions or functions defined on $\mathbb{R}^n$. We recall the definition of the Riesz potentials (or the fractional powers of the Laplacian)

$$I_{-\alpha}v(x) = (-\triangle)^{\alpha/2}v(x) = \left((2\pi|\xi|)^{\alpha}\hat{v}(\xi)\right)^{\vee}(x) = k_{\alpha} * v(x),$$

with $k_{\alpha}(x) = \left((2\pi|\xi|)^{\alpha}\right)^{\vee}(x)$, where as usual, $^{\wedge}$ indicates the Fourier transform and $^{\vee}$ indicates the inverse Fourier transform. Of special significance is the case $-n < \alpha < 0$. Then we can write the Riesz potentials as integral operators,

$$(I_{-\alpha}v)(x) = \frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n} |x-y|^{-n-\alpha}v(y)dy,$$

where $\gamma(\alpha)$ is a normalization constant depending on the dimension n and on α.

We also recall the Riesz transforms of a function f in two dimensions:

$$\widehat{(R_j f)}\,(\xi) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi), \quad j = 1, 2,$$

having the property

$$(R_1)^2 + (R_2)^2 = -I,$$

where I is the identity operator. We note that the Riesz operators R_j are bounded in BMO [43], [40]: there is a positive constant C_0 such that, for any $f \in BMO$, we have

$$\|R_j f\|_{BMO} \leq C_0 \|f\|_{BMO}.$$

Our main motivation of this work is the following lemma, which provides an isotropic equivalent norm for F, easier to be used in practice. This will also lead to generalizations.

Lemma 1. *The norm $\|v\|_F$ is equivalent with the norm $\|I_1 v\|_{BMO} = \|(-\Delta)^{-1/2} v\|_{BMO}$.*

Proof. Again, we note that the Riesz operators R_j are bounded in BMO,

$$\|R_j f\|_{BMO} \leq C_0 \|f\|_{BMO},$$

for some positive constant C_0.

We have:

$$\begin{aligned} v &= -((R_1)^2 + (R_2)^2)v = -(-\triangle)^{1/2}(-\triangle)^{-1/2}((R_1)^2 + (R_2)^2)v \\ &= R_1(-\triangle)^{1/2}(-R_1(-\triangle)^{-1/2}v) + R_2(-\triangle)^{1/2}(-R_2(-\triangle)^{-1/2}v) \\ &= R_1(-\triangle)^{1/2}g_1 + R_2(-\triangle)^{1/2}g_2 = div(g_1, g_2), \end{aligned}$$

with $g_j = -R_j((-\triangle)^{-1/2}v)$.

Then $\|g_j\|_{BMO} = \| - R_j((-\triangle)^{-1/2}v)\|_{BMO} \leq C_0\|(-\triangle)^{-1/2}v\|_{BMO}$.

Therefore,

$$\|v\|_F := \inf_{\vec{g} \in BMO \times BMO,\ div\vec{g} = v} \Big[\|g_1\|_{BMO} + \|g_2\|_{BMO}\Big] \leq 2C_0\|(-\triangle)^{-1/2}v\|_{BMO}.$$

For the converse inequality, suppose $v = div(g_1, g_2)$, with $g_1, g_2 \in BMO$. Then

$$v = div(g_1, g_2) = (-\triangle)^{1/2}(R_1 g_1 + R_2 g_2),$$

therefore

$$(-\triangle)^{-1/2} v = R_1 g_1 + R_2 g_2,$$

and then

$$\begin{aligned}\|(-\triangle)^{-1/2} v\|_{BMO} &= \|R_1 g_1 + R_2 g_2\|_{BMO} \leq \|R_1 g_1\|_{BMO} + \|R_2 g_2\|_{BMO} \\ &\leq C_0 \|g_1\|_{BMO} + C_0 \|g_2\|_{BMO} = C_0 \Big[\|g_1\|_{BMO} + \|g_2\|_{BMO} \Big].\end{aligned}$$

We conclude that

$$\|(-\triangle)^{-1/2} v\|_{BMO} \leq C_0 \inf_{\vec{g} \in BMO \times BMO,\ div\vec{g} = v} \Big[\|g_1\|_{BMO} + \|g_2\|_{BMO} \Big] = C_0 \|v\|_F,$$

and therefore the two norms are equivalent, since we have obtained

$$\frac{1}{2C_0} \|v\|_F \leq \|(-\triangle)^{-1/2} v\|_{BMO} \leq C_0 \|v\|_F.$$

□

Thus, for $v \in F$, the quantity $\|I_1 v\|_{BMO} = \|(-\Delta)^{-1/2} v\|_{BMO}$ provides an equivalent norm for $\|v\|_F$ introduced in Definition 1. This isotropic norm can be used as an alternative way to the models proposed and solved in [28]. Moreover, we are led to consider more general cases, when v is modeled by the space $B\dot{M}O^{\alpha}$, $\alpha < 0$, defined below.

Definition 3. (See Strichartz [44] and [45]) We say that a function (or distribution) v belongs to the homogeneous space $B\dot{M}O^{\alpha} = I_\alpha(BMO)$, $\alpha \in \mathbb{R}$, if

$$\|v\|_{B\dot{M}O^{\alpha}} := \|I_{-\alpha} v\|_{BMO} < \infty.$$

Equipped with $\|\cdot\|_{B\dot{M}O^{\alpha}}$, $B\dot{M}O^{\alpha}$ becomes a Banach space.

Elements in BMO or $B\dot{M}O^{\alpha}$ that are different by a constant are identified. In other words, we can assume that v has zero mean ($\int v(x)dx = 0$) if $v \in BMO$ or $v \in B\dot{M}O^{\alpha}$.

The space $\dot{BMO}^\alpha$ coincides with the classical Triebel-Lizorkin homogeneous space $\dot{F}^\alpha_{\infty,2}$ [48]. An equivalent norm for $\dot{BMO}^\alpha$ can also be obtained, as in [25]: let $\Phi(x) = Ce^{-2\pi|x|^2}$, where C is chosen so that $\int \Phi(x)\, dx = 1$. Define $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$, $x \in \mathbb{R}^n$. For each $v \in L^1_{loc}$, let $w_t(x) = \Phi_{\sqrt{4t}} * v(x)$. We have the following characterization of BMO [25], [43].

Definition 4. We say that $v \in BMO$ if

$$\begin{aligned} \|v\|_{BMO} &:= \sup_{x,R}\left(\frac{4\pi}{Q(x,R)}\int_{Q(x,R)}\int_0^R t|\nabla(\Phi_t * v)|^2\, dtdy\right)^{1/2} \\ &= \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} |\nabla w_t|^2\, dtdy\right)^{1/2} \\ &\approx \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} \left|(-\Delta)^{1/2} w_t\right|^2\, dtdy\right)^{1/2} < \infty, \end{aligned} \tag{7}$$

where $Q(x,R)$ denotes a square centered at x with side length R, and "$\approx$" denotes equivalent norms.

Similarly, we have the following characterization of $\dot{BMO}^\alpha$, which could be another alternative approach to the work in [28].

Definition 5. We say that v belongs to $\dot{BMO}^\alpha$, $\alpha \in \mathbb{R}$, if

$$\begin{aligned} \|I_{-\alpha}v\|_{BMO} &= \sup_{x,R}\left(\frac{4\pi}{Q(x,R)}\int_{Q(x,R)}\int_0^R t|\nabla(\Phi_t * (I_{-\alpha}v))|^2\, dtdy\right)^{1/2} \\ &= \sup_{x,R}\left(\frac{4\pi}{Q(x,R)}\int_{Q(x,R)}\int_0^R t|\nabla(I_{-\alpha}(\Phi_t * v))|^2\, dtdy\right)^{1/2} \\ &= \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} |\nabla(I_{-\alpha}w_t)|^2\, dtdy\right)^{1/2} \\ &\approx \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} \left|(-\triangle)^{1/2}(I_{-\alpha}w_t)\right|^2\, dtdy\right)^{1/2} < \infty. \end{aligned} \tag{8}$$

Again, "$\approx$" denotes equivalent norms.

In the remaining part of this paper, we use Definition 3 for $\dot{BMO}^{\alpha}$. For comparison, substituting BMO in Definition 3 by L^p, $1 \le p < \infty$, we arrive to the homogeneous potential Sobolev spaces, which we recall here.

Definition 6. We say that a function (or distribution) v belongs to the homogeneous potential Sobolev space $\dot{W}^{\alpha,p}$, for $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, if

$$\|v\|_{\dot{W}^{\alpha,p}} := \|I_{-\alpha} v\|_{L^p} < \infty.$$

Equipped with $\|\cdot\|_{\dot{W}^{\alpha,p}}$, $\dot{W}^{\alpha,p}$ becomes a Banach space.

Note that if $g \in \dot{W}^{\alpha,p}$, $\alpha < 0$, then $\int_\Omega g(x)\, dx = 0$. Some useful properties of $\dot{BMO}^{\alpha}$ and $\dot{W}^{\alpha,p}$ are recalled below:

- I_{-s} is an isometry from $\dot{BMO}^{\alpha}$ and $\dot{W}^{\alpha,p}$ to $\dot{BMO}^{\alpha-s}$ and $\dot{W}^{\alpha-s,p}$, respectively, for all $s, \alpha \in \mathbb{R}$.
- Let $\tau_\delta f(x) = f(\delta x)$, $\delta > 0$, $x \in \mathbb{R}^n$, be the dilation operator. We have

$$\|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$
$$\|\tau_\delta f\|_{\dot{BMO}^{\alpha}(\mathbb{R}^n)} = \delta^{\alpha} \|f\|_{\dot{BMO}^{\alpha}(\mathbb{R}^n)}, \quad \|\tau_\delta f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = \delta^{-\frac{n}{p}+\alpha} \|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}.$$

From this dilation property, we see that if $\alpha < 0$, $\|\cdot\|_{\dot{W}^{\alpha,p}}$ provides a better separation among different oscillations compared to $\|\cdot\|_{L^p}$, and for the same $\alpha < 0$, $\|\cdot\|_{\dot{W}^{\alpha,p}}$ provides a better separation among different oscillations compared to $\|\cdot\|_{\dot{BMO}^{\alpha}}$ (in other words, taking the same function f but with two different oscillating levels, $\tau_{\delta 1} f$ and $\tau_{\delta_2} f$, with $1 \le \delta_1 < \delta_2$, the difference in their norms is larger using $\|\cdot\|_{\dot{W}^{\alpha,p}}$, due to larger exponents in absolute value: $\frac{n}{p} < |\frac{n}{p} + \alpha|$ and $|\alpha| < |\frac{n}{p} + \alpha|$, thus larger exponent in absolute value will distinguish better between two levels of oscillations). The experimental results in this paper will support these remarks.

