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H^p-theory for Quasiconformal Mappings

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Abstract: We develop an H^p-theory for quasiconformal mappings in space.

Keywords: quasiconformal mapping, H^p-space.

1. Introduction

The H^p-theory of analytic functions is well understood. The purpose of this paper is to show that also quasiconformal mappings in space admit a rich H^p-theory. Naturally, much of the powerful machinery of the plane is not available to us and thus our approach is a combination of the analytic and geometric aspects of the theory of quasiconformal mappings together with a number of tools from harmonic analysis.

This paper is an edited version of a manuscript under the same name that has been circulating since the early 1990's; some of the results were already announced in [A]. Since then some of the results in the manuscript have been also obtained independently by different methods by other authors, see for example the sequence of papers by Nolder in the list of references. For the sake of completeness, such results have not been removed from this paper. We have tried to include in our list of references all the papers related to our topic that we are aware of, even though the potential overlap with our work has not necessarily been pointed out in the text.

The paper is organized as follows. Section 2 contains preliminary material. We introduce the quasiconformal H^p-classes in Section 3. Section 4 includes a

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new proof of a theorem due to Zinsmeister and its consequences. In Section 5 we produce various characterizations for the membership in the H^p-classes. These contain an extension of a classical theorem of Littlewood and Paley. We establish a version of the Riesz theorem in Section 6, and, in Section 7, we discuss results related to BMO. Section 8 deals with H^p-theory for Df; since the differential Df is only defined almost everywhere we, in fact, consider an averaged version of Df. In Section 9 we present connections between H^p-classes and A^p-classes for both f and Df. Finally, Section 10 contains a list of open problems related to the contents of this paper.

2. Preliminaries

We write $B(x,r)$ for the open ball in $\mathbf{R}^n$ of radius r and centered at x, and we abbreviate $B(0,r)$ to $B(r)$ and $B(0,1)$ to $\mathbf{B}^n$. We denote the boundary of $B(x,r)$ by $S^{n-1}(x,r)$, we write $S^{n-1} = S^{n-1}(0,1)$, and we denote the surface area of S^{n-1} by ω_{n-1}. We often use the symbol $\approx$ to mean comparability, with constants that depend only on dimension n, dilatation K or other similar appropriate parameters, but not on the specific functions under consideration.

The modulus of a family Γ of paths in $\mathbf{R}^n$ is by definition

$$M(\Gamma) = \inf \int_{\mathbf{R}^n} \rho^n dx$$

where the infimum is taken over non-negative Borel functions ρ on $\mathbf{R}^n$ with $\int_\gamma \rho\, ds \geq 1$ for each locally rectifiable $\gamma \in \Gamma$. We will frequently employ the following estimates. Given two compact, connected and disjoint sets $E, F \subset \mathbf{B}^n$, denote the family of paths γ joining E and F in $\mathbf{B}^n$ by $\Gamma_{E,F}$. Then

$$M(\Gamma_{E,F}) \geq \frac{\omega_{n-1}}{2[\log(\lambda_n(1+t))]^{n-1}}, \tag{2.0}$$

where $t = \frac{d(E,F)}{\min\{\mathrm{diam}(E),\mathrm{diam}(F)\}}$ and λ_n is a constant that only depends on n. For us, an important path family is the family Γ of radial segments joining $S^{n-1}(0,r)$, $0 < r < 1$, to a set $E \subset S^{n-1}$. We have then

$$M(\Gamma) = \sigma(E)(\log(1/r))^{1-n}, \tag{2.1}$$

where $\sigma(E)$ is the surface area of E. As for uppper bounds, we always have

$$M(\Gamma) \leq \frac{\omega_{n-1}}{[\log(R/r)]^{n-1}},$$

if each $\gamma \in \Gamma$ joins $S^{n-1}(x,r)$ to $S^{n-1}(x,R)$, $0 < r < R$. For all these estimates see [Ge],[V].

A homeomorphism of a domain Ω in $\mathbf{R}^n$ into $\mathbf{R}^n$ is K-quasiconformal if f belongs to the Sobolev class $W^{1,n}_{\text{loc}}(\Omega; \mathbf{R}^n)$ and $|Df(x)|^n \le KJ_f(x)$ for almost every $x \in \Omega$. It then follows [V] that $M(\Gamma)/K \le M(f\Gamma) \le K^{n-1}M(\Gamma)$ for all path families $\Gamma \subset \Omega$; here $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$.

In the sequel we shall always assume that f is a quasiconformal homeomorphism of $\mathbf{B}^n$ into $\mathbf{R}^n$. Also, x denotes a generic point in $\mathbf{B}^n$, and ω in S^{n-1}, and we write

$$f(\omega) = \lim_{r \to 1} f(rw)$$

whenever this limit exists. By Beurling's theorem for quasiconformal mappings, this is the case for almost all $\omega \in S^{n-1}$. Indeed, the radial limit exists if the image of the radial line segment from ω to $S^{n-1}(0, 1/2)$ is rectifiable. It is easy to check that $M(f\Gamma) = 0$ for the path family Γ consisting of those radial segments for which $f(\gamma)$ fails to be rectifiable. Thus $M(\Gamma) = 0$ and the claim follows from (2.1).

In what follows, T always denotes a Möbius transformation of $\mathbf{B}^n$ onto itself. Given $0 \ne x \in \mathbf{B}^n$ we let $S(x) = S^{n-1} \cap B(x, 3(1 - |x|))$ be the cap with center $x/|x|$ and T_x be the Möbius transformation of $\mathbf{B}^n$ with $T_x(x) = 0$,

$$T_x(y) = \frac{(1 - |x|^2)(y - x) - |y - x|^2 x}{|x|^2 |y - (x/|x|^2)|^2}.$$

Then $T_x(S(x))$ always contains a hemisphere and

$$|\omega - \omega'|/9 \le |T_x(\omega) - T_x(\omega')|(1 - |x|) \le 2|\omega - \omega'|$$

for all $\omega, \omega' \in S(x)$. For each $\omega \in S^{n-1}$ we let

$$\Gamma(\omega) = \{x \in \mathbf{B}^n : |x - \omega| \le 3(1 - |x|)\}$$

be the cone with vertex ω. Clearly, $S(x) = \{\omega \in S^{n-1} : x \in \Gamma(\omega)\}$. Moreover, for a continuous function u on $\mathbf{B}^n$, we define its non-tangential maximal function $u^*(\omega)$ by

$$u^*(\omega) = \sup\{|u(x)| : x \in \Gamma(\omega)\}.$$

Next, given $x \in \mathbf{B}^n$, we write $B_x = B(x, (1 - |x|)/2)$.

2.1. Lemma. *Let $f : \mathbf{B}^n \to \Omega$ be K-quasiconformal. There is a constant C, which depends only on n, K, so that for each $x \in \mathbf{B}^n$*

$$\operatorname{diam}(f(B_x))/C \le d(f(x), \partial\Omega) \le C \operatorname{diam}(f(B_x)) \le C^2 d(f(B_x), \partial\Omega).$$

Moreover, $f(B_x)$ contains a ball of radius $d(f(x), \partial\Omega)/C$, centered at $f(x)$.

This lemma is a well-known consequence of the modulus estimates above. See for example [V]. We continue with another consequence of the basic modulus estimates.

2.2 Lemma. *Let $f : \mathbf{B}^n \to \Omega$ be K-quasiconformal and assume that $f(x) \neq 0$ for all $x \in \mathbf{B}^n$. Then*

$$(1-|x|)^b/C \leq \frac{|f(x)|}{|f(0)|} \leq C(1-|x|)^{-b},$$

where $b = (2K)^{1/(n-1)}$ and C depends only on K, n. Moreover, if Ω is contained in a half space H, then we may replace b above in the upper estimate with $K^{1/(n-1)}$, provided we allow C also depend on $d(f(0), H)$.

If Γ denotes the family of paths joining $B(0,1/2)$ to B_x in $\mathbf{B}^n$ with $B_x \cap B(0,1/2) = \emptyset$, then $2M(\Gamma) \geq w_{n-1}(\log \frac{C}{1-|x|})^{1-n}$, where $M(\Gamma)$ is the modulus of Γ. As $M(\Gamma) \leq KM(f\Gamma)$ by the quasiconformality of f, and since $M(\Gamma') \leq w_{n-1}(\log \frac{R}{r})^{1-n}$ for the family Γ' of paths joining $S(f(0), r)$ to $S(f(0), R)$, $r < R$, in Ω, the desired growth estimate follows from Lemma 2.1. The lower decay estimate is obtained analogously. Regarding the half-space claim, simply notice that then $2M(\Gamma') \leq w_{n-1}(\log \frac{R}{Mr})^{1-n}$. For the details see [K1], [M].

Following Astala and Gehring [AG1] we write

$$\mathrm{a_f}(x) = \exp[\int_{B_x} \log J_f(y) dm/(n|B_x|)],$$

where $|B_x|$ is the n-measure of B_x. Notice that if f is conformal, then the mean value property implies that $\mathrm{a_f} = |Df|$. For the proof of the following lemma see [AG2].

2.3. Lemma. *Let $f : \mathbf{B}^n \to \Omega$ be K-quasiconformal. There is a constant C, which depends only on n, K, so that for each $x \in \mathbf{B}^n$*

$$d(f(x), \partial\Omega)/C \leq \mathrm{a_f}(x)(1-|x|) \leq Cd(f(x), \partial\Omega)$$

and

$$\frac{1}{C}[\int_{B_x} |Df(y)|^n dm/|B_x|]^{1/n} \leq \mathrm{a_f}(x) \leq C[\int_{B_x} |Df(y)|^n dm/|B_x|]^{1/n}.$$

Combining Lemma 2.1 with the first part of Lemma 2.3 easily gives the following estimate (cf. [K1]).

