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Some Improved Caffarelli-Kohn-Nirenberg Inequalities with General Weights and Optimal Remainders*

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Dedicated to Professor J.J.Kohn on the occasion of his 75th birthday

Abstract: In this paper, we establish some improved Caffarelli-Kohn-Nirenberg inequalities with general weights and optimal remainders. Moreover, we give a positive answer to an open problem raised by Abdellaoui et al. [1].

Keywords: Hardy-Sobolev inequality, general weight, optimal remainders

1 Introduction

Let $p > 1$ be a constant. In 1920, Hardy [7] showed that, for any positive $f(x) \in L^p(0, \infty)$,

$$\int_0^\infty \left[\frac{F(x)}{x}\right]^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p \,\mathrm{d}x,$$

where $F(x) = \int_0^x f(t)\,\mathrm{d}t$, and the constant $\left(\frac{p}{p-1}\right)^p$ is optimal.

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In 1933, Leray [8] gave the following multidimensional version of Hardy's inequality

$$\int_{\mathbb{R}^2\setminus B_1(0)} \frac{u^2}{|x|^2 \ln^2 |x|}\,\mathrm{d}x \le 4\int_{\mathbb{R}^2\setminus B_1(0)} |\nabla u|^2\,\mathrm{d}x, \quad u\in C_0^\infty(\mathbb{R}^2\setminus B_1(0)) \tag{1.1}$$

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,\mathrm{d}x \le \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x, \quad u\in C_0^\infty(\mathbb{R}^N),\ N\ge 3 \tag{1.2}$$

We may call the above two inequalities Hardy-Leray inequality, which is called Hardy-Sobolev inequality in the literature (see [1]). For any bounded domain $\Omega \subset B_R(0)$ including origin, $B_R(0)$ denotes a ball in $\mathbb{R}^N$ with radius R and centered at 0, Shen [9] obtained (1.1) with $\ln^2|x|$ being replaced by $\ln^2 R/|x|$.

Brézis and Vázques [3] obtained a remainder term for the Hardy-Leray's inequality. More precisely, if $1 \le q < \frac{2N}{N-2}$, $N \ge 3$, there exists a constant $C(q,|\Omega|) > 0$ such that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x \ge C(q,|\Omega|)\left(\int_\Omega |u|^q\,\mathrm{d}x\right)^{2/q}, \quad u\in H_0^1(\Omega) \tag{1.3}$$

They raised some open problems in [3], and the second one states whether there is a further improvement in the direction of this inequality.

Vázquez and Zuazua [16], among other results, improved the previous inequality by showing that if $1 < q < 2$, there exists a constant $C(q,|\Omega|) > 0$ such that, for each $u \in H_0^1(\Omega)$,

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}\,\mathrm{d}x \ge C(q,|\Omega|)\left(\int_\Omega |\nabla u|^q\,\mathrm{d}x\right)^{2/q} \tag{1.4}$$

The Caffarelli-Kohn-Nirenberg inequality [4] shows that, if $1 < p < N$ and $\gamma < \frac{N-p}{p}$, for any $u \in C_0^\infty(\Omega)$,

$$c_p \int_\Omega |x|^{-p(\gamma+1)}|u|^p\,\mathrm{d}x \le \int_\Omega |x|^{-p\gamma}|\nabla u|^p\,\mathrm{d}x \tag{1.5}$$

where Ω is allowed to be the whole space $\mathbb{R}^N$.

Wang and Willem [17] obtained the Caffarelli-Kohn-Nirenberg inequality with optimal remainder, that is, if $0 \in \Omega \subset B_R(0)$, then for any $u \in H_0^1(\Omega)$,

$$\int_\Omega |x|^{-2\gamma}|\nabla u|^2\,\mathrm{d}x - \left[\frac{N-2(\gamma+1)}{2}\right]^2 \int_\Omega |x|^{-2(\gamma+1)}|u|^2\,\mathrm{d}x \geq C\int_\Omega |x|^{-2\gamma}(\ln R/|x|)^{-2}|\nabla u|^2\,\mathrm{d}x \quad (1.6)$$

It is optimal in the sense that $(\ln R/|x|)^{-1}$ can not be replaced by $g(x)(\ln R/|x|)^{-1}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$. If $\gamma = 0$, (1.6) gives a positive answer to the second open problem of [3] in some sense. The authors proved another result which works for bounded domains as well as exterior domains, that is,

$$\int_\Omega |x|^{-2\gamma}|\nabla u|^2\,\mathrm{d}x - \left[\frac{N-2(\gamma+1)}{2}\right]^2 \int_\Omega |x|^{-2(\gamma+1)}|u|^2\,\mathrm{d}x \geq \frac{1}{4}\int_\Omega |x|^{-2(\gamma+1)}(\ln R/|x|)^{-2}u^2\,\mathrm{d}x,$$

where $\gamma \leq \frac{N-p}{p}$, $\Omega \subset B_R(0)$ or $\Omega \subset B_R^C(0)$. Moreover, the constant $\frac{1}{4}$ is also sharp.

Abdellaoui et al. [1] proved that if $1 < q < p < N$, then for any $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |x|^{-\gamma p}|\nabla u|^p\,\mathrm{d}x - \left[\frac{N-p(\gamma+1)}{p}\right]^p \int_\Omega \frac{|u|^p}{|x|^{p(\gamma+1)}}\,\mathrm{d}x \geq C\int_\Omega |x|^{-\gamma r}|\nabla u|^q\,\mathrm{d}x \quad (1.7)$$

where $q < r < +\infty$ if $\gamma \leq 0$, or $r < p + \rho(N,p,q,\gamma)$ for some positive constant ρ if $\gamma > 0$. The authors point out that it seems to be an open problem to obtain the best weight for (1.7) as in (1.6), in the case $p \neq 2$. In this paper, we give a positive answer to this open problem. In fact, we obtain the Caffarelli-Kohn-Nirenberg inequality with general weights and remainder term. Because the weight is general, we also obtain the corresponding inequality with weight $|x|^{-\gamma p}$ in the case of $N = p > 1$. When $N = p = 2$, this problem has been discussed in [14].

Now we introduce the weighted Sobolev space. Let ϕ be a positive continuous function with $\phi(|x|) \in L(B_\delta(0))$ for some positive δ, and define

$$\bar{h}(r_1, r_2) = c_0\int_{r_1}^{r_2} (\phi r^{N-1})^{-1/(p-1)}\,\mathrm{d}r$$

for $0 \leq r_1 \leq r_2 \leq \infty$, where c_0 is a given positive constant. In this paper, we consider the following two cases:

(A_1) $\bar{h}(r,\infty) < \infty$ for all $r > 0$ and $\bar{h}(0,\infty) = \infty$;

(A_2) $\bar{h}(r,\infty) = \infty$ and $\bar{h}(0,D) = \infty$ for some r, $D > 0$.

