Pure and Applied Mathematics Quarterly
Volume 7, Number 2
(*Special Issue: In honor of
Frederick W. Gehring, Part 2 of 2*)
253—273, 2011

Inequalities for The Carathéodory and Poincaré Metrics in Open Unit Balls

Clifford J. Earle and Lawrence A. Harris

Abstract: We generalize a known inequality relating the Euclidean and hyperbolic metrics in Poincaré's unit ball model of hyperbolic space. Our generalization applies to Schwarz-Pick metrics in the open unit ball of any complex Banach space. We study the case of equality, both in the general case and the case when the open unit ball is homogeneous. For open unit balls of Hilbert spaces, we explicitly determine the case of equality, and we prove a distortion theorem that quantifies the failure of equality when the inequality is strict. An analogous distortion theorem for real hyperbolic space follows readily.
Keywords: hyperbolic space, Schwarz-Pick systems, complex geodesics, bounded symmetric domains, JB^*-triples

1. Introduction

Let Δ be the open unit disk in the complex plane $\mathbb{C}$. The Poincaré metric

$$\rho(a,b) := \tanh^{-1} \frac{|a-b|}{|1-\overline{a}b|}, \qquad a \text{ and } b \text{ in } \Delta, \tag{1}$$

makes Δ a model for the real hyperbolic plane with curvature -4. We have informally posed the following question about that model to a number of people.

Received March 26, 2007.
2000 *Mathematics Subject Classification.* Primary 32F45, Secondary 30F45, 46G20.

Given $r > 0$, let $[a, b]$ be any hyperbolic line segment in Δ with length $\rho(a, b) = r$ and let c be the hyperbolic midpoint of $[a, b]$. Where in Δ should we place c in order to maximize the Euclidean distance $|a - b|$ from a to b?

All agree that c should be placed at the (Euclidean) center of Δ. This intuitively obvious answer is correct and is easily expressed as an analytic inequality. Observe first that if $c = 0$, then $a = -b$ and

$$|a - b| = 2|a| = 2 \tanh \rho(a, 0) = 2 \tanh \frac{\rho(a, b)}{2}.$$

Therefore, placing c at 0 is optimal if and only if

$$|a - b| \le 2 \tanh \frac{\rho(a, b)}{2} \qquad \text{for all } a \text{ and } b \text{ in } \Delta. \tag{2}$$

Although (2) is easy to prove and answers a very natural question, it is surprisingly hard to find in the literature. The geometric version of (2) is stated on p. 65 of [4], but no proof is offered there. The earliest statement and proof of the analytic version that we know of are in Chapter I of Vuorinen [18]. Inequality (2) is proved there not only for Δ but for the unit ball model of real hyperbolic space of any dimension $n \ge 2$. Equality continues to hold when $a = -b$. (See inequality (2.27) in [18], where the hyperbolic metric has curvature -1.)

We shall return to real hyperbolic space in §7. Our chief purpose in this paper is to generalize (2) in a quite different direction. We think of Δ as a subset of $\mathbb{C}$, not $\mathbb{R}^2$, and for us the crucial property of the Poincaré metric is that holomorphic maps of Δ into itself satisfy the Schwarz-Pick lemma. A natural replacement for Δ is the open unit ball of a complex Banach space. The Hahn-Banach theorem immediately implies the first part of the following theorem.

Theorem 1. *Let B be the open unit ball of a complex Banach space X, and let d be any metric on B that satisfies $d(a, b) \ge \rho(\ell(a), \ell(b))$ for all a and b in B and all continuous linear functionals ℓ on X of norm one. Then*

$$\|a - b\| \le 2 \tanh \frac{d(a, b)}{2} \qquad \text{for all } a \text{ and } b \text{ in } B. \tag{3}$$

Equality holds in (3) if and only if either $a = b$ or there is a continuous linear functional ℓ on X satisfying

$$\|\ell\| = 1,\ \ell(b) = -\ell(a),\ \text{and } \rho(\ell(a), \ell(b)) = d(a, b). \tag{4}$$

Any ℓ that satisfies (4) also satisfies $|\ell(a - b)| = \|a - b\|$.

To prove (3), observe that (2) and the assumption on d give

$$2\tanh\frac{d(a,b)}{2} \geq 2\tanh\frac{\rho(\ell(a),\ell(b))}{2} \geq |\ell(a-b)|$$

for all linear functionals ℓ of norm one, and apply the Hahn-Banach theorem. The remainder of Theorem 1 will be proved in §2.

When Theorem 1 is applied to Δ with the metric ρ, it yields the statement that (2) holds, with equality if and only if $a = \pm b$. That statement was proved in [5], where it is Lemma 4.3. We shall give a more elementary proof in §2.

The following corollaries of Theorem 1 will also be proved in §2. They are easy consequences of the added hypotheses on d and B.

Corollary 1. *Suppose the metric d in Theorem 1 satisfies the additional condition that $d(0,x) = \rho(0,\|x\|)$ for all x in B. Then equality holds in* (3) *whenever a and b in B satisfy $\|a\| = \|b\|$ and $\|a-b\| = \|a\| + \|b\|$, and in particular when $a = -b$.*

Corollary 2. *Let the metric d be as in Theorem 1 and Corollary 1. If $a = \pm b$ whenever a and b are points in B such that equality holds in* (3)*, then every unit vector in X is an extreme point of the closed unit ball.*

Any metric on B that belongs to a Schwarz-Pick system (see §3) satisfies the hypotheses of both Theorem 1 and its corollaries. The smallest such metric is the Carathéodory metric.

Our opening paragraphs interpreted (2) as a statement about segments in Δ. Our second theorem provides a similar interpretation of (3) for certain points a and b in B. We shall prove it in §3, where the undefined terms in its statement are explained.

Theorem 2. *Let d be the Carathéodory metric on the open unit ball B, and let $f\colon \Delta \to B$ be a complex geodesic. If a and b are distinct points in $f(\Delta)$, and if c is the midpoint of the geodesic segment $[a,b]$ in $f(\Delta)$, then the following statements are equivalent.*

(a) *If a nonzero vector v in X is tangent to $[a,b]$ at c, then its infinitesimal Carathéodory length $\alpha(c,v)$ satisfies $\alpha(c,v) = \|v\|$,*

(b) $\|a-b\| = 2\tanh\dfrac{d(a,b)}{2}$.

