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An Entire Function with Simply and Multiply Connected Wandering Domains

Walter Bergweiler

Dedicated to Professor Frederick W. Gehring on the occasion of his 80th birthday

Abstract: We modify a construction of Kisaka and Shishikura to show that there exists an entire function f which has both a simply connected and a multiply connected wandering domain. Moreover, these domains are contained in the set $A(f)$ consisting of the points where the iterates of f tend to infinity fast. The results answer questions by Rippon and Stallard.

1. Introduction and results

Let f be an entire or rational function. The *Fatou set* $F(f)$ is defined as the set where the iterates f^n of f form a normal family. If U_0 is a component of $F(f)$, then $f^n(U_0)$ is contained in a component U_n of $F(f)$. If all U_n are different, then U_0 is called a *wandering domain* of f. While a famous theorem of Sullivan [14] says that rational functions do not have wandering domains, it had been shown already earlier by Baker [2] that such domains may exist for transcendental entire functions. While the wandering domain in Baker's example was multiply connected, examples of simply connected wandering domains were given later by various authors; see [9, p. 106], [14, p. 414], [3, p. 564, p. 567]

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and [7, p. 222]. Baker [4, Theorem 2] showed that his construction can be modified to yield wandering domains of infinite connectivity. Recently Kisaka and Shishikura [11] constructed an example with a multiply connected wandering domain of finite connectivity, thereby answering a question of Baker. In fact, they showed that the connectivity may take any preassigned value. Here we modify the construction of Kisaka and Shishikura to prove the following result.

Theorem 1. *There exists an entire function which has both a simply connected and a multiply connected wandering domain.*

The question whether an entire function with this property exists had been raised by Rippon and Stallard [13, p. 1125, Remark 3]. In the same paper, Rippon and Stallard also asked a question about the set $A(f)$ introduced in [6]. This is defined by

$$A(f) := \{z : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^n(z)| > M(R, f^{n-L}) \text{ for } n > L\},$$

where $M(r, f) := \max_{|z|=r} |f(z)|$ and $R > \min_{z\in J(f)} |z|$. Roughly speaking, $A(f)$ consists of the points z where $f^n(z)$ tends to infinity "as fast as possible." Rippon and Stallard showed that $A(f)$ has no bounded components and that the closure of every multiply connected wandering domain is contained in $A(f)$. They also showed that if a simply connected wandering domain intersects $A(f)$, then it must lie entirely in $A(f)$, and they ask [13, p. 1126, Remark 4] whether an entire function f with such a simply connected wandering domain exists. It turns out that an example with this property is provided by the function constructed in Theorem 1.

Theorem 2. *There exists an entire function f for which $A(f)$ contains a simply connected wandering domain.*

As mentioned, our construction is largely based on that of Kisaka and Shishikura. We state two of their lemmas in §2 and then repeat their construction in §3. There is only one minor change in the construction, which will be explained at the beginning of §4. In the remainder of §4 we then show that the function constructed has a simply connected wandering domain, thereby proving Theorem 1. In §5 we prove Theorem 2.

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2. Two lemmas of Kisaka and Shishikura

Kisaka and Shishikura first construct a quasiregular map $g : \mathbb{C} \to \mathbb{C}$ and then obtain the entire function f with the following lemma.

Lemma 1. [11, Theorem 3.1] *Let g be a quasiregular mapping from $\mathbb{C}$ to $\mathbb{C}$. Suppose that there are (disjoint) measurable sets $E_j \subset \mathbb{C}$ $(j = 1, 2, ...)$ satisfying:*

(a) *For almost every $z \in \mathbb{C}$, the g-orbit of z passes E_j at most once for every j;*
(b) *g is K_j-quasiregular on E_j;*
(c) $K_\infty := \prod_{j=1}^{\infty} K_j < \infty$;
(d) *g is holomorphic a.e. outside $\bigcup_{j=1}^{\infty} E_j$ (i.e. $\frac{\partial g}{\partial \bar{z}} = 0$ a.e. on $\mathbb{C} \setminus \bigcup_{j=1}^{\infty} E_j$).*

Then there exists a K_∞-quasiconformal map φ such that $f = \varphi \circ g \circ \varphi^{-1}$ is an entire function

In order to construct g they need to "interpolate" two polynomials given on circles by a quasiregular map with small dilatation. This is done with the following result, where log denotes the principal branch of the logarithm.