2.4. Lemma. *Let $f : \mathbf{B}^n \to \Omega$ be K-quasiconformal. If $\gamma \subset \mathbf{B}^n$ is a rectifiable path with $l(\gamma) \geq d(\gamma, S^{n-1})$, then*

$$\operatorname{diam}(f\gamma) \leq C \int_\gamma \mathrm{a_f}(x) ds.$$

Here C depends only on n, K.

We conclude this section with a lemma that compares the integrals of $|Df|$ and $\mathrm{a_f}$.

2.5. Lemma. *Let $f : \mathbf{B}^n \to \Omega$ be K-quasiconformal. Suppose that $u > 0$ satisfies*

$$u(y)/C \leq u(x) \leq Cu(y)$$

for each $x \in \mathbf{B}^n$ and all $y \in B_x$. Let $0 < q \leq n$ and $p \geq q$. Then

$$\int_{\mathbf{B}^n} \mathrm{a_f}^p \, u \, dx \approx \int_{\mathbf{B}^n} \mathrm{a_f}^{p-q} \, |Df|^q u \, dx$$

with constants that only depend on p, q, n, C, K.

Notice that $\int_{B_x} \mathrm{a_f}^q \approx \int_{B_x} |Df|^q$ whenever $0 < q \leq n$ by Lemma 2.3. Because of the estimate on u and the analogous estimate on $\mathrm{a_f}$ that follows from Lemma 2.1 and Lemma 2.3, we may multiply the integrands by $\mathrm{a_f}^{p-q} \, u$ in this semi-global setting. The global estimate then follows by covering $\mathbf{B}^n$ using a suitable collection of the balls B_x.

3. Quasiconformal mappings and H^p-classes

Adopting the classical definition we say that a quasiconformal mapping f of $\mathbf{B}^n$, $n \geq 2$, belongs to the class H^p provided

$$||f||_{H^p} = \sup_{0<r<1} (\int_{S^{n-1}} |f(r\omega)|^p d\sigma)^{1/p} < \infty.$$

According to a theorem of Jerison and Weitsman [JW], each quasiconformal mapping f belongs to some H^p-class.

3.1. Theorem (Jerison-Weitsman). *There exists a constant $p_0 = p_0(n, K) > 0$ so that every K-quasiconformal mapping f of $\mathbf{B}^n$ belongs to H^p whenever $p < p_0$.*

By the classical theorem of Prawitz [Pr], all conformal mappings f of the unit disk belong to H^p for $p < 1/2$, and the Koebe mapping $f(z) = z/(1-z)^2$ shows that this bound is sharp. The exponent p_0 obtained by Jerison and Weitsman is not the best possible. We give a new proof for Theorem 3.1 that yields the sharp exponent in the plane. In higher dimensions, our estimate is optimal for mappings into a half space, but we do not know if our bound is also best possible in the general situation.

For each $K \geq 1$ and $n \geq 2$, let $a(n, K)$ be the infimum of the numbers a such that

$$\sup_{|x|<1} (1 - |x|)^a |f(x)| < \infty$$

for every K-quasiconformal mapping f of $\mathbf{B}^n$. Then we have:

3.2. Theorem. *The best possible bound $p_0(n,K)$ in Theorem 3.1 is*

$$p_0(n,K) = (n-1)/a(n,K).$$

In particular, $p_0(2,K) = 1/(2K)$, and for $n \geq 3$

$$(n-1)/(2K)^{1/(n-1)} \leq p_0(n,K) \leq (n-1)/K^{1/(n-1)}.$$

Moreover, for the subclass of f mapping into a half space,

$$p_0(n,K) = (n-1)/K^{1/(n-1)}.$$

When $n = 2$ and $K = 1$, $p_0 = 1/2$, and we obtain Prawitz's theorem. The upper bounds on p_0 in higher dimensions are realized by the simple K-quasiconformal mappings $f(x) = (x-e_1)|x-e_1|^{-1+a}$, $a = K^{1/(n-1)}$, composed with a Möbius transformation which map $\mathbf{B}^n$ into a half space. This mapping also works for the half space setting in dimension two, and the mapping $g(x) = f(k(x))$, where k is the Koebe function, achieves the general upper bound in the plane.

The theorem of Prawitz and its proof combined with estimates due to Hardy and Littlewood yield estimates for individual functions as well. Indeed, if we denote

$$M(r,f) = \sup\{|f(x)| : |x| = r\}, \quad 0 < r < 1,$$

then these results show that a conformal mapping of the disk belongs to H^p if and only if $\int_0^1 M(r,f)^p dr < \infty$. This characterization does not hold for analytic functions, see [D], but it generalizes to quasiconformal mappings in any dimension.

3.3. Theorem. *The following two conditions are equivalent for quasiconformal mappings of $\mathbf{B}^n$.*

$$f \in H^p. \tag{3.4}$$

$$\int_0^1 (1-r)^{n-2} M(r,f)^p dr < \infty. \tag{3.5}$$

Proof. We show first that the inequality

$$\int_{S^{n-1}} |f(\omega)|^p d\sigma \leq C(n,p)K \int_0^1 (1-r)^{n-2} M(r,f)^p dr \tag{3.6}$$

holds for each K-quasiconformal mapping with $f(0)=0$. The conclusion that (3.5) implies (3.4) immediately follows from this estimate: one may assume that $f(0)=0$, and it suffices to estimate $\int_{S^{n-1}}|f(r\omega)|^p d\sigma$ for $1/2 \le r < 1$; for this one simply applies (3.6) to the quasiconformal mapping $g(x)=f(rx)$ of $\mathbf{B}^n$.

Fix a number $\lambda > 0$ and let $E=\{\omega \in S^{n-1} : |f(\omega)| > \lambda\}$. As $M(0,f)=0$, there is a unique $r=r(\lambda)$ such that

$$2M(r,f)=\lambda.$$

Let next Γ_E be the path family consisting of the radial segments connecting $B(0,r)$ to E. Then,

$$M(\Gamma_E)=\sigma(E)(\log(1/r))^{1-n} \ge \sigma(E)2^{1-n}(1-r)^{1-n}$$

for $1/2 < r < 1$. On the other hand, if $\Gamma'_E = f\Gamma_E$ is the image family, then each γ in Γ'_E connects $B(0,\lambda/2)$ to $\mathbf{R}^n \setminus B(0,\lambda)$, and so $M(\Gamma'_E) \le \omega_{n-1}(\log 2)^{1-n}$. As $M(\Gamma_E) \le KM(\Gamma'_E)$,

$$\sigma(\{\omega \in S^{n-1} : |f(\omega)| > \lambda\}) \le C_1(n)K(1-r)^{n-1}, \tag{3.7}$$

whenever $\lambda = 2M(r,f)$ and $1/2 < r < 1$. If ν is the measure on $[0,1]$ defined by $d\nu = (1-r)^{n-2}dr$, then

$$\nu(\{t \in [0,1] : M(t,f) > \lambda/2\}) = (1-r(\lambda))^{n-1}/(n-1),$$

and so

$$\int_{S^{n-1}} |f(\omega)|^p d\sigma = p\int_0^\infty \sigma(\{\omega \in S^{n-1} : |f(\omega)| > \lambda\})\lambda^{p-1}d\lambda$$

$$\le \sigma(S^{n-1})2^p M(1/2,f)^p + C_2(n)Kp\int_0^\infty \nu(\{t \in [0,1] : M(t,f) > \lambda/2\})\lambda^{p-1}d\lambda$$

$$\le C_3(n)2^p K \int_0^1 (1-r)^{n-2}M(r,f)^p dr.$$

For the converse direction, choose points $x_k \in \mathbf{B}^n$ with $|x_k| = r_k = 1-2^{-k}$ and $|f(x_k)| = M(r_k,f)$, $k=1,2,\cdots$. Then

$$\int_0^1 (1-r)^{n-2}M(r,f)^p dr \le 2^n \Sigma_{k=1}^\infty (2^{-k})^{n-1} M(r_k,f)^p$$

$$= 2^n \int_{\mathbf{B}^n} |f(x)|^p d\mu,$$

where

$$d\mu(x) = \Sigma_{k=1}^{\infty}(1-|x|)^{n-1}\delta_{x_k}.$$

To conclude that (3.4) implies (3.5) we require the quasiconformal version of Carleson's theorem whose proof will be given in the next section in Theorem 4.5: Since μ is a Carleson measure,

$$\mu(\mathbf{B}^n \cap B(\omega,\rho)) \le C\rho^{n-1}$$

for $\omega \in S^{n-1}$ and $\rho > 0$, Theorem 4.5 implies that

$$\int_{\mathbf{B}^n} |f(x)|^p d\mu \le C(n,K)\int_{S^{n-1}} |f(\omega)|^p d\sigma.$$

Proof of Theorem 3.2. Let $f \in H^p$, and assume that $f(0)=0$. Then, by Lemma 2.1 and Theorem 3.3, for every $x \in B^n$

$$(1-|x|)^{n-1}|f(x)|^p \le C\int_0^1 (1-r)^{n-2}M(r,f)^p dr \le C < \infty$$

and hence $|f(x)| \le C(1-|x|)^{-a}$, $a=(n-1)/p$.

Conversely, if $|f(x)| \le C(1-|x|)^{-a}$ for all $x \in \mathbf{B}^n$ and $p < (n-1)/a$, then

$$\int_0^1 (1-r)^{n-2}M(r,f)^p dr \le C\int_0^1 (1-r)^{n-2}(1-r)^{-ap} dr < \infty.$$

Consequently, we have shown that $p_0(n,K) = (n-1)/a(n,K)$.

The numerical estimates now follow from the estimate on $a(n,K)$ in Lemma 2.2 and the discussion after Theorem 3.2.

4. Zinsmeister's theorem and its consequences

One of the cornerstones of the modern development of H^p-spaces is the theorem of Hardy and Littlewood that characterizes H^p-functions in terms of the nontangential maximal function

$$f^*(\omega) = \sup_{x\in\Gamma(\omega)} |f(x)|, \quad \omega \in S^{n-1}.$$

According to this result a (holomorphic) function f of the disk belongs to H^p if and only if $f^* \in L^p(S^1)$. Zinsmeister [Z] has extended this maximal function characterization to quasiconformal mappings in space.

