Injectivity Criteria for Holomorphic Curves in $\mathbb{C}^n$

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To Professor F.W. Gehring on his 80th Birthday.

Abstract: Combining the definition of Schwarzian derivative for conformal mappings between Riemannian manifolds given by Osgood and Stowe with that for parametrized curves in Euclidean space given by Ahlfors, we establish injectivity criteria for holomorphic curves $\phi : \mathbb{D} \to \mathbb{C}^n$. The result can be considered a generalization of a classical condition for univalence of Nehari.
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1. INTRODUCTION

Let $f : \mathbb{D} \to \mathbb{C}$ be a locally injective holomorphic mapping defined in the unit disk, and let

$$\mathcal{S}f = (f''/f')' - (1/2)(f''/f')^2$$

be its Schwarzian derivative. A classical univalence criterion of Nehari [9] relates the size of $|\mathcal{S}f|$ to the univalence of f. Nehari stated the result in the form:

$$(1) \qquad\qquad |\mathcal{S}f(z)| \leq 2p(|z|)$$

implies that f is injective in $\mathbb{D}$, if $p(x)$ is a positive, even, and continuous function defined for $x \in (-1, 1)$, with the properties

    (a) $(1 - x^2)^2 p(x)$ is non-increasing for $x \in [0, 1)$;

(b) the differential equation $u'' + pu = 0$ has no nontrivial solutions with more than one zero in $(-1, 1)$.

Condition (1) includes the criteria $|\mathcal{S}f(z)| \leq \pi^2/2$ and $|\mathcal{S}f(z)| \leq 2(1 - |z|^2)^{-2}$ from [8], but also many others. A function p satisfying the hypotheses above will be referred to as a *Nehari function*. Later, in Section 5, we will also introduce the notion of an *extremal Nehari function*.

Before its connection with univalence was understood, the attributes of the Schwarzian that made it interesting are that it vanishes identically precisely for Möbius transformations,

$$\mathcal{S}f(z) = 0 \quad \text{if and only if} \quad f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

and that it is invariant under post-composition with a Möbius transformation,

$$S(f \circ g) = Sg$$

if f is Möbius. More generally, one has the chain rule

$$(2) \qquad \qquad \mathcal{S}(f \circ g) = ((\mathcal{S}f) \circ g)(g')^2 + \mathcal{S}g.$$

Consider now a locally injective holomorphic curve $\phi : \mathbb{D} \to \mathbb{C}^n$, $n \geq 1$. Write $\phi = (f_1, \ldots, f_n)$, with each f_k holomorphic in $\mathbb{D}$, and define the smooth real-valued function σ on $\mathbb{D}$ by

$$\sigma = \frac{1}{2} \log(|f_1'|^2 + \cdots |f_n'|^2).$$

We define the Schwarzian derivative of ϕ to be

$$(3) \qquad \qquad \mathcal{S}\phi = 2(\sigma_{zz} - \sigma_z^2),$$

where

$$\sigma_z = \frac{1}{2} \left(\frac{\partial \sigma}{\partial x} - i \frac{\partial \sigma}{\partial y} \right).$$

This reduces to the classical Schwarzian when $n = 1$, so there is no ambiguity in using the same name and symbol. Further background on this definition is in Section 5; it derives from a generalization of the Schwarzian to conformal mappings of Riemannian manifolds.

A straightforward calculation based only on the definition (3) together with $\mathcal{S}$ vanishing on Möbius transformations shows that

$$(4) \qquad \qquad \mathcal{S}(\phi \circ T) = ((\mathcal{S}\phi) \circ T)(T')^2$$

when T is a Möbius transformation of $\mathbb{D}$. We will need this in a number of places. We do not consider $M \circ \phi$ when M is a Möbius transformation of $\mathbb{R}^{2n}$, nor do we have $\mathcal{S}(M \circ \phi) = \mathcal{S}\phi$, since, in general, $M \circ \phi$ is not holomorphic and so its Schwarzian is not defined (at least not so simply). However, there is a substitute for Möbius invariance that we will also need. It involves a version of the Schwarzian introduced by Ahlfors, discussed in Section 2. A very general version of (2) is in Section 5.

Let $\Sigma = \phi(\mathbb{D})$, which we can regard as a (real) 2-dimensional surface in $\mathbb{R}^{2n}$. The Gaussian curvature, $K(\phi(z))$, of Σ at $\phi(z)$ is given by

$$(5) \qquad\qquad K(\phi(z)) = -e^{-2\sigma(z)}\Delta\sigma(z)\,,$$

and so is nonpositive. We shall prove:

Theorem 1. *Let p be a Nehari function and let $\phi : \mathbb{D} \to \mathbb{C}^n$ be holomorphic with $\phi' \neq 0$. If*

$$(6) \qquad\qquad |\mathcal{S}\phi(z)| + \frac{3}{4}|\phi'(z)|^2|K(\phi(z)| \leq 2p(|z|)\,, \quad z \in \mathbb{D}\,,$$

then ϕ is injective.

We recover Nehari's theorem if $n = 1$, since then $\phi(\mathbb{D}) \subset \mathbb{C}$ and $K(\phi(z)) = 0$.

We are also able to describe just how injectivity fails on $\partial\mathbb{D}$, and it does so in a rather special way. To have such a statement make sense it is first necessary to know that a mapping satisfying Theorem 1 extends continuously to the boundary. In Section 6 we will give a precise analysis of the situation, but as a preliminary result we now state:

Theorem 2. *A holomorphic curve $\phi : \mathbb{D} \to \mathbb{C}^n$ satisfying Theorem 1 has an extension to $\overline{\mathbb{D}}$ that is uniformly continuous in the spherical metric.*

Thus a function ϕ satisfying the condition (6) maps the unit circle to a continuous closed curve $\Gamma \subset \mathbb{C}^n \cup \{\infty\}$. We say that ϕ is an *extremal function* for the criterion if Γ is *not* a simple closed curve. In this case there is a pair of points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ with $f(\zeta_1) = f(\zeta_2) = P$, and one says that P is a *cut point* of Γ. We now have the following characteristic property of extremal mappings.

Theorem 3. *Under the hypotheses of Theorem 1, suppose the closed curve $\Gamma = \phi(\partial\mathbb{D})$ is not simple and let P be a cut point. Then there exists a Euclidean*

circle or line C such that $C \setminus \{P\} \subset \Sigma$. Furthermore, equality holds in (6) along $\phi^{-1}(C \setminus \{P\})$.

In addition to these results, in Section 4 we will derive a covering theorem for holomorphic curves satisfying (6) that generalizes some one-dimensional results. In Section 7 we will construct examples showing that the criterion is sharp.

When we began our work on generalizing Nehari's theorem it was in a rather different context, namely a harmonic mapping f of $\mathbb{D}$ and its Weierstrass-Enneper lift $\widetilde{f}$ mapping $\mathbb{D}$ to a minimal surfaces in $\mathbb{R}^3$, see [2]. The lift $\widetilde{f}$ is a conformal mapping of $\mathbb{D}$, say with conformal factor e^σ, and again there is a generalization of the Schwarzian derivative, $\mathcal{S}f = 2(\sigma_{zz} - \sigma_z^2)$, just as in (3). One then has: If p is a Nehari function and

$$(7) \qquad\qquad |\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \le 2p(|z|), \quad z \in \mathbb{D},$$

then $\widetilde{f}$ is injective in $\mathbb{D}$. Here $K(\widetilde{f}(z))$ is the Gaussian curvature of the minimal surface $\widetilde{f}(\mathbb{D})$ at $\widetilde{f}(z)$. There are also results analogous to Theorems 2 and 3.

It was very surprising, to us at least, that such similar statements hold in these two different settings. Comparing (6) and (7) the most visible difference is in the multiple of the curvature terms, 3/4 in the former and 1 in the latter. This is a reflection of the shared nature of the proofs, with small changes in the constants in some of the preliminary results. Things did not have to be this way, one might have thought. The cause of this commonality comes from the differential geometry of the holomorphic curve as a surface in $\mathbb{R}^{2n}$, to be explained in Section 2.

We have certainly borrowed from the exposition in [2], but while the two papers run a parallel course in many – but not all – ways, we have tried to make this paper reasonably self-contained. There are a few instances where we refer to [2] for details that would have been reproduced verbatim here.

2. Ahlfors' Schwarzian and the Second Fundamental Form

We begin the same way as in [2], with Ahlfors' Schwarzian for curves in $\mathbb{R}^m$ and its relationship to curvature, but here we find an important difference with what was done in the case of harmonic maps. Here to make use of the properties

of Ahlfors' operator to study injectivity we must relate the second fundamental form of the holomorphic curve as a surface in $\mathbb{R}^{2n}$ to its Gaussian curvature.