Lemma 2. [11, Lemma 6.3] *Let $k \in \mathbb{N}$, $b, \omega \in \mathbb{C} \setminus \{0\}$ and $\rho^\sharp, \lambda^\sharp, \rho^\flat, \lambda^\flat \in \mathbb{R}$ with $0 < \lambda^\flat < \rho^\flat < 1 < \rho^\sharp < \lambda^\sharp$.*

(a) *Suppose that these constants satisfy*

$$\rho^\sharp \geq 2|\omega|, \quad \lambda^\sharp \geq e\rho^\sharp, \quad C^\sharp := 1 - \frac{1}{k+1}\left(\frac{|\log b|}{\log(\lambda^\sharp/\rho^\sharp)} + \frac{4|\omega|}{\rho^\sharp}\right) > 0,$$

Then the map $bz^k(z - \omega)$ on $|z| = \rho^\sharp$ and z^{k+1} on $|z| = \lambda^\sharp$ can be interpolated on $\rho^\sharp \leq |z| \leq \lambda^\sharp$ with a K-quasiregular map g where $K \leq 1/C^\sharp$.

(b) *Suppose that these constants satisfy*

$$|\omega| \geq 2\rho^\flat, \quad \rho^\flat \geq e\lambda^\flat, \quad C^\flat := 1 - \frac{1}{k}\left(\frac{|\log(-b\omega)|}{\log(\rho^\flat/\lambda^\flat)} + \frac{4\rho^\flat}{|\omega|}\right) > 0.$$

Then the map $bz^k(z-\omega)$ on $|z| = \rho^\flat$ and z^k on $|z| = \lambda^\flat$ can be interpolated on $\lambda^\flat \leq |z| \leq \rho^\flat$ with a K-quasiregular map g where $K \leq 1/C^\flat$.

3. Construction of f

As mentioned, we follow closely the ideas of Kisaka and Shishikura and will first construct a quasiregular map $g : \mathbb{C} \to \mathbb{C}$ and then obtain f via Lemma 1.

We denote by $\operatorname{ann}(r, R)$ the open annulus with inner radius r and outer radius R; that is, $\operatorname{ann}(r, R) := \{z \in \mathbb{C} : r < |z| < R\}$. The idea is to choose sequences (a_n) and (R_n) such that the map $z \mapsto a_n z^{n+1}$ maps $\operatorname{ann}(R_n, R_{n+1})$ onto $\operatorname{ann}(R_{n+1}, R_{n+2})$. The map g will then be defined by $g(z) = a_n z^{n+1}$ on a large subannulus of $\operatorname{ann}(R_n, R_{n+1})$, and will interpolate the mappings $z \mapsto a_{n-1} z^n$ and $z \mapsto a_n z^{n+1}$ in an annulus containing the circle $\{z : |z| = R_n\}$.

Choosing $R_1 > R_0 := 1$ we obtain sequences (R_n) and (a_n) as required by putting

$$R_{n+1} := \frac{R_n^{n+1}}{R_{n-1}^n}$$

and

$$a_n := \frac{R_{n+1}}{R_n^{n+1}} = \frac{1}{R_{n-1}^n}.$$

Various estimates in the sequel will require that R_1 has been chosen large enough. Note that with $\gamma := \log R_1$ we have

$$\log \frac{R_{n+1}}{R_n} = n \log \frac{R_n}{R_{n-1}} = ... = n! \log \frac{R_1}{R_0} = \gamma n!.$$