4.1. Theorem (Zinsmeister). *The following conditions are equivalent for each quasiconformal mapping f of $\mathbf{B}^n$, $n \geq 2$, and for all $0 < p < \infty$.*

(1) $f(\omega) \in L^p(S^{n-1})$.
(2) $f(x) \in H^p$.
(3) $f^*(\omega) \in L^p(S^{n-1})$.

In addition, the corresponding "norms" are equivalent with constants depending only on n, K, p.

Recall that $f(\omega)$ denotes the radial limit of f at $\omega \in S^{n-1}$ whenever it exists; this is the case for almost every $\omega \in S^{n-1}$. A holomorphic function f of the disk may have L^p-boundary values without being in the space H^p. On the other hand, $H^q \cap L^p = H^p$ also in the holomorphic setting and thus, by Theorem 3.1, it is reasonable to expect that conditions (1) and (3) of Theorem 4.1 are equivalent, i.e., that the quasiconformal H^p-theorems are really results of the boundary values.

The original proof of 4.1 in [Z] was based on a result of Jones [J] on Carleson measures and quasiconformal mappings. Below we describe a different approach, more directly tied to the the geometric nature of quasiconformal mappings. Indeed, Lemma 4.2, which was in [Z] deduced from the work of Jones, will be applied in Section 6 to give a simple proof of Jones' theorem according to which $\log|f(\omega)| \in BMO(S^{n-1})$ whenever f is quasiconformal and $f(x) \neq 0$ for all $x \in \mathbf{B}^n$.

4.2. Lemma. *Suppose f is quasiconformal with $f(x) \neq 0$ for all $x \in \mathbf{B}^n$. Then, for each $x \in \mathbf{B}^n$ and all $M > 1$,*

$$\sigma(\{\omega \in S(x) : |f(\omega)| < |f(x)|/M\}) \leq C(n,K)\sigma(S(x))(\log M)^{1-n}. \tag{4.2}$$

Proof. Let us first consider the case where in (4.2) we have $x = 0$. We may assume that $d(f(0), f(S^{n-1})) = 1$. After this normalization it follows from a simple modulus estimate that $|f(x) - f(0)| \leq 1/2$ for $|x| \leq r_0$; here r_0 depends only on n, K. As

$$1 = d(f(0), f(S^{n-1})) \leq |f(0)|,$$

the set $fB(0, r_0)$ cannot intersect $B(0, |f(0)|/2)$.

Let next $E = \{\omega \in S^{n-1} : |f(\omega)| < |f(0)|/M\}$ and choose Γ_E to be the path family of radial segments connecting $B(0, r_0)$ to E. Then Γ_E has modulus $M(\Gamma_E) = \sigma(E)\log(1/r_0)^{1-n}$. If $M > 3$, the paths in the image family $\Gamma_E' = f\Gamma_E$ connect the complement of $B(0, |f(0)|/2)$ to $B(0, |f(0)|/M)$ and therefore $M(\Gamma_E') \leq \omega_{n-1}(\log(M/2))^{1-n} \leq C(n)(\log M)^{1-n}$. As $M(\Gamma_E) \leq KM(\Gamma_E')$, we obtain

$$\sigma(E) \leq C(n,K)(\log M)^{1-n}.$$

Finally, for $1 < M \leq 3$, clearly $\sigma(E) \leq \sigma(S^{n-1})(\log 3)^{n-1}(\log M)^{1-n}$.

Let then $x \in \mathbf{B}^n$ be general. The desired estimate follows by mapping x to 0 by the Möbius transformation T_x defined in Section 2, and applying the estimate from the first part of the proof to $g = f \circ T_x^{-1}$.

Remark. *For further reference, let us record here that a simple modification to the proof of Lemma 4.2 gives the estimates: if $M > 1$ and $x \in \mathbf{B}^n$, then*

$$\sigma(\{w \in S(x) : |f(w) - f(x)| > Md(f(x), \partial f(\mathbf{B}^n))\}) \leq C\sigma(S(x))(\log M)^{1-n}$$

and

$$\sigma(\{w \in S(x) : |f(w)| > M|f(x)|\}) \leq C\sigma(S(x))(\log M)^{1-n}.$$

Note also that the first estimate holds without assuming $f(x) \neq 0$ in $\mathbf{B}^n$.

4.3. Corollary. *If f is quasiconformal in $\mathbf{B}^n$, then*

$$|f(x)|^q \leq C\frac{1}{\sigma(S(x))}\int_{S(x)} |f(\omega)|^q d\sigma$$

for all $x \in \mathbf{B}^n$ and each $0 < q < \infty$. The constant C depends only on n, K, q.

Proof. Assume first that $f(x) \neq 0$ for all $x \in \mathbf{B}^n$. We apply Lemma 4.2. If M_0 is so large that $C(n,K)(\log M_0)^{1-n} = 1/2$, the estimate in Lemma 4.2 gives

$$2\sigma(\{\omega \in S(x) : |f(\omega)| \geq |f(x)|/M_0\}) \geq \sigma(S(x)).$$

Consequently,

$$\begin{aligned}\int_{S(x)} |f(\omega)|^q d\sigma &\geq |f(x)|^q M_0^{-q}\sigma(\{\omega \in S(x) : |f(\omega)| \geq |f(x)|/M_0\}) \\ &= C(n,K)|f(x)|^q\sigma(S(x)).\end{aligned}$$

If $f(x) = 0$ for some $x \in \mathbf{B}^n$, we may choose a point y in the complement of $f\mathbf{B}^n$ so that $|y| \leq |f(\omega)|$ for all $\omega \in S^{n-1}$. Applying the above estimate to $f - y$ we get

$$\begin{aligned}\sigma(S(x))|f(x)|^q &\leq 2^{q+1}C(n,K)(\int_{S(x)} |f(\omega)|^q d\sigma + |y|^q\sigma(S(x))) \\ &\leq C\int_{S(x)} |f(\omega)|^q d\sigma.\end{aligned}$$

Zinsmeister's theorem follows now immediately. It is enough to prove that condition (1) implies condition (3): As $|f(r\omega)|, |f(\omega)| \leq |f^*(\omega)|$, (3) gives (1) and

(2), and (2) yields (1) by the Fatou lemma. So, assume (1). Applying Corollary 4.3 we see that $f^*(\omega)^q \leq CM(|f|^q)(\omega)$, where M denotes the Hardy-Littlewood maximal function on the sphere S^{n-1}. Since M is bounded on $L^s(S^{n-1})$ for all $1 < s < \infty$, we obtain for $q < p$

$$\int_{S^{n-1}} f^*(\omega)^p d\sigma = \int_{S^{n-1}} (f^*(\omega)^q)^{p/q} d\sigma$$

$$\leq C \int_{S^{n-1}} (M(|f|^q))^{p/q}(\omega) d\sigma \leq C \int_{S^{n-1}} |f(\omega)|^p d\sigma,$$

which proves Theorem 4.1.

It is well known that many properties of the H^p-spaces can be obtained as consequences of the Hardy-Littlewood maximal characterization. Therefore also Theorem 4.1 has similar corollaries.

4.4. Corollary. *If f is quasiconformal in $\mathbf{B}^n$ and $f(\omega) \in L^p(S^{n-1})$, then*

$$\int_{S^{n-1}} |f(r\omega) - f(\omega)|^p d\sigma \to 0 \quad \text{as } r \to 0.$$

Proof. As $|f(rw) - f(w)| \leq 2f^*(\omega)$, the claim follows from Theorem 4.1 and the dominated convergence theorem.

A measure μ on $\mathbf{B}^n$ is called an α-Carleson measure, $1 \leq \alpha < \infty$, if

$$\gamma_\alpha(\mu) := \sup\{r^{-\alpha(n-1)} \mu(\mathbf{B}^n \cap B(\omega, r)) : \omega \in S^{n-1}, r > 0\} < \infty.$$

For $\alpha = 1$ the definition reduces to the usual notion of a Carleson measure. The fundamental theorem, due to Carleson, states that the Poisson integral defines a bounded operator from $L^p(S^1)$ to $L^p_\mu(\mathbf{B}^2)$ if and only if μ is a Carleson measure on $\mathbf{B}^2$. We utilize a version of Stein's proof [Ste] for this theorem to obtain a quasiconformal counterpart.

4.5. Corollary. *Let f be a quasiconformal mapping of $\mathbf{B}^n$ and μ an α-Carleson measure on $\mathbf{B}^n$. Then*

$$\int_{\mathbf{B}^n} |f(x)|^{\alpha p} d\mu \leq C\Big(\int_{S^{n-1}} |f(\omega)|^p d\sigma\Big)^\alpha, \quad 0 < p < \infty, \tag{4.6}$$

where C depends only on $K, n, \gamma_\alpha(\mu)$. Conversely, if $K \geq 1$ and $p > (n-1)/K^{1/(n-1)}$ are fixed and (4.6) holds for all K-quasiconformal mappings, then μ is an α-Carleson measure. In particular, if $p > n-1$, the converse part holds for any K.