In [18], the authors have numerically considered the case when the oscillatory component v belongs to $\dot{B}^{\alpha}_{p,\infty}$, $\alpha < 0$, as a generalization of the space E proposed by Y. Meyer. The following remark shows that $\dot{B}^{\alpha}_{p,q}$ and $\dot{W}^{\alpha,p}$ are in fact close [49].

Remark 1. If $\alpha \in \mathbb{R}$ and $p \ge 1$, then

$$\dot{B}^{\alpha}_{p,1} \subset \dot{W}^{\alpha,p} \subset \dot{B}^{\alpha}_{p,\infty}. \tag{9}$$

3 Modeling oscillations with $B\dot{M}O^{\alpha}$ and $\dot{W}^{\alpha,p}$

Given an image f, we would like to decompose it into $u+v$, where $u \in BV$, and v is an element of $B\dot{M}O^{\alpha}$ or $\dot{W}^{\alpha,p}$, for $\alpha < 0$ and $1 \le p < \infty$. In other words, we consider modeling oscillatory component v (of zero mean) as Δg, where $g \in B\dot{M}O^{s}$ or $\dot{W}^{s,p}$, for $s < 2$, $1 \le p < \infty$, in the minimization problems for image decomposition

$$\inf_{u,g} \left\{ |u|_{BV} + \mu \|f - u - \Delta g\|_{L^2}^2 + \lambda \|g\|_{B\dot{M}O^s} \right\}, \text{ and} \tag{10}$$

$$\inf_{u,g} \left\{ |u|_{BV} + \mu \|f - u - \Delta g\|_{L^2}^2 + \lambda \|g\|_{\dot{W}^{s,p}} \right\}. \tag{11}$$

The model (10), when $s = 1$, is equivalent with the model (6). Since v belongs to $B\dot{M}O^{\alpha}$ or $\dot{W}^{\alpha,p}$ with $\alpha = s-2$, we will also consider the exact decomposition models,

$$\inf_{u} \left\{ |u|_{BV} + \lambda \|f - u\|_{B\dot{M}O^{\alpha}} \right\}, \text{ and} \tag{12}$$

$$\inf_{u} \left\{ |u|_{BV} + \lambda \|f - u\|_{\dot{W}^{\alpha,p}} \right\}. \tag{13}$$

Thus, when $\alpha = -1$ in (12), we recover Meyer's model (4). Theorems 1 and 2 from [18] can be exactly carried out here to show existence of minimizers for the above models (10), (11), (12) and (13).

We discuss next scaling properties of the proposed minimization models. Recall the dilating operator $\tau_\delta f(x) = f(\delta x)$, $\delta > 0$. We have

$$|\tau_\delta f|_{BV(\mathbb{R}^n)} = \delta^{-n+1} |f|_{BV(\mathbb{R}^n)}, \quad \|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-n/p} \|f\|_{L^p(\mathbb{R}^n)},$$
$$\|\tau_\delta f\|_{B\dot{M}O^{\alpha}(\mathbb{R}^n)} = \delta^{\alpha} \|f\|_{B\dot{M}O^{\alpha,p}(\mathbb{R}^n)}, \quad \|\tau_\delta f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = \delta^{-n/p+\alpha} \|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$$

Following [18], we would like to characterize the parameters μ and λ in the proposed models (10), (11), (12) and (13) when the image f is being dilated by a factor δ (zoom in when $0 < \delta < 1$ and zoom out when $\delta > 1$).

Proposition 1. *Denote*

$$\mathcal{J}_{f,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \lambda \|f - u\|_{B\dot{M}O^{\alpha}(\mathbb{R}^n)}$$

For a fixed f and $\lambda > 0$, let $(u_\lambda, v_\lambda = f - u_\lambda)$ be a minimizer for the energy $\mathcal{J}_{f,\lambda}$. Then for $\lambda' = \lambda \delta^{-n+1-\alpha}$, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ minimizes the energy $\mathcal{J}_{\tau_\delta f, \lambda'}$.

Proof. Since $(u_\lambda, v_\lambda = f - u_\lambda)$ is a minimizer, this implies

$$\mathcal{J}_{f,\lambda}(u_\lambda) = |u_\lambda|_{BV(\mathbb{R}^n)} + \lambda\|v_\lambda\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}$$

is minimal. Applying τ_δ to f, u_λ and v_λ using λ', we have

$$\begin{aligned}\mathcal{J}_{\tau_\delta f,\lambda'}(\tau_\delta u_\lambda) &= |\tau_\delta u_\lambda|_{BV(\mathbb{R}^n)} + \lambda'\|\tau_\delta v_\lambda\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}\\ &= \delta^{-n+1}|u_\lambda|_{BV(\mathbb{R}^n)} + \lambda'\delta^\alpha\|v_\lambda\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}.\end{aligned}$$

We have $\delta^{n-1}\mathcal{J}_{\tau_\delta f,\lambda'}(\tau_\delta u_\lambda)$ is minimized when $\lambda' = \lambda\delta^{-n+1-\alpha}$. Therefore, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ is a minimizer for $\mathcal{J}_{\tau_\delta f,\lambda'}$ with $\lambda' = \lambda\delta^{-n+1-\alpha}$. □

Similarly, when $\|\cdot\|_{B\dot{M}O^\alpha}$ is replaced by $\|\cdot\|_{\dot{W}^{\alpha,p}}$, we have the following result.

Proposition 2. *For a fixed f and $\lambda > 0$, let $(u_\lambda, v_\lambda = f - u_\lambda)$ be a minimizer for the energy,*

$$\mathcal{K}_{f,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \lambda\|f - u\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$$

Then for $\lambda' = \lambda\delta^{(-n+1)-(-n/p+\alpha)}$, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ minimizes $\mathcal{K}_{\tau_\delta f,\lambda'}$.

Using the same techniques, we obtain the following results for the models (10) and (11).

Proposition 3. *Fix an f, $\mu > 0$, and $\lambda > 0$.*

1. *Let $(u_{\mu,\lambda}, v_{\mu,\lambda})$ be a minimizer for the energy from (10), which can be rewritten as*

$$\mathcal{J}_{f,\mu,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \mu\|f - u - v\|^2_{L^2(\mathbb{R}^n)} + \lambda\|v\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}$$

   Then for $\mu' = \mu\delta$ and $\lambda' = \lambda\delta^{-n+1-\alpha}$, $(\tau_\delta u_{\mu,\lambda}, \tau_\delta v_{\mu,\lambda})$ minimizes $\mathcal{J}_{\tau_\delta f,\mu',\lambda'}$.

2. *Let $(u_{\mu,\lambda}, v_{\mu,\lambda})$ be a minimizer for the energy from (11), which can be rewritten as*

$$\mathcal{K}_{f,\mu,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \mu\|f - u - v\|^2_{L^2(\mathbb{R}^n)} + \lambda\|v\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$$

   Then for $\mu' = \mu\delta$ and $\lambda' = \lambda\delta^{(-n+1)-(-n/p+\alpha)}$, $(\tau_\delta u_{\mu,\lambda}, \tau_\delta v_{\mu,\lambda})$ minimizes $\mathcal{K}_{\tau_\delta f,\mu',\lambda'}$.

4 Characterization of minimizers

In this section, we would like to show some results regarding the characterization of minimizers for the exact decompositions (12) and (13) under some assumptions or minor modifications. These can be seen as extensions and generalizations of the results from Lemma 4, Thm. 3 (page 32), Proposition 4 (page 33) and Thm. 4 (page 4) from [34].

4.1 The case $|u|_{BV} + \lambda \|I_{-\alpha}(f-u)\|^2_{BMO}$

We have the following equivalent formulations of BMO for different values of $p \in [1, \infty)$, see [43] for example. For $f \in L^2_{loc}$, we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| dx \leq \Big(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2 dx\Big)^{1/2},$$

thus if $\sup_Q \Big(\frac{1}{|Q|}|f(x) - f_Q|^2 dx\Big)^{1/2} \leq C$, then $f \in BMO$. Conversely, if $f \in BMO$ according to Definition 2, then for any $p < \infty$, f is in L^p_{loc} and $\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx \leq c_p \|f\|^p_{BMO}$, for all squares Q.

Thus consider the problem with $p = 2$ in the definition of the equivalent BMO norm, and we substitute (12) by

$$\inf_u \mathcal{F}(u),$$

where

$$\mathcal{F}(u) = |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * (f-u) - (k_\alpha * (f-u))_Q|^2 dx, \text{ or}$$
$$\mathcal{F}(u) = |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * f - (k_\alpha * f)_Q - (k_\alpha * u - (k_\alpha * u)_Q)\|^2_{L^2(Q)}.$$

Denote $\langle f, g\rangle_{L^2(Q)} := \int_Q fg\, dx$. Consider the quantity $\|\cdot\|_{\alpha,*}$, (possibly attains ∞), defined as

$$\|f\|_{\alpha,*} = \sup_{\bar{Q}, h\in BV, |h|_{BV}\neq 0} \frac{\frac{1}{|\bar{Q}|}\left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}}{|h|_{BV}},$$

where the supremum is taken over all squares $\bar{Q}$ satisfying

$$\bar{Q} = \arg\max_{Q} \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx. \tag{14}$$

Remark 2. Note that, if the functions are not sufficiently smooth, there may not be a square $\bar{Q}$ realizing the maximum in (14). In such cases, the results presented below can still be verified, working with a sequence of maximizing squares $\bar{Q}_{\epsilon_n}$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx = \lim_{\epsilon_n \to 0} \frac{1}{|\bar{Q}_{\epsilon_n}|} \int_{\bar{Q}_{\epsilon_n}} |k_\alpha * f - (k_\alpha * f)_{\bar{Q}_{\epsilon_n}}|^2 dx.$$

Definition 7. Given an $\alpha \in \mathbb{R}$, we say f satisfies property (**P**) if for any $h \in BV$ and any square $\bar{Q}$ satisfying (14), we have

$$\begin{aligned} \liminf_{\epsilon_n \to 0} \frac{1}{|Q_{\epsilon_n}|} &\left\langle k_\alpha * f - (k_\alpha * f)_{Q_{\epsilon_n}}, k_\alpha * h - (k_\alpha * h)_{Q_{\epsilon_n}} \right\rangle_{L^2(Q_{\epsilon_n})} \\ &\geq \frac{1}{|\bar{Q}|} \left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \end{aligned} \tag{15}$$

for some sequence of squares Q_{ϵ_n} and of small parameters $\epsilon_n > 0$ converging to zero, such that

$$Q_{\epsilon_n} = \arg\max_Q \frac{1}{|Q|} \int_Q |k_\alpha * (f - \epsilon_n h) - (k_\alpha * (f - \epsilon_n h))_Q|^2 dx.$$

Proposition 4.