We define sequences $(P_n), (Q_n), (S_n)$ and (T_n) by

$$\log \frac{T_n}{S_n} = \log \frac{S_n}{R_n} = \log \frac{R_n}{Q_n} = \log \frac{Q_n}{P_n} = \sqrt{\log \frac{R_{n+1}}{R_n}} = \sqrt{\gamma n!}$$

Choosing $R_1 > e$ we have $\gamma > 1$ and thus

$$\frac{T_n}{S_n} = \frac{S_n}{R_n} = \frac{R_n}{Q_n} = \frac{Q_n}{P_n} > e.$$

We also have

$$\begin{aligned}
\log \frac{P_{n+1}}{T_n} &= -\log \frac{Q_{n+1}}{P_{n+1}} - \log \frac{R_{n+1}}{Q_{n+1}} + \log \frac{R_{n+1}}{R_n} - \log \frac{S_n}{R_n} - \log \frac{T_n}{S_n} \\
&= -2\sqrt{\gamma (n+1)!} + \gamma n! - 2\sqrt{\gamma n!} \\
&> 0
\end{aligned}$$

for all $n \in \mathbb{N}$, provided that R_1 and hence γ is large enough. Thus

$$P_n < Q_n < R_n < S_n < T_n < P_{n+1}$$

for all $n \in \mathbb{N}$. We now define

$$b_n := -\frac{(n+1)^2}{n+2}\left(\frac{n+1}{n}\right)^n a_n = -\frac{(n+1)^2}{n+2}\left(\frac{n+1}{n}\right)^n \frac{R_{n+1}}{R_n^{n+1}}$$

for $n \in \mathbb{N}$. We also put $E_1 := \operatorname{ann}(P_2, Q_2)$ and

$$E_n := \operatorname{ann}(S_n, T_n) \cup \operatorname{ann}(P_{n+1}, Q_{n+1})$$

for $n \geq 2$.

We shall show that there exists a quasiregular map $g : \mathbb{C} \to \mathbb{C}$ with the following properties:

(i) $g(z) = a_1 z^2$ for $|z| \leq P_2$;
(ii) $g(z) = a_n z^{n+1}$ for $T_n \leq |z| \leq P_{n+1}$ and $n \geq 2$;
(iii) $g(z) = b_n(z - R_n)z^n$ for $Q_n \leq |z| \leq S_n$ and $n \geq 2$;
(iv) g is K_n-quasiregular in E_n for $n \geq 1$, with $K_n := 1 + 1/n^2$;
(v) $g(\operatorname{ann}(S_n, Q_{n+1})) \subset \operatorname{ann}(S_{n+1}, Q_{n+2})$ for $n \geq 1$.

Since $E_n \subset \operatorname{ann}(S_n, Q_{n+1})$ and since the annuli $\operatorname{ann}(S_n, Q_{n+1})$ are pairwise disjoint it then follows that g satisfies the hypothesis of Lemma 1. Thus there exists a quasiconformal map φ such that $f := \varphi \circ g \circ \varphi^{-1}$ is entire. This function f then has the desired properties.

In order to show that a map g with the properties stated exists, we simply define g by (i), (ii), (iii) in the ranges given there and thus have defined g in $\mathbb{C} \setminus \bigcup_{n=1}^{\infty} E_n$.

To define g in $\operatorname{ann}(P_n, Q_n)$, where $n \geq 2$, we consider $G(z) := g(R_n z)/R_{n+1}$. For $|z| = \lambda^\flat := P_n/R_n$ we then have

$$G(z) = a_{n-1}\frac{(R_n z)^n}{R_{n+1}} = \frac{R_n}{R_{n-1}^n}\frac{R_n^n z^n}{R_{n+1}} = z^n$$

and for $|z| = \varrho^\flat := Q_n/R_n$ we have

$$G(z) = b_n \frac{(R_n z - R_n)(R_n z)^n}{R_{n+1}} = b_n \frac{R_n^{n+1}}{R_{n+1}}(z-1)z^n = c_n (z-1) z^n$$

with

$$c_n := \frac{b_n R_n^{n+1}}{R_{n+1}} = -\frac{(n+1)^2}{n+2}\left(\frac{n+1}{n}\right)^n.$$