Definition 1. Let $p > 1$, we denote by $W_0^{1,p}(\Omega,\phi)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p,\phi} = \left(\int_\Omega \phi(r)|\nabla u|^p \,\mathrm{d}x\right)^{1/p}$$

where $r = |x|$.

Example 1. Let $\phi = r^{-p\gamma}$ and $0 \in \Omega \subset B_D(0)$. If $\gamma < \frac{N-p}{p}$, then (A_1) happens, and $W_0^{1,p}(\Omega, |x|^{-p\gamma})$ is identical with $D_{0,\gamma}^{1,p}(\Omega)$ in [1]. If $\gamma = \frac{N-p}{p}$, then (A_2) happens, and $W_0^{1,p}(\Omega, |x|^{-p\gamma})$ has not been discussed before.

In what follows, for short, we use ϕ for $\phi(r)$ or $\phi(|x|)$, etc.

Set

$$\bar{h} = \begin{cases} \bar{h}(r,\infty), & \text{if } (A_1) \text{ holds} \\ \bar{h}(r,D), & \text{if } (A_2) \text{ holds} \end{cases}$$

If $N > p$ and $\phi \equiv 1$, then (A_1) holds, therefore $\bar{h}(|x|) = |x|^{\frac{p-N}{p-1}}$ is a fundamental solution for the p-Laplace operator. For general weight ϕ, function $h = \bar{h}^{(p-1)/p}$ satisfies in the sense of distribution

$$-\Delta_{\phi,p} u =: \mathrm{div}(\phi|\nabla u|^{p-2}\nabla u) = \psi|u|^{p-2}u \tag{1.8}$$

where $\psi = \left(\frac{p-1}{p}\right)^p \phi \left(-\frac{\bar{h}'}{\bar{h}}\right)^p = \phi\left(-\frac{h'}{h}\right)^p$, that is, h is a weak solution of the Euler-Lagrange equation (1.8) of the functional

$$I_{1,\phi}(u) =: \int_\Omega \left(\phi|\nabla u|^p - \psi|u|^p\right) \mathrm{d}x \tag{1.9}$$

In [10][11][12] it has been proved that if ϕ, ψ are positive functions in $C^1(0,a)$ and satisfy the Bernoulli equation

$$(\phi^{1/p}\psi^{1-1/p})' + \frac{N-1}{r}\phi^{1/p}\psi^{1-1/p} = p\psi \tag{1.10}$$

then for any $u \in C_0^\infty(\Omega)$,

$$\int_\Omega \psi|u|^p \,\mathrm{d}x \le \int_\Omega \phi|\nabla u|^p \,\mathrm{d}x,$$

and the constant 1 is optimal, where $a = +\infty$ and $\Omega = \mathbb{R}^N$ if (A$_1$) holds, or $a = D$ and $\Omega \subset B_D(0)$ if (A$_2$) holds. Because $\bar{h}$ is a fundamental solution of operator $-\Delta_{p,\phi}$, in other words, h is a distribution solution of equation (1.8), we know ψ can be expressed by $\bar{h}$ and ϕ or by h and ϕ as follows

$$\psi = \left(\frac{p-1}{p}\right)^p \phi \left(-\frac{\bar{h}'}{\bar{h}}\right)^p = \phi \left(-\frac{h'}{h}\right)^p$$

Theorem 1.1 ([5], Theorem 1.1)**.** *Let Ω be $\mathbb{R}^N$ if (A$_1$) holds or Ω be a bounded domain included in $B_D(0)$ if (A$_2$) holds. Suppose that ϕ is continuous and set*

$$h = \left(c_0 \int_r^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{(p-1)/p} \tag{1.11}$$

where $a = +\infty$ if (A$_1$) holds or $a = D$ if (A$_2$) holds. Then for any $u \in W_0^{1,p}(\Omega, \phi)$

$$\int_\Omega \phi \left(-\frac{h'}{h}\right)^p |u|^p \,\mathrm{d}x \le \int_\Omega \phi |\nabla u|^p \,\mathrm{d}x$$

where the constant 1 is optimal.

Remark 1.1. (A$_1$) or (A$_2$) implies the integrability of $\phi\left(-\frac{h'}{h}\right)^p$ in $B_\delta(0)$.

Theorem 1.2. *Let $p > 1$ and Ω be a bounded domain in $\mathbb{R}^N$. Suppose ϕ is continuous satisfying (A$_1$) or (A$_2$), h is defined by (1.11). Set*

$$h_1 = \begin{cases} \frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)}, & \text{if } (A_1) \text{ holds} \\ \frac{p}{(p-1)c_0} \ln h(r), & \text{if } (A_2) \text{ holds} \end{cases} \tag{1.12}$$

then

(1) There exists a positive constant $D_0 \le D$ such that for any $u \in W_0^{1,p}(\Omega, \phi)$

$$\int_\Omega \phi |\nabla u|^p \,\mathrm{d}x - \int_\Omega \psi |u|^p \,\mathrm{d}x \ge \frac{p}{2(p-1)c_0^2} \int_\Omega \psi h_1^{-2} |u|^p \,\mathrm{d}x \tag{1.13}$$

where $\psi = \phi\left(-\frac{h'}{h}\right)^p$.

(2) The constants in (1.13) are optimal, that is,

$$\frac{p}{2(p-1)c_0^2} = \inf_{W_0^{1,p}(\Omega,\phi)} \frac{I_\phi(u)}{\int_\Omega \psi h_1^{-2} |u|^p \,\mathrm{d}x}$$

Remark 1.2. Let $\phi = r^{-p\gamma}$ with $\gamma < \frac{N-p}{p}$ and $c_0 = \frac{N-p(\gamma+1)}{p-1}$. It follows from (1.11) and (1.12) that

$$h = r^{-\frac{N-p(\gamma+1)}{p}}, \quad \psi = \left(\frac{N-p(\gamma+1)}{p}\right)^p r^{-p(\gamma+1)}, \quad h_1 = \ln\frac{D}{r}$$

hence we obtain by (1.13)

$$\int_\Omega \left(|x|^{-p\gamma}|\nabla u|^p - \left(\frac{N-p(\gamma+1)}{p-1}\right)^p |x|^{-p(\gamma+1)}|u|^p\right) \mathrm{d}x \geq \frac{p-1}{2p}\left(\frac{N-p(\gamma+1)}{p-1}\right)^{p-2} \int_\Omega |x|^{-p(\gamma+1)}(\ln D/|x|)^{-2}|u|^p \,\mathrm{d}x$$

which is identical with Theorem A in [2] when $\gamma = 0$.