(i) If $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$, then $u = 0$ and $v = f$ is a minimizer.

*(ii) If $u = 0$ and $v = f$ is a minimizer and if, in addition, f satisfies property (**P**) from (15), then $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$.*

Proof.

(i) Let $h \in BV$ such that

$$\mathcal{F}(h) = |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * (f - h) - (k_\alpha * (f - h))_Q|^2 dx < +\infty.$$

Since $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$, we have for all $h \in BV$ and all squares $\bar{Q}$ satisfying

$$\bar{Q} = \arg\max_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx, \tag{16}$$

that

$$-2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha * f-(k_\alpha * f)_{\bar{Q}}, k_\alpha * h-(k_\alpha * h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})} \geq -|h|_{BV}.$$

Then

$$\begin{aligned}\mathcal{F}(h) &= |h|_{BV}+\lambda\sup_Q\frac{1}{|Q|}\|k_\alpha * f-(k_\alpha * f)_Q-(k_\alpha * h-(k_\alpha * h)_Q)\|^2_{L^2(Q)}\\ &= |h|_{BV}+\lambda\sup_Q\frac{1}{|Q|}\Big[\|k_\alpha * f-(k_\alpha * f)_Q\|^2_{L^2(Q)}\\ &-2\left\langle k_\alpha * f-(k_\alpha * f)_Q, k_\alpha * h-(k_\alpha * h)_Q\right\rangle_{L^2(Q)}+\|k_\alpha * h-(k_\alpha * h)_Q\|^2_{L^2(Q)}\Big].\end{aligned}$$

With $\bar{Q}$ defined as in (16), we have

$$\begin{aligned}\mathcal{F}(h) &\geq |h|_{BV}+\lambda\frac{1}{|\bar{Q}|}\|k_\alpha * f-(k_\alpha * f)_{\bar{Q}}\|^2_{L^2(\bar{Q})}+\frac{1}{|\bar{Q}|}\|k_\alpha * h-(k_\alpha * h)_{\bar{Q}}\|^2_{L^2(\bar{Q})}\\ &-2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha * f-(k_\alpha * f)_{\bar{Q}}, k_\alpha * h-(k_\alpha * h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}\\ &= |h|_{BV}+\lambda\sup_Q\frac{1}{|Q|}\|k_\alpha * f-(k_\alpha * f)_Q\|^2_{L^2(Q)}+\frac{1}{|\bar{Q}|}\|k_\alpha * h-(k_\alpha * h)_{\bar{Q}}\|^2_{L^2(\bar{Q})}\\ &-2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha * f-(k_\alpha * f)_{\bar{Q}}, k_\alpha * h-(k_\alpha * h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}\\ &\geq F(0)+\frac{1}{|\bar{Q}|}\|k_\alpha * h-(k_\alpha * h)_{\bar{Q}}\|^2_{L^2(\bar{Q})}\geq F(0).\end{aligned}$$

Therefore, $u=0$ is a minimizer.

(ii) Suppose now $u=0$ and $v=f$ is a minimizer and f satisfies property (**P**) from (15). We have

$$\begin{aligned}&|h|_{BV}+\lambda\sup_Q\frac{1}{|Q|}\int_Q|(k_\alpha *(f-h)-(k_\alpha *(f-h))_Q)|^2dx\\ &\quad\geq\lambda\sup_Q\frac{1}{|Q|}\int_Q|k_\alpha * f-(k_\alpha * f)_Q|^2dx.\end{aligned}$$

Thus

$$\begin{aligned}|h|_{BV}&+\lambda\sup_Q\frac{1}{|Q|}\Big\{\int_Q|k_\alpha * k-(k_\alpha * f)_Q|^2dx\\ &-2\left\langle k_\alpha * f-(k_\alpha * f)_Q, k_\alpha * h-(k_\alpha * h)_Q\right\rangle_{L^2(Q)}\\ &+\int_Q|k_\alpha * h-(k_\alpha * h)_Q|^2dx\Big\}\geq\lambda\sup_Q\frac{1}{|Q|}\int_Q|k_\alpha * f-(k_\alpha * f)_Q|^2dx.\end{aligned}\tag{17}$$

Let $\hat{Q}$ be defined as the square depending on f and h that achieves the maximum in $\sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * (f-h) - (k_\alpha * (f-h))_Q|^2 dx$. Then we can rewrite (17) as

$$
\begin{aligned}
&|h|_{BV} + \lambda \frac{1}{|\hat{Q}|}\Big\{ \int_{\hat{Q}} |k_\alpha * f - (k_\alpha * f)_{\hat{Q}}|^2 dx \\
&\quad - 2\left\langle k_\alpha * f - (k_\alpha * f)_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})} \\
&\quad + \int_{\hat{Q}} |k_\alpha * h - (k_\alpha * h)_{\hat{Q}}|^2 dx \Big\} \geq \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|h|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx \\
&\quad - 2\lambda \frac{1}{|\hat{Q}|}\left\langle k_\alpha * f - (k_\alpha * f)_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})} \\
&\quad + \lambda \frac{1}{|\hat{Q}|}\int_{\hat{Q}} |k_\alpha * h - (k_\alpha * h)_{\hat{Q}}|^2 dx \geq \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx.
\end{aligned}
\tag{18}
$$

Changing h into ϵh in (18), dividing both sides by $\epsilon > 0$, and taking $\epsilon \to 0$, we obtain that for any $h \in BV$ and any $\bar{Q}$ satisfying (14),

$$
\frac{\frac{1}{|\bar{Q}|}\left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})}}{|h|_{BV}} \leq \frac{1}{2\lambda}.
$$

Therefore, $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$. □

Proposition 5. *Assume now* $\|f\|_{\alpha,*} > \frac{1}{2\lambda}$.

(i) Suppose u *is a minimizer and* $f-u$ *satisfies the property* (**P**) *from (15) (with equality if* $h = u$*). Then* u *satisfies*

$$
\begin{aligned}
\frac{1}{2\lambda}|u|_{BV} &= \frac{1}{|\bar{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * u - (k_\alpha * u)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \\
&\text{and } \|k_\alpha * (f-u)\|_* = \frac{1}{2\lambda},
\end{aligned}
\tag{19}
$$

where $\bar{Q} = \mathrm{argmax}_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}$.

(ii) If u *satisfies the properties in (19), then* u *is a minimizer.*

Proof.

(i) Assume that u is a minimizer. Then, for any small ϵ and any $h \in BV$, we have

$$\begin{aligned}&|u+\epsilon h|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-(u+\epsilon h)) - (k_\alpha * (f-(u+\epsilon h)))_Q\|^2_{L^2(Q)} \\ &\geq |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}.\end{aligned} \tag{20}$$

Let $\hat{Q}$ be the square that achieves the maximum in the left-hand-side of the above equation (20), which depends on $k_\alpha * (f-(u+\epsilon h))$. By triangle inequality we obtain

$$\begin{aligned}|u|_{BV} + |\epsilon||h|_{BV} + \lambda \frac{1}{\hat{Q}}\Big[&\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(\hat{Q})} \\ &- 2\epsilon \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})} \\ &+ \epsilon^2 \|k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\|^2_{L^2(\hat{Q})}\Big] \\ &\geq |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}.\end{aligned}$$

Thus,

$$\begin{aligned}|\epsilon||h|_{BV} + \lambda \sup_Q \frac{1}{|Q|}&\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)} \\ &- 2\lambda\epsilon \frac{1}{|\hat{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})} \\ &+ \lambda\epsilon^2 \frac{1}{|\hat{Q}|}\|k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\|^2_{L^2(\hat{Q})} \\ &\geq \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\epsilon||h|_{BV} + \lambda\epsilon^2 \frac{1}{|\hat{Q}|}&\|k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\|^2_{L^2(\hat{Q})} \\ &\geq 2\lambda\epsilon \frac{1}{|\hat{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})}.\end{aligned} \tag{21}$$

Taking in (21) $\epsilon > 0$, dividing by ϵ, and letting $\epsilon \to 0$, we have, for any $h \in BV$,

$$|h|_{BV} \geq 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle. \tag{22}$$

Similarly, with $h = u$ in (20), $-1 < \epsilon < 0$, dividing by ϵ, and letting $\epsilon \to 0$, we get

$$|u|_{BV} \leq 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * u - (k_\alpha * u)_{\bar{Q}} \right\rangle. \tag{23}$$

Therefore, (22) and (23) imply (19).