Now

$$\varrho^\flat = \frac{Q_n}{R_n} = \exp(-\sqrt{\gamma n!})$$

and

$$\lambda^\flat = \frac{P_n}{R_n} = \exp(-2\sqrt{\gamma n!}) = (\varrho^\flat)^2.$$

Thus $\varrho^\flat \geq e\lambda^\flat$ since $\gamma \geq 1$ and also $2\varrho^\flat \leq 1$. By Lemma 2, (b), there exists a K-quasiregular map $G_n : \{z \in \mathbb{C} : \lambda^\flat \leq |z| \leq \varrho^\flat\} \to \mathbb{C}$ such that $G_n(z) = z^n$ for $|z| = \lambda^\flat$ and $G_n(z) = c_n(z-1)z^n$ for $|z| = \varrho^\flat$, with $K \leq 1/C^\flat$ where

$$C^\flat := 1 - \frac{1}{n}\left(\frac{|\log(-c_n)|}{\log(\varrho^\flat/\lambda^\flat)} + 4\varrho^\flat\right),$$

provided that $C^\flat > 0$. But since

$$|\log(-c_n)| = \log\left(\frac{(n+1)^2}{n+2}\left(\frac{n+1}{n}\right)^n\right) \leq \log((n+1)e) = 1 + \log(n+1)$$

we may in fact achieve that

$$C^\flat \geq 1 - \frac{1}{n}\left(\frac{1+\log(n+1)}{\sqrt{\gamma n!}} + 4\exp\left(-\sqrt{\gamma n!}\right)\right) \geq 1 - \frac{1}{(n-1)^2+1}$$

for all $n \geq 2$ by choosing γ large enough. Thus

$$K \leq \frac{1}{C^\flat} \leq 1 + \frac{1}{(n-1)^2}.$$

Putting

$$g(z) := R_{n+1} G_n\left(\frac{z}{R_n}\right)$$

for $z \in \text{ann}(P_n, Q_n) \subset E_{n-1}$ we see that (iv) holds for $z \in E_{n-1} \cap \text{ann}(P_{n+1}, Q_{n+1})$. Similarly we define g in the remaining part of E_n; that is, in $E_n \cap \text{ann}(S_n, T_n)$. Here we use the first part of Lemma 2.

To prove (v) we note that if $z \in \text{ann}(S_n, Q_{n+1})$, then, by the maximum principle,

$$\begin{aligned}
|g(z)| &\le \max_{|\zeta|=Q_{n+1}} |g(\zeta)| \\
&= \max_{|\zeta|=Q_{n+1}} \left|b_{n+1}(\zeta - R_{n+1})\zeta^{n+1}\right| \\
&= |b_{n+1}|\,(R_{n+1}+Q_{n+1})\,Q_{n+1}^{n+1} \\
&= \frac{(n+2)^2}{n+3}\left(\frac{n+2}{n+1}\right)^{n+1} R_{n+2}\left(1+\frac{Q_{n+1}}{R_{n+1}}\right)\left(\frac{Q_{n+1}}{R_{n+1}}\right)^{n+1} \\
&\le 2e(n+2)R_{n+2}\left(\frac{Q_{n+1}}{R_{n+1}}\right)^{n+1} \\
&= 2e(n+2)R_{n+2}\exp\left(-(n+1)\sqrt{\gamma(n+1)!}\right) \\
&\le R_{n+2}\exp\left(-\sqrt{\gamma(n+2)!}\right) \\
&= Q_{n+2}
\end{aligned}$$

if γ is large enough. Similarly, noting that g has no zeros in $\text{ann}(S_n, Q_{n+1})$ and using the minimum principle, we find that $|g(z)| \ge S_{n+1}$ for $z \in \text{ann}(S_n, Q_{n+1})$. We deduce that (v) holds.