Corollary 3. *If* 0 *is the midpoint of the geodesic segment* $[a,b]$ *in* $f(\Delta)$*, then statement (b) holds,* $\|a\| = \|b\|$ *and* $\|a-b\| = \|a\| + \|b\|$.

The example in §3 shows that condition (a) in Theorem 2 can hold even if $c \neq 0$ and $\|a\| \neq \|b\|$. In particular, the conditions for equality in (3) stated in Corollaries 1 and 3 are sufficient but not necessary. The example in §4 shows that these sufficient conditions do not imply that $a = \pm b$.

In §4 and §5, we study the case when B is homogeneous (or, equivalently, a bounded symmetric domain). The open unit ball of a Hilbert space plays a special role. By Theorem 3 in §4, Hilbert balls are precisely the homogeneous unit balls such that equality holds in (3) if and only if $a = \pm b$. They are also distinguished by uniqueness properties of their complex geodesics (see Theorem 4 in §5).

We continue our study of Hilbert balls in §6, where we estimate the failure of equality in (3) in terms of the norm of the midpoint of the geodesic segment $[a,b]$. That estimate leads to a similar distortion theorem for inequality (2.27) in Vuorinen [18]. We prove it in §7.

Our work on this paper began in October 2005, when we were both visiting University College Dublin. We thank Seán Dineen for making our visits possible and for his gracious hospitality during our stays.

2. Proof of Theorem 1 and its Corollaries

We start with the promised elementary treatment of the one-dimensional case.

Lemma 1. *Inequality* (2) *holds, with equality if and only if* $a = \pm b$.

Proof of Lemma 1. First we prove (2). Given a and b in Δ, put $x := \tanh\rho(a,b)$ and $y := \dfrac{|a-b|}{2}$, so that (2) takes the equivalent forms

$$y \le \tanh\frac{\tanh^{-1}x}{2}, \quad \tanh^{-1}y \le \frac{\tanh^{-1}x}{2}, \quad \text{and} \quad \tanh(2\tanh^{-1}y) \le x.$$

The identity

$$\tanh\left(2\tanh^{-1}y\right) = \frac{2y}{1+y^2}, \quad -1 < y < 1, \tag{5}$$

therefore implies that (2) is equivalent to

$$\frac{4|a-b|}{4+|a-b|^2} \le \tanh \rho(a,b) \qquad \text{for all } a \text{ and } b \text{ in } \Delta. \tag{6}$$

To prove (6), assume $b \neq 0$ and put $r = |1 - a/b|/2$. Then since

$$1 - \bar{b}a = \left(1 - \frac{a}{b}\right)|b|^2 + 1 - |b|^2,$$

it follows from the triangle inequality that

$$|1-\bar{b}a| \le 2r|b|^2 + 1 - |b|^2 \le (1+r^2)|b|^2 + 1 - |b|^2 = 1 + \left|\frac{b-a}{2}\right|^2. \tag{7}$$

Hence,

$$\frac{4|a-b|}{4+|a-b|^2} \le \left|\frac{a-b}{1-\bar{b}a}\right| = \tanh \rho(a,b) \qquad \text{for all } a \text{ and } b \text{ in } \Delta,$$

since this inequality is obvious when $b = 0$. That proves (6), hence (2).

We already know that equality holds in (2) if $a = \pm b$. Suppose, conversely, that equality holds. Without loss of generality we may suppose that $|a| \le |b|$. If $a \neq b$ and $b \neq 0$, then equality holds in (7) and thus $r = 1$. It follows that $a = -b$ since the closed unit disk intersects the circle $|z - 1| = 2$ only at $z = -1$. □

Proof of Theorem 1. We proved (3) in the introduction. If equality holds in (3) and $a \neq b$, choose a linear functional ℓ of norm one with $\ell(a-b) = \|a-b\|$. Since the first and last terms in the string of inequalities

$$2\tanh\frac{d(a,b)}{2} \ge 2\tanh\frac{\rho(\ell(a),\ell(b))}{2} \ge |\ell(a-b)|$$

are equal, the middle term equals both of them. By Lemma 1, ℓ satisfies (4).

Finally, suppose the linear functional ℓ satisfies (4). Then

$$\|a-b\| \le 2\tanh\frac{d(a,b)}{2} = 2\tanh\frac{\rho(\ell(a),\ell(b))}{2} = |\ell(a)-\ell(b)| \le \|a-b\|,$$

so $|\ell(a-b)| = \|a-b\|$ and equality holds in (3). □

Proof of Corollary 1. We may assume $a \neq b$. By Theorem 1, it suffices to find a linear functional that satisfies (4).

Choose ℓ so that $\|\ell\| = 1$ and $\ell(a-b) = \|a-b\|$. By hypothesis,

$$|\ell(a) + \ell(-b)| = \|a-b\| = \|a\| + \|b\| \ge \|a\| + |\ell(-b)| \ge |\ell(a)| + |\ell(-b)|,$$

which is possible only if $\ell(-b)$ lies on the ray from 0 to ∞ through $\ell(a)$ in $\mathbb{C}$. It follows that $\ell(a) = \ell(-b)$, as $|\ell(a)| = \|a\| = \|b\| = |\ell(-b)|$.

It remains to prove that $\rho(\ell(a), \ell(b)) = d(a, b)$. This is where the added hypothesis on d will be used. We have

$$\begin{aligned} d(a,b) \geq \rho(\ell(a), \ell(b)) &= \rho(\ell(a), -\ell(a)) = \rho(\|a\|, -\|a\|) \\ &= \rho(0, \|a\|) + \rho(0, \|b\|) = d(0,a) + d(0,b) \geq d(a,b). \qquad \square \end{aligned}$$

Proof of Corollary 2. Suppose the unit vector u in X is not an extreme point of the closed unit ball. Then there are vectors x and y in the closed unit ball, distinct from u, such that $u = (x + y)/2$. Hence

$$2 = 2\|u\| = \|x + y\| \leq \|x\| + \|y\| \leq \|x\| + 1 \leq 2,$$

so $\|x\| = \|y\| = 1$ and $\|x\| + \|y\| = \|x + y\|$.