Remark 1.3. Theorem 1.2 improves the results of [13][15].

Theorem 1.3. *Under the hypothesis of Theorem 1.2, we have*

i)

$$\int_\Omega \phi|\nabla u|^p - \psi|u|^p \,\mathrm{d}x \geq C\int_\Omega \phi h_1^{-2}|\nabla u|^p \,\mathrm{d}x, \quad \forall u \in W_0^{1,p}(\Omega,\phi) \qquad (1.14)$$

ii) The inequality (1.14) is optimal in the sense that h_1^{-2} can not be replaced by any weight of the form $g(x)h_1^{-2}$ where $g(x)$ is a positive function such that $g(x) \to \infty$ as $x \to 0$.

Remark 1.4. Taking $\phi = r^{-p\gamma}$ in Theorem 1.3, if $\gamma < \frac{N-p}{p}$, then

$$\int_\Omega \left(|x|^{-p\gamma}|\nabla u|^p - \left(\frac{N-p(\gamma+1)}{p-1}\right)^p |x|^{-p(\gamma+1)}|u|^p\right) \mathrm{d}x \geq C\int_\Omega |x|^{-p\gamma}(\ln D/|x|)^{-2}|\nabla u|^p \,\mathrm{d}x$$

for any $u \in W_0^{1,p}(\Omega,\phi)$. This is a positive answer to the open problem in [1]. If $\gamma = \frac{N-p}{p}$, then

$$\int_\Omega \left(|x|^{-p\gamma}|\nabla u|^p - \left(\frac{p-1}{p}\right)^p |x|^{-p(\gamma+1)}(\ln D'/|x|)^{-p}|u|^p\right) \mathrm{d}x \geq C\int_\Omega |x|^{-p(\gamma+1)}(\ln D'/|x|)^{-p}(\ln\ln D'/|x|)^{-2}|\nabla u|^p \,\mathrm{d}x$$

for any $u \in W_0^{1,p}(\Omega,\phi)$, where $D' > \mathrm{e}D$. This solves the problem for the case of $\gamma = \frac{N-p}{p}$ which has not been discussed before.

Remark 1.5. Wang and Willem [17] proved (1.6) by using a change of variable that appear in [6]. However, to prove Theorem 1.3, we use a change of variables that appear in [14] ($p = 2$), which involves the function $\bar{h}$ or the distribution solution h.

Remark 1.6. Theorem 1.3 gives a positive answer to the second open problem of [3] in the case of general weights.

2 Some Lemmas and Corollaries

Lemma 2.1 ([1]). *For all ζ_1, $\zeta_2 \in \mathbb{R}^N$, the following inequalities hold*

i) if $p \le 2$,

$$|\zeta_2|^p - |\zeta_1|^p - p|\zeta_1|^{p-2}\langle \zeta_1, \zeta_2 - \zeta_1\rangle \ge c(p)\frac{|\zeta_2 - \zeta_1|^2}{(|\zeta_1| + |\zeta_2|)^{2-p}} \tag{2.1}$$

ii) if $p > 2$,

$$|\zeta_2|^p - |\zeta_1|^p - p|\zeta_1|^{p-2}\langle \zeta_1, \zeta_2 - \zeta_1\rangle \ge c(p)|\zeta_2 - \zeta_1|^p \tag{2.2}$$

Direct calculations give the following results:

Lemma 2.2. *Assume h satisfies (1.11). If (A_1) or (A_2) happens, then*

$$\operatorname{div}\left(\phi h^\alpha (-h')^{p-1}\frac{x}{|x|}\right) = (1-\alpha)\phi h^{\alpha-1}(-h')^p \tag{2.3}$$

Lemma 2.3. *Let $h = \left(c_0 \int_r^\infty (\phi r^{N-1})^{-1/(p-1)}\,\mathrm{d}r\right)^{(p-1)/p}$. Then*

i) the function h satisfies the Euler-Lagrange equation

$$-\operatorname{div}(\phi|\nabla h|^{p-2}\nabla h) = \psi h^{p-1}, \quad x \in \mathbb{R}^N \setminus \{0\}$$

and in weak sense,

$$\int_{\mathbb{R}^N} \phi|\nabla h|^{p-2}\nabla h \nabla\zeta\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi h^{p-1}\zeta\,\mathrm{d}x, \quad \zeta \in C_0^\infty(\mathbb{R}^N)$$

where $\psi = \phi\left(-\frac{h'}{h}\right)^p$;

ii) the function

$$\bar{h} = h^{p/(p-1)} = c_0 \int_r^\infty (\phi r^{N-1})^{-1/(p-1)}\,\mathrm{d}r$$

satisfies in the sense of distribution

$$-\operatorname{div}(\phi|\nabla \bar{h}|^{p-2}\nabla \bar{h}) = \left(\frac{p}{p-1}\right)^{p-1} \omega_N \delta(x)$$

where $\delta(x)$ is the Dirac measure and ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. In other words, $\bar{h}$ is a fundamental solution for operator $-\Delta_{\phi,p}$ defined as before.

Corollary 2.4. *Under the hypothesis of Theorem 1.2, if $\alpha > 0$, then for any $u \in W_0^{1,p}(\Omega, \phi)$,*

$$\int_\Omega \psi h_1^{-\alpha}|u|^p\,\mathrm{d}x \le \int_\Omega \phi h_1^{-\alpha}|\nabla u|^p\,\mathrm{d}x$$

Proof. Assume (A_1) holds. Set $\bar{\phi} = \phi h_1^{-\alpha}$, then

$$-\frac{\bar{h}'}{\bar{h}} = \frac{p-1}{p}\frac{(\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)}}{\int_r^D (\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)}\,\mathrm{d}r}$$

By Theorem 1.1, we have

$$\int_\Omega \bar{\psi}|u|^p\,\mathrm{d}x \le \int_\Omega \bar{\phi}|\nabla u|^p\,\mathrm{d}x$$

where $\bar{\psi} = \bar{\phi}(-\frac{\bar{h}'}{\bar{h}})^p$. We claim that

$$\psi h_1^{-\alpha} \le \bar{\psi}$$

that is

$$\phi h_1^{-\alpha}(-\frac{h'}{h})^p \le \phi h_1^{-\alpha}(-\frac{\bar{h}'}{\bar{h}})^p$$

and this complete the proof. In the following we prove this claim. Since h_1 is decreasing, we have

$$\int_r^D (\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)}\,\mathrm{d}r \le h_1^{\alpha/(p-1)} \int_r^D (\phi r^{N-1})^{-1/(p-1)}\,\mathrm{d}r$$