As in the paper of Kisaka and Shishikura we deduce from (v) that $g^n(z) \to \infty$ as $n \to \infty$ for $z \in \text{ann}(S_1, Q_2)$, while $g(0) = 0$ by (i). This implies that $\varphi(\text{ann}(S_1, Q_2))$ lies in a multiply connected component U_1 of the Fatou set of f, with $f^n|_{U_1} \to \infty$ as $n \to \infty$. Since multiply connected components of the Fatou set are always bounded by a result of Baker [1, Theorem 1], this implies that f has a multiply connected wandering domain.

Remark. The Fatou set and the other concepts of complex dynamics can also be defined for quasiregular maps, by carrying over the definitions from the holomorphic case literally. In order to retain the basic features of the theory one has to require, however, that all iterates of g are K-quasiregular with the same K. Such maps are called *uniformly quasiregular*. That our function g is uniformly quasiregular follows directly from the definition of g, or trivially from the representation $g = \varphi^{-1} \circ f \circ \varphi$. Lemma 1 says, essentially, that uniformly quasiregular selfmaps of the plane are quasiconformally conjugated to entire functions; see also [8, 10] for this result.

4. Proof of Theorem 1: f has a simply connected wandering domain

The sequence (ξ_n) of critical points of g is given by $\xi_1 := 0$ and $\xi_n := \frac{n}{n+1} R_n$. The only difference between the present construction and that of Kisaka and Shishikura concerns the orbits of these points. While we have chosen the values b_n such that

$$g(\xi_n) = b_n(\xi_n - R_n)\xi_n^n = -b_n \frac{R_n}{n+1} \left(\frac{n}{n+1}\right)^n R_n^n = \frac{n+1}{n+2} R_{n+1} = \xi_{n+1}$$

for $n \geq 2$, Kisaka and Shishikura worked with different values of b_n which yielded $g(\xi_n) = R_{n+1}$ and hence $g^2(\xi_n) = 0$.

Denote by $D(a, r)$ the disk of radius r around a. Let $\delta > 0$ and define $D_n := D(\xi_n, \delta R_n / n^4)$ for $n \geq 2$. We shall show that if δ is sufficiently small, then $g(D_n) \subset D_{n+1}$ for all n. This implies that $D_n \subset F(g)$ for $n \geq 2$. We will then show that D_n lies in a simply connected wandering domain of g and thus $\varphi(D_n)$ lies in a simply connected wandering domain V_n of f. Moreover, we will see in §5 that $V_n \subset A(f)$ for all n.

First we note that if δ is small enough, then $D_n \subset \operatorname{ann}(Q_n, R_n)$ so that g is holomorphic in D_n and

$$\begin{aligned} |g''(z)| &= \left| b_n \left(n(n+1) z^{n-1} - R_n n(n-1) z^{n-2} \right) \right| \\ &\leq |b_n| (n(n+1) + n(n-1)) R_n^{n-1} \\ &= 2n^2 |b_n| R_n^{n-1} \end{aligned}$$

for $z \in D_n$. Thus

$$\begin{aligned} |g'(z)| &= |g'(z) - g'(\xi_n)| \\ &\leq \int_{\xi_n}^{z} |g''(\zeta)| \, |d\zeta| \\ &\leq 2n^2 |b_n| R_n^{n-1} \frac{\delta R_n}{n^4} \\ &= \frac{2\delta}{n^2} |b_n| R_n^n \end{aligned}$$

for $z \in D_n$. It follows that if $z \in D_n$, then

$$\begin{aligned}|g(z) - \xi_{n+1}| &= |g(z) - g(\xi_n)| \\ &\leq \int_{\xi_n}^{z} |g'(\zeta)|\,|d\zeta| \\ &\leq \frac{2\delta}{n^2}|b_n|R_n^n \frac{\delta R_n}{n^4} \\ &= \frac{2\delta^2}{n^6}\frac{(n+1)^2}{n+2}\left(\frac{n+1}{n}\right)^n R_{n+1} \\ &\leq \frac{2\delta^2 e(n+1)}{n^6} R_{n+1} \\ &\leq \frac{\delta R_{n+1}}{(n+1)^4},\end{aligned}$$