(ii) Let $w \in BV$ arbitrary, and let $h = w - u \in BV$, or $w = u + h$. We have

$$\begin{aligned}
\mathcal{F}(w) &= |w|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f-w) - (k_\alpha * (f-w))_Q\|^2_{L^2(Q)} \\
&= |u+h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \|k_\alpha * (f-(u+h)) - (k_\alpha * (f-(u+h)))_Q\|^2_{L^2(Q)} \\
&= |u+h|_{BV} + \lambda \sup_Q \left\{ \frac{1}{|Q|} \|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)} \right. \\
&\quad - 2\frac{1}{|Q|} \langle k_\alpha * (f-u) - (k_\alpha * (f-u))_Q, k_\alpha * h - (k_\alpha * h)_Q \rangle_{L^2(Q)} \\
&\quad \left. + \frac{1}{|Q|} \|k_\alpha * h - (k_\alpha * h)_Q\|^2_{L^2(Q)} \right\}.
\end{aligned} \tag{24}$$

Let $\bar{Q}$ be the square that achieves the supremum in

$$\sup_Q \frac{1}{|Q|} \|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}.$$

We have

$$\begin{aligned}
|u+h|_{BV} \geq 2\lambda \frac{1}{|\bar{Q}|} \Big\langle & k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, \\
& k_\alpha * (u+h) - (k_\alpha * (u+h))_{\bar{Q}} \Big\rangle_{L^2(\bar{Q})}.
\end{aligned}$$

This implies

$$
\begin{aligned}
\mathcal{F}(w) &\geq 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * u - (k_\alpha * u)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \\
&+ 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \\
&+ \lambda \frac{1}{|\bar{Q}|} \| k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}} \|^2_{L^2(\bar{Q})} \\
&- 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \\
&+ \lambda \frac{1}{|\bar{Q}|} \| k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \|^2_{L^2(\bar{Q})} \\
&= |u|_{BV} + \lambda \frac{1}{|\bar{Q}|} \| k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}} \|^2_{L^2(\bar{Q})} \\
&+ \lambda \frac{1}{|\bar{Q}|} \| k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \|^2_{L^2(\bar{Q})} \geq \mathcal{F}(u).
\end{aligned}
$$

Therefore, u is a minimizer. □

Property (**P**) from Definition 7 could hold (even with equality) for distributions f, when k_α is a sufficiently smoothing kernel. If k_α is not sufficiently smooth, we can introduce a very small amount of smoothing by additional convolution with another kernel, say the Poisson kernel. In other words, the quantity $k_\alpha * f$ could be substituted by $P_\delta * k_\alpha * f$, where P_δ is the Poisson kernel with some small $\delta > 0$, thus making $P_\delta * k_\alpha * f$ analytic.

The following counter-examples in one dimension show that property (**P**) (with equality or inequality) may not hold for instance for discontinuous functions f when $\alpha = 0$ (thus when $k_\alpha * f = f$).

Example 1. Consider on $\mathbb{R}$ the intervals $I_n = [2^{-n-1}, 2^{-n}]$, $n \geq 0$, and let c_n be the midpoint of I_n. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ outside of $[0, 1]$, and

$$
f(x) = \begin{cases} +(1 - 2^{-n}) \text{ if } x \in [2^{-n-1}, c_n] \quad (n \geq 1), \\ -(1 - 2^{-n}) \text{ if } x \in [c_n, 2^{-n}] \quad (n \geq 1), \\ +1 \text{ if } x \in [\frac{1}{2}, \frac{3}{4}], \\ -1 \text{ if } x \in [\frac{3}{4}, 1]. \end{cases}
$$

Then $\|f\|_{BMO} = 1$ and $[\frac{1}{2}, 1]$ is the interval where the norm is attained. Now let $h = -f$ on $[0, \frac{1}{2}]$ and $h \equiv 0$ otherwise. Then if $\epsilon > 0$, $f - \epsilon h$ attains its BMO

norm on one of the intervals $[2^{-n-1}, 2^{-n}]$, $n \geq 1$ (actually the norm increases to $1+\epsilon$ as $n \to \infty$).

But, for $n \geq 1$,

$$\frac{1}{2^{-n}} \int_{I_n} (f - f_{I_n})(h - h_{I_n}) dx = 1,$$

while

$$\frac{1}{2} \int_{[\frac{1}{2},1]} (f - f_{I_0})(h - h_{I_0}) dx = 0,$$

thus (15) with equality "=" instead of inequality "$\geq$" does not hold. A similar counter-example can be constructed in two dimensions.

The following counter-example shows that inequality "$\geq$" also may not hold in (15) for discontinuous functions.

Example 2. Similarly, let $I_n = [2^{-n-1}, 2^{-n})$, $n \geq 0$, and c_n be the midpoint of the interval I_n. Let $J_n = [2^{-n-1}, c_n]$ and $K_n = [c_n, 2^{-n})$. On I_0 let $f = h = \chi_{J_0} - \chi_{K_0}$ and on I_n for $n \geq 1$ let $f = (1 - \frac{1}{n})(\chi_{J_n} - \chi_{K_n})$. Splitting each J_n and each K_n into two half intervals denoted A_n and B_n, let $h = \chi_{A_n} - \chi_{B_n}$. Then f and h have mean zero over all intervals I_n. Again we assume that f and h are zero otherwise. We have $\bar{Q} = I_0$ and

$$\frac{1}{|\bar{Q}|} \int_{\bar{Q}} f(x)h(x) dx = 1,$$

but for $n \geq 1$,

$$\frac{1}{|I_n|} \int_{I_n} |f(x) - \epsilon h(x)|^2 dx = (1 - \frac{1}{n})^2 + \epsilon^2$$

and

$$\frac{1}{|I_n|} \int_{I_n} f(x)h(x) dx = 0.$$

The following example shows that, at least in one dimension, if f and h are sufficiently smooth (for example polynomials or analytic functions), then property (**P**) from Definition 7 holds with equality in (15).

Example 3. Let f and h be polynomials or analytic functions on a bounded interval I in $\mathbb{R}$. Let $Q = [x_0 - r, x_0 + r]$, be an arbitrary interval included in I. Then the quantities $\frac{1}{2r} \int_Q |f - f_Q|^2 dx$, $\frac{1}{2r} \langle f - f_Q, h - h_Q \rangle_{L^2(Q)}$, and $\frac{1}{2r} \int_Q |h - h_Q|^2 dx$ remain polynomials or analytic functions of (x_0, r). Let $P(\epsilon, x_0, r) =$

$\frac{1}{2r}\int_Q |(f-\epsilon h)-(f-\epsilon h)_Q|^2 dx$, polynomial or analytic function in (x_0, r) and quadratic polynomial in ϵ. If (x_0^0, r^0) achieves the maximum of $P(0, x_0, r)$, and if $(x_0^\epsilon, r^\epsilon)$, a bounded sequence, achieves the maximum of $P(\epsilon, x_0, r)$, then there is a subsequence $(x_0^{\epsilon_n}, r^\epsilon)$ and $\epsilon_n \to 0$ such that $\lim_{\epsilon_n\to 0} P(\epsilon_n, x_0^{\epsilon_n}, r^{\epsilon_n}) = P(0, x_0^0, r^0)$, thus property (**P**) is satisfied in this case.

4.2 The case $|u|_{BV} + \lambda\|f-u\|_{\dot{W}^{\alpha,p}}$

Consider here the minimization

$$\inf_u \left\{\mathcal{F}(u) = |u|_{BV} + \lambda\|f-u\|_{\dot{W}^{\alpha,p}}\right\}, \tag{25}$$

for some $\alpha < 0$, and $1 \le p < \infty$. Thus $\mathcal{F}(u)$ is the sum of two non-differentiable functionals at the origin. Assume that we "regularize" the second term (this is often done in practice, for valid computational calculations) by smoothing at the origin the L^p norm; thus substituting $\|f-u\|_{L^p}$ by $R_\delta(f-u) = \left\{\int \sqrt{\delta^2 + |f-u|^{2p}}dx\right\}^{1/p}$, for very small $\delta > 0$.

Therefore, substitute the problem (25) by the regularized functional

$$\inf_u \left\{\mathcal{F}_\delta(u) = |u|_{BV} + \lambda R_\delta(I_\alpha(f-u))\right\}. \tag{26}$$

Let $f \in V = \dot{W}^{\alpha,p}$, and let V' be the topological dual of V. We have $V' = \dot{W}^{-\alpha,p'}$, where p' is the conjugate of p. Denote by $\langle\cdot,\cdot\rangle$ the duality pairing for V and V'.

Problem (26) can be seen as a particular case of a more general case, where R_δ is a Gateaux-differentiable functional on the Banach space V, with continuous Gateaux derivative. For (any) fixed $f \in V$, we have $R'_\delta(f) \in V'$ and $\langle R'_\delta(f), -v\rangle = \lim_{\epsilon\to 0}\frac{R_\delta(f-\epsilon v)-R_\delta(f)}{\epsilon}$, for any $v \in V$. For any $f \in V$, define now the quantity $\|\cdot\|_{\alpha,*}$ (in $[0,+\infty]$) as

$$\|f\|_{\alpha,*} = \sup_{h\in BV,\ |h|_{BV}\neq 0} \frac{\langle R'_\delta(k_\alpha * f), k_\alpha * h\rangle}{|h|_{BV}}.$$

We also assume that for any $f, h \in V$,

$$R_\delta(f-\epsilon h) = R_\delta(f) + \epsilon\left\langle R'_\delta(f), -h\right\rangle + O(\epsilon^2)$$

in a neighborhood of the origin. Using the notation $g(\epsilon) = R_\delta(f - \epsilon h)$ for fixed f and h, this is equivalent with

$$g(\epsilon) = g(0) + \epsilon g'(0) + O(\epsilon^2),$$

where $g'(0) = \langle R'_\delta(f), -h\rangle$.

We have the following characterizations of minimizers for (26), a more general case than (25). Note that these are more general than the quadratic case considered in [34]. For the converse implications below, to show that some u is a minimizer, we need more conditions on R_δ related to convexity. The functional $R_\delta(f-u) = \left\{ \int \sqrt{\delta^2 + |f-u|^{2p}} dx \right\}^{1/p}$, defined in the particular case of interest to us, satisfies the assumptions mentioned above and the additional ones that are given below.

Proposition 6.

(i) Assume that $u = 0$ is a minimizer of (26). Then $\|f\|_{\alpha,*} \leq \frac{1}{\lambda}$.

(ii) Assume that $\|f\|_{\alpha,*} \leq \frac{1}{\lambda}$, and assume that R''_δ exists and it is a continuous bilinear form on V, satisfying $R''_\delta(v)(h,h) \geq 0$, for any $v, h \in V$. Moreover, we assume that in a neighborhood of the interval $[-1,1]$ we have

$$g(\epsilon) = g(0) + \epsilon g'(0) + \frac{\epsilon^2}{2} g''(\xi_\epsilon),$$

with ξ_ϵ between 0 and ϵ, and $g''(\xi_\epsilon) = R''_\delta(f - \xi_\epsilon h)(-h,-h) \geq 0$. Then $u = 0$ is a minimizer of (26).

Proof.

