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Landau's Theorem for Holomorphic Curves in Projective Space and The Kobayashi Metric on Hyperplane Complements

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To Frederick W. Gehring, with affection

Abstract: We prove an effective version of a theorem of Dufresnoy: For any set of $2n+1$ hyperplanes in general position in $\mathbf{P}^n$, we find an explicit constant K such that for every holomorphic map f from the unit disc to the complement of these hyperplanes, we have $f^{\#}(0) \leq K$, where $f^{\#}$ denotes the norm of the derivative measured with respect to the Fubini-Study metric. This result gives an explicit lower bound on the Royden function, *i.e.,* the ratio of the Kobayashi metric on the hyperplane complement to the Fubini-Study metric. Our estimate is based on the potential-theoretic method of Eremenko and Sodin.
Keywords: Landau's Theorem, holomorphic curves, Kobayashi metric, hyperplane complement, subharmonic functions.

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1. Introduction

Let $H_1, \ldots, H_q$ be hyperplanes in general position in complex projective space $\mathbf{P}^n$, $q \geq 2n+1$. Being in general position simply means that any $n+1$ hyperplanes have empty intersection. By a theorem of Dufresnoy [Duf2, Th. VIII], there exists a constant K, depending only on the hyperplanes $\{H_1, \ldots, H_q\}$ and the dimension n, such that if f is a holomorphic map from the unit disc to $\mathbf{P}^n$ which omits the hyperplanes $H_1, \ldots, H_q$, then $f^\#(0) \leq K$, where $f^\#$ denotes the norm of the derivative of f with respect to the Fubini-Study metric on $\mathbf{P}^n$ and can be defined by

$$(f^\#)^2 = \frac{\displaystyle\sum_{j<k} |f_j f_k' - f_k f_j'|^2}{\|f\|^4},$$

where $\|f\|^2 = |f_0|^2 + \ldots + |f_n|^2$ for some choice of homogeneous coordinate functions $[f_0 : \ldots : f_n]$ for f. In modern terms (see for instance [Ko, §3.10] or [Lang, §VII.2]), Dufresnoy's theorem says that the complement of $2n+1$ hyperplanes in general position in $\mathbf{P}^n$ is complete hyperbolic and hyperbolically embedded in $\mathbf{P}^n$.

In [Duf2, p. 25], Dufresnoy remarks that the constant K depends on the hyperplanes H_j in a "completely unknown" way. Our purpose is to give an explicit estimate for the constant K, and therefore an explicit lower bound on the ratio of the infinitesimal Kobayashi metric (a. k. a. Royden's function [Lang]) on the hyperplane complement as compared to the Fubini-Study metric on projective space.

To that end, let $[X_0 : \ldots : X_n]$ be homogeneous coordinates on $\mathbf{P}^n$, and let $H_0, \ldots, H_n$ be $n+1$ hyperplanes in general position in $\mathbf{P}^n$ given by linear defining forms

$$H_j(X_0, \ldots, X_n) = a_{j0}X_0 + \ldots + a_{jn}X_n$$

normalized such that

$$||H_j||^2 = |a_{j0}|^2 + \ldots + |a_{jn}|^2 = 1.$$

Consider the $(n+1) \times (n+1)$-matrix $A = (a_{jk})$ and let

$$0 < \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$$

be the $n+1$ eigenvalues of the matrix AA^* listed in non-decreasing order. We then define the following quantities:

$$\begin{aligned} &\lambda(H_0,\dots,H_n) \overset{\text{def}}{=} \lambda_0 &&\text{and} && \Lambda(H_0,\dots,H_n) \overset{\text{def}}{=} \lambda_n \\ &\lambda^\#(H_0,\dots,H_n) \overset{\text{def}}{=} \frac{\sqrt{\lambda_0\lambda_1}}{\lambda_n} &&\text{and} && \Lambda^\#(H_0,\dots,H_n) \overset{\text{def}}{=} \frac{\sqrt{\lambda_{n-1}\lambda_n}}{\lambda_0}. \end{aligned} \tag{1.1}$$

We can now state our main theorem as

Theorem. *Let $H_0,\dots,H_{2n}$ be $2n+1$ hyperplanes in general position in $\mathbf{P}^n$. Let*

$$G = \max_{0\le j_0<\dots<j_n\le 2n} \max\{\log\Lambda(H_{j_0},\dots,H_{j_n}), \log(n+1) - \log\lambda(H_{j_0},\dots,H_{j_n})\},$$

and

$$G^\# = \min_{0\le j_0<\dots<j_n\le 2n} \frac{1}{\lambda^\#(H_{j_0},\dots,H_{j_n})}.$$

Let B be the binomial coefficient

$$B = \binom{2n+1}{n+1},$$

and

$$K = 12{,}672\left(2.6\cdot 10^7\log B + 10^8\right)^{6(4\log B+20)} GG^\#.$$

Let f be a holomorphic map from the unit disc $|z|<1$ to $\mathbf{P}^n$ omitting $H_0,\dots,H_{2n}$. Then,

$$f^\#(0) \le K.$$

When $n=1$, the classical theorem of Landau [Land] gives effective upper bounds, and very good bounds were obtained in [Hem] and [Jen]. Effective, though very non-sharp, estimates can also be derived from Nevanlinna's theory with precise error terms [CY, §5.7–5.8]. For some symmetrical arrangements of the omitted points, one even knows sharp estimates [BC1], [BC2].

Dufresnoy's proof of his theorem was based on the deep results of Bloch and Cartan [Ca], [Lang, Ch. V]. It is not clear whether this approach can give an effective estimate. One attempt in this direction is [Hal].

The theorem of Dufresnoy can also be obtained from Borel's theorem [Lang] by using a compactness argument called the Zalcman-Brody rescaling lemma [Ko, §3.6], [Lang, §III.2]. This proof does not give any effective estimate of K.

If f omits $2^n + 1$, instead of $2n + 1$, hyperplanes, there is an estimate of Cowen [Co1] (see also [Co2]), which he gets by constructing negatively curved metrics on the hyperplane complement. It seems difficult to adapt this method to the case of $2n + 1$ omitted hyperplanes, except of course for $n \leq 2$ when they are the same. For arbitrary n, there is also a negative curvature approach of Babets [Ba1], [Ba2], but Babets's estimates involve higher order derivatives or higher associated curves.

