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Quasiconformal Maps and Substantial Boundary Points

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Dedicated to Professor Fred Gehring, in admiration

Abstract: Let D be a bounded domain in $\mathbb{C}$ with $\zeta_0 \in \partial D$. We say that ζ_0 is a **substantial boundary point** of D for the affine stretch $x+iy \mapsto Kx+iy$, where $K > 1$, if for every neighbourhood U of ζ_0 and for every component V of $U \cap D$ with $\zeta_0 \in \partial V$, the maximal dilatation of f is at least K for every quasiconformal map f of V such that $f(x+iy) = Kx+iy$ for all $x+iy \in \partial V \cap \partial D$.
We give here a criterion for a point ζ_0 to be a substantial boundary point for the affine stretch in D — Theorem 1.1 below. This will depend on the "narrowness" of D near ζ_0 though the particular way that D is narrow may vary, as we shall show.
Keywords: quasiconformal mappings, substantial boundary points, affine stretch.

1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$, with boundary $\mathbb{T}$. Let D be an arbitrary simply connected domain in $\mathbb{C}$ other than $\mathbb{C}$ itself. The

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affine stretch of D,

$$f_K:\ x+iy \mapsto Kx+iy, \qquad \text{where } K>1,$$

is the simplest example of a non-conformal quasiconformal mapping of D onto $f_K(D)$ with complex dilatation

$$\mu(f_K) \equiv \frac{K-1}{K+1}.$$

Such a mapping induces a quasiconformal mapping of $\mathbb{D}$ onto itself given by

$$\tilde{f}_K = \phi_2 \circ f_K \circ \phi_1^{-1}$$

where ϕ_1 and ϕ_2 are conformal mappings of D and $f_K(D)$ onto $\mathbb{D}$, respectively.

Of course, $\tilde{f}_K$ induces a quasisymmetric boundary homeomorphism on $\mathbb{T}$. For the basic facts on quasiconformal mappings we refer to [6].

Suppose that $\zeta_0 \in \partial D$ and write $B(\zeta_0, r) = \{w : |w-\zeta_0| < r\}$. Given a neighbourhood U of ζ_0 we let U_0 be a component of $D \cap U$ with $\zeta_0 \in \partial U_0$. Let f_K be the affine stretch restricted to $\partial U_0 \cap \partial D$ and let f be a quasiconformal mapping of U_0 with $f(x+iy) = f_K(x+iy) = Kx+iy$ on $\partial U_0 \cap \partial D$, with maximal dilatation $K(f)$.

We set

$$H(\zeta_0) = H(\zeta_0, f_K) = \inf K(f)$$

where the infimum runs over all neighbourhoods U of ζ_0 and all such quasiconformal mappings f. Clearly $H(\zeta_0) \le K$. We say that ζ_0 is a **substantial boundary point** of D for f_K if $H(\zeta_0) = K$. The same definition applies to the function $\tilde{f}_K$ restricted to $\mathbb{T}$. The function

$$H(\zeta, \tilde{f}_K), \qquad \zeta \in \mathbb{T},$$

is, of course, in $L^\infty(\mathbb{T})$. Although it need not be continuous, it has been shown by Fehlmann [2] that it is upper semicontinuous and so attains its supremum on $\mathbb{T}$. There are interesting connections between substantial boundary points and degenerating Hamilton sequences. We refer to [4] for details.

Criteria have been obtained in terms of the smoothness of a boundary function for the function to have no substantial boundary points [2]. If the geometry of the domain close to the boundary point ζ_0 is regular enough then ζ_0 is not a substantial boundary point for the affine stretch. For example, if D has a non-zero angle at ζ_0 then this is the case (see [7], pp. 123–124). One can ask whether ζ_0 is a substantial boundary point for the affine stretch provided that D is sufficiently irregular at ζ_0. Note that obviously the substantial boundary points for the affine stretch form a closed subset of ∂D so that in order to find interesting criteria, it may be a good idea to look at isolated substantial boundary points. We give below a sufficient condition for ζ_0 to be a substantial boundary point. This condition does not preclude the possibility of there being other substantial boundary points arbitrarily close to ζ_0 but it does not require it either, as our examples will show.

We prove the following result. Recall that a **ring domain** is a doubly connected domain in $\mathbb{C}$, both of whose boundary components contain more than one point. If R is a ring domain then there is a conformal map φ of R onto the annulus $\{ z : 1 < |z| < R_1 \}$ for a unique $R_1 > 1$, and, as usual, we define the conformal **module** $M(R)$ of R by $M(R) = (\log R_1)/(2\pi)$. We denote the closed line segment joining the points w_1 and w_2 in $\mathbb{C}$ by $[w_1, w_2]$, and the open line segment joining w_1 and w_2 by (w_1, w_2). The area of a set $E \subset \mathbb{C}$ is denoted by $|E|$, and the Euclidean diameter of E is denoted by $\operatorname{diam} E$.