Proof. Let $E(\lambda) = \{x \in \mathbf{B}^n : |f(x)| > \lambda\}$ and $U(\lambda) = \{\omega \in S^{n-1} : f^*(\omega) > \lambda\}$. We apply the generalized form of the Whitney decomposition, cf. [G], to the open set $U(\lambda) \subset S^{n-1}$. More precisely, we can write

$$U(\lambda) = \bigcup_{k=1}^{\infty} S(x_k),$$

where the points $x_k \in \mathbf{B}^n$ are chosen so that each $\omega \in U(\lambda)$ is contained in at most $N = N(n)$ caps $S(x_k)$ and $(1-|x_k|)/C \leq d(S(x_k), \partial U(\lambda)) \leq C(1-|x_k|)$. Here C is an absolute constant and the distance is measured in the spherical distance of S^{n-1}. If $|f(x)| > \lambda$, then $f^*(\omega) > \lambda$ for all $\omega \in S(x)$ and we see that $E(\lambda)$ is contained in the union of the balls $B(x_k/|x_k|, C(1-|x_k|))$, $k = 1, 2, \cdots$, where C is an absolute constant. Hence

$$\mu(E(\lambda)) \leq \sum_{k=1}^{\infty} \mu(B(x_k/|x_k|, C(1-|x_k|)))$$

$$\leq C \sum_{k=1}^{\infty} \sigma(S(x_k))^{\alpha} \leq C[\sum_{k=1}^{\infty} \sigma(S(x_k))]^{\alpha} \leq C\sigma(U(\lambda))^{\alpha}.$$

Therefore

$$\begin{aligned}
\int_{\mathbf{B}^n} |f(x)|^{\alpha p} d\mu &= \alpha p \int_0^{\infty} \lambda^{\alpha p - 1} \mu(E(\lambda)) d\lambda \\
&\leq C \int_0^{\infty} \lambda^{\alpha p - 1} \sigma(U(\lambda))^{\alpha} d\lambda \\
&\leq C \sum_{j=-\infty}^{\infty} \sigma(U(2^j))^{\alpha} 2^{j\alpha p} \\
&\leq C(\sum_{j=-\infty}^{\infty} \sigma(U(2^j)) 2^{jp})^{\alpha} \\
&\leq C(\int_{S^{n-1}} f^*(\omega)^p d\sigma)^{\alpha},
\end{aligned}$$

where C depends only on p, n, α, and the Carleson norm of μ. The sufficiency part of the claim now follows from Theorem 4.1.

For the necessity of the Carleson measure condition, fix $\omega \in S^{n-1}$ and $r > 0$. We may assume that $r \leq 2$. Define then a K-quasiconformal mapping f of $\mathbf{B}^n$ as follows. First, consider a Möbius transformation

$$\Phi(x) = \frac{A(x-y)}{|x-y|^2}.$$

When A is a sense reversing isometry of $\mathbf{R}^n$ we have $J_\Phi(x) \geq 0$, and then Φ is 1-quasiconformal. Composing Φ with a radial stretching gives us the K-quasiconformal mapping

$$f(x) = A(x-y)|x-y|^{-1-a}, \qquad a = K^{1/(1-n)}.$$

We choose here $y = (1+r)\omega$. Then a simple calculation and the assumption $p > (n-1)/a$ show that

$$\int_{S^{n-1}} |f|^p d\sigma \leq Cr^{n-1-pa}$$

and

$$\int_{\mathbf{B}^n} |f|^{\alpha p} d\mu \geq C^{-1} r^{-pa\alpha} \mu(\mathbf{B}^n \cap B(\omega, r)),$$

where C is independent of ω, r. The claim follows.

4.6. Corollary. *Let $L \subset \mathbf{R}^n$ be an $(n-1)$-dimensional plane through the origin. Then*

$$\int_{L \cap \mathbf{B}^n} |f(x)|^p dH^{n-1} \leq C \int_{S^{n-1}} |f(\omega)|^p d\sigma$$

for all $0 < p < \infty$ and all quasiconformal mappings f of $\mathbf{B}^n$. The constant C depends only on K, n, p.

5. Characterizations for the H^p class

In this section we present more characterizations for the membership in H^p.

Recall that, by an old theorem of Littlewood and Paley [LP],

$$\int_{\mathbf{B}^2} |Df|^p (1-|x|)^{p-1} dx < \infty$$

whenever $f \in H^p$ is analytic and $p \geq 2$. It is known, see [Gi], that this result does not hold for $0 < p < 2$. Conversely, if the above integral converges for an analytic f for some $1 \leq p \leq 2$, then f belongs to H^p, and, again, the restriction on p is necessary. Our next observation shows, in particular, that the convergence of this integral is a test for the membership in H^p for univalent functions for all $p > 0$. Since $|Df|^p$ is not necessarily locally integrable for large values of p for a fixed quasiconformal mapping, we formulate our result in terms of the averaged derivative.

5.1. Theorem. *Let f be a quasiconformal mapping of $\mathbf{B}^n$ and fix $0 < p < \infty$. Then the following conditions are equivalent.*

(1) $f(w) \in L^p(S^{n-1})$.
(2) $\int_{\mathbf{B}^n} \mathrm{a_f}^p(x)(1-|x|)^{p-1} dx < \infty$.
(3) $\sup_{x \in \Gamma(\omega)} (\mathrm{a_f}(x)(1-|x|)) \in L^p(S^{n-1})$

Proof. Suppose first that $f(w) \in L^p(S^{n-1})$. Assume that $0 < p \leq 1$. We may assume that $f \neq 0$ in $\mathbf{B}^n$. Then Lemma 5.6 below and Corollary 4.5 yield that

$$\int_{\mathbf{B}^n} |f|^p |Df|^p |f|^{-p} (1-|x|)^{p-1} dx \leq C ||f||^p_{H^p},$$

and we conclude by Lemma 2.5 that

$$\int_{\mathbf{B}^n} \mathrm{a_f}(x)^p (1-|x|)^{p-1} dx < \infty,$$

as desired. Let now $p > 1$, and assume that $f(0) = 0$. Fix $y \in \partial f(\mathbf{B}^n)$ with $|y| = d(f(0), \partial f(\mathbf{B}^n))$. Then, by Lemma 2.5 and Lemma 2.3,

$$\int_{\mathbf{B}^n} \mathrm{a_f}(x)^p (1-|x|)^{p-1} dx \leq C \int_{\mathbf{B}^n} |Df| \, \mathrm{a_f}(x)^{p-1} (1-|x|)^{p-1} dx$$

$$\leq C \int_{\mathbf{B}^n} |Df(x)| d(f(x), \partial f(\mathbf{B}^n))^{p-1} dx \leq C \int_{\mathbf{B}^n} |Df||f-y|^{p-1} dx.$$

Since $f - y \neq 0$ in $\mathbf{B}^n$ we may again apply Lemma 5.6 and Corollary 4.5 to conclude that also in this case

$$\int_{\mathbf{B}^n} \mathrm{a_f}(x)^p (1-|x|)^{p-1} dx < \infty. \tag{5.1}$$

Assume then that (5.1) holds. Notice first that for any function u, integrable on $\mathbf{B}^n$, Fubini's theorem gives

$$\int_{\mathbf{B}^n} u \, dx \approx \int_{S^{n-1}} \int_{\Gamma(w)} u(y)(1-|y|)^{1-n} dy d\sigma. \tag{5.2}$$

Set

$$v(w) = (\int_{\Gamma(w)} \mathrm{a_f}(x)^p (1-|x|)^{p-n} dx)^{1/p}.$$

Then (5.2) ensures that $v \in L^p(S^{n-1})$. Fix $w \in S^{n-1}$. If $x \in \Gamma(w)$, then Lemma 2.1 and Lemma 2.3 give

$$\mathrm{a_f}(x)(1-|x|) \leq C(\int_{B_x} \mathrm{a_f}(y)^p dy)^{1/p} (1-|x|)^{1-n/p}$$

$$\leq C(\int_{\Gamma(w)} \mathrm{a_f}(x)^p (1-|x|)^{p-n} dx)^{1/p},$$

and we conclude that

$$\sup_{x\in\Gamma(w)} (\mathrm{a_f}(x)(1-|x|)) \in L^p(S^{n-1}),$$

as desired.

Assume finally that (3) holds. Set

$$v(w) = \sup_{x\in\Gamma(w)} (\mathrm{a_f}(x)(1-|x|)).$$

Then $v \in L^p(S^{n-1})$, and appealing to Lemma 2.3 we see that for each $x \in \Gamma(w)$

$$d(f(x), \partial f(\mathbf{B}^n)) \leq Cv(w),$$

where C depends only on n, K. Thus, Lemma 5.5 below shows that (1) holds.

We continue with a characterization that involves $|Df|$ instead of $\mathrm{a_f}$.

5.3. Lemma. *Suppose that f is quasiconformal in $\mathbf{B}^n$ and $f(x) \neq 0$ for all $x \in \mathbf{B}^n$. Then $f \in H^p$, if and only if*

$$\int_{\mathbf{B}^n} |f(x)|^{p-1}|Df(x)|dx < \infty.$$

Proof. If $f \in H^p$, then the desired integrability condition follows from Theorem 4.1, Corollary 4.5 and Lemma 5.6. On the other hand, if $A_r = B(r) \setminus B(1/2)$, then

$$\begin{aligned}\int_{S^{n-1}} |f(rw)|^p d\sigma - \int_{S^{n-1}} |f(w/2)|^p d\sigma &\leq C \int_{A_r} \frac{d}{dt}|f(tw)|^p dx \\ &\leq pC \int_{A_r} |f|^{p-1}|Df|dx\end{aligned}$$

and so the indicated integrability condition ensures that $f \in H^p$.

5.4. Corollary. *Let f be a quasiconformal mapping of $\mathbf{B}^n$ with $f(x) \neq 0$ in $\mathbf{B}^n$, and fix $0 < p < \infty$. Then the following conditions are equivalent.*

(1) $f \in H^p$.
(2) $f(w) \in L^p(S^{n-1})$.
(3) $\int_{\mathbf{B}^n} |f|^{p-1}|Df|dx < \infty$.
(4) $\int_{\mathbf{B}^n} \mathrm{a_f}^p(x)(1-|x|)^{p-1}dx < \infty$.
(5) $\sup_{x\in\Gamma(\omega)}(\mathrm{a_f}(x)(1-|x|)) \in L^p(S^{n-1})$.

It remains to establish the Lemmas that we used in the proof of Theorem 5.1. First we have

5.5. Lemma. *Let f be a quasiconformal mapping of $\mathbf{B}^n$ with $f(0) = 0$. Suppose that we are given a function $v \in L^p(S^{n-1})$ such that*

$$\sup_{x \in \Gamma(\omega)} d(f(x), \partial f(\mathbf{B}^n)) \leq v(\omega)$$

for almost every $\omega \in S^{n-1}$. Then

$$\int_{S^{n-1}} |f(w)|^p d\sigma \leq C \int_{S^{n-1}} v(w)^p d\sigma,$$

where C depends only on n, K, p.