In this paper, we obtain an explicit estimate by putting in a quantitative form the argument from [Er1], where another proof of Dufresnoy's theorem was given. The proof in [Er1] was based on potential-theoretic considerations which also give a form of the Second Main Theorem for holomorphic curves [ES]. A survey of other results obtained with this method is [Er3].

To prove our main result we combine the method of [Er1] with two new ingredients. The first one is a quantitative form of the uniqueness theorem for harmonic functions in the form of a generalization of the three circles theorem of Hadamard by Nadirashvili [Nad]. The second ingredient is Theorem 4.2, which gives an estimate for $f^{\#}(0)$ in terms of the Fubini–Study area of the image of f, in the case that f omits $n + 1$ hyperplanes in general position. This estimate seems to be new even for $n = 1$, [Er2].

The plan of this paper is as follows. In section 2 we recall some formulas for the Fubini-Study derivative and some of its basic properties. In section 3, we recall the Riesz theorem and Jensen's formula. In section 4, we show using the Poisson formula and Harnack's inequality that if a holomorphic map f from a disc to $\mathbf{P}^n$ omits $n + 1$ hyperplanes in general position and covers finite area σ measured in the Fubini-Study metric, then $f^{\#}(0)$ can be effectively bounded in terms of σ, the dimension n, and the omitted hyperplanes. In Section 5, we recall Cartan's Lemma in a form we will need. In Section 6, we state Nadirashvili's generalization of the three circles theorem. In Section 7, we recall a covering lemma of Rickman that was also used in [Er1]. Finally, in section 8, we put the ingredients together to give our Landau and Schottky theorems.

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2. Fubini-Study Derivatives and Hermitian Linear Algebra

If

$$H_j(X_0, \ldots, X_n) = a_{j0}X_0 + \ldots + a_{jn}X_n$$

are $n+1$ linear forms defining $n+1$ hyperplanes in general position in $\mathbf{P}^n$ normalized as in the introduction so that

$$||H_j||^2 = |a_{j0}|^2 + \ldots + |a_{jn}|^2 = 1,$$

then for any point $[w_0 : \ldots : w_n]$ in $\mathbf{P}^n$, we have

$$\lambda(H_0, \ldots, H_n) \leq \frac{\displaystyle\sum_{j=0}^{n} |H_j(w_0, \ldots, w_n)|^2}{\displaystyle\sum_{j=0}^{n} |w_j|^2} \leq \Lambda(H_0, \ldots, H_n), \tag{2.1}$$

where λ and Λ are defined as in equation (1.1).

Let $f = [f_0 : \ldots : f_n]$ be a holomorphic map from a domain in $\mathbf{C}$ to $\mathbf{P}^n$ given by homogeneous coordinate functions f_j which are holomorphic without common zeros. Let $f' = [f_0' : \ldots : f_n']$. We recall the following formulas for the Fubini-Study derivative $f^{\#}$ of f :

$$(f^{\#})^2 = \frac{\partial^2}{\partial z \partial \bar{z}} \log \sum_{j=0}^{n} f_j \bar{f}_j \tag{2.2}$$

$$= \frac{\displaystyle\sum_{j=0}^{n}\sum_{k=0}^{n} \bar{f}_j' \bar{f}_k [f_j' f_k - f_j f_k']}{\|f\|^4} = \frac{\displaystyle\sum_{j=0}^{n}\sum_{k=0}^{n} |f_j|^2 |f_k|^2 \frac{\bar{f}_j'}{\bar{f}_j}\left[\frac{f_j'}{f_j} - \frac{f_k'}{f_k}\right]}{\|f\|^4} \tag{2.3}$$

$$= \frac{\displaystyle\sum_{j=0}^{n-1}\sum_{k=j+1}^{n} |f_j f_k' - f_k f_j'|^2}{\|f\|^4} \tag{2.4}$$

$$= \frac{||f \wedge f'||^2}{||f||^4}. \tag{2.5}$$

Proposition 2.1. *Let $H_0, \ldots, H_n$ be $n+1$ hyperplanes in general position in $\mathbf{P}^n$ given by homogeneous forms normalized so that the coefficient vectors have norm 1. Let $f = [f_0 : \ldots : f_n]$ be a holomorphic curve in $\mathbf{P}^n$ with homogeneous coordinate functions $f_0, \ldots, f_n$. Let $g = [g_0 : \ldots : g_n]$ be the holomorphic curve in $\mathbf{P}^n$ given by $g_j = H_j(f_0, \ldots, f_n)$. Then,*

$$\lambda^{\#}(H_0, \ldots, H_n) \leq \frac{g^{\#}}{f^{\#}} \leq \Lambda^{\#}(H_0, \ldots, H_n),$$

where $\lambda^{\#}$ and $\Lambda^{\#}$ are defined as in equation (1.1).

Remark. Only the left-hand inequality involving $\lambda^{\#}$ will be used in the sequel.

Proof. With notation as defined preceding equation (1.1), by linear algebra,

$$\lambda_0 \leq \frac{||g||^2}{||f||^2} \leq \lambda_n \qquad \text{and} \qquad \sqrt{\lambda_0\lambda_1} \leq \frac{||g \wedge g'||}{||f \wedge f'||} \leq \sqrt{\lambda_{n-1}\lambda_n},$$

and hence the proposition follows from (2.5). □

3. The Riesz Theorem and the Jensen Formula

In this section we recall the Riesz Theorem and the Jensen Formula, and we derive a corollary of the Jensen Formula that says a subharmonic function with large Riesz mass cannot stay close to a harmonic function.

Let v be a C^2 function on a domain in $\mathbf{C}$. We recall the differential operator dd^c which can be defined by

$$dd^c v = \frac{\partial^2 v}{\partial z \partial \bar{z}} \frac{dA}{\pi} = \frac{1}{4} \Delta v \frac{dA}{\pi},$$

where $dA = dx \wedge dy$ is the Lebesgue area form on $\mathbf{C}$. One of the advantages of the dd^c notation is that the factors 2 and π do not appear in fundamental formulas. As usual the operator dd^c is extended to the space $\mathscr{D}'$ of Schwartz distributions. When v is subharmonic, $2dd^c v$ defines a (locally finite, positive) Borel measure known as the Riesz measure of v.