Choose any r in $\mathbb{R}$ with $0 < r < 1$, and set $a := rx$ and $b := -ry$. Then a and b belong to B, $\|a\| = \|b\| = r$, and $\|a - b\| = 2r\|u\| = 2r$, so equality holds in (3), by Corollary 1. Since $a - b = 2ru$ and $x \neq y$, neither $a - b$ nor $a + b$ is zero. $\square$

3. Schwarz-Pick Metrics, Complex Geodesics, and Proof of Theorem 2

If U and V are complex Banach manifolds, we shall denote the set of all holomorphic maps of U into V by $\mathcal{O}(U, V)$. By definition, a *Schwarz-Pick system* assigns a pseudometric d_U to each complex Banach manifold U so that the following conditions hold.

(a) The pseudometric d_Δ is the Poincaré metric ρ.

(b) If U and V are complex Banach manifolds and $f \in \mathcal{O}(U, V)$, then

$$d_V(f(u_1), f(u_2)) \leq d_U(u_1, u_2) \quad \text{for all } u_1 \text{ and } u_2 \text{ in } U. \tag{8}$$

In particular, if $f\colon U \to U$ is a biholomorphic map of U onto itself, then

$$d_U(f(u_1), f(u_2)) = d_U(u_1, u_2) \quad \text{for all } u_1 \text{ and } u_2 \text{ in } U.$$

Schwarz-Pick systems were first studied systematically in Harris [7], and have been extensively studied since then (see for example [12]).

If B is the open unit ball of a complex Banach space X, we shall call the metric d on B a *Schwarz-Pick metric* if $d = d_B$ for some Schwarz-Pick system. All such metrics satisfy

$$\rho(f(x_1), f(x_2)) \le d(x_1, x_2) \quad \text{and} \quad d(g(z_1), g(z_2)) \le \rho(z_1, z_2) \tag{9}$$

for all f in $\mathcal{O}(B, \Delta)$, g in $\mathcal{O}(\Delta, B)$, x_1 and x_2 in B, and z_1 and z_2 in Δ. Applied to appropriate complex linear maps f and g, (9) implies the well-known formula

$$d(0, x) = \rho(0, \|x\|) = \tanh^{-1} \|x\| \quad \text{for all } x \text{ in } B, \tag{10}$$

satisfied by every Schwarz-Pick metric on B. More generally, if $a \in B$, $b \in B$, and f is a biholomorphic map of B onto itself satisfying $f(b) = 0$, then

$$d(a, b) = d(f(a), 0) = \tanh^{-1} \|f(a)\|$$

for every Schwarz-Pick metric d on B (see [7, p. 357]). Hence d is uniquely determined if B is homogeneous (i.e., for every a and b in B there is a biholomorphic map f of B onto itself with $f(a) = b$).

By (9) and (10), every Schwarz-Pick metric on B satisfies the hypotheses of both Theorem 1 and Corollary 1, as we stated in the introduction.

The *Carathéodory pseudometric* d on a complex Banach manifold U is defined by the formula

$$d(u_1, u_2) := \sup\{\rho(f(u_1), f(u_2)) : f \in \mathcal{O}(U, \Delta)\}, \quad u_1 \text{ and } u_2 \text{ in } U. \tag{11}$$

Carathéodory introduced this pseudometric in [2] for bounded regions U in $\mathbb{C}^2$, on which it is a metric.

It is easy to see that assigning its Carathéodory pseudometric to each U produces a Schwarz-Pick system, so the Carathéodory metric on an open unit ball B is a Schwarz-Pick metric. By (9) and (11), it is the smallest such metric on B.

By definition, the *Carathéodory length* of a vector v in X at the point x in B is the number

$$\alpha(x, v) := \sup\{|Df(x)v| : f \in \mathcal{O}(B, \Delta)\}.$$

Clearly, $\alpha(x, zv) = |z|\alpha(x, v)$ for all x in B, v in X, and z in $\mathbb{C}$. Since every linear functional of norm one on X belongs to $\mathcal{O}(B, \Delta)$, the inequality

$$\alpha(x, v) \ge \|v\| \quad \text{for all } x \text{ in } B \text{ and } v \text{ in } X \tag{12}$$

follows immediately from the Hahn-Banach theorem. Also, the Cauchy estimate for the first derivative (see [10, p. 111]) implies the formula

$$\alpha(0, v) = \|v\| \quad \text{for all } v \text{ in } X. \tag{13}$$

It is well known (see for example Corollary 4.5 in [5]) that the Carathéodory distance d and length α satisfy

$$\alpha(f(0), f'(0)) = \lim_{z\to 0} \frac{d(f(0), f(z))}{|z|} \quad \text{for all } f \text{ in } \mathcal{O}(\Delta, B). \tag{14}$$

We call f in $\mathcal{O}(\Delta, B)$ a *complex geodesic* if and only if

$$d(f(z_1), f(z_2)) = \rho(z_1, z_2) \quad \text{for all } z_1 \text{ and } z_2 \text{ in } \Delta. \tag{15}$$

For example, if u is any unit vector in X, the function $f(z) := zu$, $z \in \Delta$, is a complex geodesic. To see this, choose a linear functional ℓ such that $\|\ell\| = \ell(u) = 1$, and observe that

$$\rho(z_1, z_2) = \rho(\ell(f(z_1)), \ell(f(z_2))) \le d(f(z_1), f(z_2)) \le \rho(z_1, z_2)$$

for all z_1 and z_2 in Δ.

For a complex geodesic f, (14) gives

$$\alpha(f(0), f'(0)) = \lim_{z\to 0} \frac{\rho(0, z)}{|z|} = 1. \tag{16}$$

Conversely, any f in $\mathcal{O}(\Delta, B)$ that satisfies (16), or satisfies

$$d(f(z_1), f(z_2)) = \rho(z_1, z_2)$$

for a single pair of distinct points in Δ, is a complex geodesic (see Propositions 3.2 and 3.3 in [17] or Proposition 6.11 in [3]). We shall not use that fact here.

Our interest is in geodesic segments in $f(\Delta)$ and the vectors tangent to their midpoints. Let $f\colon \Delta \to B$ be a complex geodesic, and let a and b be distinct points in $f(\Delta)$. Let r be the real number such that $0 < r < 1$ and $\rho(-r, r) = d(a, b)$. There is a conformal automorphism ϕ of Δ that maps $f^{-1}(a)$ and $f^{-1}(b)$ to $-r$ and r respectively. Replacing f by the complex geodesic $f \circ \phi^{-1}$, we may assume that $a = f(-r)$ and $b = f(r)$.