Proof. Define $U(\lambda) = \{w \in S^{n-1} : f^*(w) > \lambda\}$. Then, as in the proof of Corollary 4.5, we can write $U(\lambda)$ as the union of caps $S(x_j)$

$$U(\lambda) = \cup S(x_j)$$

so that the caps have uniformly bounded overlap and

$$d(S(x_j), \partial U(\lambda)) \leq C(1 - |x_j|).$$

If $v(w) < \gamma$ for some $w \in S(x_j)$, then Lemma 2.1 implies that

$$|f(x_j)| \leq \lambda + C\gamma,$$

and, by assumption,

$$d(f(x_j), \partial f(\mathbf{B}^n)) \leq \gamma,$$

where C depends only on K, n. Let now $\omega \in S(x_j)$ satisfy $|f(w)| > 2\lambda$ and $v(w) \leq \gamma$, where $\lambda = (M+1)C\gamma$. Then we conclude that

$$|f(w) - f(x_j)| > Md(f(x_j), \partial f(\mathbf{B}^n)).$$

Hence we have the good λ-inequality

$$\begin{aligned} \sigma(\{w \in S(x_j) : |f(w)| > 2\lambda,\ v(w) \leq \gamma\}) \\ \leq \sigma(\{w \in S(x_j) : |f(w) - f(x_j)| > Md(f(x_j), \partial f(\mathbf{B}^n))\}) \\ \leq C\sigma(S(x_j))(\log M)^{1-n}, \end{aligned}$$

where C depends only on n, K. Here we used the remark after Lemma 4.2 for the last estimate. By continuity, $f^*(w) > 2\lambda$ provided $|f(w)| > 2\lambda$, and hence each such w belongs to $U(\lambda)$. Thus

$$\sigma(\{w \in S^{n-1} : |f(w)| > 2\lambda\})$$

$$\leq \sigma(\{w \in U(\lambda) : |f(w)| > 2\lambda,\ v(w) \leq \gamma\}) + \sigma(\{w \in S^{n-1} : v(w) > \gamma\})$$
$$\leq C\Sigma_j \sigma(S(x_j))(\log M)^{1-n} + \sigma(\{w \in S^{n-1} : v(w) > \gamma\})$$
$$\leq C\sigma(U(\lambda))(\log M)^{1-n} + \sigma(\{w \in S^{n-1} : v(w) > \gamma\}).$$

Integrating we obtain (recall that $\gamma = \lambda/(C(M+1))$)

$$\int_{S^{n-1}} |f|^p d\sigma \leq C(\log M)^{1-n} \int_{S^{n-1}} f^*(w)^p d\sigma + M^p C \int_{S^{n-1}} v(w)^p d\sigma.$$

We want to combine the integral involving $f^*(\omega)$ with the left hand side of the estimate. In principle both terms could be infinite, but by scaling and considering $f_t(x) = f(tx)$, $0 < t < 1$, we force the integrals to converge. By Theorem 4.1

$$\int_{S^{n-1}} f^*(w)^p d\sigma \leq C \int_{S^{n-1}} |f|^p d\sigma,$$

and hence, taking M sufficiently large and then letting $t \to 1$, we conclude that

$$\int_{S^{n-1}} |f|^p d\sigma \leq C \int_{S^{n-1}} v^p d\sigma.$$

We close this section with the following version of a result of Jones [J] that was employed above.

5.6. Lemma (Jones). *If f is quasiconformal in $\mathbf{B}^n$, $0 < p < n$, and $f(x) \neq 0$ for all $x \in \mathbf{B}^n$, then the measure μ defined by $d\mu = |Df(x)|^p |f(x)|^{-p}(1-|x|)^{p-1}dx$ is a Carleson measure on $\mathbf{B}^n$.*

Proof. Given a quasiconformal mapping f as above, we first notice that, given $\epsilon > 0$,

$$\int_{\mathbf{B}^n} |Df(x)|^p |f(x)|^{-p}(1-|x|)^{p-1}dx \leq$$
$$\leq \Big(\int_{\mathbf{B}^n} |Df(x)|^n |f(x)|^{-n}(1-|x|)^{\epsilon n/p}dx\Big)^{p/n}\Big(\int_{\mathbf{B}^n}(1-|x|)^{(p-1-\epsilon)n/(n-p)}dx\Big)^{(n-p)/n}$$

by Hölder's inequality. We can now choose $\epsilon > 0$, depending only on p, n so that the latter integral converges. Applying the distortion inequality $|Df(x)|^n \leq KJ_f(x)$, a change of variables, splitting the resulting integral over $f(\mathbf{B}^n)$ into two integrals, one over $f(\mathbf{B}^n) \cap B(0, |f(0)|)$ and the second over $f(\mathbf{B}^n) \setminus B(0, |f(0)|)$, and inserting the estimate from Lemma 2.2, we conclude that there is a constant $M = M(p, n, K)$ so that

$$\int_{\mathbf{B}^n} |Df(x)|^p |f(x)|^{-p}(1-|x|)^{p-1}dx \leq M$$

for each f as in the claim. Let then g satisfy the assumptions of our lemma, $\omega \in S^{n-1}$, and $r > 0$. By choosing a point x on the radius to ω appropriately and applying the Möbius transformation T_x from Section 2, we conclude that

$$\int_{B^n \cap B(\omega,r)} |Dg(y)|^p |g(y)|^{-p} (1-|y|)^{p-1} dy \leq C r^{n-1} \int_{B^n} |Df(z)|^p |f(z)|^{-p} (1-|z|)^{p-1} dz,$$

where $f(z) = g(T_x(y))$. The claim follows.

6. Conjugate functions

According to the Riesz theorem, an analytic function f belongs to H^p for some $1 < p < \infty$, if and only if the real part of f belongs to H^p. Moreover, a theorem of Burkholder, Gundy and Silverstein [BGS] shows that the $L^p(S^1)$ norm of the harmonic conjugate v of a harmonic function u is bounded by a constant multiple (depending on $p > 0$) of the $L^p(S^1)$ norm of the maximal function u^* of u. Notice that, naturally, the use of a maximal function is essential as seen for example by considering the real and imaginary parts of a Möbius transformation that maps the unit disk onto the upper half plane. We begin by giving a quasiconformal analog of this result.

6.1. Theorem. *Let f be a quasiconformal mapping of $\mathbf{B}^n$, and let $u = f_j$ be one of its components. Then $f \in H^p$ if and only if $u^*(w) \in L^p(S^{n-1})$.*

Proof. If $f \in H^p$, then $u^*(w) \in L^p(S^{n-1})$ by Theorem 4.1. For the converse, assume that $u^*(w) \in L^p(S^{n-1})$. If $u^*(w) < \gamma$ then Lemma 2.1 implies that for each $x \in \Gamma(w)$

$$d(f(x), \partial f(\mathbf{B}^n)) \leq C\gamma,$$

where C depends only on K, n. Thus we may apply Lemma 5.5 to conclude that $f \in H^p$.

The quasiconformal mapping $f(x) = (x - e_1)|x - e_1|^{-1-(n-1)/p}$, $p > 0$, shows that the Riesz theorem does not extend as such to the class of quasiconformal mappings. Indeed, this mapping does not belong to H^p, but the first coordinate function belongs to H^p. In view of the following theorem this is in some sense the worst possible situation.

6.2. Theorem. *Let f be a quasiconformal mapping of $\mathbf{B}^n$. If one of the coordinate functions of f belongs to H^p, then f belongs to H^q for all $q < p$. Moreover, f belongs to the space weak-$L^p(S^{n-1})$.*

Proof. We begin by showing that f belongs to H^q for all $q < p$. By Theorem 3.3 it suffices to show that $M(r, f) \leq C(1-r)^{(1-n)/p}$. To this end, let f_i be the coordinate function of f belonging to H^p. Then the Fubini theorem shows that

$$\int_{B_x} |f_i(y)|^p dy \leq C(1 - |x|)$$

for all $x \in \mathbf{B}^n$. Hence, by the weak Harnack inequality [HKM,3.34]

$$|f_i(x)| \le C(\fint_{B_x} |f_i(y)|^p dy)^{1/p} \le C(1-|x|)^{(1-n)/p}.$$

By Lemma 2.1 and Lemma 2.3 we conclude that

$$a_f(x) \le Cd(f(x), \partial f(\mathbf{B}^n))/(1-|x|) \le C(1-|x|)^b,$$

where $b = -1 + (1-n)/p$. Integrating and applying Lemma 2.4 we arrive at the desired growth estimate.

Now we turn to the weak-L^p-result. We may assume that $f(0) = 0$. As in the proof of Theorem 3.3, we pick for a fixed $\lambda > 0$ the unique $r > 0$ with $2M(r, f) = \lambda$. We may assume that $r > 1/2$. Then, by inequality (3.7),

$$\sigma(\{w \in S^{n-1} : |f(w)| > \lambda\}) \le C(1-r)^{n-1}. \tag{6.3}$$

On the other hand,

$$\lambda = 2M(r, f) \le C(1-r)^{(1-n)/p}$$

by the first part of the proof. Hence

$$(1-r)^{n-1} \le C\lambda^{-p},$$

and the weak-L^p-estimate follows from (6.3).

Notice that there is no restriction on p in the above theorem, whereas the assumption $p > 1$ is essential for analytic functions. Indeed, there exist analytic functions that belong to no H^p but whose real parts belong to H^p for all $0 < p < 1$.

6.4. Remark. Since the gradients of the coordinate functions of a quasiconformal mapping f are comparable almost everywhere and the coordinate functions satisfy a Caccioppoli-type inequality, it is reasonable to expect that the best exponents of integrability over $\mathbf{B}^n$ for the coordinate functions coincide. This conclusion has been verified by Iwaniec and Nolder [IN].