provided δ is sufficiently small. Thus $g(D_n) \subset D_{n+1}$ for $n \geq 2$. As already mentioned, this implies that D_n lies in a Fatou component V_n' of g and thus $\varphi(D_n)$ lies in the Fatou component $V_n := \varphi(V_n')$ of f. By U_n' we denote the multiply connected Fatou component of g which contains $\operatorname{ann}(S_n, Q_{n+1})$, and by $U_n := \varphi(U_n')$ the corresponding Fatou component of f. As mentioned at the end of §3, the U_n are wandering domains. In fact, we have $U_m \neq U_l$ for $m \neq l$ (and thus $U_m' \neq U_l'$ for $m \neq l$). We shall show that $U_m' \neq V_l'$ (and thus $U_m \neq V_l$) for all m and l. Since V_l' lies "between" the annuli $\operatorname{ann}(S_{l-1}, Q_l)$ and $\operatorname{ann}(S_l, Q_{l+1})$ and thus "between" the domains U_{l-1}' and U_l', it suffices to show that $U_m' \neq V_l'$ for $l = m$ and $l = m + 1$.

Suppose that $U_m' = V_l'$ where $l = m$ or $l = m + 1$. Since $D_l \subset V_l'$ and $\operatorname{ann}(T_m, P_{m+1}) \subset U_m'$ there exists a simply connected domain Ω_m with

$$D_l \cup \operatorname{ann}(T_m, P_{m+1}) \backslash (-P_{m+1}, T_m) \subset \Omega_m \subset U_m'.$$

Since $g^n(z) \neq 0$ for $z \in \Omega_m$ and since the g^n are K-quasiregular for some K, we may define for $n > m$ a K-quasiregular map $h_n : \Omega_m \to \mathbb{C}$ by

$$h_n(z) = \left(\frac{g^{n-m}(z)}{R_n}\right)^{m!/n!},$$

for some branch of the root. We will show that the h_n form a normal family so that (h_n) has a convergent subsequence, say $h_{n_k} \to h$. Next we will show that $h_n(z) = h(z) = z/R_m$ for $z \in \operatorname{ann}(T_m, P_{m+1}) \backslash (-P_{m+1}, T_m)$, if the branch of the root has been suitably chosen. This implies in particular that h is nonconstant.

On the other hand, we will see that h is constant in D_l so that we obtain a contradiction.

To prove that (h_n) is normal we note that if $z \in \Omega_m \subset U'_m$, then $g^{n-m}(z) \in U'_n$ and thus $|g^{n-m}(z)| \leq S_{n+1}$, since $\text{ann}(S_{n+1}, Q_{n+2}) \subset U'_{n+1}$. Hence

$$|h_n(z)| \leq \left|\frac{S_{n+1}}{R_n}\right|^{m!/n!} = \left|\frac{S_{n+1}}{R_{n+1}}\frac{R_{n+1}}{R_n}\right|^{m!/n!}$$

for $z \in \Omega_m$. We deduce that

$$\begin{aligned}\log|h_n(z)| &\leq \frac{m!}{n!}\left(\log\frac{S_{n+1}}{R_{n+1}} + \log\frac{R_{n+1}}{R_n}\right)\\ &= \frac{m!}{n!}\left(\sqrt{\gamma(n+1)!} + \gamma n!\right)\\ &\leq 2\gamma m!\end{aligned}$$

for $z \in \Omega_m$ and large n, and this yields the desired normality.