Ahlfors [1] defined a Schwarzian derivative for mappings $\varphi : (a, b) \to \mathbb{R}^m$ of class C^3 with $\varphi'(x) \neq 0$ by generalizing separately the real and imaginary parts of the Schwarzian for analytic functions. We only need the operator corresponding to the real part, which is

$$S_1\varphi = \frac{\langle \varphi', \varphi''' \rangle}{|\varphi'|^2} - 3\frac{\langle \varphi', \varphi'' \rangle^2}{|\varphi'|^4} + \frac{3}{2}\frac{|\varphi''|^2}{|\varphi'|^2}\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. If T is a Möbius transformation of $\mathbb{R}^m$ then, as Ahlfors showed,

$$(8) \qquad\qquad\qquad S_1(T \circ \varphi)(t) = S_1\varphi(t)\,.$$

We also record the fact that if $\gamma(t)$ is a smooth function with $\gamma'(t) \neq 0$ then

$$S_1(\varphi \circ \gamma) = ((S_1\varphi) \circ \gamma)(\gamma')^2 + \mathcal{S}\gamma\,,$$

analogous to the chain rule (2) for the analytic Schwarzian.

As in the introduction, let $\phi : \mathbb{D} \to \mathbb{C}^n$ be a holomorphic curve, and let $\Sigma = \phi(\mathbb{D})$. Ahlfors' Schwarzian enters the proof of Theorem 1 for two reasons. First, it is related to $\mathcal{S}\phi$ via the geometry of Σ as a surface in $\mathbb{R}^{2n}$. Second, bounds on $S_1\varphi$ imply injectivity along curves, and this will be sufficient to prove injectivity in $\mathbb{D}$. We take up the first point in this section and the second point in Section 3.

We recall the notion of the second fundamental form of a submanifold. Let M be a submanifold of $\mathbb{R}^m$ with the metric induced by the Euclidean metric on $\mathbb{R}^m$. Let D be the covariant derivative on $\mathbb{R}^m$ and let D' be the covariant derivative on M. If X and Y are vector fields tangent to M then D_XY need not be tangent to M but rather has components tangent and normal to M:

$$D_XY = D'_XY + II(X, Y)\,.$$

The normal component is the second fundamental form, $II(X, Y)$. It is a tensor.

For holomorphic curves the second fundamental form is related to the Gaussian curvature of Σ in the following way.

Lemma 1. *Let* $\phi : \mathbb{D} \to \mathbb{C}^n$ *be holomorphic with* $\phi' \neq 0$, *and let* $V(x) = \phi'(x)/|\phi'(x)|$, $x \in (-1, 1)$. *Then along* ϕ *the second fundamental form of* $\Sigma =$

$\phi(\mathbb{D})$ *satisfies*

$$(9) \qquad\qquad |II(V,V)|^2 = \frac{1}{2}|K(\phi)|\,.$$

Proof. We need to find the components of $D_V V$ tangent and normal to Σ. First, using $V(x) = \phi'(x)/|\phi'(x)| = e^{-\sigma(x)}\phi'(x)$ we have, along $(-1,1)$,

(10)

$$D_V V = e^{-\sigma}(e^{-\sigma}\phi')' = e^{-\sigma}(e^{-\sigma}\phi'' - e^{-\sigma}\sigma_x\phi') = e^{-2\sigma}(\phi'' - \sigma_x\phi') = e^{-2\sigma}(\phi_{xx} - \sigma_x\phi_x)\,.$$

Next let Y be the vector field $e^{-\sigma}\phi_y$ along $(-1,1)$. Since ϕ is conformal the pair $\{V,Y\}$ is an orthonormal frame for Σ along the curve $\phi((-1,1))$. In order to determine the component of $D_V V$ normal to Σ we will compute $\langle D_V V, V\rangle$ and $\langle D_V V, Y\rangle$.

From $\langle V, V\rangle = 1$ it follows that $\langle D_V V, V\rangle = 0$. Then,

$$\langle D_V V, Y\rangle = e^{-3\sigma}\langle \phi_{xx} - \sigma_x\phi_x, \phi_y\rangle = e^{-3\sigma}\langle \phi_{xx}, \phi_y\rangle\,,$$

because $\langle \phi_x, \phi_y\rangle = 0$. But ϕ is also harmonic, hence

$$\langle D_V V, Y\rangle = -e^{-3\sigma}\langle \phi_{yy}, \phi_y\rangle = -\frac{1}{2}e^{-3\sigma}\frac{\partial}{\partial y}\langle \phi_y, \phi_y\rangle$$

$$= -\frac{1}{2}e^{-3\sigma}\frac{\partial}{\partial y}(e^{2\sigma}) - e^{-\sigma}\sigma_y\,.$$

It follows that

$$D_V V = -e^{-\sigma}\sigma_y Y + II(V,V)\,,$$

that is, from (10),

$$II(V,V) = e^{-2\sigma}(\phi_{xx} - \sigma_x\phi_x + \sigma_y\phi_y)\,.$$

Therefore

$$e^{4\sigma}|II(V,V)|^2 = |\phi_{xx}|^2 + e^{2\sigma}\sigma_x^2 + e^{2\sigma}\sigma_y^2 - 2\sigma_x\langle \phi_{xx}, \phi_x\rangle + 2\sigma_y\langle \phi_{xx}, \phi_y\rangle$$

$$= |\phi_{xx}|^2 + e^{2\sigma}(\sigma_x^2 + \sigma_y^2) - 2\sigma_x\langle \phi_{xx}, \phi_x\rangle - 2\sigma_y\langle \phi_{yy}, \phi_y\rangle$$

$$= |\phi_{xx}|^2 + e^{2\sigma}(\sigma_x^2 + \sigma_y^2) - \sigma_x\frac{\partial}{\partial x}(e^{2\sigma}) - \sigma_y\frac{\partial}{\partial y}(e^{2\sigma})$$

$$= |\phi_{xx}|^2 - e^{2\sigma}(\sigma_x^2 + \sigma_y^2)\,.$$

More compactly,

$$(11) \qquad\qquad e^{4\sigma}|II(V,V)|^2 = |\phi_{xx}|^2 - e^{2\sigma}|\nabla\sigma|^2\,.$$

On the other hand, using $\phi = (f_1, \ldots, f_n)$ and $\sigma = (1/2 \log \left(|f_1'|^2 + \cdots + |f_n'|^2 \right)$, together with $\phi_x = \phi' = (f_1', \ldots, f_n')$ and $\phi_{xx} = (f_1'', \ldots, f_n'')$, we obtain

$$
(12) \qquad 2\sigma_z = \frac{\overline{f_1'} f_1'' + \cdots \overline{f_n'} f_n''}{|f_1'|^2 + \cdots + |f_n'|^2} \, ,
$$

and after some algebra,

$$
|\nabla \sigma|^2 = |2\sigma_z|^2 = e^{-4\sigma} \left| \sum_i \overline{f_i'} f_i'' \right|^2 .
$$

This inserted in (11) gives

$$
(13)
$$
$$
e^{6\sigma} |\Pi(V,V)|^2 = \sum_i |f_i''|^2 \sum_j |f_j'|^2 - \sum_i \overline{f_i'} f_i'' \sum_j f_j' \overline{f_j''} = \sum_{i<j} |f_i' f_j'' - f_j' f_i''|^2 .
$$

Finally we compute $K(\phi(x)) = -e^{-2\sigma(x)} \Delta\sigma(x)$. From (12) it follows that

$$
\frac{1}{2} \Delta\sigma = 2\sigma_{z\bar{z}} = \frac{|f_1''|^2 + \cdots + |f_n''|^2}{|f_1'|^2 + \cdots + |f_n'|^2} - \frac{\left| \overline{f_1'} f_1'' + \cdots \overline{f_n'} f_n'' \right|^2}{\left(|f_1'|^2 + \cdots + |f_n'|^2 \right)^2} \, ,
$$

and after some manipulation one obtains

$$
(14) \qquad \Delta\sigma = 2e^{-4\sigma} \sum_{i<j} \left| f_i' f_j'' - f_j' f_i'' \right|^2 .
$$

Comparing (13) with (14) we deduce that

$$
|\Pi(V,V)|^2 = \frac{1}{2} |K(\phi)| \, ,
$$

as desired.

$\square$

Ahlfors' Schwarzian S_1 and the holomorphic Schwarzian $\mathcal{S}$ from (3) appear together in the following relationship.