7. BMO and VMO

In this section, we first give a geometric proof for a result of Jones [J] according to which $\log|f(w)|$ belongs to $BMO(S^{n-1})$ for each quasiconformal mapping f of $\mathbf{B}^n$ that omits 0. Prior to Jones' work this result was established for univalent functions by Baernstein [B] and by Cima and Schober [CS]. Then we produce a long list of characterizations for the membership in BMO and comment on various function classes related to the boundary values.

7.1. Theorem (Jones). *Suppose that f is quasiconformal in $\mathbf{B}^n$ and $f(x) \neq 0$ for all $x \in \mathbf{B}^n$. Then* $\log |f(w)| \in BMO(S^{n-1})$.

Proof. By Lemma 4.2 and the remark after it

$$\sigma(\{w \in S(x) : |\log |f(w)| - \log |f(x)|| > \log M\}) \leq C\sigma(S(x))(\log M)^{1-n},$$

and so

$$\sigma(S(x))^{-1} \int_{S(x)} |\log |f(w)| - \log |f(x)||^p d\sigma$$

$$= \sigma(S(x))^{-1} p \int_0^\infty t^{p-1} \sigma(\{w \in S(x) : |\log |f(w)| - \log |f(x)|| > t\}) dt$$

$$\leq 1 + C \int_1^\infty p t^{p-1} t^{1-n} dt < C_1$$

whenever $p < n-1$. When $n \geq 3$ we may take $p = 1$ and the claim follows. We can also apply alternative arguments which work also when $n = 2$, for example the theorem of Strömberg [Str].

It is well known [G] that a holomorphic function f belongs to BMO(S^1) if and only if $|Df(z)|(1-|z|)$ is a Carleson measure. Moreover, a holomorphic function f belongs to BMO with respect to area if and only if f is a Bloch function, i.e.,

$$\sup_{|z|<1} |f'(z)|(1-|z|) < \infty.$$

Furthermore, by a result of Pommerenke [P1], a univalent holomorphic function f belongs to BMO(S^1) if and only if f is a Bloch function. Theorem 7.2 below extends these observations.

7.2. Theorem. *The following conditions are equivalent for a quasiconformal mapping of $\mathbf{B}^n$.*

(1) $f \in$*BMO(*S^{n-1}*).*
(2) $\sup_T \int_{S^{n-1}} |f \circ T - f(T(0))| d\sigma < \infty$, *where the supremum is taken over the Möbius transforms $T : \mathbf{B}^n \to \mathbf{B}^n$.*
(3) $|Df(x)|^n (1-|x|)^{n-1}$ *is a Carleson measure.*
(4) $\mathrm{a_f}(x) \leq M(1-|x|)^{-1}$ *for all $x \in \mathbf{B}^n$.*
(5) $d(f(x), \partial f(\mathbf{B}^n)) \leq C < \infty$ *for all $x \in \mathbf{B}^n$.*
(6) $f_j \in$*BMO for some (each) coordinate function f_j of f.*
(7) $f \in$*BMO.*

Proof. It easily follows from the argument in [G] that (1) implies (2). Next, (3) follows from (2) by Lemma 7.5. Then Lemma 2.3 shows that (3) yields (4) and that (4) implies (5). From Lemma 2.3 and Lemma 7.6 below we observe that (1) is a consequence of (5). Moreover, (5), Lemma 2.3 and the Poincaré inequality guarantee that the BMO-estimate holds for f for each ball B with $2B \subset \mathbf{B}^n$. This seemingly weaker version of the BMO-condition implies that f belongs to BMO (cf., [Sta]). Thus (7) follows from (5). Next, clearly, (6) is a consequence of (7).

Finally, by the John-Nirenberg inequality we deduce from (6) that

$$\int_{B_x} |f_j - a|^n dy \leq C(1-|x|)^n$$

for each $x \in \mathbf{B}^n$, where a denotes the average of f_j in B. The Caccioppoli inequality [HKM, (3.33)] then states that

$$\int_{B_x} |\nabla f_j|^n dx \leq C,$$

and (4) follows using Lemma 2.3 (the gradients of the coordinate functions of f are comparable almost everywhere). This completes the desired string of implications.

For analytic functions the growth of $|f'(z)|$ determines whether f satisfies a Lipschitz condition or not. As an appropriate radial stretching indicates, for quasiconformal mappings one has to look at Lipschitz conditions for the boundary values. We write $f \in \mathrm{Lip}_\alpha(E)$ if there is a constant M such that

$$|f(x) - f(y)| \leq M|x-y|^\alpha$$

for all $x, y \in E$.

7.3. Theorem. *The following are equivalent for* $0 < \alpha \leq 1$.

(1) $f \in \mathrm{Lip}_\alpha(\partial \mathbf{B}^n)$.
(2) $\mathrm{a_f}(x) \leq M(1-|x|)^{\alpha-1}$ *for all* x *in* $\mathbf{B}^n$.

Proof. A standard modulus estimate shows that (1) implies (2) (use Lemma 2.1 and Lemma 2.3). Furthermore, (2) yields (1) by Lemma 2.3.

Since $\mathrm{a_f}$ is essentially a constant in each B_x, we have the following corollary that gives a version of the Sobolev embedding theorem for $p > n$; notice that one cannot conclude local Hölder continuity with exponent α as seen by considering, for example, the quasiconformal mapping of the type $f(x) = x|x|^{-1/2}$ for which $\mathrm{a_f}$ is bounded.

7.4. Corollary. *If* $\int_{\mathbf{B}^n} \mathrm{a_f}^p \, dx < \infty$, $p > n$, *then* $f \in \mathrm{Lip}_\alpha(\partial \mathbf{B}^n)$, *where* $\alpha = 1 - n/p$.

We close this section with the following two lemmas that were used in the proof of Theorem 7.2 above.

7.5. Lemma. *If $f \in BMO(S^{n-1})$, then $|Df(x)|^n(1-|x|)^{n-1}$ is a Carleson measure.*

Proof. Since $f \in BMO(S^{n-1})$, it follows from the argument in [G], Corollary VI.1.4, that

$$\sup_T \int_{S^{n-1}} |f \circ T - f(T(0))|^n d\sigma = M < \infty.$$

From Lemma 9.4 below we thus deduce that

$$\int_{\mathbf{B}^n} |D(f \circ T)|^n (1-|x|)^{n-1} dx \le CM$$

for each T. Now

$$\int_{\mathbf{B}^n} |D(f \circ T)|^n (1-|x|)^{n-1} dx = \int_{\mathbf{B}^n} |Df|^n (1-|T^{-1}(x)|)^{n-1} dx$$

$$\approx \int_{\mathbf{B}^n} |Df|^n (1-|x|)^{n-1} |D(T^{-1})|^{n-1} dx.$$

Hence

$$\sup_T \int_{\mathbf{B}^n} |Df|^n (1-|x|)^{n-1} |DT|^{n-1} dx \le CM < \infty,$$

and hence it follows from an n-dimensional version of Lemma VI 3.3 in [G] that $|Df|^n(1-|x|)^{n-1}$ is a Carleson measure.

7.6. Lemma. *If $d(f(x), \partial D) \le M < \infty$ for all $x \in \mathbf{B}^n$, then $f \in BMO(S^{n-1})$.*

Proof. Suppose first that g is a quasiconformal mapping of $\mathbf{B}^n$ with $g(0) = 0$ and $d(g(x), \partial D) \le M < \infty$ for all $x \in \mathbf{B}^n$. Then

$$\int_{S^{n-1}} |g| d\sigma \le CM$$

by Lemma 5.5, where C depends only on n, K. The claim follows easily from this estimate by applying it, for a fixed $x \in \mathbf{B}^n$, to $g(y) = f(T_x^{-1}(y)) - f(T_x^{-1}(x))$, where T_x is as in Section 2.

8. H^p-theory for Df and the growth of integral means

The derivative of an analytic function is analytic and hence a number of results in the H^p-theory for the derivative of an analytic function immediately follow. This approach naturally fails in the context of quasiconformal mappings. Another drawback is the lack of smoothness. For example, the image of a circle under a quasiconformal mapping of the disk can fail to be rectifiable and the quantity $Df(x)$ is in general only defined almost everywhere. Hence a reasonable attempt is to replace $Df(x)$ by the averaged derivative $a_f(x)$. This choice has turned out to be fruitfull. We begin by recording a result due to Hanson [H]. Also see [BK].

8.1. Theorem [H]. *Let f be quasiconformal in B^2. Then $\mathrm{a_f} \in H^1$ if and only if the length of the boundary $\mathcal{H}^1(\partial f(B^2)) < \infty$.*

Theorem 8.1 does not extend as such to higher dimensions. If $\mathcal{H}^{n-1}(\partial f\mathbf{B}^n) < \infty$, then still $\mathrm{a_f} \in H^{n-1}$, but in the opposite direction, one can only control the size of the porous part of $\partial f\mathbf{B}^n$. For this see [BK].

As Theorem 8.1 suggests, the growth of the integral means of $\mathrm{a_f}$ is related to the size of the boundary of the image domain. We continue in this direction by establishing a quasiconformal analog of a result of Ch. Pommerenke [P2] for conformal mappings of the disk.

8.2. Theorem. *Suppose that $f \in \mathrm{Lip}_\alpha(\mathbf{B}^n)$ for some $0 < \alpha \leq 1$. Then the following two conditions are equivalent.*

(a) $\dim_M(\partial f\mathbf{B}^n) \leq \lambda$.
(b) $\int_{S^{n-1}} \mathrm{a_f}(tw)^\beta d\sigma \leq C(1-t)^{n-\beta-1}$ *for each $\beta > \lambda$ for some constant C for all $0 < t < 1$.*

Pommerenke proved the equivalence of (a) and (b) for conformal mappings of the disk onto so called John domains. Since a (quasi)conformal mapping of the ball onto a John domain is always uniformly Hölder continuous, Theorem 8.2 extends Pommerenke's result. He also constructs an example that shows that (b) does not, in general, imply (a) without additional assumptions on f.