Theorem 1.1. *Let D be a bounded domain in $\mathbb{C}$ with $\zeta_0 \in \partial D$. Suppose that there is a neighbourhood W of ζ_0 such that $W \cap \partial D$ is a Jordan arc. Suppose further that D has a sequence of subdomains D_n for $n \geq 1$ such that*

(i) $\zeta_0 \in \partial D_n$ *for all n;*

(ii) $D_{n+1} \subset D_n$ *for all n;*

(iii) $a_n = \operatorname{diam} D_n \to 0$ *as $n \to \infty$;*

(iv) $(\partial D_n) \setminus (\partial D) = L_n$ *is an open line segment of length b_n;*

(v) $b_n^2/|D_n| \to 0$ *as $n \to \infty$; and*

(vi) *there exist positive constants $C_0 > 1$ and $\delta_0 \leq 1/10$, depending only on D and independent of n, such that whenever the distinct points c_1 and c_2 lie in $L_n = (z_1, z_2)$ with c_1 between z_1 and c_2 and with*

$$|c_1 - c_2| < \delta_0 \min\{|c_1 - z_1|, |c_2 - z_2|\},$$

and $c_3 = (c_1 + c_2)/2$, $r = \min\{|c_3 - z_1|, |c_3 - z_2|\}$, and further B' is the component of $B(c_3, r) \cap D$ containing c_3, then

$$M(B \setminus [c_1, c_2]) \leq C_0 M(B' \setminus [c_1, c_2]).$$

Then ζ_0 is a substantial boundary point of D for the affine stretch $f_K : x+iy \mapsto Kx+iy$ for each $K>1$.

The assumption of Theorem 1.1 that there is a neighbourhood W of ζ_0 such that $W \cap \partial D$ is a Jordan arc is imposed to ensure that the domains D_n give us all the information that is necessary concerning the behaviour of D close to ζ_0. Clearly this assumption is satisfied, for example, if D is a Jordan domain.

The assumptions of Theorem 1.1 mean that there is a sequence of short cross cuts L_n tending to ζ_0, such that D is very narrow at these cross cuts. However, the particular way that D is narrow may vary, as we now show by examples. The condition (v) means that a disk with diameter L_n has a small area if compared to the area of D_n. The condition (vi) states that there is, in a sense, enough space in D (also outside D_n) close to L_n. We could formulate geometric conditions guaranteeing that (vi) holds, for example by requiring that D contains the disk with diameter L_n, but that would be more restrictive than necessary. Therefore we have opted for including the module condition that we actually use in the proof.

Examples. Let P be a non-rectangular parallelogram with one pair of sides parallel to the $x-$axis and of length a (we call them the upper and lower sides of P). Let the other pair of sides have length sa, where $s>0$. For $n \geq 1$, let P_n be a parallelogram congruent to P and with one pair of sides parallel to the $x-$axis and of length α_n. Then the other pair of sides has length $s\alpha_n$. Suppose that the points on the upper side of P_{n+1} have imaginary part at most that of those on the lower side of P_n. Let L_n be an open segment of length b_n contained in the upper side of P_n for $n \geq 2$. We assume that the right end point of L_n coincides with the right end point of the upper side of P_n. Let D be a bounded Jordan domain formed by taking the parallelograms P_n and joining each P_n to P_{n+1} by forming a connection between L_{n+1} and a segment of length at most cb_{n+1}, for some positive absolute constant c, on the lower side of P_n. We assume that $\alpha_n \to 0$ monotonically and $\sum_n \alpha_n < +\infty$, so that the P_n converge to a single point $\zeta_0 \in \partial D$. Let D_n be the natural subdomain of D whose boundary contains L_n, with $\zeta_0 \in \partial D_n$. Suppose that $b_n/\alpha_n \to 0$. Then also $b_n^2/|D_n| \to 0$, and D satisfies the conditions of Theorem 1.1, possibly apart from (vi). However, it is easy to adjust the construction to make sure that also (vi) is satisfied.

We now see that small modifications of this example would lead to others, still satisfying the conditions of Theorem 1.1, but in a different way. Above, the segment L_n is horizontal. We could take L_n to be part of the right hand side boundary segment of P_n. This would lead to the set V_n considered in Section 6 being non-empty. Next, the examples would look somewhat different depending on whether L_n is only a small part of the right hand side boundary segment of P_n, or almost all of it. The latter can still be compatible with the assumptions of Theorem 1.1 provided that b_n is small compared to a_n and $\sqrt{|D_n|}$. With a suitable connection of P_n to P_{n-1}, condition (vi) of Theorem 1.1 can still be satisfied. These choices may **vary** with n, so we can obtain a domain that is narrow infinitely often but in somewhat different ways. Obviously we can construct D so that ζ_0 is the only substantial boundary point of D.

In Section 2 we obtain a lemma that shows how one can estimate the dilatation of a quasiconformal extension of the affine stretch even when there is part of the boundary of the domain where the boundary values are not given. This is done by adapting the usual length-area method used in the proof of the Main Inequality of Reich and Strebel [7].

In Section 3 we adopt notation to discuss the kind of domains that satisfy the assumptions of Theorem 1.1.