Lemma 2. *Let* $\phi : \mathbb{D} \to \mathbb{C}^n$ *be holomorphic with* $\phi' \neq 0$. *Let* $\gamma(t)$ *be a Euclidean arc-length parametrized curve in* $\mathbb{D}$ *with curvature* $\kappa(t)$, *and let* $\varphi(t) = \phi(\gamma(t))$ *be the corresponding parametrization of* $\Gamma = \phi(\gamma)$ *on* $\Sigma = \phi(\mathbb{D})$. *Let* $V(t)$ *be the Euclidean unit tangent vector field along* $\varphi(t)$, *given by*

$$
V(t) = \frac{\varphi'(t)}{|\varphi'(t)|} = \frac{\phi'(\gamma(t))\gamma'(t)}{|\phi'(\gamma(t))|} \, .
$$

Then

(15) $S_1\varphi(t) = \mathrm{Re}\{\mathcal{S}\phi(\gamma(t))(\gamma'(t))^2\} + \dfrac{3}{4}|\phi'(\gamma(t))|^2 K(\varphi(t)) + \dfrac{1}{2}\kappa^2(t)\,.$

This is the most general form of the relationship between S_1 and $\mathcal{S}$. Compare this formula to the one in Lemma 1 in [2]. We will also need (15) in the special case when $\gamma(t)$ is a diameter of $\mathbb{D}$, say from -1 to 1. In this case $\kappa = 0$ and we can write the equation as

(16) $S_1\phi(x) = \mathrm{Re}\{\mathcal{S}\phi(x)\} + \dfrac{3}{4}|\phi'(x)|^2|K(\phi(x))|\,.$

Here, by $S_1\phi(x)$ we mean S_1 applied to ϕ restricted to the interval $(-1,1)$.

Proof of Lemma 2. Let $v(t) = |\varphi'(t)|$. The proof is based on a formula of Chuaqui and Gevirtz in [3], according to which

(17) $S_1\varphi = \left(\dfrac{v'}{v}\right)' - \dfrac{1}{2}\left(\dfrac{v'}{v}\right)^2 + \dfrac{1}{2}v^2k^2\,,$

where k is the Euclidean curvature of the curve $\varphi(t)$ in $\mathbb{R}^{2n}$. We compute the terms on the right-hand side.

First, by definition,

$$v(t) = |\varphi'(t)| = |\phi'(\gamma(t))| = e^{\sigma(\gamma(t))}\,,$$

and hence

$$\dfrac{v'}{v} = \langle \nabla\sigma\,,\,\gamma'\rangle\,, \quad \left(\dfrac{v'}{v}\right)' = \mathrm{Hess}\,\sigma(\gamma',\gamma') + \langle \nabla\sigma\,,\,\gamma''\rangle\,,$$

where

$$\mathrm{Hess}\,\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

is the Hessian matrix regarded as a bilinear form and γ' is identified with the unit tangent vector $(x'(t), y'(t))$. Moreover, with a similar identification, $\gamma'' = \kappa\mathbf{n}$, where $\mathbf{n}$ is the unit normal to $\gamma(t)$. Thus

(18) $\left(\dfrac{v'}{v}\right)' - \dfrac{1}{2}\left(\dfrac{v'}{v}\right)^2 = \mathrm{Hess}\,\sigma(\gamma',\gamma') + \kappa\langle \nabla\sigma\,,\,\mathbf{n}\rangle - \dfrac{1}{2}\langle \nabla\sigma\,,\,\gamma'\rangle^2\,.$

Next we work with the curvature term k^2v. We can write

(19) $k^2 = k_i^2 + |II(V,V)|^2\,,$

where k_i is the intrinsic (geodesic) curvature of $\varphi(\gamma(t))$ on the surface $\varphi(\mathbb{D}) = \Sigma \subset \mathbb{R}^{2n}$. Furthermore, $\varphi\colon (\mathbb{D}, e^{2\sigma}\mathbf{g}_0) \to (\Sigma, \mathbf{g}_0)$ is a local isometry between Σ with the Euclidean metric, denoted here by $\mathbf{g}_0$, and $\mathbb{D}$ with the conformal metric $e^{2\sigma}\mathbf{g}_0$, and so $k_i = \hat{\kappa}$, the curvature of $\gamma(t)$ in the metric $e^{2\sigma}\mathbf{g}_0$. In turn, a classical formula in conformal geometry states that

$$e^{\sigma}\hat{\kappa} = \kappa - \langle \nabla\sigma\,, \mathbf{n}\rangle;$$

see, for example, Section 3 in [10]. Combining these formulas gives us

$$(20) \qquad v^2 k^2 = k^2 e^{2\sigma} = \kappa^2 - 2\kappa\langle\nabla\sigma\,, \mathbf{n}\rangle + \langle\nabla\sigma\,, \mathbf{n}\rangle^2 + e^{2\sigma}|II(V,V)|^2\,.$$

For $S_1\varphi$ in (17) we combine (18) and (20) and manipulate some terms to write

$$S_1\varphi = \mathrm{Hess}(\sigma)(\gamma',\gamma') - \frac{1}{2}\langle\nabla\sigma\,, \gamma'\rangle^2 + \frac{1}{2}\langle\nabla\sigma\,, \mathbf{n}\rangle^2 + \frac{1}{2}\kappa^2 + \frac{1}{2}e^{2\sigma}|II(V,V)|^2$$

$$= \mathrm{Hess}(\varphi)(\gamma',\gamma') + \frac{1}{2}|\nabla\sigma|^2 - \langle\nabla\sigma\,, \gamma'\rangle^2 + \frac{1}{2}\kappa^2 + \frac{1}{2}e^{2\sigma}|II(V,V)|^2$$

$$= \mathrm{Hess}(\sigma)(\gamma',\gamma') - \langle\nabla\sigma\,, \gamma'\rangle^2 - \frac{1}{2}(\Delta\sigma - |\nabla\sigma|^2) + \frac{1}{2}\Delta\sigma + \frac{1}{2}\kappa^2 + \frac{1}{2}e^{2\sigma}|II(V,V)|^2$$

Next, one finds by straight calculation (see also Section 5) that

$$\mathrm{Hess}(\sigma)(\gamma',\gamma') - \langle\nabla\sigma\,, \gamma'\rangle^2 - \frac{1}{2}(\Delta\sigma - |\nabla\sigma|^2) = \mathrm{Re}\{\mathcal{S}\phi(\gamma)(\gamma')^2\}$$

while from (5),

$$\frac{1}{2}\Delta\sigma = -\frac{1}{2}e^{2\sigma}K(\phi) = \frac{1}{2}e^{2\sigma}|K(\phi)|\,.$$

Substituting these and $e^{2\sigma} = |\phi'|^2$ gives

$$S_1\varphi(t) = \mathrm{Re}\{\mathcal{S}\phi(\gamma(t))(\gamma'(t))^2\} + \frac{1}{2}|\phi'(\gamma(t))|^2(K(\varphi(t)) + |II(V(t),V(t))|^2) + \frac{1}{2}\kappa^2(t)\,.$$

Now appealing to Lemma 1 brings this into final form,

$$S_1\varphi(t) = \mathrm{Re}\{\mathcal{S}\phi(\gamma(t))(\gamma'(t))^2\} + \frac{3}{4}|\phi'(\gamma(t))|^2 K(\varphi(t)) + \frac{1}{2}\kappa^2(t)\,.$$

$\square$

3. Injectivity, Extremal Functions, and the Proofs of Theorems 1 and 3

We recall the hypothesis of Theorem 1, that ϕ satisfies (6),

$$|\mathcal{S}\phi(z)| + \frac{3}{4}|\phi'(z)|^2|K(\phi(z))| \leq 2p(|z|)\,, \quad z \in \mathbb{D}\,.$$

The proof of injectivity rests on a result of Chuaqui and Gevirtz [3] giving a criterion for univalence on (real) curves in terms of S_1.

Theorem A. *Let $p(x)$ be a continuous function such that the differential equation $u''(x) + p(x)u(x) = 0$ admits no nontrivial solution $u(x)$ with more than one zero in $(-1, 1)$. Let $\varphi : (-1, 1) \to \mathbb{R}^m$ be a curve of class C^3 with tangent vector $\varphi'(x) \neq 0$. If $S_1\varphi(x) \leq 2p(x)$, then φ is univalent.*

We pass immediately to

Proof of Theorem 1. If ϕ satisfies (6) then, from (16), along the diameter $(-1, 1)$,

$$S_1\phi(x) \leq 2p(x),$$

and so ϕ is injective there by Theorem A. The same holds for any rotation $\phi(e^{i\theta}z)$ of ϕ and hence ϕ is injective along any diameter of $\mathbb{D}$.

Suppose now that z_1 and z_2 are distinct points not on a diameter. Let γ be the hyperbolic geodesic through z_1 and z_2. By a rotation of $\mathbb{D}$ we may assume that γ meets the imaginary axis orthogonally at a point $i\rho$. The Möbius transformation

$$T(z) = \frac{z - i\rho}{1 + i\rho z}$$

maps $\mathbb{D}$ onto itself, preserves the imaginary axis, and carries γ to the diameter $(-1, 1)$. The function

$$\psi(z) = \phi(T(z))$$

is a holomorphic reparametrization of $\Sigma = \phi(\mathbb{D})$ with $\psi' \neq 0$ and we claim that

$$|\mathcal{S}\psi(x)| + \frac{3}{4}|\psi'(x)|^2|K(\psi(x))| \leq 2p(x), \quad -1 < x < 1.$$

If so, then $S_1\psi(x) \leq 2p(x)$ as above, whence ψ is injective along $(-1, 1)$ and $\phi(z_1) \neq \phi(z_2)$.