Theorem 3.1 (**Riesz Theorem**)**.** *Let v be subharmonic on the unit disc $|z| \leq 1$ with finite Riesz mass. Then,*

$$v(z) \quad - \int\limits_{|z|<1} \log |z - \zeta|^2 dd^c v(\zeta)$$

is harmonic on $|z| < 1$.

Theorem 3.2 (**Jensen Formula**). *Let $v(z)$ be a difference of two subharmonic functions on $|z| \leq R$ and let $0 < r < R$. Then,*

$$\int_r^R \frac{dt}{t} \int\limits_{|z|\leq t} dd^c v = \frac{1}{2}\left[\int_0^{2\pi} v(Re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} v(re^{i\theta})\frac{d\theta}{2\pi}\right].$$

Corollary 3.3. *Let v be a subharmonic function on $|z| \leq R$ and let u be harmonic on $|z| \leq R$. Let $r < R$ and assume*

$$(a) \quad \int\limits_{|z|\leq r} dd^c v = 1 \qquad \text{and} \qquad (b) \quad |v - u| < \varepsilon \quad \text{for } |z| \leq R.$$

Then $\varepsilon \geq \log \dfrac{R}{r}$.

Proof. On the one hand by (a) and the Jensen Formula,

$$\frac{1}{2}\left[\int_0^{2\pi} v(Re^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} v(re^{i\theta})\frac{d\theta}{2\pi}\right] = \int_r^R \frac{dt}{t} \int\limits_{|z|\leq t} dd^c v \geq \log\frac{R}{r}.$$

On the other hand by (b) since u is harmonic,

$$\frac{1}{2}\int_0^{2\pi} [v(Re^{i\theta}) - v(re^{i\theta})]\frac{d\theta}{2\pi} \leq \frac{1}{2}\int_0^{2\pi} [u(Re^{i\theta}) - u(re^{i\theta}) + 2\varepsilon]\frac{d\theta}{2\pi} \leq \varepsilon. \qquad \square$$

4. From an Area Estimate to a Derivative Estimate

For a holomorphic curve omitting $2n+1$ hyperplanes in general position, our eventual goal is to bound $f^{\#}(0)$ explicitly in terms of the $2n+1$ given omitted hyperplanes. What we will first bound is the integral of $(f^{\#})^2$ on a disc centered at the origin. In this section, we see how to obtain a derivative bound from such an area bound. This section is based on [Er2] and appears to be new even in dimension one.

Proposition 4.1. *Let f be a holomorphic function without zeros on $|z| < 1$ such that $|f(z)| < 1$. Then,*

$$\left|\frac{f'(0)}{f(0)}\right| \leq -\log|f(0)|^2.$$

Proof. Let $r < 1$. Since

$$\frac{f'(z)}{f(z)} = \frac{\partial}{\partial z}\log|f(z)|^2,$$

differentiating the Poisson formula for $\log|f(z)|^2$ gives us

$$\left|\frac{f'(0)}{f(0)}\right| = \frac{1}{r}\left|\int_0^{2\pi} e^{-i\theta}\log|f(re^{i\theta})|^2\frac{d\theta}{2\pi}\right|$$
$$\leq \frac{1}{r}\int_0^{2\pi} -\log|f(re^{i\theta})|^2\frac{d\theta}{2\pi} = -\frac{1}{r}\log|f(0)|^2$$

because $|\log|f(z)|^2| = -\log|f(z)|^2$. The statement follows letting $r \to 1$. □

Theorem 4.2. *Let $f = [f_0 : \ldots : f_n]$ be a holomorphic map from the unit disc $|z| < 1$ to $\mathbf{P}^n$ which omits the $n+1$ hyperplanes $H_0, \ldots, H_n$ in general position. If*

$$\int\limits_{|z|<1} (f^{\#})^2\,\frac{dA}{\pi} \leq \sigma,$$

then

$$f^{\#}(0) \leq \frac{3\sqrt{2}}{\lambda^{\#}(H_0,\ldots,H_n)}\left[(2\log 2)\sigma + \log(n+1) + \log\frac{\Lambda(H_0,\ldots,H_n)}{\lambda(H_0,\ldots,H_n)}\right].$$

Before giving a proof, we make some comments. The above theorem is really only of interest when $\sigma \geq 1$. When $\sigma < 1$ and $n = 1$, a theorem of Dufresnoy [Duf1] (or see also [Hay1, Th. 6.1] or [CY, Th. 2.8.3]) says

$$[f^{\#}(0)]^2 \leq \frac{\sigma}{1-\sigma},$$

even without the assumption that f is zero free. The linear functions $f(z) = az$ show the Dufresnoy result is sharp in that setting. The same example shows that one cannot remove the assumption on the omitted hyperplanes from Theorem 4.2. Note also that our estimate does not even tend to 0 as $\sigma \to 0$.

We briefly comment on the sharpness of the coefficient $6\sqrt{2}\log 2 < 5.89$ in front of σ. Consider the case $n = 1$ and a zero free holomorphic function. Let

$$f_m(z) = \left(\frac{z-1}{z+1}\right)^m.$$

Then, $f_m^{\#}(0) = m$, and because f_m wraps the disc around the sphere so as to cover m hemispheres,

$$\sigma_m = \int\limits_{|z|<1} (f_m^{\#})^2\,\frac{dA}{\pi} = \frac{m}{2}.$$

Hence,

$$f_m^{\#}(0) = 2\sigma_m$$

no matter how large m is, and thus the 5.89 we obtain in front of σ is at worst 2.95 times too big.