In Sections 4 and 5 we perform auxiliary estimates required to control the behaviour of the function on that part of the boundary where boundary values are not prescribed. This is based on estimating moduli of quadrilaterals and the well known connection between such moduli and geometric quantities.

In Section 6 we put our previous results together and prove Theorem 1.1. In the concluding Section 7 we make some general remarks on possible generalisations.

2. Horizontal foliations

Before we get to the proof of Theorem 1.1, we make some preliminary observations. Here the notation is not restricted to that of Theorem 1.1.

Given a bounded domain D and the affine stretch $f_K : x+iy \mapsto Kx+iy$ of D into D_1, say, we may consider the boundary correspondence $f_K|\partial D$ on ∂D and ask for all possible extensions f of $f_K|\partial D$ into D. This problem, in the general case, has been much studied, see, for example, [1], [3], [5], [7], [9], [10], [11], [12],

[13]. We wish here to consider the extensions f, say, with $f = f_K$ on $\partial D \setminus L$ where L is in some appropriate sense a small subset of ∂D.

Let D be a bounded domain, and L a small line segment contained in the boundary of D. So we are looking at those quasiconformal mappings f of D such that $f = f_K$ on $\partial D \setminus L$ and the values of f on L are not prescribed. If L is horizontal we may foliate D by horizontal segments whose end points are in $\partial D \setminus L$ so that $f = f_K$ at all these end points. If $f = f_K$ on all of ∂D then $K(f) \geq K$. But since there is an exception on L we can only assert that $K(f) \geq K_1$, where K_1 is a quantity which may be smaller than K but will be close to K if the length of L is small.

To deal with the values of f on L, we show by means of techniques employing estimates of modules of ring domains and the connection of such modules to distances that $f|L$ cannot deviate very far from $f_K|L$ if L is short and $K(f) \leq K$. If L is not horizontal, then we cannot foliate all of D by horizontal segments such that $f = f_K$ at both end points of every segment. However we may still foliate a subset of D in this way, namely, the set obtained from D by removing the segments with one end point on L. Even in the part of D that was so removed, it is possible to emulate the standard arguments by again using the fact that $f|L$ will be close to $f_K|L$ if $K(f) \leq K$.

In the proof below, we will apply these ideas to the domains D_n described in Theorem 1.1, and then the role of L will be taken by L_n.

We introduce some standard notation. For a general quasiconformal mapping f we write $f_z = \partial f/\partial z$, $f_{\overline{z}} = \partial f/\partial \overline{z}$. The quotient $f_{\overline{z}}/f_z$ is the complex dilatation μ_f and the Jacobian J_f is $|f_z|^2 - |f_{\overline{z}}|^2$. These are functions of z, but this dependence is frequently suppressed.

The following Lemma is based on an elementary application of the method of proof of the $M-$inequality due to Reich and Strebel, as given, for example, in [7], p. 110. It gets us started. Later, we will revisit the proof of this lemma to study a more general situation involving a non-horizontal line segment L.

Lemma 2.1. *Let D be a bounded open set in the plane. Let f be a quasiconformal homeomorphism of D such that for some $K > 1$, we have $f = f_K$ on $\partial D \setminus L$ where $L \subset \partial D$ is a horizontal line segment. Then*

$$K^2|D| \leq K(f)|f(D)|.$$

We assume in Lemma 2.1 that D is an open set rather than a domain since we may wish to apply Lemma 2.1 to the situation where D is obtained by removing a rectangle-like domain from a larger domain.

Lemma 2.1 gives a lower bound for $K(f)$ in terms of the quantity $|D|/|f(D)|$, which we should expect to be close to $1/K$.

Proof of Lemma 2.1. For any $\eta \in \mathbb{R}$ let $\Gamma(\eta) = D \cap \{\, x + i\eta \,:\, x \in \mathbb{R}\,\}$. The set $\Gamma(\eta)$, if not empty, is the union of at most countably many open horizontal line segments. Each such segment has both of its boundary points in $\partial D \setminus L$. Let $L(\Gamma)$ denote the total length of paths comprising the set Γ.

Suppose that for a fixed η, the set $\Gamma(\eta)$ consists of only one segment. We write $\Gamma'(\eta) = f_K(\Gamma(\eta))$, $\Gamma''(\eta) = f(\Gamma(\eta))$. Then $\Gamma'(\eta)$ is a horizontal line segment with the same end points as the arc $\Gamma''(\eta)$ so that $KL(\Gamma(\eta)) = L(\Gamma'(\eta)) \le L(\Gamma''(\eta))$. Since f is absolutely continuous on lines, we can deduce that for almost every η

$$\begin{aligned} K\int_{\Gamma(\eta)} dx &\le \int_{\Gamma(\eta)} |f_z\, dz + f_{\overline{z}}\, d\overline{z}| \\ &= \int_{\Gamma(\eta)} |f_z + f_{\overline{z}}|\, dx = \int_{\Gamma(\eta)} |f_z|\, |1 + \mu(f)|\, dx. \end{aligned}$$

If $\Gamma(\eta)$ consists of more than one segment, the above argument applies to each of them, and adding up we obtain the conclusion

$$K\int_{\Gamma(\eta)} dx \le \int_{\Gamma(\eta)} |f_z|\, |1 + \mu(f)|\, dx. \tag{2.1}$$

for the entire set $\Gamma(\eta)$.