For this, first note that

$$|\psi'(x)| = |\phi'(T(x))|\, |T'(x)|$$

while also

$$\mathcal{S}\psi(x) = \mathcal{S}\phi(T(x))T'(x)^2,$$

from (4). Next, by hypothesis,

$$|\mathcal{S}\phi(T(x))|\, |T'(x)|^2 + \frac{3}{4}|\phi'(T(x))|^2|T'(x)|^2|K(\phi(T(x)))| \leq 2p(|T(x)|)|T'(x)|^2,$$

and so the claim will be established if we show

(21) $$p(|T(x)|)|T'(x)|^2 \le p(|x|) , \quad -1 < x < 1 .$$

But now a simple calculation gives that

(22) $$|x| \le |T(x)|$$

for this particular Möbius transformation, whence

$$(1 - |T(x)|^2)^2 p(|T(x)|) \le (1 - x^2)^2 p(|x|)$$

by the assumption that $(1-x^2)^2 p(x)$ is non-increasing for $x \in [0,1)$. Furthermore

$$\frac{|T'(x)|}{1 - |T(x)|^2} = \frac{1}{1 - x^2}$$

for any Möbius transformation of $\mathbb{D}$ onto itself. Thus

$$|T'(x)|^2 p(|T(x)|) = \frac{(1 - |T(x)|^2)^2}{(1 - x^2)^2} p(|T(x)|) \le p(|x|) ,$$

finishing the proof. $\qquad\qquad\square$

The trick of passing from injectivity along diameters to the general case via this special Möbius transformation, using (22) to obtain (21), goes back to Nehari. We used the same argument in [2].

Theorem 3 states that if ϕ satisfies the univalence criterion (6) and fails to be injective on $\partial\mathbb{D}$ (assuming continuous extension) then it must do so in a particular way, that the surface $\Sigma = \phi(\mathbb{D})$ contains a Euclidean circle minus the cut point where injectivity fails. Moreover, equality holds in (6) along the preimage of the circle. These properties of such extremal functions depend upon another result of Chuaqui and Gevirtz in the same paper [3]. To state it we need one additional construction, which will be used again in later sections.

If the function $p(x)$ of Theorem A is even, as will be the case for a Nehari function, then the solution u_0 of the differential equation $u'' + pu = 0$ with initial conditions $u_0(0) = 1$ and $u_0'(0) = 0$ is also even, and therefore $u_0(x) \ne 0$ on $(-1, 1)$, since otherwise it would have at least two zeros. Thus the function

(23) $$\Phi(x) = \int_0^x u_0(t)^{-2} \, dt , \qquad -1 < x < 1 ,$$

is well defined and has the properties $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi''(0) = 0$, $\Phi(-x) = -\Phi(x)$, and $\mathcal{S}\Phi = 2p$. Furthermore, $S_1\Phi = \mathcal{S}\Phi$ since Φ is real-valued, and so $S_1\Phi = 2p$ as well.

In terms of Φ, the second Chuaqui-Gevirtz theorem is as follows.

Theorem B. *Let $p(x)$ be an even function with the properties assumed in Theorem A, and let Φ be defined as above. Let $\varphi : (-1, 1) \to \mathbb{R}^m$ satisfy $S_1\varphi(x) \leq 2p(x)$ and have the normalization $\varphi(0) = 0$, $|\varphi'(0)| = 1$, and $\varphi''(0) = 0$. Then $|\varphi'(x)| \leq \Phi'(|x|)$ for $x \in (-1, 1)$, and φ has an extension to the closed interval $[-1, 1]$ that is continuous with respect to the spherical metric. Furthermore, there are two possibilities:*

(i) If $\Phi(1) < \infty$, then φ is univalent in $[-1, 1]$ and $\varphi([-1, 1])$ has finite length.

(ii) If $\Phi(1) = \infty$, then either φ is univalent in $[-1, 1]$ or $\varphi = R \circ \Phi$ for some rotation R of $\mathbb{R}^m$.

We can now proceed with

Proof of Theorem 3. Suppose that ϕ satisfies (6) and fails to be injective on $\partial\mathbb{D}$, and let $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ be points for which $\phi(\zeta_1) = \phi(\zeta_2) = P$. We first show that we can form $\psi = \phi \circ T$ for a suitable Möbius transformation of $\mathbb{D}$ onto itself, with ψ still satisfying (6) and with $\psi(1) = \psi(-1) = P$. We refer to the calculations in the proof of Theorem 1, and we distinguish two cases.

Suppose first that $(1 - x^2)^2 p(x)$ is constant. Then the condition (6) is fully invariant under the Möbius transformations of $\mathbb{D}$, and for a Möbius modification $\psi = \phi \circ T$ with $T(1) = \zeta_1$, $T(-1) = \zeta_2$ we obtain $\psi(1) = \psi(-1) = P$.

Now suppose that $(1 - x^2)^2 p(x)$ is not constant. Suppose also, by way of contradiction, that ζ_1 and ζ_2 are not on a diameter. Since (6) is, in any case, invariant under rotations of $\mathbb{D}$, we may assume that ζ_1 and ζ_2 are both in the upper half plane and are symmetric in the imaginary axis. Just as in the proof of Theorem 1, let $T(z) = (z - i\rho)/(1 + i\rho z)$ with $T(-1) = \zeta_1$, $T(1) = \zeta_2$, and let $\psi = \phi \circ T$. Then (6) holds for ψ, together with $\psi(1) = \psi(-1) = P$ and $S_1\psi(x) \leq 2p(x)$. Moreover we must have a strict inequality $S_1\psi(x) < 2p(x)$ on some interval in $(-1, 1)$ because $(1 - x^2)^2 p(x)$ is not constant. However, this last statement stands in contradiction with Theorem B. To be precise, there is a Möbius transformation M of $\mathbb{R}^{2n}$ such that $M \circ \psi$ satisfies the hypotheses of

Theorem B with $(M \circ \psi)(-1) = (M \circ \psi)(1)$, and because ψ is not injective on $[-1, 1]$ we have, using (4), $S_1(M \circ \psi) = S_1 \psi = 2p$. The contradiction shows that ζ_1 and ζ_2 must lie on a diameter.

In all cases, by a suitable modification we can now assume that the injectivity of ϕ on $\partial \mathbb{D}$ fails by ϕ mapping the interval $[-1, 1]$ to a closed curve on $\phi(\overline{\mathbb{D}})$ with $\phi(-1) = \phi(1) = P$, and by a post composition with a Möbius transformation of $\mathbb{R}^{2n}$ we may further assume that ϕ is normalized as in Theorem B. Then again by Theorem B, $\phi = V \circ \Phi$ for some Möbius transformation of $\mathbb{R}^{2n}$. Hence $S_1 \phi = S_1 \Phi = 2p$ and ϕ maps $[-1, 1]$ to a Euclidean circle or a line, since Φ maps $[-1, 1]$ onto $\mathbb{R} \cup \{\infty\}$ and V preserves circles. Finally, since $S_1\phi(x) = 2p(x)$ and ϕ satisfies (6), it follows from (16) that

$$|\mathcal{S}\phi(x)| + \frac{3}{4}|\phi'(x)|^2|K(\phi(x))| = 2p(x), \quad x \in [-1, 1].$$

This concludes the proof.   □

4. A Covering Theorem

We continue to assume that ϕ satisfies the injectivity criterion (6). In this section we will derive a lower bound for the radius of a metric disk centered at $\phi(0)$ on Σ. We will assume an additional normalization of $|\phi'(0)| = 1$, and then the lower bound will depend on $|\phi''(0)|$ and on a second extremal function associated with the Nehari function p. We also need to assume that p is nondecreasing on $[0, 1)$, which is the case for many examples. The result we obtain is very much in line with those from classical geometric function theory, see, for example, [6].

Let U be the solution of

$$U'' - pU = 0, \qquad U(0) = 1, \ U'(0) = 0$$

(note the minus sign in the differential equation) and define

$$\Psi(x) = \int_0^x U(t)^{-2}\,dt.$$

Theorem 4. *Let ϕ be a holomorphic curve satisfying* (6) *with $|\phi'(0)| = 1$, and suppose that $p(x)$ is nondecreasing on $[0, 1)$. Then*

(24) $$\min_{|z|=r} d_\Sigma(\phi(z), \phi(0)) \geq \frac{2\Psi(r)}{2 + |\phi''(0)|\Psi(r)},$$

where d_Σ denotes distance on $\Sigma = \phi(\mathbb{D})$. In particular, Σ contains a metric disk of radius

$$\frac{2\Psi(1)}{2 + |\phi''(0)|\Psi(1)}$$

centered at $\phi(0)$.

Proof. The proof relies ultimately on comparing solutions of two differential equations, one involving the extremal Ψ, where the relevant inequalities come to us by way of the formulas from Lemma 2 and its proof. This requires some preparation.