By definition, the *geodesic segment* $[a, b]$ *in* $f(\Delta)$ is the set $[a, b] := f([-r, r])$. The closed interval $[-r, r]$ is a hyperbolic geodesic segment in Δ, and its hyperbolic midpoint is 0; i.e., $\rho(-r, 0) = \rho(0, r) = \rho(-r, r)/2$ by (5). We define the *midpoint of* $[a,b]$ to be the point $c := f(0)$. Clearly,

$$d(a,c) = d(c,b) = \frac{1}{2}d(a,b). \tag{17}$$

The vector $f'(0)$ in X is tangent to $[a,b]$ at c. We call it the *unit tangent vector to* $[a,b]$ *at* c because its Carathéodory length $\alpha(f(0), f'(0))$ equals one. The vectors tangent to $[a,b]$ at c are the positive scalar multiples of $f'(0)$.

In general, distinct points a and b in B may lie in the images of complex geodesics f and g such that the geodesic segments $[a,b]$ in $f(\Delta)$ and $g(\Delta)$ are not the same. This does not happen if X is a Hilbert space (see §6).

Proof of Theorem 2. We are given a complex geodesic $f\colon \Delta \to B$ and distinct points a and b in $f(\Delta)$. As above, we assume that $a = f(-r)$ and $b = f(r)$, with $0 < r < 1$, so $c := f(0)$ is the midpoint of $[a,b]$, and $f'(0)$ is the unit tangent vector to $[a,b]$ at c.

By a vector-valued version of the Schwarz lemma (see e.g. [10, p. 100]), if $g : \Delta \to B$ is any holomorphic function with $g(0) = 0$, then $\|g(z)\| = |z|$ for all z in Δ when this equality holds for some nonzero z in Δ or when $\|g'(0)\| = 1$. We shall apply this to

$$g(z) := \frac{f(z) - f(-z)}{2}, \quad z \in \Delta.$$

Observe that $g'(0) = f'(0)$.

If equality holds in (3), then

$$2\|g(r)\| = \|a - b\| = 2\tanh\frac{d(a,b)}{2} = 2\tanh\frac{\rho(-r,r)}{2} = 2\tanh\rho(0,r) = 2r,$$

so Schwarz's lemma and (16) give $\|f'(0)\| = \|g'(0)\| = 1 = \alpha(f(0), f'(0))$. Therefore (b) implies (a).

Conversely, if $\|f'(0)\| = \alpha(f(0), f'(0))(= 1)$, then $\|g'(0)\| = 1$, so Schwarz's lemma gives $\|g(r)\| = r$. Therefore,

$$\|a - b\| = 2r = 2\tanh\frac{\rho(-r,r)}{2} = 2\tanh\frac{d(a,b)}{2},$$

and (a) implies (b). □

Proof of Corollary 3. The equality $\|a\| = \|b\| = \tanh d(a,b)/2$ follows from (10) and (17) with $c = 0$. Since $\alpha(0,v) = \|v\|$ for all v, statement (a) holds. Hence (b) holds as well, so $\|a - b\| = 2\tanh d(a,b)/2 = \|a\| + \|b\|$. □

Example. Here we show that the equivalent statements of Theorem 2 can hold with $c \neq 0$ and $\|a\| \neq \|b\|$. Let X be $\mathbb{C}^2$ with the norm

$$\|(z, w)\| = \max\{|z|, |w|\} \qquad \text{for all } z \text{ and } w \text{ in } \mathbb{C}.$$

The open unit ball is the unit polydisk Δ^2 in $\mathbb{C}^2$. It is well known and easy to verify that the Carathéodory metric d and length function α on Δ^2 satisfy

$$d((z, w), (z', w')) = \max\{\rho(z, z'), \rho(w, w')\}, \quad (z, w) \text{ and } (z', w') \text{ in } \Delta^2, \text{ and}$$

$$\alpha((z, w), (u, v)) = \max\left\{\frac{|u|}{1 - |z|^2}, \frac{|v|}{1 - |w|^2}\right\}, \quad (z, w) \text{ in } \Delta^2 \text{ and } (u, v) \text{ in } \mathbb{C}^2.$$

Define f in $\mathcal{O}(\Delta, \Delta^2)$ by

$$f(z) := \left(z, \frac{z + 1}{2}\right), \quad z \text{ in } \Delta.$$

Then f is a complex geodesic, and $\|f'(0)\| = \alpha(f(0), f'(0)) = 1$. Choose any real number r with $0 < r < 1$, and put $a := f(-r)$ and $b := f(r)$. By Theorem 2, equality holds in (3), even though

$$\|a\| = \max\left\{r, \frac{1 - r}{2}\right\} < \frac{1 + r}{2} = \|b\|.$$

4. Bounded Symmetric Domains

By definition, a bounded symmetric domain D in a complex Banach space X is a bounded domain in X where for each x in D there exists a biholomorphic mapping S of D (called a symmetry) such that $S^2 = I$ and x is an isolated fixed point of S, where I is the identity map on D. By a deep classification theorem of Kaup [14], every such D is holomorphically equivalent to the open unit ball B of a JB*-triple.

By definition, a JB*-triple U is a complex Banach space where there is a continuous sesquilinear mapping $<\cdot, \cdot>$ of $U \times U$ into the bounded linear operators on U such that for any x, y, z, w in U,

i) $<x, y>z = <z, y>x$,
ii) $<<x, y>z, w> - <z, <y, x>w> = <x, y><z, w> - <z, w><x, y>$,
iii) the spectrum $\sigma(<x, x>)$ is contained in the interval $[0, \infty)$,
iv) $\exp(i<x, x>)$ is an isometry of U,
v) $\|<x, x>x\| = \|x\|^3$.

For example, if U is finite dimensional then $(x, y) = \mathrm{tr}{<}x, y{>}$ is an inner product on U and ${<}x, y{>}^* = {<}y, x{>}$ by (ii).

It is shown in [11, p. 285] that for each $b \in B$ there is a biholomorphic mapping T_b of B, defined in a simple way in terms of the JB* structure, such that $T_b(0) = b$ and $T_{-b} = T_b^{-1}$. Hence every bounded symmetric domain is homogeneous. Conversely, every homogeneous open unit ball B is a bounded symmetric domain. If b is in B and T is a biholomorphic mapping of B with $T(0) = b$, then $S(x) = T(-T^{-1}(x))$ is the required symmetry of B at b.