We note that

$$|f_z| = \left(\frac{J_f}{1 - |\mu_f|^2}\right)^{1/2}.$$

Integrating (2.1) with respect to η between the limits arising from the bounded open set D we obtain

$$K|D| \le \int_D \left(\frac{J_f}{1 - |\mu_f|^2}\right)^{1/2} |1 + \mu(f)|\, dxdy.$$

Applying the Cauchy–Schwarz inequality we obtain

$$K^2|D|^2 \le \left(\int_D J_f\, dxdy\right)\left(\int_D \frac{|1 + \mu(f)|^2}{1 - |\mu(f)|^2}\, dxdy\right) \le |f(D)| K(f) |D|.$$

This last inequality follows from the fact that

$$\frac{|1+\mu(f)|^2}{1-|\mu(f)|^2} \le \frac{(1+|\mu(f)|)^2}{1-|\mu(f)|^2} = \frac{1+|\mu(f)|}{1-|\mu(f)|} \le K(f)$$

almost everywhere in D. This proves Lemma 2.1.

3. The set-up related to the domain D

Let D satisfy the assumptions of Theorem 1.1. Let U be a neighbourhood of ζ_0 and let U_0 be a component of $D \cap U$ with $\zeta_0 \in \partial U_0$ such that $U \cap \partial D$ is a Jordan arc. Let f be a quasiconformal map of U_0 with $f = f_K$ on $(\partial U_0) \cap (\partial D)$. There exists a positive integer m such that $D_m \subset U_0$. We now view f as a quasiconformal map of D_m. Then $f = f_K$ on $\partial D_m \setminus L_m$. To get a lower bound for $K(f)$, we may assume that $K(f) \le K$.

We shall show that for all sufficiently large values of m, we have

$$\frac{|f(D_m)|}{|D_m|} \le K + \pi C_1^2 \frac{b_m^2}{|D_m|} \tag{3.1}$$

for a suitable positive constant C_1.

If L_m were horizontal, we could combine this with Lemma 2.1 and obtain

$$K(f) \ge K\left(1 + \frac{\pi C_1^2}{K}\frac{b_m^2}{|D_m|}\right)^{-1} \to K \text{ as } m \to \infty.$$

However, since L_m need not be horizontal, a more careful argument will be needed.

Clearly the part of D which is far from ζ_0 plays no role in the argument.

We discuss the above selection of m more carefully. First note that U_0 contains $D \cap B(\zeta_0, r)$ for some $r > 0$, and hence there exists $m_1 \ge 2$ such that $D_n \subset U_0$ for all $n \ge m_1$. We choose $m > m_1$ such that if D_m is the subdomain of D which lies "below" the segment L_m and D_{m_1} is the subdomain of D which lies "below" the segment L_{m_1}, then there exists $\rho > 0$ such that $D_m \subset D \cap B(\zeta_0, \rho/2)$ and L_{m_1} lies outside $B(\zeta_0, \rho)$. It suffices to find a lower bound for $K(f|D_m)$. In what follows, L_m need not be horizontal.

We denote the end points of L_m by z_1 and z_2, respectively, suppressing the dependence on m. Recall that we denote the closed line segment joining the points w_1 and w_2 by $[w_1, w_2]$, and the open line segment joining w_1 and w_2 by (w_1, w_2). Thus $L_m = (z_1, z_2)$.

The assumption that $D_n \subset U_0$ for all $n \geq m_1$ and not only for all $n \geq m$ means that there is some extra space in the domain of definition of f "above" D_m. We will make some use of this later. Note that $D_m \subset D_{m_1}$, and f is defined and is $K-$quasiconformal in D_{m_1}, with $f = f_K$ on $\partial D_{m_1} \cap \partial D$. The set $D_{m_1} \setminus D_m$ contains at least $D_{m-1} \setminus \overline{D_m}$.

Furthermore, by the assumption (vi) of Theorem 1.1, there exist fixed positive constants $C_0 > 1$ and $\delta_0 \leq 1/10$, depending only on D and independent of m, such that the following holds. Pick any points $c_1 \in (z_1, z_2) = L_m$ and $c_2 \in (c_1, z_2)$ such that

$$|c_1 - c_2| < \delta_0 \min\{|c_1 - z_1|, |c_2 - z_2|\}.$$

Recall from condition (vi) of Theorem 1.1 that $c_3 = (c_1 + c_2)/2$ and $B = B(c_3, r)$ where $r = \min\{|c_3 - z_1|, |c_3 - z_2|\}$. We need not have $B \subset D$. Write B' for the component of $B \cap D$ containing c_3. Then clearly $(z_1, c_2) \subset B'$ if $|c_3 - z_1| \leq |c_3 - z_2|$, and $(c_1, z_2) \subset B'$ if $|c_3 - z_2| \leq |c_3 - z_1|$. Depending on the precise shape of D, it is conceivable that $B \setminus B'$ could have two or more components.