Multiplying by $(\phi r^{N-1})^{-1/(p-1)}$, we obtain

$$\frac{(\phi r^{N-1})^{-1/(p-1)}}{\int_r^D (\phi r^{N-1})^{-1/(p-1)}\,\mathrm{d}r} \le \frac{(\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)}}{\int_r^D (\phi h_1^{-\alpha} r^{N-1})^{-1/(p-1)}\,\mathrm{d}r}$$

Hence

$$-\frac{h'}{h} \leq -\frac{\bar{h}'}{\bar{h}}$$

that is, the claim is true. □

3 Proof of Theorem

Proof of Theorem 1.2 (1). We proceed to make use of a suitable vector field as in [2]. Define a vector field as follows

$$T = \phi \left(-\frac{h'}{h}\right)^{p-1} (1 + c_0^{-1}\eta + a\eta^2)\nabla r$$

where a is a free parameter to be chosen later and $\eta = h_1^{-1}$. By Lemma 2.1, we have

$$\operatorname{div} T \geq \phi \left(-\frac{h'}{h}\right)^{p} \left[(p + pc_0^{-1}\eta + ap\eta^2) + \frac{p\eta^2}{(p-1)c_0^2} + \frac{2ap\eta^3}{(p-1)c_0}\right] \tag{3.1}$$

Next we compute $(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)}$. We set for convenience

$$g(\eta) = (1 + c_0^{-1}\eta + a\eta^2)^{p/(p-1)}$$

When $\eta > 0$ is small, the Taylor expansion of $g(\eta)$ about $\eta = 0$ gives

$$g(\eta) = 1 + \frac{p}{(p-1)c_0}\eta + \frac{1}{2}\left(\frac{p}{(p-1)^2c_0^2} + \frac{2pa}{p-1}\right)\eta^2 + \frac{1}{6}\left(\frac{p(2-p)}{(p-1)^3c_0^3} + \frac{6pa}{(p-1)^2c_0}\right)\eta^3 + O(\eta^4)$$

and so

$$(p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} = \phi\left(-\frac{h'}{h}\right)^p \left[(p-1) + \frac{p}{c_0}\eta + \left(\frac{p}{2(p-1)c_0^2} + pa\right)\eta^2 + \left(\frac{p(2-p)}{(p-1)^2c_0^3} + \frac{pa}{(p-1)c_0}\right)\eta^3 + O(\eta^4)\right]$$

Hence

$$\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \geq \phi\left(-\frac{h'}{h}\right)^p \left[1 + \frac{p\eta^2}{2(p-1)c_0^2} + \left(\frac{pa}{(p-1)c_0} - \frac{p(2-p)}{(p-1)^2c_0^3}\right)\eta^3 + O(\eta^4)\right]$$

If we show

$$\frac{ap}{(p-1)c_0} \geq \frac{p(2-p)}{(p-1)^2 c_0^3} + O(\eta) \tag{3.2}$$

then we obtain

$$\operatorname{div} T - (p-1)\phi^{-1/(p-1)}|T|^{p/(p-1)} \geq \phi\left(-\frac{h'}{h}\right)^p \left[1 + \frac{p\eta^2}{2(p-1)c_0^2}\right] \tag{3.3}$$

If $1 < p < 2$, we assume that η is small for the case (A_1). Since

$$h_1 = \frac{p}{(p-1)c_0} \ln \frac{h(r)}{h(D)}$$

and $\Omega \subset B_{D_0}(0)$ is bounded, we can choose D_0 large enough such that $h_1^{-1}(D_0)$ is small enough. Then $\eta = h_1^{-1}$ is small. Hence, we have (3.2) for a big enough. The same argument gives (3.2) for the case (A_2).

If $p \geq 2$, we choose $a = 0$, then

$$(1 + c_0^{-1}\eta)^{\frac{p}{p-1}} = 1 + \frac{p}{(p-1)c_0}\eta + \frac{p}{2(p-1)^2 c_0^2}\eta^2 + \frac{p(2-p)}{6(p-1)^3 c_0^3}(1 + c_0^{-1}\xi)^{\frac{3-2p}{p-1}}\eta^3$$

for some $\xi \in (0, \eta)$, without any smallness assumption. Since $2 - p \leq 0$, we have

$$(1 + c_0^{-1}\eta)^{\frac{p}{p-1}} \leq 1 + \frac{p}{(p-1)c_0}\eta + \frac{p}{2(p-1)^2 c_0^2}\eta^2$$

Hence we prove (3.3).

Let $u \in C_0^\infty(\Omega)$. For $\epsilon > 0$, it follows from integration by parts that

$$\int_{\Omega \setminus B_\epsilon(0)} |u|^p \operatorname{div} T \,\mathrm{d}x = -p \int_{\Omega \setminus B_\epsilon(0)} (T \cdot \nabla u)|u|^{p-2} u \,\mathrm{d}x - \int_{\partial B_\epsilon(0)} |u|^p T \cdot \nabla r \,\mathrm{d}S$$

Note that

$$\phi\left(-\frac{h'}{h}\right)^{p-1} = r^{-(N-1)}\left(\int_r^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{-(p-1)} = r^{-(N-1)} h^{-p/(p-1)^2}(r)$$

then

$$\left|\int_{\partial B_\epsilon(0)} |u|^p T \cdot \nabla r \,\mathrm{d}S\right| \leq \int_{\partial B_\epsilon(0)} |u|^p \epsilon^{-(N-1)} h^{-p/(p-1)^2}(\epsilon) \,\mathrm{d}S$$

which tends to 0 as $\epsilon \to 0$ since $h^{-1}(0) = 0$. Hence we obtain

$$\int_\Omega |u|^p \operatorname{div} T \,\mathrm{d}x = -p \int_\Omega (T \cdot \nabla u)|u|^{p-2} u \,\mathrm{d}x$$

By Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\int_\Omega |u|^p \operatorname{div} T \,\mathrm{d}x &\le p\left(\int_\Omega \phi|\nabla u|^p \,\mathrm{d}x\right)^{1/p}\left(\int_\Omega |T\phi^{-1/p}|^{p/(p-1)}|u|^p \,\mathrm{d}x\right)^{(p-1)/p}\\ &\le \int_\Omega \phi|\nabla u|^p \,\mathrm{d}x + (p-1)\int_\Omega |T\phi^{-1/p}|^{p/(p-1)}|u|^p \,\mathrm{d}x\end{aligned}$$

that is,

$$\int_\Omega \phi|\nabla u|^p \,\mathrm{d}x \ge \int_\Omega (\operatorname{div} T - (p-1)|T\phi^{-1/p}|^{p/(p-1)})|u|^p \,\mathrm{d}x$$

This complete the proof by (3.3). □

Proof of Theorem 1.2 (2). We complete the proof by four steps.