Let z_r be a point on $|z| = r$ for which the minimum on the left hand side of (24) is attained. Since ϕ is injective the minimum is positive and $z_r \neq 0$. The geodesic, Γ, on Σ that joins $\phi(0)$ with $\phi(z_r)$ is contained in $\phi(\{|z| \leq r\})$ and we let $\gamma = \phi^{-1}(\Gamma)$. Write $\gamma(t)$ for the parametrization of γ by Euclidean arc-length, and let $\varphi(t) = \phi(\gamma(t))$ be the corresponding parametrization of Γ. Further, let

$$v(t) = |\phi'(\gamma(t))|.$$

Now compare the expressions for $S_1\varphi(t)$ in (15) and (17), using (19) relating the Euclidean and intrinsic curvatures of Γ and the second fundamental form of Σ:

$$(v'/v)' - \frac{1}{2}(v'/v)^2 + \frac{1}{2}v^2\left(k_i^2 + |II(V,V)|^2\right) =$$

$$\mathrm{Re}\{S\phi(\gamma)(\gamma')^2\} + \frac{1}{2}v^2\left(|K(\phi(\gamma))| + |II(V,V)|^2\right) + \frac{1}{2}\kappa^2,$$

where κ is the Euclidean curvature of γ and V is the Euclidean unit tangent vector field along Γ; all expressions are to be evaluated at t. Since Γ is a geodesic $k_i = 0$ and this becomes

$$(v'/v)' - \frac{1}{2}(v'/v)^2 = \mathrm{Re}\{\mathcal{S}\phi(\gamma)(\gamma')^2\} + \frac{1}{2}v^2|K(\phi(\gamma))| + \frac{1}{2}\kappa^2.$$

But the univalence criterion (6) implies that

$$\mathrm{Re}\{S\phi(\gamma)(\gamma')^2\} \geq -|S\phi(\gamma)| \geq -2p(\gamma) + \frac{3}{4}v^2|K(\phi(\gamma))|,$$

hence

$$(v'/v)' - \frac{1}{2}(v'/v)^2 \geq -2p(|\gamma|) + \frac{5}{4}v(t)^2|K(\phi(\gamma))| + \frac{1}{2}\kappa^2$$

$$\geq -2p(|\gamma|).$$

Now let

$$h(t) = \int_0^t v(\tau)\,d\tau.$$

Then $\mathcal{S}h = (v'/v)' - \frac{1}{2}(v'/v)^2$, and from the preceding estimate,

$$\mathcal{S}h(t) \geq -2p(|\gamma(t)|) \geq -2p(t) \, ,$$

the final inequality holding because $|\gamma(t)| \leq t$ and we have assumed that $p(x)$ is nondecreasing on $[0, 1)$.

The result $\mathcal{S}h(t) \geq -2p(t)$ is the main inequality we need in order to apply the Sturm comparison theorem. For this, first note that the function $w = v^{-1/2}$ is the solution of

$$w'' + \frac{1}{2}(\mathcal{S}h)w = 0 \, , \quad w(0) = 1 \, , \, w'(0) = -\frac{1}{2}v'(0) \, .$$

Next, consider also the solution $y(t)$ of

$$y'' - py = 0 \, , \quad y(0) = 1 \, , \, y'(0) = \frac{1}{2}|\phi''(0)| \, .$$

Since $-\frac{1}{2}\mathcal{S}h(t) \leq p(t)$, and also

$$w'(0) = -\frac{1}{2}v'(0) \leq \frac{1}{2}|\phi''(0)| \, ,$$

it follows by comparison that

$$w(t) \leq y(t) \, .$$

Finally, observe that, explicitly, $y = (H')^{-1/2}$, where

$$H = \frac{2\Psi}{2 + |\phi''(0)|\Psi} \, .$$

Consequently,

$$h(t) = \int_0^t w^{-2}(\tau) \, d\tau \geq \int_0^t y^{-2}(\tau) \, d\tau = \int_0^t H'(\tau) \, d\tau = H(t) \, ,$$

and hence

$$d_\Sigma(\phi(z_r), \phi(0)) = \int_\gamma v \geq \int_0^r v(t) \, dt = h(r) \geq H(r) \, .$$

This completes the proof.

$\square$

5. Conformal Schwarzians, Extremal Nehari Functions, and Convexity

To advance further in the analysis of mappings satisfying (6), in particular to study continuous extension to the boundary, we need estimates based on convexity. This requires a notion of the Schwarzian for conformal metrics and an associated differential equation. This section generally follows the treatment of these ideas in [2], abbreviated somewhat and modified to serve the case of holomorphic maps rather than lifts of harmonic maps. We refer to that paper and also to [10] for (many) more details.

Let $\mathbf{g}$ be a Riemannian metric on the disk $\mathbb{D}$. We may assume that $\mathbf{g}$ is conformal to the Euclidean metric, $\mathbf{g}_0 = dx \otimes dx + dy \otimes dy = |dz|^2$. Let σ be a smooth function on $\mathbb{D}$ and form the symmetric 2-tensor

(25)
$$\mathrm{Hess}_{\mathbf{g}}(\sigma) - d\sigma \otimes d\sigma.$$

Here Hess denotes the Hessian operator. If $\gamma(s)$ is an arc-length parametrized geodesic for $\mathbf{g}$, then

$$\mathrm{Hess}_{\mathbf{g}}(\sigma)(\gamma', \gamma') = \frac{d^2}{ds^2}(\sigma \circ \gamma).$$

The Hessian depends on the metric, and since we will be changing metrics we indicate this dependence by the subscript $\mathbf{g}$.

We now form

$$B_{\mathbf{g}}(\sigma) = \mathrm{Hess}_{\mathbf{g}}(\sigma) - d\sigma \otimes d\sigma - \frac{1}{2}(\Delta_{\mathbf{g}}\sigma - ||\,\mathrm{grad}_{\mathbf{g}}\,\sigma||^2)\mathbf{g}.$$

The final term has been subtracted to make the trace zero.

This is the Schwarzian tensor of σ. Before explaining its connection to conformal maps, metrics and the Schwarzian derivative, first note that in standard Cartesian coordinates one can represent $B_{\mathbf{g}}(\sigma)$ as a symmetric, traceless 2×2 matrix, say of the form

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}.$$

Further identifying such a matrix with the complex number $a + bi$ then allows us to associate the tensor $B_{\mathbf{g}}(\sigma)$ with $a + bi$, and then

$$||B_{\mathbf{g}_0}(\sigma)(z)||_{\mathbf{g}_0} = |a + bi|.$$

A locally injective holomorphic curve $\phi : \mathbb{D} \to \mathbb{C}^n$ is a conformal mapping of $\mathbb{D}$ with the Euclidean metric into $\mathbb{R}^{2n}$ with the Euclidean metric and if, as before, we write $\phi = (f_1, \ldots, f_n)$ then the conformal factor is

$$\phi^*(\mathbf{g_0}) = e^{2\sigma}\mathbf{g_0}\,, \quad \sigma = \frac{1}{2}\log(|f_1'|^2 + \cdots |f_n'|^2)\,.$$

The Schwarzian derivative of ϕ is defined to be

$$\mathcal{S}_\mathbf{g}\phi = B_\mathbf{g}(\sigma)\,.$$

When $n = 1$ and $\sigma = \log|\phi'|$ we find that

$$B_{\mathbf{g_0}}(\log|\phi'|) = \begin{pmatrix} \operatorname{Re}\mathcal{S}\phi & -\operatorname{Im}\mathcal{S}\phi \\ -\operatorname{Im}\mathcal{S}\phi & -\operatorname{Re}\mathcal{S}\phi \end{pmatrix}\,,$$

writing the tensor in matrix form as above, where $\mathcal{S}\phi$ is the classical Schwarzian derivative of ϕ. When $n \geq 1$, identifying the tensor with a complex number leads to

$$\mathcal{S}\phi = 2(\sigma_{zz} - \sigma_z^2)\,,$$

the definition that we gave in Section 1.

The tensor $B_\mathbf{g}\sigma$ changes in a simple way if there is a conformal change in the background metric $\mathbf{g}$. Specifically, if $\widehat{\mathbf{g}} = e^{2\rho}\mathbf{g}$ then

$$B_\mathbf{g}(\rho + \sigma) = B_\mathbf{g}(\rho) + B_{\widehat{\mathbf{g}}}(\sigma).$$

This is actually a generalization of the chain rule (2) for the classical Schwarzian. An equivalent formulation is

(26) $$B_{\widehat{\mathbf{g}}}(\sigma - \rho) = B_\mathbf{g}(\sigma) - B_\mathbf{g}(\rho)\,,$$

which is what we will need in later calculations.