Our assumption implies that

$$M(B' \setminus [c_1, c_2]) \leq M(B \setminus [c_1, c_2]) \leq C_0 M(B' \setminus [c_1, c_2]), \tag{3.2}$$

where $M(R)$ denotes the module of the ring domain R.

Note that if $\infty \in \overline{f(D_{m_1})}$ then there is a point $\zeta \in D_{m_1} \cup L_{m_1}$ such that $f(\zeta) = \infty$ (since the set $f(\partial D_{m_1} \cap \partial D) = f_K(\partial D_{m_1} \cap \partial D)$ is bounded). Taking m_1 larger, we manage to exclude ζ from $\overline{D_{m_1}}$. We assume that this has been done, if necessary. Thus we assume that $\overline{f(D_{m_1})}$ is a compact subset of the finite complex plane $\mathbb{C}$.

4. Auxiliary results

To use Lemma 2.1 we must find a lower bound for $|D_m|/|f(D_m)|$, or equivalently an upper bound for $|f(D_m)|/|D_m|$. For fixed m we write $\gamma_0 = \partial D_m \setminus L_m$. Then $f(D_m)$ is a Jordan domain with $\partial f(D_m) = f_K(\gamma_0) \cup f(L_m)$. If $f(D_m) \subset f_K(D_m)$, we have $|f(D_m)| \leq |f_K(D_m)| = K|D'|$. So we may assume that

$f(D_m) \setminus f_K(D_m) \neq \emptyset$. Then we can find an upper bound for $|f(D_m)|$ by estimating

$$s_1 = \sup\{|f(z) - f(z_1)| : z \in L_m, f(z) \notin f_K(D_m)\}. \tag{4.1}$$

Note that $f(z_1) = f_K(z_1)$ and $\operatorname{Im} f(z_1) = \operatorname{Im} z_1$. We clearly have $|f(D_m)| \leq |f_K(D_m)| + \pi s_1^2$, and $|f_K(D_m)| = K|D_m|$.

The set $f(L_m) \setminus \overline{f_K(D_m)}$, possibly empty, is the union of at most countably many open arcs, $\cup_k \gamma_k$, say. We fix k and suppress the dependence on k in the notation. Then $\overline{\gamma_k}$ contains at least one point w such that

$$t = |f(w) - f(z_1)| = \max\{\, |f(z) - f(z_1)| \,:\, z \in \overline{\gamma_k}\,\}. \tag{4.2}$$

If $f(w) \in [f_K(z_1), f_K(z_2)]$ then $t \leq K b_m$. Hence we may assume that $f(w) \notin \overline{f_K(D_m)}$.

We require the following two results. The first one can be found in [6], Lemma 4.1, p. 23.

Lemma 4.1. *Let Q be a quadrilateral in the Riemann sphere $\overline{\mathbb{C}}$. Let $s_a = s_a(Q)$ denote the distance of the $a-$sides of Q measured in Q, that is, s_a is the infimum of the lengths of closed Jordan arcs γ such that γ lies in the interior of Q apart from its end points, and γ has one end point on each of the $a-$sides of Q. Similarly let $s_b = s_b(Q)$ denote the distance of the $b-$sides of Q measured in Q. Let $M(Q)$ denote the module of Q. Then*

$$M(Q) \leq \pi \frac{1+2L}{L^2} \qquad \textit{where} \qquad L = \log\left(1 + \frac{2s_a}{s_b}\right). \tag{4.3}$$

The **conjugate quadrilateral** Q' of Q is obtained from Q by interchanging the roles of the $a-$sides and $b-$sides. We have $M(Q') = 1/M(Q)$. Hence Lemma 4.1 implies that

$$M(Q) \geq \frac{1}{\pi}\frac{(L')^2}{1+2L'} \qquad \text{where} \qquad L' = \log\left(1 + \frac{2s_b(Q)}{s_a(Q)}\right). \tag{4.4}$$

The second result is known as Teichmüller's module theorem, and it can be found, e.g., in [6], p. 56.

Lemma 4.2. *If the ring domain R separates the points 0 and ζ_1 from ζ_2 and ∞ then*

$$M(R) \leq 2\mu\left(\sqrt{\frac{|\zeta_1|}{|\zeta_1| + |\zeta_2|}}\right).$$

Here, as usual, $\mu(r) = M(B(0,1)\backslash[0,r])$ is the module of the Grötzsch ring (see [6]). The function $\mu(r)$ is decreasing for $0 < r < 1$ and has an inverse function $\mu^{-1}(r)$ defined on $(0,\infty)$.