Step 1. Let $\theta \in C_0^\infty(B_\delta)$ be such that $0 \le \theta \le 1$ in B_δ and $\theta = 1$ in $B_{\delta/2}$, where B_δ denotes the ball of radius δ centered at the origin. We fix small positive parameters α_0, α_1 and define the functions

$$w(x) = h^{1-\frac{\alpha_0}{(p-1)c_0}} h_1^{\frac{1-\alpha_1}{p}}$$

and

$$u(x) = \theta(x)w(x)$$

Let (A_1) or (A_2) happen. Hence $u \in W_0^{1,p}(\Omega, \phi)$. To prove the proposition we shall estimate the corresponding Rayleigh quotient of u in the limit of the order $\alpha_0 \to 0$, $\alpha_1 \to 0$.

It is easily seen that

$$\nabla w = \frac{p}{(p-1)c_0} h^{-\frac{\alpha_0}{(p-1)c_0}} h' Y_1^{\frac{-1+\alpha_1}{p}} \left(\frac{(p-1)c_0}{p} + \frac{\eta}{p}\right)\nabla r$$

where $Y_1 = h_1^{-1}$ and $\eta = -\alpha_0 + (1-\alpha_1)Y_1$.

Now $\nabla u = \theta\nabla w + w\nabla\theta$ and hence, using the elementary inequality

$$|a+b|^p \le |a|^p + c_p(|a|^{p-1}|b| + |b|^p), \quad a, b \in \mathbb{R}^N$$

for $p > 1$, we obtain

$$\begin{aligned}\int_\Omega \phi|\nabla u|^p \,\mathrm{d}x &\le \int_\Omega \phi\theta^p|\nabla w|^p \,\mathrm{d}x \\ &\quad + c_p\int_\Omega \phi\theta^{p-1}|\nabla\theta||w||\nabla w|^{p-1}\,\mathrm{d}x + c_p\int_\Omega \phi|\nabla\theta|^p|w|^p\,\mathrm{d}x \qquad (3.4)\\ &=: I_1 + I_2 + I_3 \qquad (3.5)\end{aligned}$$

We claim that

$$I_2, I_3 = O(1) \quad \text{uniformly as } \alpha_0, \alpha_1 \text{ tend to zero.} \tag{3.6}$$

Let us give the proof for I_2. In fact,

$$\begin{aligned} I_2 &\le C\int_{B_\delta} \phi h^{-\frac{\alpha}{c_0}}|h'|^{p-1}Y_1^{\frac{(-1+\alpha_1)(p-1)}{p}}\big[(p-1)c_0+\alpha_0+(1-\alpha_1)Y_1\big]^{p-1} \\ &\qquad\qquad \cdot h^{1-\frac{\alpha_0}{(p-1)c_0}}Y_1^{\frac{-1+\alpha_1}{p}}\,\mathrm{d}x \\ &\le C\int_{B_\delta}\phi h^{1-\frac{\alpha_0 p}{(p-1)c_0}}|h'|^{p-1}Y_1^{-1+\alpha_1}\big[(p-1)c_0+\alpha_0+(1-\alpha_1)Y_1\big]^{p-1}\,\mathrm{d}x\end{aligned}$$

It follows from the definition of h (1.11) that

$$\phi|h'|^{p-1}h = Cr^{1-N} \tag{3.7}$$

hence

$$I_2 \le C\int_{B_\delta} r^{1-N}h^{-\frac{\alpha_0 p}{(p-1)c_0}}Y_1^{-1+\alpha_1}\big[(p-1)c_0+\alpha_0+(1-\alpha_1)Y_1\big]^{p-1}\,\mathrm{d}x$$

Then the boundedness of h^{-1} together with the fact $Y_1(0) = 0$ implies that I_2 is uniformly bounded. The integral I_3 is treated similarly.

Step 2. Define

$$\begin{aligned} A_0 &= \int_\Omega \theta^p\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^pY_1^{-1+\alpha_1}\,\mathrm{d}x\\ A_1 &= \int_\Omega \theta^p\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^pY_1^{1+\alpha_1}\,\mathrm{d}x\\ \Gamma_{01} &= \int_\Omega \theta^p\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^pY_1^{\alpha_1}\,\mathrm{d}x\end{aligned}$$

By Lemma 2.1, we have

$$\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p = \frac{(p-1)c_0}{p\alpha_0}\,\mathrm{div}(\phi h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^{p-1}\nabla r)$$

Multiplying the above equality by $\theta^p Y_1^{-1+\alpha_1}$ and integrating over Ω, we obtain

$$\begin{aligned}
A_0 &= \frac{(p-1)c_0}{p\alpha_0}\int_\Omega \theta^p Y_1^{-1+\alpha_1}\,\mathrm{div}(\phi h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^{p-1}\nabla r)\,\mathrm{d}x \\
&= \frac{(p-1)c_0}{p\alpha_0}\int_\Omega \phi h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^{p-1}\nabla(\theta^p Y_1^{-1+\alpha_1})\,\mathrm{d}x \\
&= \frac{(p-1)c_0}{p\alpha_0}\Big(-\frac{p(1-\alpha_1)}{(p-1)c_0}\int_\Omega \theta^p \phi h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{\alpha_1}\,\mathrm{d}x \\
&+ \int_\Omega (\theta^p)'\phi h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^{p-1}Y_1^{-1+\alpha}\,\mathrm{d}x\Big) \\
&= (1-\alpha_1)\Gamma_{01} + O(1)
\end{aligned}$$

Step 3. We proceed to estimate I_1.

$$\begin{aligned}
I_1 &= \int_\Omega \phi\theta^p|\nabla w|^p\,\mathrm{d}x \\
&\le \Big(\frac{p}{(p-1)c_0}\Big)^p\int_\Omega \theta^p\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{-1+\alpha_1}\Big(\frac{(p-1)c_0}{p}+\frac{\eta}{p}\Big)^p\,\mathrm{d}x
\end{aligned}$$

where $\eta = -\alpha_0 + (1-\alpha_1)Y_1$. Since η is small compared to $(p-1)c_0/p$, we may use Taylor's expansion to obtain

$$\begin{aligned}
\Big(\frac{(p-1)c_0}{p}+\frac{\eta}{p}\Big)^p \le \Big(\frac{(p-1)c_0}{p}\Big)^p + \Big(\frac{(p-1)c_0}{p}\Big)^{p-1}\eta \\
+ \frac{p-1}{2p}\Big(\frac{(p-1)c_0}{p}\Big)^{p-2}\eta^2 + C\eta^3
\end{aligned}$$