Next, just as the linear differential equation $w'' + (1/2)pw = 0$ is associated with $Sf = p$, there is also a linear differential equation associated with the Schwarzian tensor. If

$$B_\mathbf{g}(\sigma) = p\,,$$

where p is a symmetric, traceless 2-tensor, then $\eta = e^{-\sigma}$ satisfies

(27) $$\operatorname{Hess}_\mathbf{g}(\eta) + \eta p = \frac{1}{2}(\Delta_\mathbf{g}\eta)\mathbf{g}\,.$$

We now turn to convexity. In this setting, a function η is convex relative to the metric $\mathbf{g}$ if

$$\mathrm{Hess}_{\mathbf{g}}\,\eta \geq \alpha\mathbf{g}\,,$$

where α is a nonnegative function. This is equivalent to

$$\frac{d^2}{ds^2}(\eta \circ \gamma) \geq \alpha \geq 0$$

for any arc-length parametrized geodesic γ.

Convexity is an important notion for us because we will find that an upper bound for $\mathcal{S}_{\mathbf{g}}\phi$ coming from the injectivity criterion (6) leads via (26) and (27) to just such a positive lower bound for the Hessian of an associated function, and this is what we need to study boundary behavior. This fact obtains, however, not relative to the Euclidean metric but when the background metric $\mathbf{g}$ is a complete, radial metric coming from an *extremal* Nehari function. We explain this now.

It follows from the Sturm comparison theorem that if p is a Nehari function then so is a multiple kp for any k with $0 < k < 1$. This need not be so if $k > 1$ and we say that p is an *extremal Nehari function* if kp is *not* a Nehari function for *any* $k > 1$. For example, $p(x) = 1/(1-x^2)^2$ and $p(x) = \pi^2/4$ are both extremal Nehari functions. In [5] it was shown that some constant multiple of each Nehari function is an extremal Nehari function. Observe that since a holomorphic curve ϕ satisfying a condition of the type $|\mathcal{S}\phi| + \cdots \leq 2p$, as in (6), also then satisfies $|\mathcal{S}\phi| + \cdots \leq 2kp$ for any $k > 1$ we may always assume when (6) is in force that p is an extremal Nehari function.

There is another way to describe this situation in terms of the extremal *function* associated with a given p. Recall the definition from (23),

$$\Phi(x) = \int_0^x u_0(t)^{-2}\,dt\,, \qquad -1 < x < 1\,,$$

where u_0 is the solution of $u'' + pu = 0$ with initial conditions $u_0(0) = 1$ and $u_0'(0) = 0$. We use Φ to form the radial conformal metric

(28) $$\mathbf{g}_\Phi = \Phi'(|z|)^2|dz|^2$$

on $\mathbb{D}$. It is implicit in [5], without the terminology, that the following conditions are equivalent:

(a) p is an extremal Nehari function.
(b) $\Phi(1) = \infty$.

(c) The metric $\Phi'(|z|)^2|dz|^2$ is complete.

We recall that for a complete metric any two points can be joined by a geodesic and that a geodesic can be extended indefinitely. We let $d_{\mathbf{g}_\Phi}$ be the distance in the $\mathbf{g}_\Phi$ metric and note that since $\mathbf{g}_\Phi$ is radial

$$d_{\mathbf{g}_\Phi}(0, z) = \Phi(|z|)\,.$$

The curvature of a radial metric of the form (28), complete or not, can be expressed as

$$(29) \qquad K_{\mathbf{g}_\Phi}(z) = -2\Phi'(|z|)^{-2}(A(|z|) + p(|z|))\,, \quad r = |z|\,,$$

where

$$(30) \qquad A(r) = \frac{1}{4}\left(\frac{\Phi''(r)}{\Phi'(r)}\right)^2 + \frac{1}{2r}\frac{\Phi''(r)}{\Phi'(r)}\,, \quad r \geq 0\,.$$

From the properties of Φ it follows that $A(r)$ is continuous at 0 with $A(0) = p(0)$ and that the curvature is negative. Thus, as we will need,

$$(31) \qquad |K_{\mathbf{g}_\Phi}(z)| = 2\Phi'(|z|)^{-2}(A(|z|) + p(|z|))\,.$$

If the metric (28) is complete, or equivalently comes from an extremal Nehari function, then as was shown in [5]

$$(32) \qquad p(r) \leq A(r)\,.$$

All of these comments go into the proof of the following theorem.

Theorem 5. *Let ϕ satisfy the injectivity criterion* (6) *for an extremal Nehari function p. Then*

$$(33) \qquad w(z) = \sqrt{\frac{\Phi'(|z|)}{|\phi'(z)|}}$$

satisfies

$$(34) \qquad \mathrm{Hess}_{\mathbf{g}_\Phi}(w) \geq \frac{1}{8}\, w^{-3}|K_{\mathbf{g}_\Phi}|\mathbf{g}_\Phi\,,$$

In particular, w is a convex function relative to the metric $\mathbf{g}_\Phi$.

Compare this result to the corresponding result, Theorem 4 in [2], for lifts of harmonic mappings (in [2] we did not use the term "extremal Nehari function") where the constant in the inequality bounding the Hessian from below is 1/4 rather than 1/8.

As in [2] we formulate a separate lemma.

Lemma 3. *Let ϕ satisfy the injectivity criterion* (6) *for an extremal Nehari function p and let $\rho = \log|\Phi'|$. Then*

$$(35) \qquad \|B_{\mathbf{g}_\Phi}(\sigma - \rho)\|_{\mathbf{g}_\Phi} + \frac{3}{4}e^{2(\sigma-\rho)}|K(\phi)| \leq \frac{1}{2}|K_{\mathbf{g}_\Phi}|.$$

Here recall that $K(\phi(z))$ is the Gaussian curvature of the surface $\Sigma = \phi(\mathbb{D})$ at $\phi(z)$.

Proof. The proof is very much like the proof of Lemma 2 in [2], but to show how the assumptions enter we will present the argument.

First note from (31) that, in terms of ρ, the absolute value of the curvature of $\mathbf{g}_\Phi = \phi'(|z|)^2|dz|^2 = e^{2\rho(z)}|dz|^2$ is

$$(36) \qquad |K_{\mathbf{g}_\Phi}(z)| = e^{-2\rho(z)}(A(|z|) + p(|z|)).$$

Also, from (26) we have

$$B_{\mathbf{g}_\Phi}(\sigma - \rho) = B_{\mathbf{g}_0}(\sigma) - B_{\mathbf{g}_0}(\rho).$$

Since $\mathbf{g}_\Phi = e^{2\rho}\mathbf{g}_0$, the norm scales to give

$$\|B_g(\sigma - \rho)\|_{\mathbf{g}_\Phi} = e^{-2\rho}\|B_{\mathbf{g}_0}(\sigma) - B_{\mathbf{g}_0}(\rho)\|_{\mathbf{g}_0} = e^{-2\rho}|B_{\mathbf{g}_0}(\sigma) - B_{\mathbf{g}_0}(\rho)|,$$

where in the last equation we have identified the Euclidean norm of the tensor with the magnitude of the corresponding complex number.

Next, a calculation (see also [5]) produces

$$B_{\mathbf{g}_0}(\sigma) - B_{\mathbf{g}_0}(\rho) = \zeta^2 S\phi(z) + A(|z|) - p(|z|), \quad \zeta = \frac{z}{|z|}.$$

In light of these statements, establishing (35) is equivalent to

$$\left|\zeta^2 S\phi(z) + A(|z|) - p(|z|)\right| + \frac{3}{4}e^{2\sigma(z)}|K(\phi(z))| \leq A(|z|) + p(|z|).$$

This in turn follows from the assumption that ϕ satisfies the injectivity criterion (6) and, crucially, from the inequality (32):

$$\left|\zeta^2 S\phi(z) + A(|z|) - p(|z|)\right| + \frac{3}{4}e^{2\sigma}|K| \leq |\zeta^2 S\phi(z)| + |A(|z|) - p(|z|)| + \frac{3}{4}e^{2\sigma}|K|$$

$$= |S\phi(z)| + \frac{3}{4}e^{2\sigma}|K| + A(|z|) - p(|z|)$$

$$\leq A(|z|) + p(|z|)|.$$

□

The deduction of Theorem 5 from Lemma 3, relies on (27). Write $w = e^{(\rho-\sigma)/2}$, $v = w^2 = e^{\rho-\sigma}$, and then, according to (27),

$$\mathrm{Hess}_{\mathbf{g}_\Phi} v + v B_{\mathbf{g}_\Phi}(\sigma - \rho) = \frac{1}{2}(\Delta_{\mathbf{g}_\Phi} v)\mathbf{g}.$$

With this, the proof is almost word-for-word the same as the corresponding proof in [2], and we omit the details. Instead, let us present one consequence of Theorem 5 here, with more to come in the next section.