5. Distance estimates

Recall that t is defined by (4.2) and that δ_0 and C_0 are as in Section 3 (that is, as in Theorem 1.1, (vi)). We wish to show that there exists a positive constant C_1 depending on K, δ_0 and C_0 alone, such that

$$t \le C_1 b_m. \tag{5.1}$$

Then s_1, defined by (4.1), also satisfies

$$s_1 \le C_1 b_m.$$

We define δ_1 and δ_2 by

$$\frac{2\delta_2}{1+\delta_2^2} = \mu^{-1}\left(4KC_0\mu\left(\frac{1}{4}\right)\right),$$

$$\delta_1 = \min\left\{\delta_0, \delta_2, \frac{1}{2}(\sqrt{e}-1)\right\}. \tag{5.2}$$

Then $0 < \delta_2 < 1$ and

$$4KC_0\,\mu\left(\frac{1}{4}\right) = \mu\left(\frac{2\delta_2}{1+\delta_2^2}\right) \le \mu\left(\frac{2\delta_1}{1+\delta_1^2}\right).$$

We establish (5.1) with

$$C_1 = 4K\exp\left\{\frac{8\pi^2 K}{\delta_1}\right\} > 4K. \tag{5.3}$$

We suppose now that (5.1) does not hold, where C_1 is defined by (5.3). Then there exists a first point z_3 on (z_1, z_2) when moving from z_1 to z_2 such that $|f(z_3) - f(z_1)| = C_1 b_m$. Similarly there is a last point z_4 on (z_1, z_3) when moving from z_1 to z_3 such that $|f(z_4) - f(z_1)| = C_1 b_m/2$. Thus, for all z on the segment (z_4, z_3),

$$\frac{C_1 b_m}{2} < |f(z) - f(z_1)| < C_1 b_m.$$

To get a contradiction we suppose that

$$|z_4 - z_3| < \delta_1 \min\{\,|z_4 - z_1|,\ |z_2 - z_3|\,\}$$

and define B and B' as in (3.2) using z_4 and z_3 instead of c_1 and c_2. Now, of course, f is $K-$quasiconformal in B' and so, since $\delta_1 \le \delta_0$, we obtain from (3.2) that

$$M(B \setminus [z_4, z_3]) \le C_0\, M(B' \setminus [z_4, z_3]) \le K\, C_0\, M(f(B') \setminus f([z_4, z_3])).$$

Now $\partial B'$ contains at least one of the points z_1 and z_2 so that $\partial f(B')$ contains at least one of the points $f(z_1)$ and $f(z_2)$. We denote such a point by b, so $b \in \{f(z_1), f(z_2)\}$. Since $f(B')$ is bounded, the ring domain $f(B') \setminus f([z_4, z_3])$ separates the points $f(z_3)$ and $f(z_4)$ from the points b and ∞. It follows from Lemma 4.2 and the invariance of the module under Möbius transformations that

$$M(f(B') \setminus f([z_4, z_3])) \le 2\mu(r_0) \tag{5.4}$$

where

$$r_0 = \left\{1 + \left|\frac{b - f(z_4)}{f(z_3) - f(z_4)}\right|\right\}^{-1/2}. \tag{5.5}$$

To get a lower bound for r_0, we next obtain an upper bound for

$$\left|\frac{b - f(z_4)}{f(z_3) - f(z_4)}\right|.$$

Since $|f(z_3) - f(z_1)| = C_1 b_m$ and $|f(z_4) - f(z_1)| = C_1 b_m/2$, it follows that $|f(z_3) - f(z_4)| \ge C_1 b_m/2$.

If $b = f(z_1)$ then $|b - f(z_4)| = |f(z_1) - f(z_4)| = C_1 b_m/2$, so that

$$\left|\frac{b - f(z_4)}{f(z_3) - f(z_4)}\right| \le 1.$$

If $b = f(z_2)$ then $|b - f(z_4)| = |f(z_2) - f(z_4)| \le |f(z_2) - f(z_1)| + |f(z_1) - f(z_4)| \le K b_m + C_1 b_m/2$. Hence

$$\left|\frac{b - f(z_4)}{f(z_3) - f(z_4)}\right| \le \frac{K b_m + C_1 b_m/2}{C_1 b_m/2} = \frac{2K + C_1}{C_1} = 1 + \frac{2K}{C_1} \le \frac{3}{2}$$

by (5.3).

Thus in all cases

$$r_0 \ge \{1 + 3/2\}^{-1/2} = \sqrt{\frac{2}{5}} > \frac{1}{2}.$$

On the other hand, considering a Möbius transformation taking $B \setminus [z_4, z_3]$ onto $B(0,1) \setminus [0, r_1]$, we find that

$$M(B \setminus [z_4, z_3]) \geq \mu(r_1) \tag{5.6}$$

where

$$r_1 = \frac{2\delta_1}{1+\delta_1^2}. \tag{5.7}$$

Hence, by (5.2),

$$4KC_0\mu(1/4) \leq \mu(r_1) < 2K\,C_0\,\mu(1/4), \tag{5.8}$$

which gives a contradiction. We deduce that (5.1) cannot hold, and so necessarily

$$|z_4 - z_3| \geq \delta_1 \min\{\,|z_4 - z_1|,\, |z_2 - z_3|\,\}. \tag{5.9}$$

Now let Q be the quadrilateral whose domain is D_m and whose $a-$sides are the arcs of ∂D_m going along $\partial D_m \setminus L_m$ from z_1 to z_2 and along the segment L_m from z_3 to z_4. Then we have (noting that $s_a(Q)$ is defined by means of distances measured inside Q)