(i) For any $\epsilon \in \mathbb{R}$ and any $h \in BV$, we have

$$\begin{gathered}
|\epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - \epsilon h)) \geq \lambda R_\delta(k_\alpha * f), \\
|\epsilon||h|_{BV} + \lambda \Big[R_\delta(f) + \epsilon \langle R'_\delta(k_\alpha * f), -k_\alpha * h\rangle + O(\epsilon^2) \Big] \geq \lambda R_\delta(f), \\
|\epsilon||h|_{BV} + \lambda\epsilon \langle R'_\delta(k_\alpha * f), -k_\alpha * h\rangle + \lambda O(\epsilon^2) \geq 0.
\end{gathered}$$

Taking $\epsilon > 0$, dividing by ϵ and letting $\epsilon \to 0$, we obtain

$$|h|_{BV} \geq \lambda \left\langle R'_\delta(k_\alpha * f), k_\alpha * h \right\rangle, \quad \text{thus} \quad \frac{1}{\lambda} \geq \|f\|_{\alpha,*}.$$

(ii) Conversely, take any $h \in BV$. Then using the assumptions, we have

$$\begin{aligned}|h|_{BV} + \lambda R_\delta(k_\alpha * (f-h)) &\geq \lambda \left\langle R'_\delta(k_\alpha * f), k_\alpha * h\right\rangle + \lambda R_\delta(k_\alpha * f) \\ &+ \lambda \left\langle R'_\delta(k_\alpha * f), -k_\alpha * h\right\rangle + \frac{\lambda}{2} g''(\xi_1) \\ &= \lambda R_\delta(k_\alpha * f) + \frac{\lambda}{2} g''(\xi_1) \geq \lambda R_\delta(k_\alpha * f).\end{aligned}$$

Therefore, $u = 0$ is a minimizer. ☐

Proposition 7. *Assume that* $\|f\|_{\alpha,*} > \frac{1}{\lambda}$.

(i) If u is a minimizer, then

$$\frac{1}{\lambda} = \|f-u\|_{\alpha,*} \text{ and } \frac{1}{\lambda}|u|_{BV} = \left\langle R'_\delta(k_\alpha * (f-u)), k_\alpha * u\right\rangle.$$

(ii) Suppose that $u \in BV$ satisfies

$$\frac{1}{\lambda} = \|f-u\|_{\alpha,*} \text{ and } \frac{1}{\lambda}|u|_{BV} = \left\langle R'_\delta(k_\alpha * (f-u)), k_\alpha * u\right\rangle,$$

and assume in addition the same conditions from Proposition 6 (ii) on the regularity and convexity of R_δ. Then u is a minimizer.

Proof. By the assumption and the previous result, $u = 0$ cannot be a minimizer.

(i) If $u \in BV$ is a minimizer, then

$$|u + \epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - (u + \epsilon h))) \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)).$$

Thus

$$\begin{aligned}|u+\epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f-u)) + \lambda\epsilon \left\langle R'_\delta(k_\alpha * (f-u), -k_\alpha * h\right\rangle + O(\epsilon^2) \\ \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)).\end{aligned} \tag{27}$$

By triangle inequality, we also obtain

$$\begin{aligned}|u| + |\epsilon||h|_{BV} + \lambda R_\delta(k_\alpha * (f-u)) + \lambda\epsilon \left\langle R'_\delta(k_\alpha * (f-u), -k_\alpha * h\right\rangle + O(\epsilon^2) \\ \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)).\end{aligned}$$

After terms cancellation and division by $\epsilon > 0$, taking $\epsilon \to 0$, we obtain that for any $h \in BV$,

$$|h|_{BV} \geq \lambda \left\langle R'_\delta(k_\alpha * (f-u), k_\alpha * h\right\rangle,$$

$$\text{therefore } \frac{1}{\lambda} \geq \|f-u\|_{\alpha,*}. \tag{28}$$

Taking now $h = u$ in (27), with $-1 < \epsilon < 0$, after cancellations and division by $\epsilon < 0$ and letting $\epsilon \to 0$, we obtain

$$|u|_{BV} \leq \lambda \left\langle R'_\delta(k_\alpha * (f-u), k_\alpha * u \right\rangle. \tag{29}$$

Combining (28) and (29), we obtain the desired results,

$$\frac{1}{\lambda} = \|f-u\|_{\alpha,*}, \quad \frac{1}{\lambda}|u|_{BV} = \left\langle R'_\delta(k_\alpha * (f-u), k_\alpha * u \right\rangle.$$

(ii) Conversely, by the assumptions and taking $\epsilon = 1$, we have

$$\begin{aligned}
|u+h|_{BV} + \lambda R_\delta(k_\alpha * (f-(u+h))) &= |u+h|_{BV} + \lambda R_\delta(k_\alpha * (f-u)) \\
&+ \lambda \left\langle R'_\delta(k_\alpha * (f-u), -k_\alpha * h \right\rangle + \frac{\lambda}{2} g''(\xi_1) \\
&\geq \lambda \left\langle R'_\delta(k_\alpha * (f-u)), k_\alpha * (u+h) \right\rangle + \lambda R_\delta(k_\alpha * (f-u)) \\
&+ \lambda \left\langle R'_\delta(k_\alpha * (f-u), -k_\alpha * h \right\rangle + \frac{\lambda}{2} g''(\xi_1) \\
&= |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)) + \frac{\lambda}{2} g''(\xi_1) \\
&\geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)),
\end{aligned}$$

thus u is a minimizer. □

5 Numerical minimization algorithms

For numerical studies, we consider spaces of functions or distributions that are periodic and $\overline{\Omega} = [0,1]^2$ is the fundamental domain in $\mathbb{R}^2$. We give in this section the ingredients for minimizing in practice the proposed decomposition models from Section 3, in a gradient descent and purely PDE approach, based on Uzawa's minmax algorithm [15]. We formally compute the associated Euler-Lagrange equations, which are then discretized and solved by finite differences. We do not have a convergence proof of our algorithms, but these are stable and well-behaved in practice.

5.1 Algorithms for $(BV, B\dot{M}O^{\alpha})$ decompositions

For $\alpha < 0$, recall the minimization problem (12) for exact decompositions

$$\inf_u \mathcal{E}(u) = |u|_{BV(\Omega)} + \lambda \|I_{-\alpha}(f-u)\|_{BMO(\Omega)} = |u|_{BV(\Omega)} + \lambda \|k_\alpha * (f-u)\|_{BMO(\Omega)} \tag{30}$$

where $k_\alpha(x) = ((2\pi|\xi|)^\alpha)^\vee(x)$ (here the dimension is $n=2$).

We show the steps to solve (30). Using the classical definition of the BMO norm, we re-write (30) as

$$\inf_{u\in BV(\Omega)} \Big\{\mathcal{E}(u) = \int_\Omega |\nabla u| dx + \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * (f-u) - c_Q| dx\Big\},$$

where $\alpha < 0$, Q is a square with sides parallel with the axes, and c_Q denotes a constant which depends on $k_\alpha * (f-u)$ in Q. Here we take c_Q to be the median of $k_\alpha * (f-u)$ in Q. The main steps of the algorithm are as follows (see [28]):

1. Start with an initial guess u^0.

2. If u^n is computed, $n \geq 0$, evaluate $k_\alpha * (f-u^n)$ using the Fast Fourier Transform and find a square $Q = Q^n$ that achieves the BMO norm of $k_\alpha * (f-u)$ in Ω (by one of the methods proposed in [28]; here, we use the dyadic squares and their 1/3 translations, as explained in [17]).

3. Fix Q the square obtained at the previous step, denote by χ_Q the characteristic function of this square Q, and minimize with respect to $u = u^{n+1}$ the energy

$$\mathcal{E}(u) = \int_\Omega |\nabla u| dx + \lambda \frac{1}{|Q|}\int_\Omega \Big| k_\alpha * (f-u) - c_Q \Big| \chi_Q dx. \tag{31}$$

   The associated Euler-Lagrange equation in $u = u^{n+1}$ can be computed, and we obtain using gradient descent

$$\frac{\partial u}{\partial t} = \frac{\lambda}{|Q|} k_\alpha * \text{sign}\left[(k_\alpha * (f-u) - c_Q)\,\chi_Q\right] + \text{div}\Big(\frac{\nabla u}{|\nabla u|}\Big), \tag{32}$$

   with $Q = Q^n$ and $u = u^{n+1}$. Note that c_Q is the median of $k_\alpha * (f-u)$ in Q.

4. Repeat steps 2 and 3 using equation (32), until convergence (update u^{n+1} and Q^{n+1} each time and repeat).

Similarly, for the minimization problem (10), again with $s < 2$ ($\alpha = s - 2$),

$$\inf_{u,g} \Big\{ \mathcal{A}(u,g) = |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|^2_{L^2(\Omega)} + \lambda \|I_{-s} g\|_{BMO(\Omega)} \\ = |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|^2_{L^2} + \lambda \|k_s * g\|_{BMO(\Omega)} \Big\}, \tag{33}$$

re-written as

$$\inf_{u,g} \Big\{ \mathcal{A}(u,g) = \int_\Omega |\nabla u| dx + \mu \int_\Omega |f - u - \triangle g|^2 dx + \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_s * g - c_Q| dx \Big\},$$

where c_Q is the median of $k_s * g$ over the square Q, the main steps are as follows.

1. Start with initial guess u^0, g^0.

2. If u^n and g^n are computed, $n \geq 0$, evaluate $k_s * g^n$ using the Fast Fourier Transform and find a square $Q = Q^n$ that achieves the BMO norm of $k_s * g$ in Ω (by one of the methods proposed in [28]).

3. Fix Q the square obtained at the previous step, denote by χ_Q the characteristic function of this square Q, and minimize with respect to $u = u^{n+1}$ and $g = g^{n+1}$ the energy

$$\mathcal{A}(u,g) = \int_\Omega |\nabla u| dx + \mu \int_\Omega |f - u - \triangle g|^2 dx + \lambda \frac{1}{|Q|} \int_\Omega |k_s * g - c_Q| \chi_Q dx,$$

by solving the associated Euler-Lagrange equations using gradient descent

$$\frac{\partial u}{\partial t} = 2\mu(f - u - \triangle g) + \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big),$$

$$\frac{\partial g}{\partial t} = -\frac{\lambda}{|Q|} k_s * \operatorname{sign}\left[(k_s * g - c_Q)\, \chi_Q\right] + 2\mu \triangle (f - u - \triangle g)$$

with $Q = Q^n$ and $u = u^{n+1}$, $g = g^{n+1}$. Note that c_Q is the median of $k_s * g$ in Q.