Lemma 4. *Under the assumptions of Theorem 5, if w has at least two critical points then the range of ϕ lies in a plane.*

Proof. Suppose z_1 and z_2 are critical points of w. Then, because w is convex, $w(z_1)$ and $w(z_2)$ are absolute minima, and so is every point on the geodesic segment γ (for the metric $\mathbf{g}_\Phi$) joining z_1 and z_2 in $\mathbb{D}$. Hence $\mathrm{Hess}_g(w)(\gamma', \gamma') = 0$, which from (34) implies that $|K| \equiv 0$ along $\Gamma = \phi(\gamma)$. From (14) in Section 2 we have that

$$|K| = 2e^{-6\sigma} \sum_{i<j} \left| f_i' f_j'' - f_j' f_i'' \right|^2,$$

hence $f_i' f_j'' - f_j' f_i'' = 0$ along γ, for all $i < j$. By analytic continuation, $f_i' f_j'' - f_j' f_i'' = 0$ everywhere, and using that $\phi' \neq 0$, it follows that for some i and all j there are constants a_j, b_j such that $f_j = a_j f_i + b_j$. This proves the lemma.

□

A corresponding result for harmonic maps is Lemma 3 in [2].

6. Boundary Behavior and the Proof of Theorem 2

We now study the boundary behavior for functions ϕ satisfying (6). As in the results just above we suppose that p is an extremal Nehari function and we set

$$w(z) = \sqrt{\frac{\Phi'(|z|)}{|\phi'(z)|}}$$

where Φ is the extremal function associated with p. We just saw that if w has at least two critical points then the range of ϕ lies in a plane, and so the setting is effectively that of an analytic function satisfying Nehari's criterion (1). The

boundary behavior in this case has been thoroughly studied; see [7], [4] and also the summary of the classical results in [2].

We next consider the situation when w has a unique critical point, and here the basic estimate is as follows.

Lemma 5. *If w has a unique critical point then there are positive constants a and b and a number r_0, $0 < r_0 < 1$, such that*

$$(37) \qquad |\phi'(z)| \leq \frac{\Phi'(|z|)}{(a\Phi(|z|) + b)^2}, \quad r_0 < |z| < 1.$$

Proof. Let z_0 be the unique critical point of w. Let $\gamma(s)$ be an arc-length parametrized geodesic in the metric $\mathbf{g}$ starting at z_0 in a given direction. Let $\widetilde{w}(s) = w(\gamma(s))$. Now the critical point is unique, and therefore $\widetilde{w}'(s) > 0$ for all $s > 0$. Thus there is an $s_0 > 0$ and an $a > 0$ such that $\widetilde{w}'(s) > a$ for all $s > s_0$. This implies that $\widetilde{w}(s) > as + b$ for some positive constant b and $s > s_0$. It is easy to see from compactness that the constants s_0, a, and b in this estimate can be made uniform, independent of the direction of the geodesic starting at z_0. In other words,

$$w(z) \geq a d_{\mathbf{g}_\Phi}(z, z_0) + b$$

for all z with $d_{\mathbf{g}_\Phi}(z, z_0) > s_0$. By renaming the constant b and for suitable r_0 we will then have

$$w(z) \geq a d_{\mathbf{g}_\Phi}(z, 0) + b$$

for all z with $1 > |z| > r_0$. The theorem follows from the definition of w because $d_{\mathbf{g}_\Phi}(z, 0) = \Phi(|z|)$. $\qquad\square$

The estimate in this lemma allows one to deduce that ϕ has a continuous extension to $\overline{\mathbb{D}}$, and the argument is just as in [2]. We will give only a few details here, enough for a more precise accounting of the regularity of the extension (also as in [2]).

Since the function $(1 - x^2)^2 p(x)$ is positive and decreasing on $[0, 1)$, we can form

$$\lambda = \lim_{x \to 1}(1 - x^2)^2 p(x).$$

It was shown in [5] that $\lambda \leq 1$, and that $\lambda = 1$ if and only if $p(x) = (1 - x^2)^{-2}$. In this case, the function Φ is given by

$$\Phi(z) = \frac{1}{2}\log\frac{1 + z}{1 - z}.$$

Thus (37) amounts to

$$|\phi'(z)| \le \frac{1}{(1 - |z|^2)\left(\dfrac{a}{2}\log\dfrac{1 + |z|}{1 - |z|} + b\right)^2}, \quad |z| > r_0.$$

From this, the technique of integrating along hyperbolic segments in $\mathbb{D}$, see also [7], leads to

$$(38) \qquad |\phi(z_1) - \phi(z_2)| \le C \left(\log \frac{1}{|z_1 - z_2|}\right)^{-1},$$

for some constant C and points z_1, z_2 for which the hyperbolic geodesic segment joining them is contained in the annulus $r_0 < |z| < 1$. This implies that ϕ is uniformly continuous in the closed disk, and its continuous extension also satisfies (38). Thus when $\lambda = 1$ the extension has a logarithmic modulus of continuity.

Suppose now that $\lambda < 1$. We appeal to a result from [5], according to which

$$\lim_{x \to 1}(1 - x^2)\frac{\Phi''}{\Phi'}(x) = 2(1 + \sqrt{1 - \lambda}) = 2\mu.$$

Note that $1 < \mu \le 2$. It follows that for any $\epsilon > 0$ there exists $0 < x_0 < 1$ such that

$$\frac{\mu - \epsilon}{1 - x} \le \frac{\Phi''}{\Phi'}(x) \le \frac{\mu + \epsilon}{1 - x}, \quad x > x_0,$$

which implies that

$$\frac{1}{(1 - x)^{\mu - \epsilon}} \le \Phi'(x) \le \frac{1}{(1 - x)^{\mu + \epsilon}}, \quad x > x_0,$$

Then

$$\frac{\Phi'(x)}{(a\Phi(x) + b)^2} \le \frac{C}{(1 - x)^{\alpha + 3\epsilon}},$$

where $\alpha = 2 - \mu = 1 - \sqrt{1 - \lambda}$ and C depends on a, b and the values of Φ at x_0. This estimate, together with the technique of integration along hyperbolic segments, implies that

$$|\phi(z_1) - \phi(z_2)| \le C|z_1 - z_2|^{1 - \alpha - 3\epsilon} = C|z_1 - z_2|^{\sqrt{1 - \lambda} - 3\epsilon},$$

for all points z_1, z_2 for which the hyperbolic geodesic segment joining them is contained in the annulus $\max\{r_0, x_0\} < |z| < 1$. This shows that ϕ admits a continuous extension to the closed disk, with at least a Hölder modulus of continuity.

We also point out that if one has the additional information that $x = 1$ is a regular singular point of the differential equation $u'' + pu = 0$, then from an analysis of the Frobenius solutions at $x = 1$ one can deduce that

$$\Phi'(x) \sim \frac{1}{(1-x)^\mu} \, , \ x \to 1 \, .$$

This then provides a Hölder continuous extension when $\lambda > 0$ and a Lipschitz continuous extension when $\lambda = 0$.

All of this discussion has been under the assumption that w has a unique critical point. The argument in the case where w has no critical points, though still based on convexity, requires additional work. This, too, is very close to what was done in [2], so we only sketch the key points.

As will be explained momentarily, it is necessary to consider shifts $T \circ \phi$ of the holomorphic curve ϕ by Möbius transformations T of $\mathbb{R}^{2n}$. The composition $T \circ \phi$ will not, in general, be holomorphic, though it is still conformal as a mapping of $\mathbb{D}$ into $\mathbb{R}^{2n}$. Write the corresponding conformal metric on the disk as $e^{2\tau}\mathbf{g}_0$ and, restricting τ to the radial segment $re^{i\theta}$, $0 \le r < 1$, let

$$\Upsilon_\theta(r) = e^{\tau(re^{i\theta})}$$

Also, let $s = \Phi(r)$ be the arc-length parameter of $[0,1)$ in the metric $\mathbf{g}_\Phi$, so that $r = \Phi^{-1}(s)$. Replacing Theorem 5, along radial segments, we find that the function

$$\omega_\theta(s) = \left\{ \frac{\Phi'(\Phi^{-1}(s))}{\Upsilon_\theta(\Phi^{-1}(s))} \right\}^{1/2}$$

is convex, meaning in this case simply that $\omega_\theta''(s) \ge 0$.

This is Lemma 5 in [2], in a slightly different notation, and we will not give the (identical) proof. The reason, however, why one can compose with a Möbius transformation of the range and still get a convexity result on radial segments is that $S_1(T \circ \phi) = S_1\phi$ on radial segments, from (4). Then via Lemma 2, bounds on $\mathcal{S}\phi$ entail bounds on $S_1(T \circ \phi)$ and the convexity of $\omega(s)$ can be deduced from such bounds for S_1.

Now one shows, as in [2], that for any θ_0 it is possible to choose a Möbius transformation T so that $\omega_{\theta_0}'(0) > 0$. Therefore by convexity $\omega_{\theta_0}(s) \ge as + b$, a, $b > 0$, and then

$$\Upsilon_{\theta_0}(r) \le \frac{\Phi'(r)}{(a\Phi(r) + b)^2} \, ,$$

which provides a substitute for (37). By continuity this estimate holds for θ near θ_0, so in a small angular sector about the radius $re^{i\theta_0}$. This in turn implies that $T \circ \phi$ has a continuous extension to the part of $\overline{\mathbb{D}}$ in the sector. Since θ_0 was arbitrary and since we allow for Möbius transformations of the range, we obtain an extension of ϕ to $\overline{\mathbb{D}}$ that is continuous in the spherical metric.