$$s_a(Q) \leq \min\{\,|z_4 - z_1|,\, |z_2 - z_3|\,\}$$

and $s_b(Q) = |z_3 - z_4|$. Hence

$$L_1 = \log\left(1 + \frac{2s_b(Q)}{s_a(Q)}\right) \geq \log(1 + 2\delta_1) = P_1,$$

say. Since $\delta_1 \leq (1/2)(\sqrt{e} - 1)$ by (5.2) we see that $P_1 \leq 1/2$. Thus, from (4.4),

$$M(Q) \geq \frac{1}{\pi}\frac{P_1^2}{1+2P_1} \geq \frac{P_1^2}{2\pi} \geq \frac{\delta_1^2}{2\pi}.$$

Analogously, we write

$$P_2 = \log\left(1 + \frac{2s_a(f(Q))}{s_b(f(Q))}\right)$$

so that, by (4.3),

$$\frac{M(Q)}{K} \leq M(f(Q)) \leq \pi\frac{1+2P_2}{P_2^2}.$$

We have $s_b(f(Q)) \leq K|z_1 - z_2| = Kb_m$ and $s_a(f(Q)) \geq C_1 b_m/2 - Kb_m$. Thus

$$P_2 \geq \log\left(1 + \frac{(C_1 - 2K)b_m}{2Kb_m}\right) = \log\left(1 + \frac{C_1 - 2K}{2K}\right) = L_0,$$

say. Now, from (5.3), we have $(C_1 - 2K)/(2K) > 1$ so that $L_0 \geq \log 2 > 1/2$. Hence,

$$\frac{M(Q)}{K} \leq M(f(Q)) \leq \pi \frac{1 + 2L_0}{L_0^2} \leq \frac{4\pi}{L_0}.$$

We deduce that

$$\frac{4\pi K}{L_0} \geq \frac{\delta_1^2}{2\pi}$$

and hence

$$1 + \frac{C_1 - 2K}{2K} \leq \exp\left\{ \frac{8\pi^2 K}{\delta_1^2} \right\}$$

which gives a contradiction to (5.3).

We conclude that $s_1 \leq C_1 b_m$.

We have proved that for all sufficiently large m, we have

$$\frac{|f(D_m)|}{|D_m|} \leq \frac{|f_K(D_m)| + \pi s_1^2}{|D_m|} \leq K + \pi C_1^2 \frac{b_m^2}{|D_m|}.$$

This yields (3.1).

6. Completion of the proof of Theorem 1.1

Choose $\varepsilon > 0$. Since, by the assumptions of Theorem 1.1, we have that $b_n^2/|D_n| \to 0$, there is a positive integer n_0 such that for all $n \geq n_0$, we have

$$b_n^2/|D_n| < \varepsilon^2.$$

Now pick n with $n \geq n_0$. Our aim is to get a lower bound for $K(f)$, where f is a quasiconformal map of U_0 with $f = f_K$ on $\partial U_0 \cap \partial D$ and hence on $\partial D_n \setminus L_n$, in terms of $|D_n|$ and $|f(D_n)|$. Note that $K(f) \geq K(f|D_n)$ for all large n. We begin by exploring the consequences of Lemma 2.1.

Now foliate the domain D_n by horizontal line segments. (Note that foliation by **horizontal** segments is necessary for us to be able to use the ideas of the proof of Lemma 2.1. This is because we are considering a horizontal stretch f_K.) Let V_n denote the union of those open horizontal line segments with one end point in L_n. Then V_n is either a domain or the empty set. Write $U_n = D_n \setminus \overline{V_n}$. Clearly $V_n = \emptyset$ if, and only if, L_n is horizontal. Note that $L_n \subset D$.

If $V_n = \emptyset$, we may apply Lemma 2.1 to D_n and obtain

$$K^2 |D_n| \leq K(f)|f(D_n)|.$$

If there is a sequence of values of n with $V_n = \emptyset$, we may apply the arguments of the previous sections to these n (with m there replaced by n), and conclude, using (3.1) and the assumption that $\lim_{k\to\infty} b_k^2/|D_k| = 0$, that $K(f) \geq K$. Namely, given any neighbourhood U_0 of ζ_0, we merely have to consider a value of n that is large enough for all these conditions to hold, and such that D_n is contained in U_0 by a wide margin as explained before. So we may now assume that for all large n, we have $V_n \neq \emptyset$.

Suppose next that for a certain large $n \geq n_0$, we have $V_n \neq \emptyset$. Applying Lemma 2.1 to the bounded open set U_n we obtain

$$K^2|U_n| \leq K(f)|f(U_n)|. \tag{6.1}$$

Next, at least one of the inequalities

$$\frac{b_n^2}{|V_n|} < \varepsilon \qquad \text{and} \qquad \frac{|V_n|}{|D_n|} < \varepsilon$$

must hold for this n, for if both of these inequalities fail, then

$$\frac{b_n^2}{|D_n|} = \frac{b_n^2}{|V_n|}\,\frac{|V_n|}{|D_n|} \geq \varepsilon^2,$$

which is a contradiction. Note that it may depend on n which one of these two inequalities holds.