Theorem 8.2 relates the growth of the integral means of $\mathrm{a_f}$ to the Minkowski dimension of the boundary of $f\mathbf{B}^n$. Here the Minkowski dimension $\dim_M(E)$ of a compact set set E is defined as follows. Set

$$M_s(E,r) = \inf\{kr^s : E \subset \cup_1^k B(x_i,r)\}$$

for $r > 0$ and define

$$\dim_M(E) = \inf\{s > 0 : \limsup_{r\to 0} M_s(E,r) < \infty\}.$$

Notice that the Minkowski dimension is closely related to the Hausdorff dimension; we cover the set by balls of equal radii instead of allowing variable radii. The Minkowski dimension of a set E is larger or equal to the Hausdorff dimension of the set E.

8.3. Corollary. *Suppose that $\int_{\mathbf{B}^n} \mathrm{a_f}^p(x)dx < \infty$, $p > n$. Then*

$$\dim_M(\partial f\mathbf{B}^n) \leq p(n-1)/(p-1) < n.$$

Proof. Note first that $f \in \mathrm{Lip}_\alpha(\mathbf{B}^n)$ for some $\alpha > 0$ by Corollary 7.4 and [NP]. Hence, by Theorem 8.2, it suffices to estimate the integral means of $\mathrm{a_f}$. From

Hölder's inequality and the Fubini theorem we deduce for any $\lambda < p$ and for any $0 < t < 1$ that

$$\int_{S^{n-1}} \mathrm{a_f}(tw)^{\lambda} d\sigma \leq C_0 (\int_{S^{n-1}} \mathrm{a_f}(tw)^p d\sigma)^{\lambda/p} \leq C_1 (1-t)^{-\lambda/p.}$$

Furthermore, $\lambda/p \leq \lambda - n + 1$ if and only if $\lambda \geq p(n-1)/(p-1)$ and hence Theorem 8.2 yields the claim.

Recall that for each $1/2 < \alpha \leq 1$ there is a quasiconformal mapping f of $\mathbf{B}^2$ with

$$|f(x) - f(y)| \approx |x - y|^{\alpha}$$

for all $x, y \in S^1$; see for example [K2]. For this mapping one easily computes that

$$\int_{\mathbf{B}^2} \mathrm{a_f}^p \, dx < \infty$$

for all $p < 1/(\alpha - 1)$. On the other hand, $\dim_M(\partial f \mathbf{B}^n) = 1/\alpha$ and hence the conclusion of Corollary 8.3 is sharp in the plane. In higher dimensions, this is also the case at least for all α close to 1; one can use a mapping from [DT].

We divide the proof of Theorem 8.2 into several lemmas. We begin by relating the dimension of the boundary to an integrability condition.

8.4. Lemma. *Suppose that* $f \in \mathrm{Lip}_{\alpha}(\mathbf{B}^n)$ *for some* $0 < \alpha \leq 1$. *If*

$$\int_{f(\mathbf{B}^n)} d(x, \partial f(\mathbf{B}^n))^{\lambda - n} dx < \infty,$$

then $\dim_M(\partial f(\mathbf{B}^n)) \leq \lambda$.

Proof. Because f has a continuous extension to the closure of $\mathbf{B}^n$, it easily follows that for each $y \in \partial f(\mathbf{B}^n)$ there is $\omega \in S^{n-1}$ so that the radial limit at ω is y. Fix $y \in \partial f(\mathbf{B}^n)$ and small $r > 0$. Pick a point $z \in f(I_\omega) \cap S^{n-1}(y, r/2)$, where I_ω denotes the radius terminating at ω. Because $f \in \mathrm{Lip}_{\alpha}(\mathbf{B}^n)$, we know that $|f^{-1}(z) - \omega| \geq r^{1/\alpha}/C$. Consider the balls $B_x \subset \mathbf{B}^n$ that intersect the line segment between $f^{-1}(z)$ and ω and whose images intersect $B(y, r)$. We may choose k such balls, $B_1, \cdots, B_k$ so that $k \leq 2\log(C/r^{1/\alpha})$, $\Sigma_1^k \chi_{B_i} \leq 3$ and $f(B_1), \cdots, f(B_k)$ join $S^{n-1}(y, r/2)$ to $S^{n-1}(y, r)$. Then Hölder's inequality gives

$$r/2 \leq \sum_1^k \mathrm{diam}(fB_i) \leq (\sum_1^k \mathrm{diam}(fB_i)^n)^{1/n} k^{(n-1)/n},$$

and the estimate on k, Lemma 2.1 and Lemma 2.3 give us the bound

$$|B(y, Cr) \cap f(\mathbf{B}^n)| \geq r^n \log^{1-n}(1/r)/C,$$

where C is a constant independent of y, r.

Let now $r > 0$ be small. By the Vitali covering theorem, we find pairwise disjoint balls $B(y_j, r)$, $j = 1, \cdots, l$, centered at $\partial f(\mathbf{B}^n))$ so that the balls $B(y_j, 5r)$ cover all of $\partial f(\mathbf{B}^n))$. By the previous paragraph, $|B(y_j, r) \cap f(\mathbf{B}^n)| \geq r^n \log^{1-n} /C'$, and thus

$$\int_{B(y_j,r)} d(x, \partial f(\mathbf{B}^n))^{\lambda-n} dx \geq r^\lambda \log^{1-n}(1/r)/C'.$$

The desired estimate follows.

The converse to the statement of Lemma 8.4 holds even without the Hölder continuity assumption.

8.5. Lemma. *If $f(\mathbf{B}^n)$ is bounded and $\dim_M(\partial f(\mathbf{B}^n)) \leq \lambda < n$, then*

$$\int_{f(\mathbf{B}^n)} d(x, \partial f(\mathbf{B}^n))^{\lambda-n+\varepsilon} dx < \infty$$

for each $\varepsilon > 0$.

Proof. It suffices to show that, given $\varepsilon > 0$,

$$|\{x \in f(\mathbf{B}^n) : \ d(x, \partial f(\mathbf{B}^n) \leq t\}| \leq M t^{n-\lambda-\varepsilon}$$

for some constant M for all sufficiently small $t > 0$. This immediately follows from the bound on the Minkowski dimension and a standard covering argument.

8.6. Lemma. *For all real numbers q, s we have*

$$\int_{\mathbf{B}^n} \frac{d(f(x), \partial f(\mathbf{B}^n))^{n+q}}{(1-|x|)^{n+s}} dx \approx \int_{f(\mathbf{B}^n)} \frac{d(x, \partial f(\mathbf{B}^n))^q}{(1-|f^{-1}(x)|)^s} dx$$

with constants depending only on K, n, q, s.

Proof. Since f is K-quasiconformal, a change of variables shows that

$$\int_{\mathbf{B}^n} |Df(x)|^n \frac{d(f(x), \partial f(\mathbf{B}^n))^q}{(1-|x|)^s} dx \approx \int_{f\mathbf{B}^n} \frac{d(x, \partial f(\mathbf{B}^n))^q}{(1-|f^{-1}(x)|)^s} dx.$$

Moreover, Lemma 2.1 shows that for any $B_x \subset \mathbf{B}^n$,

$$d(f(x), \partial f(\mathbf{B}^n)) \approx d(f(y), \partial f(\mathbf{B}^n))$$

for each $y \in B_x$ with constants depending only on K, n. Hence Lemma 2.5 reveals that

$$\int_{B_x} \mathrm{a_f}(z)^n \frac{d(f(z), \partial f(\mathbf{B}^n))^q}{(1-|z|)^s} dz \approx \int_{B_x} |Df(z)|^n \frac{d(f(z), \partial f(\mathbf{B}^n))^q}{(1-|z|)^s} dz$$

with constants depending only on K, n, q, s. Therefore we obtain the desired conclusion by relying on Lemma 2.3 and by covering $\mathbf{B}^n$ suitably by the balls B_x.

8.7. Lemma. *Suppose that* $\int_{f(\mathbf{B}^n)} d(x, \partial f(\mathbf{B}^n))^{\lambda-n} dx \leq M$. *Then*

$$\int_{S^{n-1}} \mathrm{a_f}(tw)^{\lambda} d\sigma \leq C(1-t)^{n-\lambda-1}$$

for all $0 < t < 1$.

Proof. Fix $0 < t < 1$. By Lemma 8.6 (with $q = \lambda - n$ and $s = 0$), Lemma 2.3, the Fubini theorem and the assumption there is a constant $C = C(K, n, \lambda, M)$ such that

$$\int_t^{(1+t)/2} (1-s)^{\lambda-n} \int_{S^{n-1}} \mathrm{a_f}(sw)^{\lambda} d\sigma ds \leq C.$$

Hence there exists $t < t_o < (1+t)/2$ with

$$\int_{S^{n-1}} \mathrm{a_f}(t_o w)^{\lambda} d\sigma \leq C(1-t_o)^{n-\lambda-1}.$$

Furthermore, $t_o - t < (1+t)/2 - t < (1-t_o)/2$, and hence, by Lemma 2.1 and Lemma 2.3, $\mathrm{a_f}(t_o w)$ is comparable to $\mathrm{a_f}(tw)$ for any $w \in S^{n-1}(1)$. Consequently,

$$\int_{S^{n-1}} \mathrm{a_f}(tw)^{\lambda} d\sigma \leq C'(1-t)^{n-\lambda-1},$$

where $C' = C'(n, K, \lambda)$.

8.8. Lemma. *Suppose that* $f \in \mathrm{Lip}_{\alpha}(\mathbf{B}^n)$ *for some* $0 < \alpha \leq 1$. *If*

$$\int_{S^{n-1}} \mathrm{a_f}(tw)^{\lambda} d\sigma \leq C(1-t)^{n-\lambda-1}$$

for all $0 < t < 1$, *then*

$$\int_{f(\mathbf{B}^n)} d(x, \partial f(\mathbf{B}^n))^{\lambda+\varepsilon-n} dx < \infty$$

for each $\varepsilon > 0$.