7. EXAMPLES

In this section we present some examples to show that the injectivity criterion (6) is sharp. In the setting of lifts of harmonic maps the corresponding examples were provided by mappings into a catenoid in $\mathbb{R}^3$. Surprisingly, the formulas are similar here, though some of the analytical details are different.

Example 1. Let $p(x) = \pi^2/4$. Then the criterion (6) becomes

$$(39) \qquad\qquad |S\phi| + \frac{3}{4}|\phi'|^2|K| \le \frac{\pi^2}{2} \, .$$

Define $\phi : \mathbb{D} \to \mathbb{C}^2$ to be

$$\phi(z) = (c\, e^{\pi z}, \, e^{-\pi z}) \, ,$$

where the constant c is to be chosen later. Then

$$e^{2\sigma} = \pi^2 (c^2\, e^{2\pi x} + e^{-2\pi x}) \, .$$

Straightforward calculations produce

$$\sigma_x = \pi\, \frac{c^2 e^{4\pi x} - 1}{c^2 e^{4\pi x} + 1} \, ,$$

and

$$\sigma_{xx} = \frac{8\pi^2 c^2 e^{4\pi x}}{(c^2 e^{4\pi x} + 1)^2} \, .$$

Hence for the Schwarzian,

$$S\phi = 2(\sigma_{zz} - \sigma_z^2) = \frac{1}{2}(\sigma_{xx} - \sigma_x^2) = \frac{4\pi^2 c^2 e^{4\pi x}}{(c^2 e^{4\pi x} + 1)^2} - \frac{\pi^2}{2}\left(\frac{c^2 e^{4\pi x} - 1}{c^2 e^{4\pi x} + 1}\right)^2 \, .$$

If c is chosen large enough, then

$$|S\phi| = \frac{\pi^2}{2}\left(\frac{c^2 e^{4\pi x} - 1}{c^2 e^{4\pi x} + 1}\right)^2 - \frac{4\pi^2 c^2 e^{4\pi x}}{(c^2 e^{4\pi x} + 1)^2} \, .$$

Thus

$$|\mathcal{S}\phi| + \frac{3}{4}|\phi'|^2|K| = |\mathcal{S}\phi| + \frac{3}{4}\sigma_{xx} = \frac{\pi^2}{2}\left(\frac{c^2e^{4\pi x} - 1}{c^2e^{4\pi x} + 1}\right)^2 + \frac{2\pi^2c^2e^{4\pi x}}{(c^2e^{4\pi x} + 1)^2} = \frac{\pi^2}{2}\,.$$

Therefore, equality holds in (39) everywhere, and ϕ is injective, but barely, since $\phi(1) = \phi(-1)$.

Example 2. The previous example can be extrapolated to give a general construction. Let p be a Nehari function with the additional property that it is the restriction to $(-1, 1)$ of an analytic function in the disk $p(z)$ that satisfies $|p(z)| \le p(|z|)$. Typical examples are $p(z) = (1 - z^2)^{-2}$ and $p(z) = 2(1 - z^2)^{-1}$. The extremal map Φ is then analytic and univalent in the disk, and satisfies $\mathcal{S}\Phi(z) = 2p(z)$ there. Moreover, the image $\Phi(\mathbb{D})$ is a parallel strip like domain, symmetric with respect to the real and imaginary axes, and containing the entire real line. Let

$$f(z) = \frac{c\Phi(z) + i}{c\Phi(z) - i}\,,$$

where $c > 0$ is to be chosen later and sufficiently small so that $i/c \notin \Phi(\mathbb{D})$ (it can be shown that the map Φ is always bounded along the imaginary axis, see [4]). The function f maps $\mathbb{D}$ onto a simply-connected domain containing the unit circle minus the point 1. The smaller the value of c the thinner the image of f.

Define $\phi : \mathbb{D} \to \mathbb{C}^2$ by

$$\phi(z) = \left(f(z), \frac{1}{f(z)}\right)\,.$$

Then

$$e^{2\sigma} = |f'(z)|^2\left(1 + \frac{1}{|f(z)|^4}\right),$$

and a lengthy calculation results in

$$(40) \qquad \mathcal{S}\phi = 2(\sigma_{zz} - \sigma_z) = \mathcal{S}\Phi + 6\frac{(\overline{f}f')^2}{(1 + |f|^4)^2}\,,$$

and

$$(41) \qquad e^{2\sigma}|K| = 8\frac{|ff'|^2}{(1 + |f|^4)^2}\,.$$

Condition (6) then reads

$$(42) \qquad \left|\mathcal{S}\Phi + 6\frac{(\overline{f}f')^2}{(1 + |f|^4)^2}\right| + 6\frac{|ff'|^2}{(1 + |f|^4)^2} \le \mathcal{S}\Phi(|z|)\,.$$

Suppose, for example, we let $p(z) = (1 - z^2)^{-2}$, for which the extremal function is

$$\Phi(z) = \frac{1}{2} \log \frac{1+z}{1-z} \,.$$

Then $\Phi'(z) = (1-z^2)^{-1}$ and $f' = -2ic\Phi'/(c\Phi - i)^2$, and after some simplifications (42) becomes

$$\left| \frac{2}{(1-z^2)^2} - \frac{24c^2(c\overline{\Phi} - i)^2}{(1-z^2)^2(c\overline{\Phi} + i)^2(c\Phi - i)^4(1 + |f|^4)} \right| + \frac{24c^2|c\Phi + i|^2}{|1 - z^2|^2|c\Phi - i|^6(1 + |f|^4)}$$
$$\leq \frac{2}{(1 - |z|^2)^2} \,,$$

which further reduces to

$$(43) \qquad \left| 1 - \frac{12c^2(1 + c^2\Phi^2)^2}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2} \right| + \frac{12c^2|1 + c^2\Phi^2|^2}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2} \leq \frac{|1 - z^2|^2}{(1 - |z|^2)^2} \,.$$

Let

$$\zeta = \frac{12c^2(1 + c^2\Phi^2)^2}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2} \,.$$

In order to guarantee (43), we need the following estimates for $|1 - \mathrm{Re}\{\zeta\}|$ and $|\mathrm{Im}\{\zeta\}|$.

Lemma 6. *If c is small then there exist absolute constants A, B, C such that*

$$(44) \qquad\qquad |1 - \mathrm{Re}\{\zeta\}| \leq 1 - |\zeta| + Ac^4|\mathrm{Im}\{\Phi\}|^2 \,,$$

$$(45) \qquad\qquad |\mathrm{Im}\{\zeta\}| \leq Bc^3|\mathrm{Im}\{\Phi\}| \,,$$

and

$$(46) \qquad\qquad |1 - \zeta| \leq 1 - |\zeta| + Cc^4|\mathrm{Im}\{\Phi\}|^2 \,.$$

Proof. It is clear that $|\zeta| < 1$ if c is small, hence $|1 - \mathrm{Re}\{\zeta\}| = 1 - \mathrm{Re}\{\zeta\}$. Thus (44) amounts to

$$(47) \qquad\qquad |\zeta| - \mathrm{Re}\{\zeta\} \leq Ac^4|\mathrm{Im}\{\Phi\}|^2 \,.$$

We have

$$|\zeta| - \mathrm{Re}\{\zeta\} = \frac{12c^2}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2} \left[|1 + c^2\Phi^2|^2 - \mathrm{Re}\{(1 + c^2\Phi^2)^2\} \right]$$

$$= \frac{12c^6(|\Phi|^4 - \mathrm{Re}\{\Phi^4\})}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2}$$

$$= -\frac{6c^6(\Phi^2 - \overline{\Phi}^2)^2}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2}$$

$$= \frac{24c^6(\Phi + \overline{\Phi})^2 |\mathrm{Im}\{\Phi\}|^2}{(|c\Phi - i|^4 + |c\Phi + i|^4)^2}$$

which shows (47) and thus (44) because $c^2(\Phi + \overline{\Phi})^2/(|c\Phi - i|^4 + |c\Phi + i|^4)^2$ is bounded for small c.

To establish (45) observe that

$$2i\,\mathrm{Im}\{(1 + c^2\Phi^2)^2\} = (1 + c^2\Phi^2)^2 - (1 + c^2\overline{\Phi}^2)^2 = c^4(\Phi^4 - \overline{\Phi}^4) + 2c^2(\Phi^2 - \overline{\Phi^2}),$$

from which (45) follows directly. Finally, (46) is a consequence of (44) and (45) because for $\zeta = x + iy$ small then $|1 - \zeta| \leq |1 - x| + 2y^2$.

$\square$

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