It is not difficult to see by induction that if $z \in \text{ann}(T_m, P_{m+1})\backslash(-P_{m+1}, T_m)$ and $n > m$, then

$$g^{n-m}(z) = R_n\left(\frac{z}{R_m}\right)^{n!/m!} \in \text{ann}(T_n, P_{n+1})\backslash(-P_{n+1}, T_n)$$

so that

$$h_n(z) = h(z) = \frac{z}{R_m}$$

if the branch of the root in the definition of h_n has been suitable chosen. In particular, h is nonconstant.

For $z \in D_l$ we have

$$g^{n-m}(z) \in D_{n-m+l}$$

and thus $g^{n-m}(z) \in D_n$ or $g^{n-m}(z) \in D_{n+1}$ depending on whether $l = m$ or $l = m+1$. In the first case we have

$$\frac{g^{n-m}(z)}{R_n} \in D\left(\frac{\xi_n}{R_n}, \frac{\delta}{n^4}\right) = D\left(\frac{n}{n+1}, \frac{\delta}{n^4}\right) \subset D\left(1, \frac{1}{2}\right)$$

for large n, and this implies that h is constant in D_l. In the second case, that is for $l = m+1$, we have

$$\frac{g^{n-m}(z)}{R_n} \in D\left(\frac{\xi_{n+1}}{R_n}, \frac{\delta R_{n+1}}{(n+1)^4 R_n}\right)$$

and hence

$$\left(1-\frac{2}{n}\right)\frac{R_{n+1}}{R_n}\leq\left|\frac{g^{n-m}(z)}{R_n}\right|\leq\frac{R_{n+1}}{R_n}$$

for large n. Since $\sqrt[n!]{1-2/n}\to 1$ as $n\to\infty$ and

$$\left|\frac{R_{n+1}}{R_n}\right|^{m!/n!}=\exp(\gamma m!)$$

we deduce that $|h(z)|=\exp(\gamma m!)$ for $z\in D_l$. Thus h is constant in D_l in this case as well. As already noted, this is a contradiction. This completes the proof that $U'_m\neq V'_l$ and hence $U_m\neq V_l$ for all l and m.

It remains to show that V_l is simply connected for all l. Suppose that this is not the case. It is not difficult to show that then all V_l are multiply connected. By a result of Baker [3, Theorem 3.1] there exists $k\in\mathbb{N}$ and a Jordan curve τ in V_k whose interior contains 0. This implies that $D(0,Q_k)$ is contained in the interior of $\tau':=\varphi^{-1}(\tau)$. The argument principle implies that the winding number of $g(\tau')$ around 0 is at least k. This winding number is equal to the winding number of $f(\tau)$ around 0, and thus the latter winding number is also at least k. Induction shows that the winding number of $\tau_n:=f^{n-k}(\tau)$ around 0 is at least $(n-1)!/(k-1)!$ for $n\geq k$. A contradiction will now be obtained from a consideration of the hyperbolic length of τ_n in V_n. We denote the hyperbolic length of a curve σ in a hyperbolic domain U by $\ell(\sigma,U)$. By the Schwarz-Pick-Lemma we have

$$\ell(\tau_n,V_n)\leq\ell(\tau,V_k)$$

for all $n\geq k$. On the other hand, we have $V'_n\subset\operatorname{ann}(Q_n,S_n)$ and thus $V_n\subset\varphi(\operatorname{ann}(Q_n,S_n))$. This implies that

$$\ell(\tau_n,V_n)\geq\ell(\tau_n,\varphi(\operatorname{ann}(Q_n,S_n))).$$

Now $\operatorname{ann}(Q_n,S_n)$ is an annulus of modulus $\log(S_n/Q_n)/(2\pi)$. Since φ is K-quasiconformal this yields that $\varphi(\operatorname{ann}(Q_n,S_n))$ has modulus at most

$$\frac{K}{2\pi}\log\left(\frac{S_n}{Q_n}\right)=\frac{1}{2\pi}\log\left(\left(\frac{S_n}{Q_n}\right)^K\right).$$