Proof. Note first that for any $\varepsilon, M > 0$ the Fubini theorem and our assumption give

$$\begin{aligned}&\int_{\mathbf{B}^n} \mathrm{a_f}(x)^{\lambda}(1-|x|)^{\lambda-n}[\log \frac{M}{1-|x|}]^{-1-\varepsilon} dx \\ &= \int_0^1 \int_{S^{n-1}} \mathrm{a_f}(tw)^{\lambda} d\sigma (1-t)^{\lambda-n} [\log \frac{M}{1-t}]^{-1-\varepsilon} t^{n-1} dt < \infty.\end{aligned}$$

On the other hand, by the Hölder continuity of f

$$\log \frac{1}{d(f(x), \partial f(\mathbf{B}^n))} \geq \alpha^{-1} \log \frac{M}{1-|x|},$$

where $M = M(f)$. Appealing to these estimates and Lemma 2.3 we conclude that

$$\int_{f\mathbf{B}^n} d(x, \partial f(\mathbf{B}^n))^{\lambda+\varepsilon-n} dx < \infty$$

for each $\varepsilon > 0$.

We conclude this section by deducing Theorem 8.2 from the above lemmas.

Proof of Theorem 8.2. Suppose first that $\dim_M(\partial f(\mathbf{B}^n)) \leq \lambda < n$. Then the growth estimate on the integral means of $\mathrm{a_f}$ follows by combining Lemma 8.5 and Lemma 8.7. Conversely, the estimate on the growth of the integral means of $\mathrm{a_f}$ in Theorem 8.2 yields the Minkowski dimension bound by Lemma 8.8 and Lemma 8.4.

9. A^p-spaces

Extending the classical definition we define

$$||f||_{A^p} = (\int_{\mathbf{B}^n} |f|^p dx)^{1/p},$$

and write $f \in A^p$ if $||f||_{A^p} < \infty$.

9.1. Theorem. *Let f be a quasiconformal mapping of* $\mathbf{B}^n$.

(1) *If $f \in H^p$, then $f \in A^{pn/(n-1)}$.*
(2) *If $f \in A^p$, then $f \in H^q$ for all $q < p(n-1)/n$.*

Proof. The first claim follows from Corollary 4.5 with the choice $d\mu = dx$. For the second claim, notice that

$$|f(x)| \leq C(\fint_{B_x} |f(y)|^p dy)^{1/p} \leq C(1-|x|)^{-n/p}$$

for each $x \in \mathbf{B}^n$. Hence we conclude from Theorem 3.3 that $f \in H^q$ for all $q < p(n-1)/n$.

Theorem 9.1(2) is sharp. To see this, consider the quasiconformal mapping

$$f(x) = (x - e_1)|x - e_1|^{-s-1}(\log \frac{3}{|x-e_1|})^{-t}, \tag{9.2}$$

where $s, t > 0$. Choosing $s = (n-1)/q$ and $t = 1/q$, we obtain a quasiconformal mapping that belongs to A^p, $p = qn/(n-1)$, but fails to belong to H^q.

For quasiconformal mappings the integrability of $|f|$ and that of $|Df|$ are closely related. The following theorem follows from results in [BuK] and [K1].

9.3. Theorem. *Let f be a quasiconformal mapping of* $\mathbf{B}^n$.

(1) *If* $|Df| \in A^p$, $0 < p < n$, *then* $f \in A^{pn/(n-p)}$.
(2) *If* $f \in A^{pn/(n-p)}$, $0 < p < n$, *then* $|Df| \in A^q$ *for all* $q < p$.

Theorem 9.1 motivates for the search of an extension of Theorem 9.3 where the A^q-condition for f is replaced by an appropriate H^s-condition. On the other hand, if one chooses for $0 < p < 1$ the exponents s, t in (9.2) by $s = -1 + n/p$ and $t = \frac{n-p}{p(n-1)}$, then $|Df| \in A^p$, but f does not belong to H^q, $q = \frac{p(n-1)}{n-p}$. Theorem 9.5 below establishes a positive result for $p \geq 1$. Another, similar choice of the exponents s, t produces a quasiconformal mapping f so that Df does not belong to A^p, $p > 1$, but with $f \in H^q$, $q = \frac{p(n-1)}{n-p}$. Again, Theorem 9.5 gives a positive conclusion for other values of p.

9.4. Lemma. *For each quasiconformal mapping f of* $\mathbf{B}^n$ *with* $f(0) = 0$,

$$\int_{\mathbf{B}^n} |Df(x)|^n (1 - |x|)^{n-1} dx \leq C||f||^n_{H^n},$$

where C depends only on n and K.

Proof. For $j = 1, 2, ...$ define $\Omega_j = B(0, 1 - 2^{-j}) \setminus \overline{B}(0, 1 - 2^{-j+1})$. Since f is quasiconformal,

$$\int_{\Omega_j} |Df(x)|^n (1 - |x|)^{n-1} dx \leq C|f(\Omega_j)|^n (1 - 2^{-j-1})^{n-1}$$
$$\leq CM(f, 1 - 2^{-j})^n (1 - 2^{-j})^{n-1}.$$

Summing over j and applying Theorem 3.3 gives the claim.

9.4. Theorem. *Let f be a quasiconformal mapping of* $\mathbf{B}^n$. *If* $|Df| \in A^p$, $0 < p < n$, *then* $f \in H^s$, *where for* $p \geq 1$ *we may take* $s = \frac{(n-1)p}{n-p}$, *and for* $0 < p < 1$ *any* $s < \frac{(n-1)p}{n-p}$. *Conversely, if* $f \in H^q$, $q = \frac{(n-1)p}{n-p}$, *then* $|Df| \in A^s$, *where for* $0 < p \leq 1$ *we may take* $s = p$ *and for* $p > 1$ *any* $s < p$.

Proof. Assume that $|Df| \in A^p$. Suppose first that $0 < p < 1$. Then we conclude from Theorem 9.3(1) and Theorem 9.1(2) that f belongs to H^s for all $s < \frac{(n-1)p}{n-p}$.

Suppose then that $p \geq 1$, and assume that $f(0) = 0$. By Hölder's inequality we have

$$\int_{\mathbf{B}^n} |f|^{q-1} |Df| dx \leq (\int_{\mathbf{B}^n} |Df|^p dx)^{1/p} (\int_{\mathbf{B}^n} |f|^{(q-1)p/(p-1)} dx)^{(p-1)/p}.$$

Now $(q-1)p/(p-1) = pn/(n-p)$ and hence the usual Sobolev-Poincaré inequality shows that the second integral is bounded by an appropriate power of the p-integral of Df; see for example [BuK]. Hence Lemma 5.3 shows that $f \in H^q$.

For the converse, suppose that $f \in H^q$ and assume that $0 < p \leq 1$. By Lemma 5.3 it suffices to consider the case $0 < p < 1$. We may clearly assume that $f(0) = 0$. Define for $s > 0$ a function g by setting $g(x) = x|x|^{s-1}$. Then g is quasiconformal and it follows that, with the choice $s = q/n$, the quasiconformal mapping

$$F(x) = g(f(x))$$

belongs to H^n. Moreover,

$$|Df(x)| \leq C|F(x)|^{(1-s)/s}|DF(x)|$$

for almost every x. By Lemma 9.4

$$\int_{\mathbf{B}^n} |DF(x)|^n(1-|x|)^{n-1}dx \leq C||f||^q_{H^q}.$$

Then the Hölder inequality gives

$$\int_{\mathbf{B}^n} |Df(x)|^p dx \leq \int_{\mathbf{B}^n} |DF(x)|^p|F(x)|^{p(1-s)/s}dx$$

$$\leq (\int_{\mathbf{B}^n} |DF(x)|^n(1-|x|)^{n-1}dx)^{p/n}(\int_{\mathbf{B}^n} |F(x)|^r(1-|x|)^t dx)^{(n-p)/n}$$

$$\leq C||f||^{qp/n}_{H^q}(\int_{\mathbf{B}^n} |F(x)|^r(1-|x|)^t dx)^{(n-p)/n},$$

where $r = \frac{pn(1-s)}{s(n-p)}$ and $t = \frac{p(1-n)}{n-p}$. Since $0 < p < 1$, $t > -1$, and hence the measure μ defined by

$$d\mu = (1-|x|)^t dx$$

is an α-Carleson measure,

$$\alpha = (n+t)/(n-1) \geq 1.$$

By Theorem 4.5

$$\int_{\mathbf{B}^n} |F(x)|^{n\alpha}d\mu \leq C(\int_{S^{n-1}} |F|^n d\sigma)^\alpha.$$

A simple but tedious computation shows that $n\alpha = r$, and the desired inequality follows.

The remaining case $p > 1$ follows by combining Theorem 9.1(1) and Theorem 9.2(2).

10. Open problems

1) Does there exist a quasiconformal analog for the Riesz theorem on conjugate functions? More precisely, given n, K, does there exist p_0 depending on n, K such that each K-quasiconformal mapping whose first coordinate function belongs to H^p for some $p > p_0$, in fact, belongs itself to H^p? See Section 6.
2) If f is conformal in the unit disk B, then

$$\int_B |f'(x)|^{2-p} dx < \infty$$

for all $p < p_0$, where $3 < p_0 \leq 4$. Brennan's conjecture states that one can take $p_0 = 4$, for estimates on p_0 see [HS]. By using factorization, one has a similar result for quasiconformal mappings. In higher dimensions, one can ask for an analog for $\mathrm{a_f}$. The initial result from [HK] has recently been improved in [R] for the so-called inner dilatation. Can one obtain such an improvement also for the distortion K considered in this paper?
3) The exponent b in Lemma 2.2 is sharp in the plane, but one expects that it can be improved in higher dimensions. This would then result in improved exponents for Theorem 3.2. An improvement in terms of the inner dilatation is given in [R]. Can one obtain improvements also in terms of the distortion considered in this paper?

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