Proof. By first working on a disc of radius $\rho < 1$ and then taking a limit as $\rho \to 1$, we may assume without loss of generality that f is analytic on $|z| \leq 1$. By abuse of notation, let H_j also denote the linear forms defining the hyperplanes, and normalize them so that $||H_j|| = 1$. Let $u_j = \log |H_j \circ f|^2$, and without loss of generality, assume the least harmonic majorant of the u_j on the unit disc $|z| \leq 1$ is 0 and that $\max u_j(0) = u_0(0)$. Let $u = \log \sum |f_j|^2$ and let $v(z) = \max_j u_j(z)$. Then u and v are subharmonic and satisfy

$$v \leq \log \sum |H_j \circ f|^2 \leq u + \log \Lambda(H_0, \ldots, H_n) \text{ and} \tag{4.1}$$

$$v \geq \log \sum |H_j \circ f|^2 - \log(n+1) \geq u + \log \lambda(H_0, \ldots, H_n) - \log(n+1). \tag{4.2}$$

Let $g_j = H_j \circ f$ and $g = (g_0, \ldots, g_n)$. Because

$$\frac{\sum_{j=0}^{n} \sum_{k=0}^{n} |g_j|^2 |g_k|^2}{\|g\|^4} = 1,$$

we have

$$\Big(\lambda^\#(H_0, \ldots, H_n) f^\#(0)\Big)^2 \leq (g^\#(0))^2 \leq 2 \max_j \left| \frac{g_j'(0)}{g_j(0)} \right|^2 \leq 2(u_0(0))^2$$

by Proposition 2.1, formula (2.3), Proposition 4.1 and our assumption that

$$u_0(0) = \max_j u_j(0).$$

Thus, it suffices to bound $u_0(0)$.

Let $0 < r < 1$. From the Jensen formula and the assumed bound on the integral of $(f^\#)^2$, we have

$$\frac{1}{2}\left[\int_0^{2\pi} u(e^{i\theta}) \frac{d\theta}{2\pi} - \int_0^{2\pi} u(re^{i\theta}) \frac{d\theta}{2\pi}\right] = \int_r^1 \frac{dt}{t} \int_{|z| \leq t} (f^\#)^2 \frac{dA}{\pi} \leq \sigma \log \frac{1}{r}.$$

Thus, there is some point $|z_0|$ with $|z_0| = r$ so that

$$\begin{aligned} v(z_0) + \log(n+1) - \log \lambda(H_0, \ldots, H_n) \geq u(z_0) &\geq -2\sigma \log \frac{1}{r} + \int_0^{2\pi} u(e^{i\theta}) \frac{d\theta}{2\pi} \\ &\geq -2\sigma \log \frac{1}{r} - \log \Lambda(H_0, \ldots, H_n), \end{aligned}$$

where the outside inequalities follow from (4.1), (4.2), and the fact that $v = 0$ when $|z| = 1$. Let j be the index such that

$$u_j(z_0) = v(z_0)$$

and then apply Harnack's Inequality to conclude

$$\begin{aligned} u_0(0) \geq u_j(0) &\geq \frac{1+r}{1-r} u_j(z_0) \\ &\geq \frac{1+r}{1-r}\left[-2\sigma \log \frac{1}{r} - \log(n+1) - \log \frac{\Lambda(H_0, \ldots, H_n)}{\lambda(H_0, \ldots, H_n)}\right]. \end{aligned}$$

Taking $r = 1/2$, we get

$$u_0(0) \geq -3\left[(2\log 2)\sigma + \log(n+1) + \log \frac{\Lambda(H_0, \ldots, H_n)}{\lambda(H_0, \ldots, H_n)}\right]. \quad \square$$

5. Cartan's Lemma

In this section we recall Cartan's Lemma, a well-known fundamental estimate whose significance in the study of function theory was recognized by Bloch and made rigorous by Cartan.

Theorem 5.1 (Cartan). *Let μ be a finite Borel measure on $|z| < 1$, and let Φ be the Blaschke potential*

$$\Phi(z) = \int\limits_{|\zeta|<1} \log \left| \frac{z - \zeta}{1 - \bar{\zeta} z} \right| d\mu(\zeta).$$

Let $0 < r < 1$, let $\eta < 1/4e$, and let $|z_0| \leq r$. Then, there is a countable family of exceptional discs D_j, the sum of whose radii does not exceed $4e\eta$, such that for those z with $|z| \leq r$ and not in any of the D_j, we have

$$\Phi(z) > C(r, \eta)\Phi(z_0),$$

where

$$C(r, \eta) = \frac{4}{(1-r)^2} \log \frac{1}{\eta}.$$

Theorem 5.1 is stated and proved for finite sums in Cartan [Ca, §II.21], which is also reproduced in [Lang, Th. VIII.3.3]. The argument for general μ with finite mass is the same, so we omit the proof here.

Corollary 5.2. *Let v be a negative subharmonic function on $|z| < 1$ such that for all $\rho < 1$,*

$$\int\limits_{|z|\leq\rho} dd^c v < \infty.$$

Let r, η, z_0 and $C(r,\eta)$ be as in the theorem. Then, for all $|z| \leq r$ outside of a countable set of discs the sum of whose radii is at most $4e\eta$, we have

$$v(z) \geq C(r,\eta)v(z_0).$$

Remark. Note that the radius of each exceptional disc is at most r, and hence the collective area of the exceptional discs is at most $4\pi e\eta r$.

In [Ca, Th. V], Cartan states this result in the case $v = \log|g|$ for an analytic function g with $|g| < 1$, but with a slightly worse expression for $C(r,\eta)$. Cartan's proof only makes use of Theorem 5.1 in the case of a discrete measure, rather than for a general measure as we shall do.

Proof. If $v(z_0) = -\infty$, the estimate is trivial, so we assume $v(z_0)$ is finite. Let M be a large constant so that

$$M \log r < C(r,\eta)v(z_0),$$

and let $\tilde{v}(z) = \max\{v(z), M\log|z|\}$. Note that $\tilde{v}$ is subharmonic, that $\tilde{v}(z) = 0$ for $|z| = 1$ by the assumption that $v(z) < 0$, and that

$$\int\limits_{|z|<1} dd^c \tilde{v} < \infty.$$

By the Riesz Theorem,

$$\tilde{v}(z) \; - \int\limits_{|z|<1} \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2 dd^c\tilde{v}(\zeta)$$

is harmonic on $|z| < 1$. However, both $\tilde{v}$ and the integral potential vanish identically for $|z| = 1$, and so $\tilde{v}(z)$ is actually equal to the integral. We then apply the theorem to conclude the existence of exceptional discs as in the statement of this corollary such that for those z outside those discs and $|z| \leq r$,

$$\frac{1}{C(r,\eta)}\tilde{v}(z) \geq \tilde{v}(z_0) > v(z_0) > M\log r \geq M\log|z|.$$

We thus conclude that for these same z, we have $\tilde{v}(z) = v(z)$. $\square$

6. Nadirashvili's Generalization of the Three Circles Theorem.

In this section we state a result of Nadirashvili [Nad] specialized to dimension 2. We slightly reformulate Nadirashvili's statement to get explicit constants convenient for our application.