Using this inequality we can obtain

$$I_1 \le I_{10} + I_{11} + I_{12} + I_{13} \tag{3.8}$$

where

$$\begin{aligned}
I_{10} &= \int_\Omega \theta^p\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{-1+\alpha_1}\,\mathrm{d}x = \int_\Omega \theta^p\psi h^{p-\frac{\alpha_0 p}{(p-1)c_0}}Y_1^{-1+\alpha_1}\,\mathrm{d}x \\
&= \int_\Omega \theta^p\psi|w|^p\,\mathrm{d}x = \int_\Omega \psi|u|^p\,\mathrm{d}x
\end{aligned} \tag{3.9}$$

$$I_{12} = \frac{p}{2(p-1)c_0^2}\int_\Omega \theta^p\phi h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{-1+\alpha_1}\eta^2\,\mathrm{d}x \tag{3.10}$$

We shall prove that

$$I_{11}, I_{13} = O(1) \quad \text{uniformly in } \alpha_0, \alpha_1. \tag{3.11}$$

Firstly,

$$\begin{aligned} I_{11} &= \frac{p}{(p-1)c_0}\Big[-\alpha_0 \int_\Omega \theta^p \phi(-h')^p h^{-\frac{\alpha_0 p}{(p-1)c_0}} Y_1^{-1+\alpha_1}\,\mathrm{d}x \\ &\quad + (1-\alpha_1)\int_\Omega \theta^p \phi(-h')^p h^{-\frac{\alpha_0 p}{(p-1)c_0}} Y_1^{\alpha_1}\,\mathrm{d}x\Big] + O(1) \\ &= \frac{p}{(p-1)c_0}(-\alpha_0 A_0 + (1-\alpha_1)\Gamma_{01}) + O(1) \end{aligned}$$

Next we estimate I_{13}.

$$\begin{aligned} I_{13} &\le \alpha_0^3 \int_\Omega \theta^p \phi(-h')^p h^{-\frac{\alpha_0 p}{(p-1)c_0}} Y_1^{-1+\alpha_1}\,\mathrm{d}x + C\int_\Omega \theta^p \phi(-h')^p h^{-\frac{\alpha_0 p}{(p-1)c_0}} Y_1^{2+\alpha_1}\,\mathrm{d}x \\ &=: I'_{13} + I''_{13} \end{aligned}$$

Since

$$Y_1^{-1} = \frac{p}{(p-1)c_0}\ln\frac{h(r)}{h(D)}$$

we have

$$\begin{aligned} I'_{13} &\le C\alpha_0^3 \int_0^\delta \left(\int_r^\infty (\phi r^{k-1})^{-1/(p-1)}\,\mathrm{d}r\right)^{-1-\alpha_0/c_0}\left[\ln\frac{h(r)}{h(D)}\right]^2 \\ &\qquad \mathrm{d}\left(\int_r^\infty (\phi r^{k-1})^{-1/(p-1)}\,\mathrm{d}r\right) \\ &\le C\alpha_0^2 c_0 \int_0^\delta \left[\ln\frac{h(r)}{h(D)}\right]^2 \mathrm{d}\left(\int_r^\infty (\phi r^{k-1})^{-1/(p-1)}\,\mathrm{d}r\right)^{-\alpha_0/c_0} \end{aligned}$$

Denote

$$s = \left(\int_d^\infty (\phi r^{k-1})^{-1/(p-1)}\,\mathrm{d}r\right)^{-\alpha_0/c_0}$$

then we have

$$I'_{13} \le C\alpha_0^2 \int_0^\delta \left[C - \frac{(p-1)c_0}{p\alpha_0}\ln s\right]^2 \mathrm{d}s \le O(1)$$

The same argument gives $I''_{13} = O(1)$ uniformly in α_0 and α_1. Hence, by (3.4), (3.6), (3.8), (3.9) and (3.11), we conclude that

$$\int_\Omega \phi|\nabla u|^p\,\mathrm{d}x - \int_\Omega \psi|u|^p\,\mathrm{d}x \le I_{12} + O(1) \tag{3.12}$$

uniformly in α_0 and α_1.

Step 4. We proceed to estimate I_{12} and complete the proof.

$$\begin{aligned} I_{12} &= \frac{p}{2(p-1)c_0^2} \int_\Omega \theta^p \phi h^{-\frac{\alpha_0 p}{(p-1)c_0}} (-h')^p Y_1^{-1+\alpha_1} \left(\alpha_0^2 + (1-\alpha_1)^2 Y_1^2 - 2\alpha_0(1-\alpha_1)Y_1\right) \mathrm{d}x \\ &= \frac{p}{2(p-1)c_0^2} \left(\alpha_0^2 A_0 - 2\alpha_0(1-\alpha_1)\Gamma_{01} + (1-\alpha_1)^2 A_1\right) \\ &= \frac{p}{2(p-1)c_0^2} A_1 + O(1) \end{aligned} \tag{3.13}$$

if α_0 and α_1 tend to 0. Because

$$\phi(-h')^p h^{-\frac{\alpha_0 p}{(p-1)c_0}} = \left(c_0 \int_r^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{-\alpha_0/c_0 - 1} \cdot c_0 \phi (\phi r^{N-1})^{-1/(p-1)}$$

we have

$$\begin{aligned} A_1 &\geq C \int_0^{\delta/2} \left(\int_r^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{-\alpha_0/c_0 - 1} \cdot c_0 (\phi r^{N-1})^{-1/(p-1)} h_1^{-1-\alpha_0} \,\mathrm{d}r \\ &\geq C \left. \frac{\left(\int_r^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{-\alpha_0/c_0}}{-\alpha_0/c_0} \right|_0^{\delta/2} \\ &= C \cdot \frac{c_0}{\alpha_0} \left[\left(\int_0^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{-\alpha_0/c_0} - \left(\int_{\delta/2}^a (\phi r^{N-1})^{-1/(p-1)} \,\mathrm{d}r\right)^{-\alpha_0/c_0} \right] \to \infty \end{aligned}$$