The main example of a JB*-triple is a J*-algebra. These were introduced in [6] and studied as generalizations of C*-algebras in [9]. By definition, a J*-algebra is a closed complex subspace X of the bounded linear operators from one Hilbert space to another such that xy^*x is in X whenever x and y are in X. Then X is a JB*-triple with ${<}x, y{>}z = (xy^*z + zy^*x)/2$ and

$$T_b(x) = (I - bb^*)^{-1/2}(x + b)(I + b^*x)^{-1}(I - b^*b)^{1/2},$$

where I denotes the identity map on one of the underlying Hilbert spaces.

In particular, every Hilbert space H is a J*-algebra since it is isometrically isomorphic to the space of bounded linear operators from $\mathbb{C}$ to H, and

$$T_b(x) = \frac{b + E_b(x) + \sqrt{1 - \|b\|^2}(I - E_b)x}{1 + (x, b)} \tag{18}$$

is a biholomorphic mapping of the open unit ball of H. Here E_b is the projection of H onto the span of b and is given by $E_b(x) := (x, b)b/\|b\|^2$ when $b \neq 0$.

A nonzero element u of a JB*-triple U is called a tripotent if ${<}u, u{>}u = u$. It follows from [15, Proposition 3.5] that every extreme point of the closed unit ball of U is a tripotent. In the case where U is a J*-algebra, the tripotents are the nonzero partial isometries of U. In this section and the next, we will need the following well-known fact about JB*-triples given in [13, Lemma (4.7)]. For completeness, at the end of the section we give a direct proof which is a straightforward extension to JB*-triples of part of Proposition 7 of [8].

Lemma 2. *If every element of a JB*-triple U is a scalar multiple of a tripotent, then U is a Hilbert space in its norm.*

A Gelfand-Naimark theorem of Friedman and Russo answers many questions about JB*-triples since it reduces their study to that of J*-algebras and the exceptional triples in dimensions 16 and 27. For further references on JB*-triples, see [16].

Theorem 3. *Suppose the open unit ball B of a complex Banach space X is a bounded symmetric domain and let d be the unique Schwarz-Pick metric on B. Then X is a Hilbert space if and only if $a = \pm b$ whenever $a, b \in B$ and*

$$\|a - b\| = 2 \tanh \frac{d(a,b)}{2}. \tag{19}$$

Proof. To prove the forward implication, suppose X is a Hilbert space and (19) holds for some $a, b \in B$ with $a \neq b$. We show that $a = -b$. By Theorem 1 there exists a continuous linear functional ℓ on X satisfying (4). By multiplying ℓ by a complex number of modulus one if necessary, we may suppose that $\ell(a - b) = \|a - b\|$. Clearly $2\ell(a) = \|a - b\|$ and ℓ has the form $\ell(x) = (x, a - b)/\|a - b\|$ for $x \in X$ by the Riesz representation theorem. Hence

$$0 = \ell(a + b)\|a - b\| = \|a\|^2 - (a, b) + \overline{(a, b)} - \|b\|^2$$

and therefore (a, b) is real and $\|a\| = \|b\|$.

Taking hyperbolic tangents in the equality $d(a, b) = \rho(\ell(a), \ell(b))$, we obtain

$$\|T_{-b}(a)\| = \left|T_{-\ell(b)}(\ell(a))\right| = \frac{2r}{1 + r^2},$$

where $r := \ell(a) = \|a - b\|/2$. It follows from the identity

$$1 - \|T_{-b}(a)\|^2 = \frac{(1 - \|a\|^2)(1 - \|b\|^2)}{|1 - (a, b)|^2}$$

that

$$\frac{1 - \|a\|^2}{1 - (a, b)} = \frac{1 - r^2}{1 + r^2}.$$

Hence,

$$[\,\|a\|^2 + (a, b)]r^2 = 2r^2 + (a, b) - \|a\|^2 = 0,$$

so $\|a + b\|^2 = 0$. Thus $a = -b$.

To prove the reverse implication, we first observe that X may be taken to be a JB*-triple. To see this, observe that B is holomorphically equivalent to the open unit ball of a JB*-triple U by the classification theorem of Kaup and thus X and U are isometrically isomorphic Banach spaces by [6, Corollary 1]. By hypothesis and Corollary 2, every unit vector of X is an extreme point of the closed unit ball of X and hence is a tripotent. Thus every element of X is a scalar multiple of a tripotent and therefore X is a Hilbert space by Lemma 2. □

By Corollary 3, equality (19) as well as the norm conditions of Corollary 1 hold if 0 is the midpoint of a geodesic segment $[a, b]$ in B. The following gives many explicit cases where this occurs but $a \neq \pm b$.

Example. Let U be a JB*-triple and let $b \in B$ be nonzero. Take $a = T_b(-\gamma b)$, where $\gamma = 2/(1 + r^2)$ and $r = \|b\|$, and define

$$f(z) = T_b\left(-\frac{T_r(z)}{r}\,b\right). \tag{20}$$

Then $f(-r) = b$, $f(0) = T_b(-b) = 0$ and $f(r) = T_b(-\gamma b) = a$.

We have shown in §3 that $g(z) = -zb/r$ is a complex geodesic in B. Hence $f = T_b \circ g \circ T_r$ is a complex geodesic in B since T_b and T_r are isometries in the Carathéodory and Poincaré metrics for B and Δ, respectively. Thus 0 is the midpoint of the geodesic segment $[a, b]$ in $f(\Delta)$, and the results of Corollary 3 apply.

Note that $a \neq b$ since b is nonzero. If U is a J*-algebra it is not difficult to show that $a = -b$ if and only if $bb^*b = \|b\|^2 b$, i.e., b is a scalar multiple of a partial isometry. This argument extends to JB*-triples since there is a J*-isomorphism of the JB*-subtriple generated by b onto a space of continuous functions [14, Corollary 1.15]. Thus one obtains another proof of the reverse implication in Theorem 3.

By (13) and (16), the unit tangent vector to $[a, b]$ at 0 is a unit vector in the norm of U and

$$v = f'(0) = DT_b(-b)\left(-\frac{T_r'(0)}{r}\,b\right) = \frac{r^2 - 1}{r} b(I - b^*b)^{-1}$$

for J*-algebras by [6, p. 20]. A direct operator-theoretic proof that $\|v\| = 1$ is less simple.

Proof of Lemma 2. Since all one dimensional JB*-triples are J*-isomorphic to the complex plane, we may suppose that U has dimension of at least two. We first show that for any x and y in U there exists a complex number a such that $<x, y>x = ax$. This is clear by hypothesis when x and y are linearly dependent.