Theorem 6.1 (Three Circles Theorem). *Let u be a harmonic function in the unit disc with $|u| < 1$. Let $0 < r < R < 1/2$ and assume $|u(z)| < r^\tau$ for $|z| < r$ and some $\tau > 0$. Then, $|u(z)| < 2(2R)^\tau$ for $|z| < R$.*

Proof. As in [KM, Lem. 2.1], the classical Hadamard three circles theorem implies,

$$\int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} \leq \rho^{2\tau} \qquad r < \rho < 1.$$

Thus,

$$\int_r^{2R} 2\rho \int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} d\rho \leq \frac{(2R)^{2\tau+2} - r^{2\tau+2}}{\tau+1} \leq (2R)^{2\tau+2} - r^{2\tau+2}.$$

Trivially,

$$\int_0^{r} 2\rho \int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} d\rho \leq r^{2\tau+2}.$$

Let z_0 be a point with $|z_0| = R$ such that $|u(z)| \leq |u(z_0)|$ for $|z| \leq R$. Then, by Jensen's Inequality,

$$\begin{aligned}(2R)^{2\tau+2} &\geq \int_0^{2R} 2\rho \int_0^{2\pi} [u(\rho e^{i\theta})]^2 \frac{d\theta}{2\pi} d\rho \\ &\geq \int_0^{R} 2t \int_0^{2\pi} [u(z_0 + te^{i\theta})]^2 \frac{d\theta}{2\pi} dt \\ &\geq \int_0^{R} 2t \left[\int_0^{2\pi} u(z_0 + te^{i\theta}) \frac{d\theta}{2\pi}\right]^2 dt = [u(z_0)]^2 R^2. \qquad \square\end{aligned}$$

Proposition 6.2 (Remez Inequality). *Let D be a disc of radius r in $\mathbf{C}$ and let U be a subset of D with area $\alpha\pi r^2$. Let $P(x,y)$ be a real polynomial of degree n on $\mathbf{R}^2$ and assume $|P(z)| < \varepsilon$ for z in U. Then, for all z in D,*

$$|P(z)| < \varepsilon \left(\frac{e}{2\alpha}\right)^n.$$

Proof. See [Nad, Lem. 2]. $\square$

If a is a complex number and $r > 0$, let $D(a, r)$ denote the disc of radius r centered at a.

Theorem 6.3 (Nadirashvili). *Let u be a harmonic function in the unit disc such that $|u| < 1$. Let $0 < \rho < 1/5$. Let U be a subset of $D(0,\rho)$ with area $\alpha\pi\rho^2 > 0$. Let*

$$r = \frac{1}{2}\left(\frac{\alpha}{36e}\right)^2,$$

and assume $r < \rho/2$. If $|u(z)| < \varepsilon < 1/9$ for all z in U, then there is a point z_0 in $D(0,\rho)$ such that

$$|u(z)| < 3\sqrt{\varepsilon}, \qquad \text{for all } z \in D(z_0, r). \tag{6.1}$$

Moreover,

$$\log|u(z)| < \frac{\log(9\varepsilon)\log(1/(5\rho))}{2\log\left[\frac{8}{5}\left(\frac{36e}{\alpha}\right)^2\right]} + \log 2 \qquad \text{for all } |z| < \rho. \tag{6.2}$$

Proof. Let z_0 be a point in $D(0,\rho)$ such that

$$\frac{\operatorname{Area}(U \cap D(z_0, r))}{\pi r^2} \geq \frac{\alpha}{72}.$$

Such a point exists because $D(0,\rho)$ is covered by at most

$$\left(3\sqrt{2}\left\lceil\frac{\rho}{r}\right\rceil\right)^2$$

discs of radius r with centers on a square lattice. Let $n = \lfloor \log_{2r} \varepsilon \rfloor$ and let P be the n-th Taylor polynomial of u. Because $D(z_0, r) \subset D(0, 1/2)$, by estimating the error in Taylor's theorem using $|u| < 1$, we know that for z in $D(z_0, r)$, we have

$$|u(z) - P(z)| \leq 2^{n+1} r^{n+1} \leq (2r)^{\log_{2r} \varepsilon} = \varepsilon. \tag{6.3}$$

Thus, $|P(z)| < 2\varepsilon$ for z in $D(z_0, r) \cap U$. By Proposition 6.2, if z is in $D(z_0, r)$, then

$$\begin{aligned} |P(z)| &\leq 2\varepsilon\left(\frac{e\pi r^2}{2\operatorname{Area}(U \cap D(z_0, r))}\right)^n \\ &\leq 2\varepsilon(36e/\alpha)^{\log_{2r}\varepsilon} = 2 \cdot \varepsilon^{1 + \log_{2r}(36e/\alpha)} = 2\sqrt{\varepsilon}, \end{aligned}$$

where the last equality follows from our choice of r. This together with (6.3) gives (6.1).

Let $v(z) = u(\frac{4}{5}z + z_0)$. Then, v is harmonic on the unit disc, $|v| < 1$, and

$$|v(z)| < 3\sqrt{\varepsilon} \qquad \text{for } |z| < \frac{5}{4}r.$$

Let τ be such that $3\sqrt{\varepsilon} = (\frac{5}{4}r)^\tau$. Then, we conclude by Theorem 6.1 that

$$|v(z)| < 2(5\rho)^\tau \qquad \text{for } |z| < \frac{5}{2}\rho.$$

Because $D(0, \rho) \subset D(z_0, 2\rho)$, we conclude that

$$|u(z)| < 2(5\rho)^\tau \qquad \text{for } |z| < \rho,$$

which gives (6.2). □

7. Rickman's Covering Lemma

This section describes a covering lemma of Rickman [Rick] as corrected in [Er1]. We provide a proof here to get a slightly better constant. If a is a complex number and $r > 0$, let $D(a, r)$ denote the disc of radius r centered at a. For a complex number b with $|b| < 1$ and an integer $m \geq 1$, denote by

$$\rho_m(b) \stackrel{\text{def}}{=} \frac{1}{2^{m+1}}(1 - |b|).$$

Proposition 7.1. *Let a be a complex number with $|a| < 1$ and for $j = 0 \dots 14$, let*

$$b_j = a + \frac{3}{2}\rho_m(a)\frac{a}{|a|}e^{2\pi i j/15},$$

where $a/|a|$ is understood to be 1 if $a = 0$. Then

$$D(a, 2\rho_m(a)) \subset D(a, \rho_m(a)) \cup \bigcup_{j=0}^{14} D(b_j, \rho_m(b_j)).$$