as α_0 tends to 0. Since

$$\begin{aligned} \int_\Omega \psi h_1^{-2} |u|^p \,\mathrm{d}x &= \int_\Omega \phi \left(-\frac{h'}{h}\right)^p h_1^{-2} \theta^p h^{p - \frac{\alpha_0 p}{(p-1)c_0}} h_1^{1-\alpha_1} \,\mathrm{d}x \\ &= \int_\Omega \theta^p \phi (-h')^p h^{-\frac{\alpha_0 p}{(p-1)c_0}} h_1^{-1-\alpha_1} \,\mathrm{d}x = A_1 \end{aligned}$$

by (3.12) and (3.13), we have

$$\frac{\int_\Omega (\phi |\nabla u|^p - \psi |u|^p) \,\mathrm{d}x}{\int_\Omega \psi h_1^{-2} |u|^p \,\mathrm{d}x} \leq \frac{\frac{p}{2(p-1)c_0^2} A_1 + O(1)}{A_1} \to \frac{p}{2(p-1)c_0^2}$$

as α_0 tends to 0. This completes the proof. □

The proof of Theorem 1.3. i) Assume (A_1) holds. Consider the case of $p \geq 2$.

Let $u \in C_0^\infty(\Omega)$ and set $v = u(x)/h(r)$. Then by Lemma 2.1 (2.2), we have

$$\begin{aligned}\int_\Omega \phi|\nabla u|^p \,\mathrm{d}x &= \int_\Omega \phi \left| vh' \frac{x}{|x|} + h\nabla v \right|^p \mathrm{d}x \\ &\geq \int_\Omega \phi |v|^p |h'|^p \,\mathrm{d}x - p \int_\Omega \phi |vh'|^{p-2} \langle vh' \frac{x}{|x|}, h\nabla v\rangle \,\mathrm{d}x \\ &+ c(p) \int_\Omega \phi h^p |\nabla v|^p \,\mathrm{d}x\end{aligned}$$

Note that

$$\int_\Omega \phi |v|^p |h'|^p \,\mathrm{d}x = \int_\Omega \psi |u|^p \,\mathrm{d}x$$

and for any $\epsilon > 0$, by Lemma 2.2, (3.7) and (A_1) (or (A_2)), we have

$$\begin{aligned}&- \int_{\Omega\setminus B_\epsilon(0)} \phi |vh'|^{p-2} \,\mathrm{d}x \\ &= \int_{\Omega\setminus B_\epsilon(0)} \phi h (-h')^{p-1} \langle \frac{x}{|x|}, h\nabla |v|^p \rangle \,\mathrm{d}x \\ &= \int_{\partial B_\epsilon(0)} \phi (-h')^{p-1} h |v|^p \,\mathrm{d}S - \int_{\Omega\setminus B_\epsilon(0)} |v|^p \operatorname{div}(\phi h (-h')^{p-2} \nabla h) \\ &= \int_{\partial B_\epsilon(0)} \phi (-h')^{p-1} h |v|^p \,\mathrm{d}S \to 0\end{aligned}$$

as $\epsilon \to 0$. Hence, we obtain

$$I_{1,\phi}(u) = \int_\Omega (\phi |\nabla u|^p - \psi |u|^p) \,\mathrm{d}x \geq c(p) \int_\Omega \phi h^p |\nabla v|^p \,\mathrm{d}x \tag{3.14}$$

Taking $C_1 > 0$ such that $C_1 h_1^{-2} \leq c(p)$, it follows from (2.2) of Lemma 2.1 that

$$\begin{aligned}c(p) \int_\Omega \phi h^p |\nabla v|^p \,\mathrm{d}x &\geq C_1 \int_\Omega \phi h^p h_1^{-2} |\nabla v|^p \,\mathrm{d}x \\ &\geq C_1 \int_\Omega \phi h_1^{-2} \left[\left| \frac{\nabla h_1}{h_1} \right| |u|^p - p \left| \frac{\nabla h}{h} u |^{p-1} \right| \nabla u| + c(p) |\nabla u|^p \right] \mathrm{d}x \\ &\geq C_1 \int_\Omega \phi h_1^{-2} \left[(c(p) - \epsilon) |\nabla u|^p - \left((p-1)\epsilon^{-1/(p-1)} - 1 \right) \left(-\frac{h'}{h} \right)^p |u|^p \right] \mathrm{d}x\end{aligned}$$

Taking $\epsilon = c(p)/2$, then by Theorem 1.2, we obtain

$$I_{1,\phi}(u) \geq C \int_\Omega \phi h_1^{-2} |\nabla u|^p \,\mathrm{d}x$$

Now let $1 < p < 2$. By using Lemma 2.1, (2.1) and arguments analogues to the case of $p \geq 2$, we have

$$\int_\Omega (\phi|\nabla u|^p - \psi|u|^p)\,\mathrm{d}x \geq c(p)\int_\Omega \frac{\phi|\nabla u - \frac{\nabla h}{h}u|^p}{(|\nabla u| + |u||\frac{h'}{h}|)^{2-p}}\,\mathrm{d}x$$
$$\geq c(p)\int_\Omega \frac{\phi h_1^{-2}|\nabla u - \frac{\nabla h}{h}u|^2}{(|\nabla u| + |u||\frac{h'}{h}|)^{2-p}}\,\mathrm{d}x$$

By Hölder's inequality and Corollary 2.4, we have

$$\int_\Omega \phi h_1^{-2}|\nabla u - \frac{\nabla h}{h}u|^p\,\mathrm{d}x \leq \left(\int_\Omega \frac{\phi h_1^{-2}|\nabla u - \frac{\nabla h}{h}u|^2}{(|\nabla u| + |u||\frac{h'}{h}|)^{2-p}}\,\mathrm{d}x\right)^{p/2}$$
$$\left(\int_\Omega \phi h_1^{-2}(|\nabla u| + |\frac{h'}{h}||u|)^p\,\mathrm{d}x\right)^{1-p/2}$$
$$\leq C(I_{1,\phi}(u))^{p/2}\left(\int_\Omega \phi h_1^{-2}|\nabla u|^p\,\mathrm{d}x\right)^{1-p/2}$$

Note that

$$\int_\Omega \phi h_1^{-2}|\nabla u|^p\,\mathrm{d}x \leq C\left(\int_\Omega \phi h_1^{-2}|\nabla u - \frac{\nabla h}{h}u|^p\,\mathrm{d}x + \int_\Omega \phi h_1^{-2}\left|\frac{\nabla h}{h}\right||u|^p\,\mathrm{d}x\right)$$
$$\leq C(I_{1,\phi}(u))^{p/2}\left(\int_\Omega \phi h_1^{-2}|\nabla u|^p\,\mathrm{d}x\right)^{1-p/2} + \int_\Omega \psi h_1^{-2}|u|^p\,\mathrm{d}x$$
$$\leq C(I_{1,\phi}(u))^{p/2}\left(\int_\Omega \phi h_1^{-2}|\nabla u|^p\,\mathrm{d}x\right)^{1-p/2} + I_{1,\phi}(u)$$