Let x and y be any linearly independent elements of U and set

$$f(z) = <x + zy, x + zy>(x + zy)$$

for $z \in \mathbb{C}$. By hypothesis, there exists a function $\phi : \mathbb{C} \to \mathbb{C}$ such that $f(z) = \phi(z)(x + zy)$ and, by the sesquilinearity of the mapping $<\cdot, \cdot>$, we may write

$$f(z) = c_0 + c_1 z + c_2 \overline{z} + c_3 z^2 + c_4 |z|^2 + c_5 z|z|^2 \tag{21}$$

for $z \in \mathbb{C}$. It is not difficult to verify that each of the coefficients $c_0, \ldots, c_5$ can be expressed as a linear combination of the values of f (at, for example, $0, \pm 1, \pm i$ and $1 \pm i$). Thus there exist complex numbers $a_0, \ldots, a_5$ and $b_0, \ldots, b_5$ such that $c_k = a_k x + b_k y$ for $k = 0, \ldots, 5$. Equating the coefficients of x and y in (21), we obtain

$$\phi(z) = a_0 + a_1 z + a_2 \overline{z} + a_3 z^2 + a_4 |z|^2 + a_5 z|z|^2,$$
$$z\phi(z) = b_0 + b_1 z + b_2 \overline{z} + b_3 z^2 + b_4 |z|^2 + b_5 z|z|^2$$

for all $z \in \mathbb{C}$. Clearly $b_0 = 0$ and ϕ is continuous at $z = 0$. Hence dividing the last equality by z and letting z approach 0 through real and imaginary values, we obtain $b_2 = 0$. Therefore, $<x, y>x = c_2 = a_2 x$, as asserted.

Define (x, y) to be the complex number with $<x, y>x = (x, y)x$ if $x \neq 0$ and define $(x, y) = 0$ if $x = 0$. Obviously, $(x, y) = 0$ if $y = 0$. It is easy to verify that (x, y) is well defined and conjugate linear in y. We next show that

$$(x, y) = \overline{(y, x)} \qquad \text{for all } x, y \in U. \tag{22}$$

This holds when either $x = 0$ or $y = 0$ so suppose $x, y \neq 0$. It follows from (ii) that $<<x, y>x, y> = <x, <y, x>y>$ so $(x, y)<x, y> = \overline{(y, x)}<x, y>$. If $<x, y> \neq 0$ then $(x, y) = \overline{(y, x)}$ as desired. If $<x, y> = 0$, let $w \in U$ and take $z = <y, x>w$ in (ii). Then $<z, z> = 0$ so $z = 0$ by (v). Hence $<y, x> = 0$. Thus $(x, y) = \overline{(y, x)} = 0$.

It follows from (22) that $(\cdot, \cdot)$ is sesquilinear. Also since $<x, x>x = (x, x)x$, it follows from (iii) that $(x, x) \geq 0$ and hence $(x, x) = \|x\|^2$ by (v). Thus U is a Hilbert space in the inner product $(\cdot, \cdot)$. □

5. Complex Geodesics in Bounded Symmetric Domains

If $f : \Delta \to B$ is a complex geodesic and $p \in f(\Delta)$, we say that f passes through p. Existence and uniqueness of complex geodesics passing through two distinct points has been thoroughly studied for convex domains in many Banach spaces (see e.g. [3, §6.2]). We consider only open unit balls that are bounded symmetric domains. Existence is trivial in that case; we include a proof for the reader's convenience.

Lemma 3. *Suppose the open unit ball B of a complex Banach space X is a bounded symmetric domain. If a and b are distinct points in B, then there is a complex geodesic passing through them.*

Proof. Let the distinct points a and b in B be given, and let T_b be a biholomorphic mapping of B with $T_b(0) = b$. Set $f(z) = zT_b^{-1}(a)/\|T_b^{-1}(a)\|$, z in Δ. Then f is a complex geodesic passing through $T_b^{-1}(a)$ and 0. Since T_b preserves Carathéodory distances, $T_b \circ f$ is a complex geodesic passing through a and b. □

A discussion of uniqueness must allow for reparametrization. Two complex geodesics f and g in an open unit ball B are said to be the same up to parametrization if there is a conformal automorphism ϕ of Δ such that $f \circ \phi = g$. There is a sharp uniqueness result for complex geodesics passing through 0 and some nonzero a in B; it depends on the notion of complex extreme points. By definition, a point x in a convex set K is a complex extreme point of K if $\{x + \zeta y : \zeta \in \Delta\} \subset K$ implies $y = 0$. In particular, every extreme point of K is a complex extreme point of K. The following statement is a paraphrase of Proposition 6.20 on page 95 of Dineen [3]. We refer to that book for its proof.

Proposition 1. *Let a be a nonzero vector in the open unit ball B. All complex geodesics passing through 0 and a are the same up to parametrization if and only if $a/\|a\|$ is a complex extreme point of the closed unit ball.*

Theorem 3 used equation (19) to characterize Hilbert balls among bounded symmetric domains. We can now give a second characterization. The following theorem follows so readily from Proposition 1, Lemma 2, and [15, Proposition 3.5] that it is undoubtedly familiar to experts. For completeness, we provide the proof.

Theorem 4. *Suppose the open unit ball B of a complex Banach space X is a bounded symmetric domain. Then X is a Hilbert space if and only if all complex geodesics that pass through any given pair of distinct points in B are the same up to parametrization.*

Proof. To prove the forward implication, suppose X is a Hilbert space, and choose distinct points a and b in B. Let T_b be the automorphism (18) of B. Consider the set of all complex geodesics $f\colon \Delta \to B$ passing through a and b. For each f, the complex geodesic $T_{-b} \circ f$ passes through $T_{-b}(a)$ and 0. By Proposition 1, all the complex geodesics $T_{-b} \circ f$ are the same up to parametrization, since every unit vector of a Hilbert space is an extreme point of its closed unit ball. Thus the complex geodesics f are the same up to parametrization.

For the converse, we shall again take X to be a JB*-triple U. By Proposition 1 and the hypothesis, every unit vector of U is a complex extreme point of the closed unit ball of U and hence is a tripotent by [15, Proposition 3.5]. Therefore, U is a Hilbert space in its norm by Lemma 2. □

The next example gives many pairs of points where there are complex geodesics passing through the points that are not the same up to parametrization.