4. Repeat steps 2 and 3 using equation (32), until convergence (update u^{n+1}, g^{n+1} and Q^{n+1} each time and repeat).

5.2 Algorithms for $(BV,\dot{W}^{\alpha,p})$ decompositions

For $\alpha < 0$, recall the minimization problem (13)

$$\inf_u \mathcal{E}(u) = |u|_{BV(\Omega)} + \lambda \|I_{-\alpha}(f-u)\|_{L^p(\Omega)} = |u|_{BV(\Omega)} + \lambda \|k_\alpha * (f-u)\|_{L^p(\Omega)}, \tag{34}$$

which is again minimized using Euler-Lagrange equation and gradient descent, as follows. Solve to steady state

$$\frac{\partial u}{\partial t} = \lambda \|k_\alpha * (f-u)\|^{1-p}_{L^p(\Omega)} k_\alpha * \Big[|k_\alpha * (f-u)|^{p-2} k_\alpha * (f-u)\Big] + \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big),$$

computing the convolutions using the Fast Fourier Transform.

Finally, for the minimization problem (11), recalled here with $s < 2$ ($\alpha = s-2$),

$$\begin{aligned}\inf_{u,g} \Big\{ \mathcal{A}(u,g) &= |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|^2_{L^2(\Omega)} + \lambda \|I_{-s} g\|_{L^p(\Omega)} \\ &= |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|^2_{L^2(\Omega)} + \lambda \|k_s * g\|_{L^p(\Omega)} \Big\},\end{aligned} \tag{35}$$

we use again the associated Euler-Lagrange equations and gradient descent, formally written as

$$\frac{\partial u}{\partial t} = 2\mu(f - u - \triangle g) + \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big),$$

$$\frac{\partial g}{\partial t} = -\lambda \|k_s * g\|^{1-p}_{L^p(\Omega)} k_s * \Big[|k_s * g|^{p-2} k_s * g\Big] + 2\mu \triangle (f - u - \triangle g).$$

In practice, the above Euler-Lagrange equations are discretized using finite differences. The calculations are stable and the numerical energy decreases versus iterations.

6 Numerical results and comparisons

Figure 1 shows three Barbara test images, to be used in our experimental calculations.

Figure 2 shows a decomposition of f_1 from Figure 1 using the Rudin-Osher-Fatemi model (3). Note the loss of intensity on the face area.

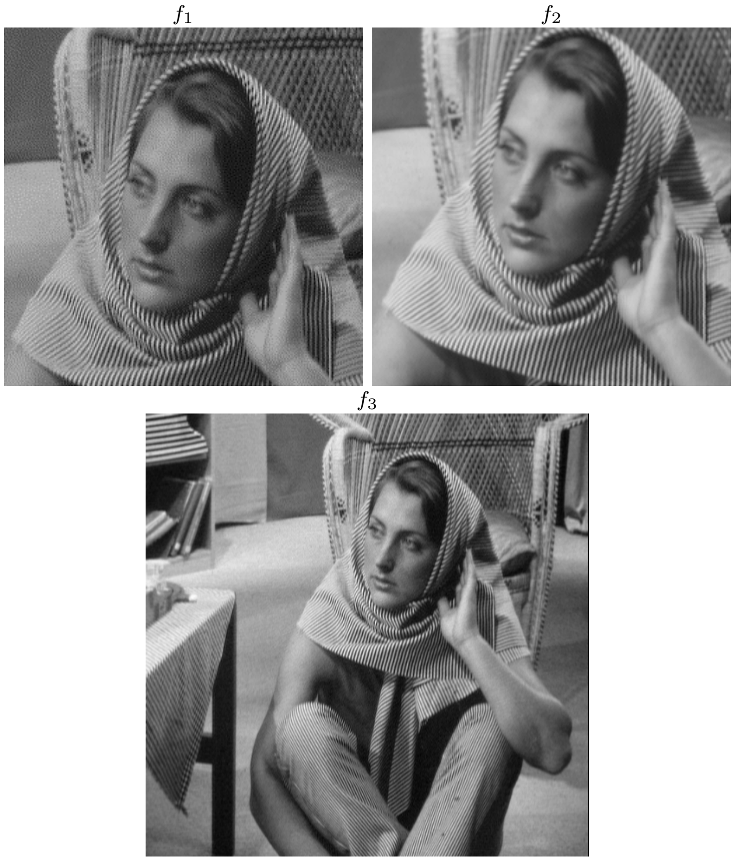

Figure 1: Test images to be decomposed.

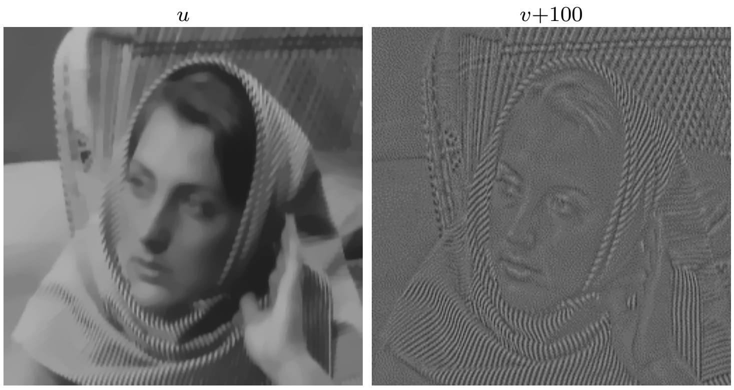

Figure 2: A decomposition of f_1 from Figure 1 using the Rudin-Osher-Fatemi model (3).

Figure 3 shows a decomposition of f_1 from Figure 1 using the model (5) from [28]. Here the oscillatory component is modeled as $v = div(\vec{g})$, $\vec{g} \in (BMO)^2$. We obtain an improvement in the loss of intensity, however vertical and horizontal textures are still kept in u.

Figure 4 shows a decomposition of f_1 from Figure 1 using the model (6) from [28]. Here the oscillatory component is modeled as $v = \Delta g$, $\nabla g \in (BMO)^2$. The decomposition is now more isotropic, textures are well captured in v including non-repeated patterns. This comes from the property of BMO.

Figure 5 shows a decomposition of f_1 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$, $s = 0.2$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$. Now, mostly repeated patterns are captured in v.

Figure 6 shows a decomposition of f_2 from Figure 1 using the model (10). Here the oscillatory component is modeled as $v = \Delta g$, $g \in B\dot{M}O^{\alpha}$ with $\alpha = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.001$. As remarked earlier, non-repeated patterns are also captured in v. We also show the numerical energy decrease versus iterations for this test.

Figures 7-8 show decompositions of f_2 and f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$,

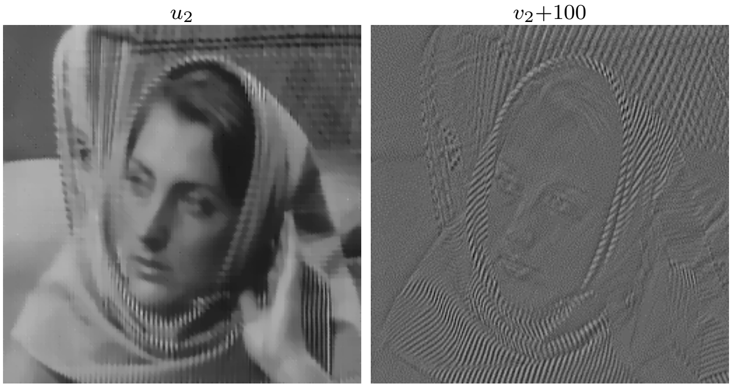

Figure 3: A decomposition of f_1 from Figure 1 using the model (5) from [28]. Here the oscillatory component is modeled as $v = div(\vec{g}),\ \vec{g} \in (BMO)^2$.

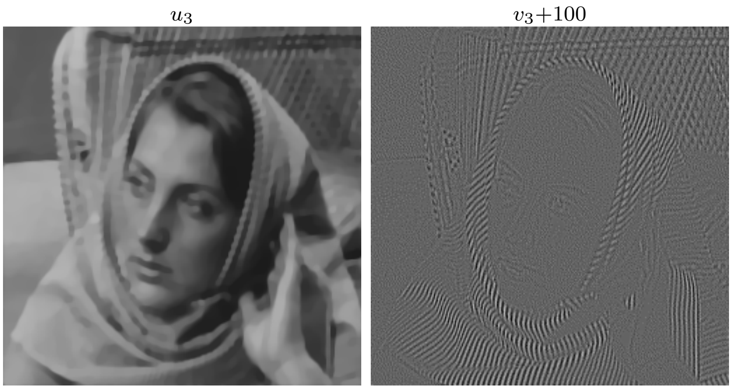

Figure 4: A decomposition of f_1 from Figure 1 using the model (6) from [28]. Here the oscillatory component is modeled as $v = \Delta g$, $\nabla g \in (BMO)^2$.

Figure 5: A decomposition of f_1 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$, $s = 0.2$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

Figures 9-10 show decompositions of f_2 and f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0005$.

Figure 11 shows a decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1.5$, and $p = 1$. The parameters used are: $\mu = 10$, and $\lambda = 0.00005$.

Figure 12 shows a decomposition of f_2 from Figure 1 using the model (12). Here the oscillatory component $v \in B\dot{M}O^{-0.5}$, $\lambda = 200$. The numerical energy versus iterations is also shown, illustrating that the algorithm is stable and well behaved in practice.

Figure 13 shows a decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.1$, $p = 1$, $\lambda = 1.25$.

Figure 14 shows a decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 15$.

Figure 15 shows a decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.6$, $p = 1$, $\lambda = 30$.