It follows that there exists a conformal map $\psi:\varphi(\operatorname{ann}(Q_n,S_n))\to\operatorname{ann}(1,r_n)$ where $r_n\leq(S_n/Q_n)^K$. We may choose ψ such that $|\psi(\varphi(z))|\to 1$ as $|z|\to Q_n$. Put $\sigma_n:=\psi(\tau_n)$. Then

$$\ell(\sigma_n,\operatorname{ann}(1,r_n))=\ell(\tau_n,\varphi(\operatorname{ann}(Q_n,S_n)))$$

and the winding number of σ_n around 0 is the same as that of τ_n and thus at least $(n-1)!/(k-1)!$. We note that the density $\varrho(z)$ of the hyperbolic metric in $\text{ann}(1, r_n)$ is given by (see, e. g., [12, p. 12])

$$\varrho(z) = \frac{\pi}{|z| \sin(\pi \log |z| / \log r_n) \log r_n}.$$

In particular we have $\varrho(z) \geq \pi/(|z| \log r_n)$ and thus we conclude that

$$\ell(\sigma_n, \text{ann}(1, r_n)) = \int_{\sigma_n} \varrho(w)|dw| \geq \frac{\pi}{\log r_n} \int_{\sigma_n} \frac{|dw|}{|w|} \geq \frac{2\pi^2}{\log r_n} \frac{(n-1)!}{(k-1)!}.$$

Since $\log r_n \leq K \log(S_n/Q_n) = 2K\sqrt{\gamma n!}$ we deduce that

$$\ell(\sigma_n, \text{ann}(1, r_n)) \geq \frac{\pi^2}{K(k-1)!} \frac{(n-1)!}{\sqrt{\gamma n!}}$$

so that

$$\ell(\sigma_n, \text{ann}(1, r_n)) \to \infty$$

as $n \to \infty$.

On the other hand, our previous estimates imply that

$$\ell(\sigma_n, \text{ann}(1, r_n)) = \ell(\tau_n, \varphi(\text{ann}(Q_n, S_n))) \leq \ell(\tau_n, V_n) \leq \ell(\tau, V_k).$$

This is a contradiction. Thus V_ℓ is simply connected for all ℓ. This completes the proof of Theorem 1.

Remark. Except for the fixed point $\varphi(0)$, the critical points of f are contained in the simply connected wandering domains V_n of f. Thus the wandering domains U_n do not contain critical points. Using this it can be shown with the arguments of Kisaka and Shishikura (in particular, [11, Proposition 4.5]) that the U_n are doubly connected.

5. Proof of Theorem 2: the V_k are in $A(f)$

We will use the following characterisation of the set $A(f)$ given by Rippon and Stallard [13, Lemma 2.4]. Here we denote for a domain U by $\tilde{U}$ the union of U and its bounded complementary components.

Lemma 3. *Let f be a transcendental entire function and let D be a domain intersecting the Julia set of f. Then*

$$A(f) = \{z : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(z) \notin \widetilde{f^n(D)} \text{ for } n \in \mathbb{N}\}.$$

We apply this result with $D := U_1$. It follows from the maximum principle that $\widetilde{f^n(U_1)} = \widetilde{U_{n+1}}$. Since $U'_m \subset D(0, S_{m+1})$ we have $V'_l \cap \widetilde{U'_m} = \emptyset$ and hence $V_l \cap \widetilde{U_m} = \emptyset$ for $l \geq m+2$. Thus we see that if $k \geq 2$ and $z \in V_k$ so that $f^{n+1}(z) \in V_{k+n+1}$, then

$$f^{n+1}(z) \notin \widetilde{U_{n+1}} = \widetilde{f^n(U_1)}.$$

Choosing $L := 1$ in Lemma 3 we see that $V_k \subset A(f)$ for $k \geq 2$. This completes the proof of Theorem 2.

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Walter Bergweiler
Mathematisches Seminar,
Christian–Albrechts–Universität zu Kiel,
Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany
E-mail: bergweiler@math.uni-kiel.de