Example. Let U be a JB*-triple, let $b \in B$ be nonzero and let $r = \|b\|$. Define complex geodesics f and g by (20) and $g(z) = -zb/r$, respectively. Then $f(0) = g(0) = 0$ and $f(-r) = g(-r) = b$. Suppose that complex geodesics passing through 0 and b are unique up to parametrization. Then $f = g \circ \phi$ where ϕ is an automorphism of Δ. Now $\phi(0) = 0$ since $g(\phi(0)) = f(0) = 0$ and $\phi(-r) = -r$ since $b = f(-r) = g(\phi(-r))$. Hence ϕ is the identity so $f = g$. In particular, $a := f(r) = g(r) = -b$ and thus b is a scalar multiple of a tripotent by the arguments given in the example of §4. Thus one obtains a proof of the reverse implication in Theorem 4 that does not require Proposition 1.

6. A Distortion Theorem for Segments in a Hilbert Ball

Theorems 3 and 4 invite a more detailed study of the open unit ball B of a Hilbert space. The unique Schwarz-Pick metric d on B is its Carathéodory metric. The following is an immediate consequence of Lemma 3 and Theorem 4.

Corollary 4. *Let B be the open unit ball of a Hilbert space H, and let a and b be distinct points in B. Up to parametrization, there is exactly one complex geodesic $f\colon \Delta \to B$ that passes through both a and b.*

We remark that the corollary is a special case of Corollary 6.17 on page 93 of Dineen [3], but that result relies on much deeper arguments.

Let a and b be distinct points in a Hilbert ball B. By Lemma 3 and Corollary 4, there are complex geodesics passing through a and b, and they all determine the same geodesic segment $[a, b]$. Hence we can speak unambiguously of the geodesic segment $[a, b]$ and its midpoint c, without specifying f. The following lemma is an immediate consequence of Theorems 2 and 3.

Lemma 4. *Let B be the open unit ball of a Hilbert space H, and let a and b be distinct points in B. The following statements are equivalent.*

(a) *The midpoint of* $[a, b]$ *is* 0,

(b) $\|a - b\| = 2\tanh\dfrac{d(a,b)}{2}$,

(c) $a = -b$.

Proof. Since $a \neq b$, Theorems 2 and 3 give (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c). Now assume (c). Since $a \neq b$, $b \neq 0$. Consider the complex geodesic $f(z) = zb/\|b\|$, z in Δ. The segment $[f(-\|b\|), f(\|b\|)] = [-b, b]$. Its midpoint is $f(0) = 0$. □

Our next theorem quantifies the failure of statement (b) when 0 is not the midpoint of $[a, b]$. The following real valued function will play a role.

Lemma 5. *Given a real number s with $0 \le s < 1$, set*

$$\Psi_s(t) = \begin{cases} \sqrt{1-s} & \textit{if} \quad 0 \le t \le \dfrac{1}{2}, \\ \sqrt{\dfrac{1-s}{4t(1-t)}} & \textit{if} \quad \dfrac{1}{2} < t < \dfrac{1}{2-s}, \\ \dfrac{1-s}{1-st} & \textit{if} \quad \dfrac{1}{2-s} \le t \le 1. \end{cases} \tag{23}$$

The function Ψ_s is C^1, $\Psi_s'(t) \ge 0$ for all t in $[0, 1]$, and $0 < \Psi_s(t) < 1$ when $0 < s < 1$ and $0 \le t < 1$.

The proof is elementary.

Theorem 5. *Let d be the unique Schwarz-Pick metric on the open unit ball B of a Hilbert space H. Let a and b be distinct points in B, and let c be the midpoint of the geodesic segment $[a, b]$. Set $r := \tanh(d(a, b)/2)$.*

If $\dim(H) = 1$, *then*

$$\frac{1 - \|c\|^2}{1 + r^2\|c\|^2} \le \frac{\|a - b\|}{2r} \le \frac{1 - \|c\|^2}{1 - r^2\|c\|^2}. \tag{24}$$

If $\dim(H) > 1$*, then*

$$\frac{1-\|c\|^2}{1+r^2\|c\|^2} \le \frac{\|a-b\|}{2r} \le \Psi_{\|c\|^2}(r^2). \tag{25}$$

These are the best possible bounds that depend only on $\|c\|$ *and* r*.*

Proof. We may assume $c \neq 0$.

The image of $[a,b]$ under the biholomorphic automorphism T_{-c} of B is the geodesic segment $[T_{-c}(a), T_{-c}(b)]$. The center of that segment is $T_{-c}(c) = 0$, so $T_{-c}(a) = -T_{-c}(b)$, by Lemma 4. Set $x := T_{-c}(b)$. Then $b = T_c(x)$ and $a = T_c(-x)$. Since T_c is an isometry and $z \mapsto zx/\|x\|$, z in Δ, is a complex geodesic, we have

$$\|x\| = \tanh(d(0,x)) = \tanh\frac{d(-x,x)}{2} = \tanh\frac{d(a,b)}{2} = r,$$

and

$$\frac{\|a-b\|}{2r} = \frac{\|T_c(-x) - T_c(x)\|}{2\|x\|}.$$

Since the last term in the numerator of (18) is orthogonal to the others, it is easy to derive the identity

$$\|T_c(x) - T_c(-x)\|^2 = 4\frac{(1-\|c\|^2)(\|x\|^2 - |(x,c)|^2)}{|1-(x,c)^2|^2}, \tag{26}$$

which yields the sharp estimates

$$\frac{(1-\|c\|^2)(\|x\|^2 - |(x,c)|^2)}{\|x\|^2(1+|(x,c)|^2)^2} \le \Big(\frac{\|a-b\|}{2r}\Big)^2 \le \frac{(1-\|c\|^2)(\|x\|^2 - |(x,c)|^2)}{\|x\|^2(1-|(x,c)|^2)^2}. \tag{27}$$

If $\dim(H) = 1$, we always have $|(x,c)| = \|x\|\,\|c\|$, so (27) reduces to (24), since $\|x\| = r$. If $\dim(H) > 1$, $|(x,c)|$ can take any value in the interval $[0, \|x\|\,\|c\|]$. The left side of (27) is obviously minimized when $|(x,c)| = \|x\|\,\|c\|$, so the lower bounds in (24) and (25) are the same. To find an upper bound that depends only on $\|c\|$ and $\|x\|$, we solve the following calculus problem.