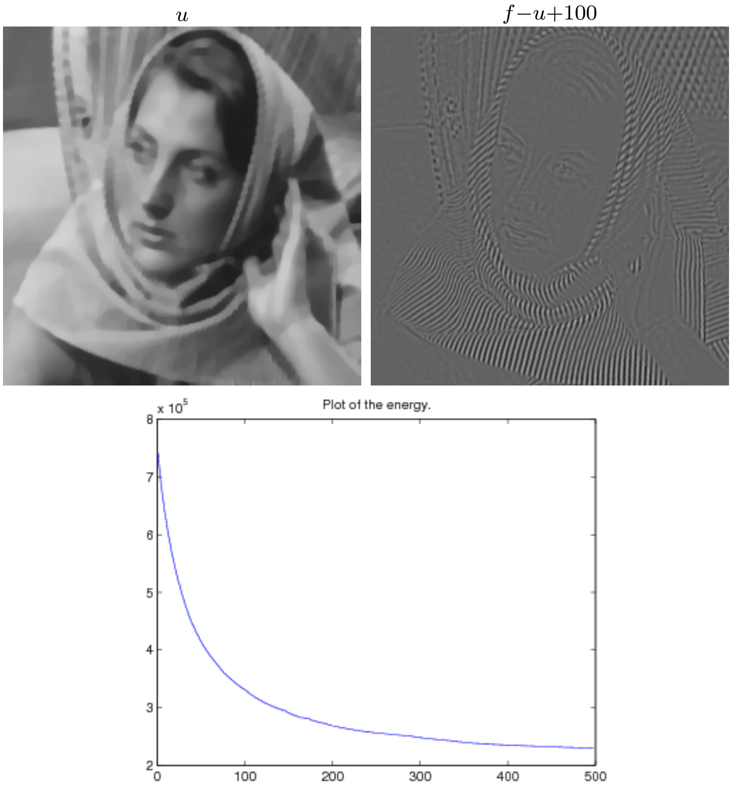

Figure 6: A decomposition of f_2 from Figure 1 using the model (10). Here the oscillatory component is modeled as $v = \Delta g$, $g \in B\dot{M}O^{\alpha}$ with $\alpha = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.001$. The numerical energy versus iterations is also shown.

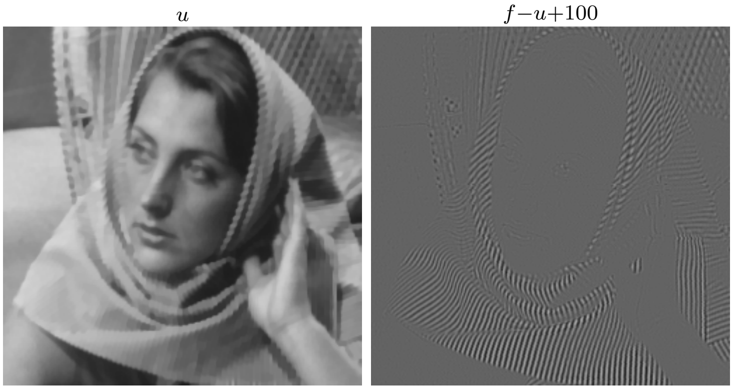

Figure 7: A decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

In conclusion, we have shown experimental results and comparisons for image decomposition $f = u + v$, with $u \in BV$ and $v \in B\dot{M}O^{\alpha}$ or $v \in W^{\alpha,p}$, for some $\alpha < 0$. The case of Sobolev spaces gives very good cartoon-texture separation.

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References

[1] S. Alliney, *Digital filters as L1-norm regularizers*, in: Sixth Multidimensional Signal Processing Workshop 6-8 Sept. 1989, p. 105.

[2] S. Alliney, *Digital filters as absolute norm regularizers*, IEEE Trans. on Signal Process. 40(6): 1548-1562, 1992.

[3] L. Alvarez, Y. Gousseau, J.-M. Morel, *Scales in natural images and a consequence on their bounded variation norm*, LNCS 1682: 247-258, 1999.

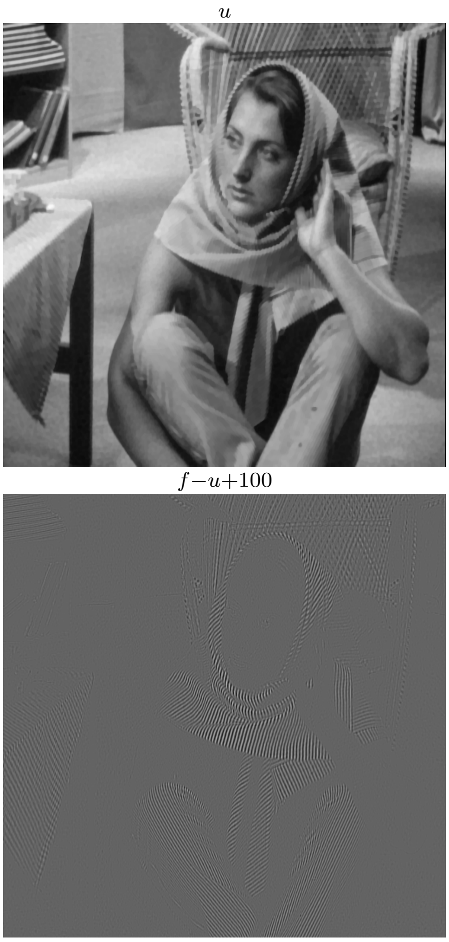

Figure 8: A decomposition of f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

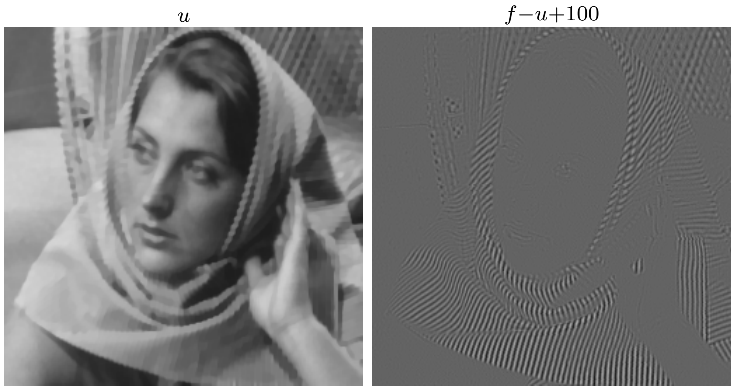

Figure 9: A decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0005$.

[4] L. Ambrosio, *Variational problems in SBV and image segmentation*, Acta Appl. Math., 17: 1-40, 1989.

[5] G. Aubert and J.-F. Aujol, *Modeling very oscillating signals. Application to image processing*, Applied Mathematics and Optimization, Vol. 51, no. 2, pp. 163-182, March 2005.

[6] J.-F. Aujol, G. Aubert, L. Blanc-Féraud, and A. Chambolle, *Image decomposition. Application to SAR images*, LNCS 2695: 297-312, 2003.

[7] J.-F Aujol and A. Chambolle, *Dual norms and image decomposition models*, Intern. J. Comput. Vision, 63(2005), pp. 85-104.

[8] J.-F. Aujol, *Contribution à l'analyse de textures en traitement d'images par méthodes variationnelles et équations aux dérivées partielles*, Thèse de Doctorat, University of Nice Sophia Antipolis, France, June 2004.

[9] G. Aubert and L. Vese, *A Variational method in Image Recovery*, SIAM Journal on Numerical Analysis, 34 (5): 1948-1979, 1997.

[10] T. F. Chan and S. Esedoglu, *Aspects of total variation regularized L^1 function approximation*, Siam J. Appl. Math., Vol. 65, No. 5, pp. 1817-1837, 2005.

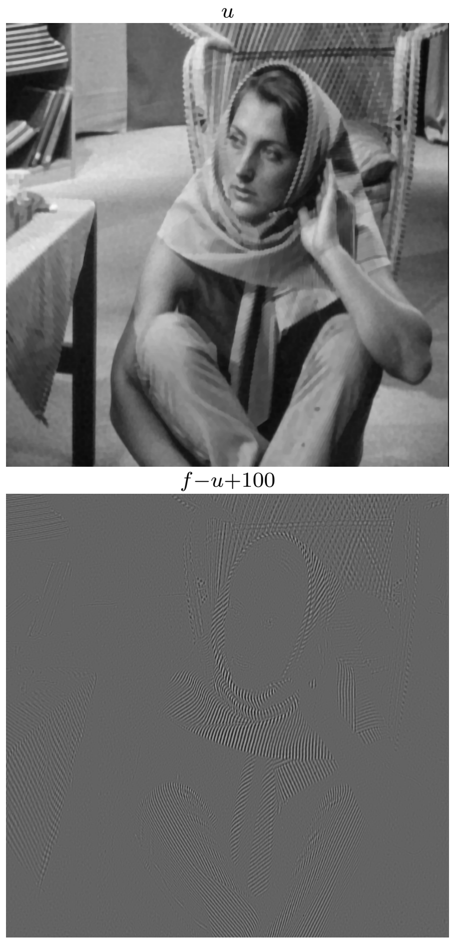

Figure 10: A decomposition of f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.00025$.

Figure 11: A decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1.5$, and $p = 1$. The parameters used are: $\mu = 10$, and $\lambda = 5e\neg 05$.

[11] E. Cheon, A. Paranjpye, L. Vese and S. Osher, *Noise Removal Project by Total Variation Minimization*, Math 199 REU project, UCLA Department of Mathematics, Spring-Summer 2002.

[12] G. Chung, T. Le, L. H. Lieu, N. Tanushev, and L. Vese, *Computational methods for image restoration, image segmentation, and texture modeling*, in Computational Imaging IV, C.A. Bouman, E.L. Miller, I. Pollak, eds., Proc. of SPIE-IS&T Electronic Imaging, SPIE 6065, pp. 60650J-1 – 60650J-15, 2006.

[13] R.R. Coifman and D. Donoho, *Translation-invariant De-Noising*, in "Wavelets and Statistics", A. Antoniadis and G. Oppenheim, eds., Springer-Verlag, New York, pp. 125-150, 1995.

[14] I. Daubechies and G. Teschke, *Wavelet based image decomposition by variational functionals*, in Wavelet Applications in Industrial Processing, Proc. SPIE 5226, F. Truchetet, ed., SPIE, Bellingham, WA, 2004, pp. 94-105.

[15] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Amsterdam, 1976.