By Young's inequality, we obtain

$$\int_\Omega \phi h_1^{-2}|\nabla u|^p\,\mathrm{d}x \leq CI_{1,\phi}(u)$$

One can prove the result for the case of (A_2) by the analogues argument.

ii) Let w and u be as those defined in the proof of Theorem 1.2 (2), and let $p > 2$. First, it follows from (3.12) and (3.13) that

$$\int_\Omega \phi|\nabla u|^p\,\mathrm{d}x - \int_\Omega \psi|u|^p\,\mathrm{d}x \leq \frac{p}{2(p-1)c_0^2}A_1 + O(1)$$

By (2.2) we have

$$|\nabla u|^p = |\theta\nabla w + w\nabla\theta|^p \geq \theta^p|\nabla w|^p - p\theta^{p-1}|\nabla w|^{p-1}|\nabla\theta|w + c(p)|w|^p|\nabla\theta|^p$$

Hence

$$\int_\Omega \phi h_1^{-2} g(x)|\nabla u|^p \,\mathrm{d}x \geq \min_{x\in B_\delta(0)} g(x) \int_{B_\delta(0)} \phi h_1^{-2}\theta^p |\nabla w|^p \,\mathrm{d}x +$$
$$c(p)\int_{B_\delta(0)} \phi h_1^{-2} g(x)|w|^p|\nabla\theta|^p \,\mathrm{d}x - p\int_{B_\delta(0)} \phi h_1^{-2} g(x)\theta^{p-1}|\nabla\theta||\nabla w|^{p-1}|w| \,\mathrm{d}x$$

Analogues to the argument of Step 3 for the proof of Theorem 1.2 (2), we obtain

$$\int_{B_\delta(0)} \phi h_1^{-2}\theta^p|\nabla w|^p \,\mathrm{d}x = \int_{B_\delta(0)} \phi h_1^{-2}\theta^p h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{-1+\alpha_1}\left(\frac{(p-1)c_0}{p}+\frac{\eta}{p}\right)^p \mathrm{d}x$$

where $\eta = -\alpha_0 + (1-\alpha_1)Y_1$. Because of $p > 2$, we have

$$\begin{aligned}\left(\frac{(p-1)c_0}{p}+\frac{\eta}{p}\right)^p &= \left(\frac{(p-1)c_0-\alpha_0}{p}+\frac{\eta+\alpha_0}{p}\right)^p \\ &\geq \left(\frac{(p-1)c_0-\alpha_0}{p}\right)^p + \left(\frac{(p-1)c_0-\alpha_0}{p}\right)^{p-1}(\eta+\alpha_0)\end{aligned}$$

Hence

$$\begin{aligned}\int_{B_\delta(0)} \phi h_1^{-2}\theta^p|\nabla w|^p \,\mathrm{d}x &\geq \left(\frac{(p-1)c_0-\alpha_0}{p}\right)^p \int_{B_\delta(0)} \phi\theta^p h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{1+\alpha_1}\,\mathrm{d}x \\ &+ \left(\frac{(p-1)c_0-\alpha_0}{p}\right)^{p-1}(1-\alpha_1)\int_{B_\delta(0)} \phi\theta^p h^{-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^p Y_1^{2+\alpha_1}\,\mathrm{d}x \\ &=: J_1 + J_2\end{aligned}$$

By Step 3 of the proof of Theorem 1.2 (2) we know

$$J_1 = \left(\frac{(p-1)c_0-\alpha_0}{p}\right)^p A_1, \quad J_2 = O(1)$$

if α_0, α_1 tend to 0. Next, we will estimate

$$J_3 := \int_{B_\delta(0)} \phi h_1^{-2} g(x)\theta^{p-1}|\nabla\theta||\nabla w|^{p-1}|w| \,\mathrm{d}x$$

In fact,

$$J_3 \leq C\int_{B_\delta(0)} g(x)\phi h_1^{-2} h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^{p-1} Y_1^{-1+\alpha_1}[(p-1)c_0-\alpha_0+(1-\alpha_1)Y_1]^{p-1}\,\mathrm{d}x$$
$$\leq \int_{B_\delta(0)} g(x)\phi h_1^{-2} h^{1-\frac{\alpha_0 p}{(p-1)c_0}}(-h')^{p-1}\left[\big((p-1)c_0-\alpha_0\big)^{p-1} Y_1^{1+\alpha_1} + (1-\alpha_1)^{p-1} Y_1^{p+\alpha_1}\right]\mathrm{d}x$$

It follows from (3.7) that

$$J_3 \le \int_0^\delta \widetilde{g}(r) h^{-\frac{\alpha_0 p}{(p-1)c_0}} \left[\big((p-1)c_0 - \alpha_0\big)^{p-1} Y_1^{1+\alpha_1} + (1-\alpha_1)^{p-1} Y_1^{p+\alpha_1} \right] \mathrm{d}r$$

Set

$$\widetilde{g}(r) = \frac{1}{N\omega_N} \int_{|\omega|=1} g(r\omega)\,\mathrm{d}\omega$$

and we may assume

$$|\widetilde{g}(r) h_1^{-1}(r)| \le C$$

Then we obtain

$$J_3 \le C \int_0^\delta h^{-\frac{\alpha_0 p}{(p-1)c_0}} \left[\big((p-1)c_0 - \alpha_0\big)^{p-1} Y_1^{\alpha_1} + (1-\alpha_1)^{p-1} Y_1^{p-1+\alpha_1} \right] \mathrm{d}r \le C$$

Hence

$$\frac{\int_\Omega (\phi|\nabla u|^p - \psi|u|^p)\,\mathrm{d}x}{\int_\Omega \phi h_1^{-2} g(x) |\nabla u|^p\,\mathrm{d}x} \le \frac{\frac{p}{2(p-1)c_0^2} A_1 + O(1)}{\min\limits_{x\in B_\delta(0)} g(x) \left(\frac{(p-1)c_0-\alpha_0}{p}\right)^p A_1 + O(1)} \to 0$$

as $\delta \to 0$ since $A_1 \to \infty$ as α_0, $\alpha_1 \to 0$ and $g(x) \to \infty$ as $x \to 0$.

We can prove our result for the case of $1 < p < 2$ by the similar argument. □

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