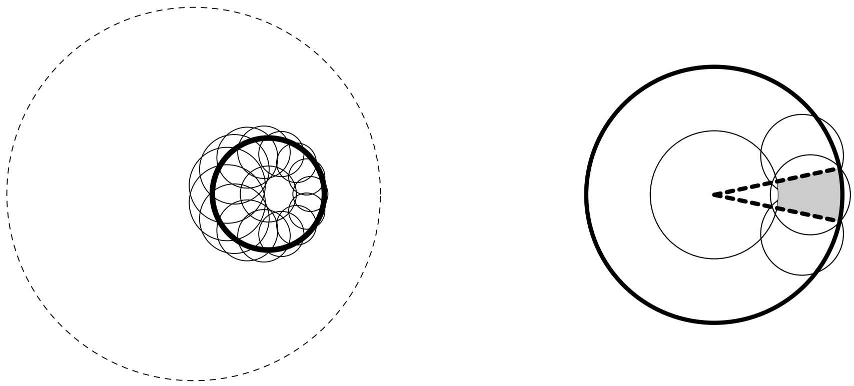

Figure 1. Illustration of Proposition 7.1.

Proof. Without loss of generality, assume $a \geq 0$. The proof is illustrated by Figure 1. The dotted circle in the left-hand picture represents the boundary of $D(0,1)$, and the bold circle is the boundary of $D(a, 2\rho_m(a))$. The picture on the right is a magnification of the bold circle on the left with only three of the covering discs of the form $D(b_j, \rho_m(b_j))$ drawn in.

The point is that all the covering discs have radius at least as big as $\rho_m(b_0)$. Thus, it suffices to show that the sector of the ring (shaded in the figure on the right)

$$\{z : |z-a| > \rho_m(a), |z-a| < 2\rho_m(a), -2\pi/30 < \arg(z-a) < 2\pi/30\}$$

is contained in the disc $D(b_0, \rho_m(b_0))$. This is true because the four corner points of the annular sector are easily seen to be contained in $D(b_0, \rho(b_0))$ using the Law of Cosines on the triangles that have as vertices a, b_0, and one of the corner points of the sector, noting that the side lengths of the triangle and $\rho(b_0)$ vary in direct proportion with $1-|a|$ as a varies. $\square$

Lemma 7.2 (**Rickman covering lemma**). *let μ be a Borel measure on $|z| < 1$ such that $|z| < 1$ has finite μ-measure. Let $m \geq 1$ be an integer and let $c > 1$. Then, there is a complex number a with $|a| < 1$ such that*

$$\mu(D(a, 2^m\rho_m(a))) \leq 16^m c\mu(D(a, \rho_m(a)))$$

and

$$\mu(D(0, 1/2^{m+1})) \leq c\mu(D(a, \rho_m(a))).$$

Proof. Choose a with $|a| < 1$ such that

$$\mu(D(a, \rho_m(a))) \geq \frac{1}{c} \sup_{|z|<1} \mu(D(z, \rho_m(z))),$$

which immediately implies the second inequality of the lemma by considering $z = 0$. The first inequality follows by iterating Proposition 7.1. $\square$

8. Landau and Schottky Theorems

We now prove our upper bound on the Fubini-Study area covered by a holomorphic curve omitting $2n+1$ hyperplanes in general position in $\mathbf{P}^n$.

Theorem 8.1. *Let f be a holomorphic map from the unit disc $|z| < 1$ to $\mathbf{P}^n$ omitting $2n+1$ hyperplanes $H_0, \ldots, H_{2n}$ in general position. Then,*

$$\int\limits_{|z|<\frac{1}{32}} (f^{\#})^2 \frac{dA}{\pi} \leq 36 \left(2.6 \cdot 10^7 \log B + 10^8\right)^{6(4\log B + 20)} G,$$

where B is the binomial coefficient

$$B = \binom{2n+1}{n+1}$$

and

$$G = \max_{0 \leq j_0 < \ldots < j_n \leq 2n} \max\{\log \Lambda(H_{j_0}, \ldots, H_{j_n}), \log(n+1) - \log \lambda(H_{j_0}, \ldots, H_{j_n})\},$$

where λ and Λ are defined in equation (1.1).

Proof. By a standard limiting argument we may assume f is holomorphic in a neighborhood of $|z| \leq 1$. Let $[f_0 : \ldots : f_n]$ be projective coordinate functions for f, let

$$u = \log\left(|f_0|^2 + \ldots + |f_n|^2\right),$$

and for $j = 0, \ldots, 2n$, let $u_j = \log |H_j \circ f|^2$, where as usual we also use H_j to denote the linear forms defining the hyperplanes normalized so $||H_j|| = 1$. For $c > 1$, by Lemma 7.2, there exists a and r such that

$$\int\limits_{|z-a|\leq 16r} dd^c u \leq 16^4 c \int\limits_{|z-a|\leq r} dd^c u$$

and such that

$$\sigma_{a,r} \stackrel{\text{def}}{=} \int\limits_{|z-a|\leq r} dd^c u \geq \frac{1}{c} \int\limits_{|z|\leq\frac{1}{32}} dd^c u. \tag{8.1}$$

Let $\tilde{u}(z)$ be the least harmonic majorant of $\max\limits_{0\leq j\leq 2n} u_j$ on $|z-a| \leq 16r$. Define

$$v(z) = \frac{1}{\sigma_{a,r}}[u(a+rz) - \tilde{u}(a+rz)] \qquad \text{and} \qquad v_j(z) = \frac{1}{\sigma_{a,r}}[u_j(a+rz) - \tilde{u}(a+rz)].$$

Then v_j are harmonic and negative on $|z| \leq 16$ with $\max v_j(z) = 0$ when $|z| = 16$. The function v is subharmonic on $|z| \leq 16$ with

$$\int\limits_{|z|\leq 1} dd^c v = 1 \qquad \text{and} \qquad \int\limits_{|z|\leq 16} dd^c v \leq 16^4 c. \tag{8.2}$$