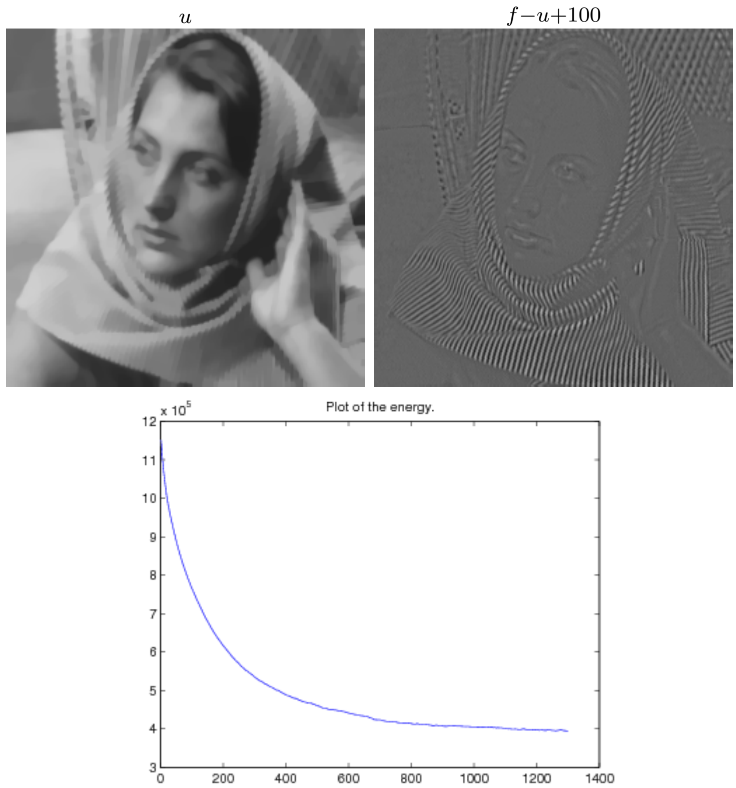

Figure 12: A decomposition of f_2 using (12) with $p = 1$. Here the oscillatory component $v \in B\dot{M}O^{\alpha}$, $\alpha = -0.5$, $\lambda = 25$. We also show a plot of the numerical energy versus iterations for this test.

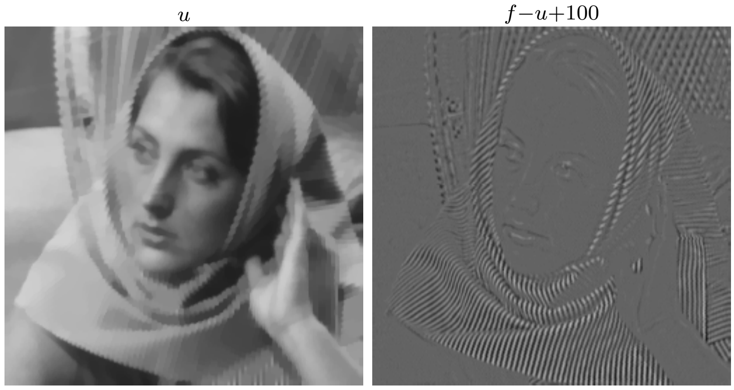

Figure 13: A decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.1$, $p = 1$, $\lambda = 1.25$.

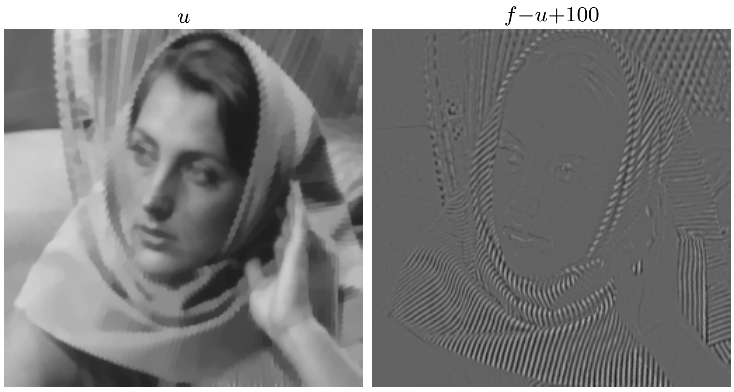

Figure 14: A decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 15$.

Figure 15: A decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 30$.

[16] L. C. Evans and R. F Gariepy, *Measure theory and fine properties of functions*, CRC Press, 1991.

[17] J. B. Garnett and P. W. Jones, *BMO from Dyadic BMO*, Pacific Journal of Mathematics, Vol. 99, No. 2, pp. 351-371, 1982.

[18] J.B. Garnett, T.M. Le, Y. Meyer and L.A. Vese, *Variational Image Decompositions Using Bounded Variations and Generalized Homogeneous Besov Spaces*, Appl. Comput. Harmon. Anal. 23, pp. 25-56, 2007.

[19] D. Geman and S. Geman, Stochastic Relaxation, Gibbs Distributions and the Bayesian Restoration of Images, IEEE Trans. Pattern Anal. Mach. Intell., 6, 721-741, 1984.

[20] J. Gilles, *Décomposition et détection de structures géométriques en imagerie*, Thèse de Doctorat, CMLA E.N.S. Cachan, France, June 2006.

[21] Y. Gousseau, J.-M. Morel, *Are natural images of bounded variation ?*, SIAM Journal of Mathematical Analysis 33 (3): 634-648, 2001.

[22] A. Haddad and Y. Meyer, *Variational Methods in Image Processing*, Contemporary Mathematics, AMS, Vol 446, pp. 273-295, 2007.

[23] A. Haddad, *Méthodes variationnelles en traitement d'image*, Thèse de Doctorat, CMLA E.N.S. Cachan, France, June 2005.

[24] F. John and L. Nirenberg, *On Functions of Bounded Mean Oscillation*, Comm. Pure Appl. Math., 14, 415-426, 1961.

[25] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes Equations*, Adv. in Math., vol. 157, pp. 22-35, 2001.

[26] T.M. Le, *A study of a few image segmentation and decomposition models in a variational approach*, Ph.D. Thesis, University of California, Los Angeles, June 2006.

[27] T.M. Le, L. Lieu and L.A. Vese, *(Φ,Φ^*) Image Decomposition Models and Minimization Algorithms*, JMIV 33(2): 135-148, 2009.

[28] T.M. Le and L.A. Vese, *Image Decomposition Using Total Variation and div(BMO)*, Multiscale Model. Simul., Vol. 4, No. 2, pp. 390-423, 2005.

[29] L. Lieu, *Contribution to Problems in Image Restoration, Decomposition, and Segmentation by Variational Methods and Partial Differential Equations*, (Ph.D. Thesis), UCLA CAM Report 06-46, June 2006.

[30] L. Lieu and L. Vese, *Image Restoration and Decomposition via Bounded Total Variation and Negative Hilbert-Sobolev Spaces*, Applied Mathematics & Optimization 58: 167-193, 2008.

[31] S. Levine, *An Adaptive Variational Model for Image Decomposition*, Energy Minimization Methods in Computer Vision and Pattern Recognition (2005), LNCS 757 Springer 2005, pp. 382-397.

[32] S. Lintner and F. Malgouyres, *Solving a variational image restoration model which involves L^∞ constraints*, Inverse Problems, vol. 20, no. 3, pp 815-831, 2004.

[33] F. Malgouyres, *Mathematical Analysis of a Model Which Combines Total Variation and Wavelet for Image Restoration*, Journal of Information Processes, vol.2, num. 1, pp 1-10, 2002.

[34] Y. Meyer, *Oscillating Patterns in Image Processing and Nonlinear Evolution Equations*, University Lecture Series, Col. 22, Amer. Math. Soc., 2001.

[35] J.-M. Morel and S. Solimini, *Variational Methods in Image Segmentation: With Seven Image Processing Experiments*, (Progress in Nonlinear Differential Equations and Their Applications), Birkhauser Boston 1994.

[36] D. Mumford and B. Gidas, *Stochastic models for generic images*, Quarterly of Applied Mathematics, Vol. 59, Issue 1, pp. 85-111, 2001.

[37] D. Mumford and J. Shah, *Optimal approximations by piecewise smooth functions and associated variational problems*, Comm. Pure Applied Mathematics, 42(5): 577-685, 1989.

[38] S. Osher and O. Scherzer, *G-Norm Properties of Bounded Variation Regularization*, Commun. Math. Sci. Vol. 2, no. 2, pp. 237-254, 2004.

[39] S. Osher, A. Sole, and L. Vese, *Image Decomposition and Restoration Using Total Variation Minimization and H^{-1} Norm*, Multiscale Modeling and Simulation 1(3): 349-370, 2003.

[40] J. Pfetre, *On convolution operators leaving $L^{p,\lambda}$ spaces invariant*, Annali di Matematica Pura ed Applicata, Volume 72, Number 1, pp. 295-304, 1966.

[41] L. Rudin, S. Osher, E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Physica D, 60, pp. 259-268, 1992.

[42] J.-L. Starck, M. Elad, and D.L. Donoho, *Image Decomposition: Separation of Texture from Piecewise Smooth Content*, SPIE conference on Signal and Image Processing: Wavelet Applications in Signal and Image Processing X, SPIE's 48th Annual Meeting, 3-8 August 2003, San Diego.

[43] E.M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.

[44] R.S. Strichartz, *Bounded Mean Oscillation and Sobolev Spaces*, Indiana Univ. Math. J., vol. 29, pp. 539-558, 1980.

[45] R.S. Strichartz, Traces of BMO-Sobolev spaces, Proc. Amer. Math. Soc., vol. 83(3), pp. 509-513, 1981.

[46] D. Strong, *Adaptive total variation minimizing image restoration*, PhD Dissertation, UCLA Mathematics Department CAM Report 97-38, 1997.

[47] D. Strong and T. Chan, *Edge-preserving and scale-dependent properties of total variation regularization*, Inverse Problems 19, S165-S187, 2003.

[48] H. Triebel, *Characterizations of Besov-Hardy-Sobolev spaces via Harmonic functions, temperatures and related means*, J. Approximation Theory, vol. 35, pp. 275-297, 1982.

[49] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Monographs in Mathematics vol. 84, 1992

[50] L. Vese, *A study in the BV space of a denoising-deblurring variational problem*, Applied Mathematics and Optimization, 44 (2):131-161, 2001.

[51] L. Vese and S. Osher, *Modeling Textures with Total variation Minimization and Oscillating Patterns in Image Processing*, Journal of Scientific Computing, 19(1-3), pp. 553-572, 2003.

[52] L.A. Vese and S.J. Osher, *Image Denoising and Decomposition with Total Variation Minimization and Oscillatory Functions*, Journal of Mathematical Imaging and Vision, 20: 7-18, 2004.

[53] W. Yin, D. Goldfarb and S. Osher, *A Comparison of Total Variation Based Texture Extraction Models*, Journal of Visual Communication and Image Representation, 18(3), 240-252, 2007.

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