Suppose that $V_n \neq \emptyset$ and $|V_n|/|D_n| < \varepsilon$. This is the easier case. Now, since $U_n \subset D_n$, we obtain from (6.1) that

$$\frac{K^2}{K(f)} \leq \frac{|f(U_n)|}{|U_n|} \leq \frac{|f(D_n)|}{|D_n| - |V_n|} = \frac{|f(D_n)|}{|D_n|}\,\frac{|D_n|}{|D_n| - |V_n|} \leq \frac{|f(D_n)|}{(1-\varepsilon)|D_n|}.$$

We may then assume that n is so large that the arguments given in the previous sections for D_m apply when m is replaced by n. Then (3.1) implies that

$$\frac{|f(D_n)|}{|D_n|} \leq K + \pi C_1^2 \frac{b_n^2}{|D_n|}.$$

Thus altogether

$$\frac{K^2}{K(f)} \leq \frac{1}{1-\varepsilon}\left(K + \pi C_1^2 \frac{b_n^2}{|D_n|}\right),$$

that is,

$$K(f) \geq (1-\varepsilon)K\left(1 + \pi C_1^2 \frac{b_n^2}{K|D_n|}\right)^{-1}.$$

It follows that if there is a sequence of positive integers n tending to infinity such that for each such n, we have $V_n \neq \emptyset$ and $|V_n|/|D_n| < \varepsilon$, then $K(f) \geq K(1-\varepsilon)$.

Suppose then that for some n, we have $V_n \neq \emptyset$ and $b_n^2/|V_n| < \varepsilon$. Now apply the proof of Lemma 2.1 to V_n. Note that by (5.1), we have

$$L(f(\Gamma(\eta))) \geq KL(\Gamma(\eta)) - C_1 b_n$$

for each η. Here we observe that even if the set $\Gamma(\eta)$ were the union of more than one line segment, at most one of those segments can have an end point on L_n. Thus for almost every η,

$$KL(\Gamma(\eta)) - C_1 b_n \leq \int_{\Gamma(\eta)} \frac{\sqrt{J_f}}{\sqrt{1-|\mu_f|^2}}\, |1+\mu_f|\, dx$$

so that integration over the appropriate values of η gives

$$K|V_n| - C_1 b_n^2 \leq K|V_n| - C_1 b_n h_n \leq \sqrt{K(f)|V_n||f(V_n)|}$$

where h_n denotes the variation of the η-coordinate over V_n so that $h_n \leq b_n$. We may assume that $\varepsilon < K/C_1$ and therefore $K|V_n| - C_1 b_n^2 > 0$. This yields

$$(K|V_n| - C_1 b_n^2)^2 \leq K(f)|V_n||f(V_n)|.$$

To get an upper bound for $|f(V_n)|$, we recall the arguments concerning D_m. Clearly (5.1) implies that $|f(V_n)| \leq K|V_n| + \pi C_1 b_n^2$. Thus

$$(K|V_n| - C_1 b_n^2)^2 \leq K(f)|V_n|(K|V_n| + \pi C_1 b_n^2).$$

It follows that

$$K(f) \geq \frac{(K|V_n| - C_1 b_n^2)^2}{|V_n|(K|V_n| + \pi C_1 b_n^2)} = K\, \frac{(1 - \frac{C_1}{K}\frac{b_n^2}{|V_n|})^2}{1 + \frac{\pi C_1}{K}\frac{b_n^2}{|V_n|}}.$$

This last expression is greater than or equal to

$$K\, \frac{(1 - \varepsilon\frac{C_1}{K})^2}{1 + \varepsilon\frac{\pi C_1}{K}}.$$

Thus $K(f) \geq K(1 - C'\varepsilon)$, where $C' > 1$ depends on K, δ_0, and C_0 only. Now, since for each $\varepsilon > 0$, there are infinitely many n for which $V_n \neq \emptyset$ and $b_n^2/|V_n| < \varepsilon$, or there are infinitely many n for which $V_n \neq \emptyset$ and $|V_n|/|D_n| < \varepsilon$, it follows that $K(f) \geq K(1 - C'\varepsilon)$ for each $\varepsilon > 0$. Thus $K(f) \geq K$, so that ζ_0 is a substantial boundary point of D. This completes the proof of Theorem 1.1.

7. General remarks

The result of Theorem 1.1 is, of course, very special but it raises the question of whether there would be results of the same type in more general situations. It would also seem reasonable to suppose that the line segment L_n can be replaced by any Jordan arc, and that the condition $b_n^2/|D_n| \to 0$ as $n \to \infty$ should be replaced by a condition involving harmonic measure. One can also ask to what extent the condition (vi) of Theorem 1.1, guaranteeing that there is a certain amount of space in D around L_n, could be relaxed. Indeed, it seems possible that a condition of narrowness of a domain D at a point $\zeta_0 \in \partial D$ might be sufficient to make this point a substantial boundary point for the affine stretch.

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