By equation 2.1, for any distinct $n+1$ indices $j_0, \ldots, j_n$, we have

$$\max u_{j_k} \le \log \sum_{k=0}^{n} |H_{j_k} \circ f|^2 \le u + \log \Lambda(H_{j_0}, \ldots, H_{j_n})$$

and

$$u + \log \lambda(H_{j_0}, \ldots, H_{j_n}) \le \log \sum_{k=0}^{n} |H_{j_k} \circ f|^2 \le \max u_{j_k} + \log(n+1).$$

Thus,

$$(8.3) \qquad |v - \max v_{j_k}| \le \varepsilon \qquad \text{for } |z| \le 16 \quad \text{where } \varepsilon = \frac{G}{\sigma_{a,r}}.$$

Because the v_j are negative, this implies

$$v(z) \le \varepsilon \quad \text{for } |z| \le 16 \qquad \text{and} \qquad v(z) \ge -\varepsilon \quad \text{for } |z| = 16.$$

Let J be an index set of cardinality $n+1$, and let

$$U_J = \{z : |z| \le 2 \text{ and } |v(z) - v_j(z)| < \varepsilon \text{ for all } j \text{ in } J\}.$$

By (8.3), because we have $2n+1$ hyperplanes total, at every point z_0 with $|z_0| \le 16$, there are (at least) $n+1$ distinct indices $j_0, \ldots, j_n$ such that

$$|v(z_0) - v_{j_k}(z_0)| \le \varepsilon \qquad \text{for } k = 0, \ldots, n,$$

and hence there is at least one such index set J_0 such that

$$\text{Area}(U_{J_0}) \ge \frac{4\pi}{B}.$$

By re-ordering the indices, without loss of generality, we will assume J_0 contains $0, \ldots, n$, from now on we will consider only these indices, and we will denote U_{J_0} simply by U. By shrinking U if necessary, we may assume

$$\text{Area}(U) = \frac{4\pi}{B}.$$

By the Jensen Formula and (8.2),

$$\frac{1}{2}\left[\int_0^{2\pi} v(16e^{i\theta})\frac{d\theta}{2\pi} - \int_0^{2\pi} v(2e^{i\theta})\frac{d\theta}{2\pi}\right] = \int_2^{16} \frac{dt}{t} \int_{|z|\le t} dd^c v \le 16^4 c \log 8.$$

Because $v(z) \ge -\varepsilon$ for $|z| = 16$, we conclude that there is a point z' with $|z'| = 2$ such that

$$v(z') \ge -2 \cdot 16^4 c \log 8 - \varepsilon.$$

We now apply Corollary 5.2 to $v(z)-\varepsilon$ with $\eta^{-1} > 16eB$ to conclude that

$$v(z)-\varepsilon \geq C(1/8,\eta)[v(z')-\varepsilon]$$

for $|z| \leq 2$ and outside a collection of discs with radii r_j such that $\sum r_j < 2/B$. As $r_j < 2$, this implies the exceptional discs have collective area less than the area of U, and so there is a point z_0 in U with

$$v(z_0) \geq v(z_0)-\varepsilon \geq C(1/8,\eta)[v(z')-\varepsilon] \geq C(1/8,\eta)[-2\cdot 16^4 c\log 8 - 2\varepsilon].$$

For $0 \leq j \leq n$, because v_j is within ε of v at z_0, we conclude

$$v_j(z_0) \geq -C(1/8,\eta)[2\cdot 16^4 c\log 8 + 3\varepsilon].$$

Applying the Harnack Inequality twice implies

$$v_j(z) \geq -9C(1/8,\eta)(2\cdot 16^4 c\log 8+3\varepsilon), \qquad \text{for } j=0,\ldots,n \text{ and } |z| \leq 12.$$

Letting $\eta^{-1} \to 16eB$, we get for $j=0,\ldots,n$ and $|z|\leq 12$,

$$v_j(z) \geq -\frac{2304}{49}\log(16eB)(2\cdot 16^4 c\log 8 + 3\varepsilon). \tag{8.4}$$

Now suppose there is a constant $\delta > 0$ such that for any two indices j and k with $0 \leq j < k \leq n$, we have

$$|v_j(z)-v_k(z)| \leq \delta \qquad \text{for all } |z| \leq 2.$$

Because at every point z, one of these $n+1$ functions v_j comes within ε of $v(z)$ and they are all within δ of each other, we have that

$$|v(z)-v_j(z)| \leq \delta+\varepsilon \qquad \text{for } 0 \leq j \leq n \text{ and for all } |z| \leq 2.$$

We can now apply Corollary 3.3 with $R=2$ and $r=1$ to conclude

$$\varepsilon+\delta \geq \log(2).$$

We then consider the case that $\varepsilon < 1/18$, and so from the above,

$$\delta > \log 2 - \varepsilon > \frac{1}{2}.$$

In this case, there are indices j and k and a point $\tilde{z}$ with $|\tilde{z}| \leq 2$ such that

$$|v_j(\tilde{z})-v_k(\tilde{z})| \geq \frac{1}{2}.$$

Now, $v_j - v_k$ is harmonic on $|z| \leq 12$, $|v_j-v_k| < 2\varepsilon$ on U, and

$$|v_j-v_k| \leq M \overset{\text{def}}{=} 2\cdot\frac{2304}{49}\log(16eB)\left(2\cdot 16^4 c\log 8+\frac{1}{6}\right) \quad \text{for } |z|\leq 12 \text{ by (8.4)}.$$

Apply Theorem 6.3 to $(v_j - v_k)/M$, which is bounded by $2\varepsilon/M < 1/9$ on U, to conclude from (6.2) that

$$\frac{1}{\varepsilon} \leq \frac{36}{M}(2M)^{\frac{2\log\frac{8}{5}(36eB)^2}{\log(6/5)}} \leq 36(2M)^{\frac{2\log\frac{8}{5}(36eB)^2}{\log(6/5)}}.$$

Letting $c \to 1$ and simplifying we get,

$$\frac{1}{\varepsilon} < 36\left(2.6 \cdot 10^7 \log B + 10^8\right)^{6(4\log B + 20)}.$$