Set $s := \|c\|^2$, $t := \|x\|^2$, and $u := |(x,c)|^2$. We wish to maximize the function

$$F(u) := \frac{(1-s)(t-u)}{t(1-u)^2}$$

when s and t are fixed numbers in $(0,1)$ and u varies over the interval $[0, st]$. If $0 < t \le 1/2$, then $F'(u) < 0$ in $(0, st)$, so the maximum value of F is $F(0) = 1 - s$, which is $\Psi_s(t)^2$. If $(2-s)^{-1} \le t < 1$, then $F'(u) > 0$ in $(0, st)$, and the maximum value of F is $F(st) = \Psi_s(t)^2$. For the remaining values of t, $F'(0) > 0$, $F'(st) < 0$, and F attains its maximum at the zero of its derivative. The critical value is $\Psi_s(t)^2$. We leave details to the reader. These bounds are sharp, by construction. □

7. A Distortion Theorem for Segments in Real Hyperbolic Space

In this section, B^n will be the open unit ball of $\mathbb{R}^n$, $n \geq 2$, with respect to the usual Euclidean norm. The Euclidean norm of x in $\mathbb{R}^n$ will be denoted by $\|x\|$. We shall interpret B^n as the Poincaré model of hyperbolic n-space, scaling the metric d to have curvature -4, as we did in the introduction. With that scaling, inequality (2.27) in Vuorinen [18] takes the now-familiar form

$$\|a - b\| \leq 2 \tanh \frac{d(a,b)}{2} \qquad \text{for all } a \text{ and } b \text{ in } B^n.$$

This inequality is strict if $a \neq \pm b$. In fact, we shall prove an analogue of Theorem 5. Its statement and proof involve the geodesic segments in hyperbolic n-space. These are discussed, for example, on pages 21–24 of [18]. The midpoint c of the geodesic segment $[a, b]$ is characterized by equation (17) in §3.

Theorem 6. *Let B^n be the open unit ball of $\mathbb{R}^n$, $n \geq 2$, with its hyperbolic metric d of curvature -4, and let a and b be distinct points of B^n. Put $r := \tanh(d(a,b)/2)$.*

If c is the midpoint of the geodesic segment $[a, b]$, then

$$\frac{1 - \|c\|^2}{1 + r^2\|c\|^2} \leq \frac{\|a - b\|}{2r} \leq \frac{1 - \|c\|^2}{1 - r^2\|c\|^2}. \tag{28}$$

Proof. The case of B^2 in $\mathbb{R}^2$ is the same as the case of Δ in $\mathbb{C}$, with the Poincaré metric ρ. This is the one-dimensional case of Theorem 5. As inequality (28) is the same as inequality (24) in that theorem, it needs no further proof.

The general case reduces to the two-dimensional case by orthogonal projection of $\mathbb{R}^n$ onto a real two-dimensional subspace V that contains a and b. The geodesic segment $[a, b]$ lies entirely in $B^n \cap V$, which is isometric to the Poincaré disk. □

Remark. Theorem 6 can also be derived from a version of (26) in which T_c is a Möbius transformation that maps B^n to itself. Formulas in §2.7 of Ahlfors [1] simplify the required computations. We have not followed this path, as discussion of these formulas is outside the scope of this paper.

References

[1] L. V. Ahlfors, *Möbius transformations in several dimensions*, Lecture Notes, School of Mathematics, Univ. of Minnesota, 1981.

[2] C. Carathéodory, *Über das Schwarzsche Lemma bei analytischen Funktionen von zwei komplexen Veränderlichen*, Math. Ann. **97** (1926), 76–98.

[3] S. Dineen, *The Schwarz Lemma*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1989.

[4] C. J. Earle and R. S. Hamilton, *A fixed point theorem for holomorphic mappings*, in 'Global Analysis', Proc. Sympos. Pure Math. **XVI**, Amer. Math. Soc., Providence, 1970, pp. 61–65.

[5] C. J. Earle, L. A. Harris, J. H. Hubbard and S. Mitra, *Schwarz's lemma and the Kobayashi and Carathéodory pseudometrics on complex Banach manifolds*, in 'Kleinian Groups and Hyperbolic 3-Manifolds (Warwick, 2001)', London Math. Soc. Lecture Note Ser. 299, Cambridge Univ. Press, Cambridge, 2003, pp. 363–384.

[6] L. A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, in 'Proceedings on Infinite Dimensional Holomorphy (Internat. Conf., Univ. Kentucky, Lexington, Ky., 1973)', Lecture Notes in Math. 364, Springer-Verlag, Berlin–Heidelberg–New York, 1974, pp. 13–40.

[7] ———, *Schwarz-Pick systems of pseudometrics for domains in normed linear spaces*, in 'Advances in Holomorphy', Math. Studies 34, North-Holland, Amsterdam, 1979, pp. 345–406.

[8] ———, *Analytic invariants and the Schwarz-Pick inequality*, Israel J. Math. **34** (1979), 177–197.

[9] ———, *A generalization of C*-algebras*, Proc. London Math. Soc. **42** (1981), 331–361.

[10] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ., Vol. 31, AMS, Providence, 1957.

[11] J. M. Isidro and W. Kaup, *Weak continuity of holomorphic automorphisms in JB*-triples*, Math. Z. **210** (1992), 277–288.

[12] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter Expositions in Mathematics 9, Walter de Gruyter, Berlin–New York, 1993.

[13] W. Kaup, *Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension. I*, Math. Ann. **257** (1981), 463–486.

[14] ———, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503–529.

[15] ——— and H. Upmeier, *Jordan algebras and symmetric Siegel domains*, Math. Z. **157** (1977), 179–200.

[16] ———, K. McCrimmon and H. P. Petersson, eds., *Jordan Algebras*, (Proceedings of the conference held in Oberwolfach, August 9–15, 1992), Walter de Gruyter, Berlin, 1994.

[17] E. Vesentini, *Complex geodesics*, Compositio Math. **44** (1981), 375–394.

[18] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. 1319, Springer-Verlag, Berlin–Heidelberg–New York, 1988.

Clifford J. Earle
Department of Mathematics
Cornell University
Ithaca, NY 14853–4201
E-mail: cliff@math.cornell.edu

Lawrence A. Harris
Department of Mathematics
University of Kentucky
Lexington, KY 40506–0027
E-mail: larry@ms.uky.edu