Because this is much greater than 18, there was no harm in our considering only $\varepsilon < 1/18$. Combining the above with (8.1) and recalling that $dd^c u = (f^\#)^2 dA/\pi$, we conclude

$$\int_{|z|\leq\frac{1}{32}} (f^\#)^2 \frac{dA}{\pi} \leq \sigma_{a,r} \leq \frac{G}{\varepsilon},$$

and hence the theorem follows. $\square$

Theorem 8.2 (Landau type theorem). *Let $H_0, \ldots, H_{2n}$ be $2n+1$ hyperplanes in general position in $\mathbf{P}^n$. Let*

$$G = \max_{0\leq j_0 < \ldots < j_n \leq 2n} \max\{\log\Lambda(H_{j_0}, \ldots, H_{j_n}), \log(n+1) - \log\lambda(H_{j_0}, \ldots, H_{j_n})\},$$

and

$$G^\# = \min_{0\leq j_0 < \ldots < j_n \leq 2n} \frac{1}{\lambda^\#(H_{j_0}, \ldots, H_{j_n})}.$$

Let B be the binomial coefficient

$$B = \binom{2n+1}{n+1},$$

and let

$$K = 12{,}672\left(2.6 \cdot 10^7 \log B + 10^8\right)^{6(4\log B + 20)} GG^\#.$$

Let f be a holomorphic map from the unit disc $|z| < 1$ to $\mathbf{P}^n$ omitting $H_0, \ldots, H_{2n}$. Then for $|z| < 1$,

$$f^\#(z) \leq \frac{K}{1 - |z|^2}.$$

When $n = 1$, Landau's original theorem gave a bound on $|f'(0)|$ in terms of $|f(0)|$ for analytic functions on the unit disc omitting 0 and 1. In this context,

$$f^\#(0) = \frac{|f'(0)|}{1 + |f(0)|^2},$$

and so Theorem 8.2 also gives a bound on $|f'(0)|$ in terms of $|f(0)|$. Of course, the bound is non-optimal in this special case.

As we discussed in the introduction, the Kobayashi metric on the complement of the hyperplanes H_j is a metric, so Theorem 8.2 can be interpreted as a global lower-bound over this hyperplane complement on the ratio of the infinitesimal Kobayashi metric to the infinitesimal Fubini-Study metric. Unfortunately, our method does not give any information in terms of the geometry of the omitted hyperplanes on where the minimum ratio occurs. For geometrically symmetric hyperplane configurations, one could hope to determine these minimum points. This was done in the one-dimensional case in [BC1] and [BC2].

Note that the dependence on G in Theorem 8.2 is essentially best possible. Indeed, consider $n = 1$ and consider the points 0, m and ∞ for m large. Then the map $f_r(z) = e^{rz}$ omits the three points provided $r < \log m$. Then, $f_r^{\#}(0) = r/2$ approaches $(\log m)/2$ as r approaches $\log m$. Because we have included 0 and ∞ among our three points, $G^{\#} = 1$ in this case. A straightforward computation shows that

$$G = \log(2) - \log\left(1 - \sqrt{\frac{m^2}{1+m^2}}\right).$$

An easy application of L'Hôpital's rule also shows that

$$\lim_{m\to\infty} \frac{G}{\log m} = 2,$$

and hence the linear appearance of G in the inequality in Theorem 8.2 is correct. Of course the enormous constant in front is not optimal.

Now consider the case where the three points are m, $-m$, and ∞ for m large. In this case Bonk and Cherry [BC1] gave a sharp upper bound on $f^{\#}(0)$. If f_1 denotes the universal covering map of $\mathbf{C} \setminus \{1, -1\}$ which sends 0 to 0, then the map $f_m(z) = mf_1(z)$ has the largest spherical derivative at the origin in comparison with all maps omitting m, $-m$ and ∞. One can even compute using Schwarz triangle functions that

$$f_m^{\#}(0) = m\frac{\Gamma(1/4)^4}{4\pi^2} \approx 4.4m.$$

One sees as above that here G is asymptotic to $2\log m$ as $m \to \infty$ and that $G^{\#}$ is asymptotic to m, and thus, the hyperplane dependence through the term $G^{\#}$ is also not too bad.

The appearance of the combinatorial coefficient B in the exponent of our estimate causes the estimate to deteriorate rather severely as the dimension n increases. It would be interesting to find out if there must be some dependence on n in the estimate beyond the fact that G depends implicitly on n. The appearance of this combinatorial constant also prevents our method from giving a better bound when f omits more than $2n+1$ hyperplanes. If f omits more than $2n+1$ hyperplanes, the best estimate our method gives is to choose the $2n+1$ hyperplanes that give the best G from among the omitted hyperplanes. This is clearly a deficiency in our approach. For maps to $\mathbf{P}^n$ omitting q hyperplanes, it would be nice to prove an estimate that improves as q increases, as in [CY, Th. 5.7.4] when $n = 1$.

Proof. By precomposing f with a Möbius automorphism of the disc, it suffices to prove the theorem for $z = 0$.

Let σ be the upper bound on

$$\int_{|z|\leq \frac{1}{32}} (f^{\#})^2 \frac{dA}{\pi}$$

obtained from Theorem 8.1. We apply Theorem 4.2 taking the best choice of $n+1$ hyperplanes among our given $2n+1$ omitted hyperplanes to conclude that

$$f^{\#}(0) \leq 3\sqrt{2}G^{\#}[(2\log 2)(32\sigma) + \log(n+1) + 2G] \leq 11 \cdot 32\sigma,$$

since $\log(n+1) + 2G$ is clearly less than σ, and the factor 32 in front of σ comes from rescaling $D(0, 1/32)$ to the unit disc before applying Theorem 4.2. $\square$

We conclude with a Schottky type theorem which bounds how close $f(z)$ can get to the omitted hyperplanes depending on the location of $f(0)$.

Theorem 8.3 (**Schottky type theorem**). *With the hypotheses and notation of Theorem 8.2, let*

$$\delta_j(z) = \frac{|H_j \circ f(z)|}{||f||},$$

recalling that we have normalized the defining forms of our hyperplanes so that $||H_j|| = 1$. *Then for* $j = 0, \ldots, n$ *and* $|z| < 1$,

$$\log \frac{1}{\delta_j(z)} < \frac{1}{1-|z|}\left[16 \log \frac{1}{\delta_j(0)} + 8K^2\right].$$

Proof. See [Duf2, §20]. □

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