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Embedding Compact Strongly Pseudoconvex CR Manifolds of Class $C^{3,\alpha}$

Mei-Chi Shaw

Dedicated to Professor Joseph J. Kohn

Abstract: In this paper we derive maximal pointwise Hölder estimates for the Kohn's Laplacian on strongly pseudoconvex CR manifolds of class C^3 using the Tanaka-Webster Pseudohermitian metric. The estimates can be used to improve the Boutet De Monvel's embedding theorem for strongly pseudoconvex compact CR manifolds of real dimension greater or equal to five with less smoothness assumption.

Keywords: Pseudohermitian CR manifolds, Kohn's Laplacian, Embedding abstract CR structures.

Introduction

A CR manifold, as first formulated in Kohn-Rossi [KR], is a smooth $2n - 1$ -dimensional real manifold equipped with an integrable CR structure where $n \geq 2$. Any boundary of a smooth bounded domain in \mathbb{C}^n is a CR manifold. It is natural to ask if every CR manifold can be embedded as a submanifold in \mathbb{C}^n . If M is a smooth compact CR manifold, we have the following well-known theorem.

Theorem (Boutet De Monvel [Bou]). *Let M be a compact C^∞ smooth strongly pseudoconvex CR manifold of real dimension $2n - 1$, $n \geq 3$. Then M is embeddable into \mathbb{C}^N by smooth CR functions for some large N .*

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The proof uses the Hölder regularity of the Szegő projection obtained in [BS]. In [Ko3], Kohn gives another proof which uses the Sobolev regularity for the Kohn's Laplacian \square_b [Ko2] which gives by the Sobolev embedding theorem the Hölder regularity for \square_b as well as the Szegő projection. When $n = 2$, Rossi [Ros] gives an example of a real analytic compact strongly pseudoconvex CR manifold of real dimension 3 which is not globally embeddable.

In this paper we study the embeddability of strongly pseudoconvex CR manifolds of class $C^{3,\alpha}$ using Hölder estimates for \square_b . Hölder and L^p estimates for \square_b and the Szegő projection for any strongly pseudoconvex CR structures of class C^∞ are well-known (see [FS, RS, BS, BG, KS]). In recent papers by Shaw-Wang [SW1,SW2], a much more direct method is used to derive estimates for \square_b and the Szegő projection under condition Y(q).

When the CR manifold is strongly pseudoconvex, one can use the pseudohermitian metric introduced in Tanaka [Tan] (see also Webster [We1]) to have a very elegant formula for the Kohn's Laplacian. Similar formula were first obtained for the Heisenberg group case by Folland-Stein [FS]. Here we derive the precise formula for \square_b using the C^3 smoothness assumption. This explicit formula for the Kohn's Laplacian can be viewed as an analogue of the simple expression for the $\bar{\partial}$ -Laplacian \square on Kähler manifold (see Wu [Wu]).

Minimal smoothness assumption is important since many applications require it. One motivation to study this problem is to construct bounded holomorphic functions on Kähler manifolds with nonpositive curvature (see [Bla]). Our results immediately reduced the smoothness assumption used in [Bla] from C^∞ to $C^{3,\alpha}$ (see Theorem 4.2). To completely solve this problem without assuming that the boundary has a $C^{3,\alpha}$ CR structure, one actually needs to study the CR manifold of only Hölder continuous topology (see Schoen-Yau [SY]). Estimates for \square_b with only Hölder smoothness assumption are also important for studying nonlinear subelliptic operators. The embeddability result is also important in the deformations of CR structures (see [Tan]). Our results are not the most general, but the beginning of more investigation on these problems.

In the case for the local embedding of almost complex manifolds, the well-known Newlander-Nirenberg theorem [NN] show that any almost complex manifold of class C^{2n} is actually embeddable. The smoothness assumption for the

almost complex structures to be embeddable is reduced to $C^{1,\alpha}$, $0 < \alpha < 1$ by Nijenhuis and Woolf [NW] (see also [We2]). An L^2 linear approach was given by Kohn [Ko1] for smooth almost complex structure using the $\bar{\partial}$ -Neumann problem where the required smoothness also depends on n since the Sobolev embedding theorem is used. Using the pointwise Hölder estimates applied to the elliptic systems immediately relaxes the assumption of the linear method to $C^{1,\alpha}$ class with the Hölder condition required only at one point (see Remarks at the end of the paper). The method used in this paper also has the potential of relaxing the smoothness requirements of local embedding problem of strongly pseudoconvex CR structure, but it is a much more difficult problem and will not be discussed here.

In Section 1, we discuss CR manifolds of class C^k , $k \geq 2$ and the pseudo-hermitian metric on strongly pseudoconvex CR manifolds. In Section 2 we derive the Folland-Stein-Tanaka formula for CR manifold with C^2 pseudo-hermitian Levi metric. This gives a very simple proof of Kohn's L^2 theory for \square_b on strongly pseudoconvex CR manifolds. Based on the L^2 theory, optimal pointwise Hölder estimates for \square_b can be obtained based on scaling arguments and the use of Campanato spaces, as was done in [SW1] and [SW2]. The embedding theorem of compact strongly pseudoconvex CR structures of class $C^{3,\alpha}$ is obtained in section 4 using arguments in [Bou].

1. Strongly pseudoconvex CR structures of class C^2 and the pseudo-hermitian metric

A manifold is of class C^k , $k \in \mathbb{N}$, if any two coordinate systems are related by a transformation of class C^k . If M is a manifold of class C^{k+1} , then the tangent bundle of M , $T(M)$, is a C^k manifold. A vector field X is of class C^k if its coefficients in any coordinates are in C^k . This is well-defined since changing coordinates will result in multiplication of coefficients with C^k functions. Similarly, we can define differential forms or other tensor fields of class C^k .

Recall that a vector bundle is of class C^k if the transition functions are of class C^k . Locally it is spanned by C^k sections.

Definition 1.1. *Let M be a real manifold of class C^{k+1} of dimension $2n - 1$, where $k \in \mathbb{N}$ and $n \geq 2$. The pair $(M, T^{1,0})$ is called a CR manifold of class C^k*

if M is equipped with a subbundle $T^{1,0}$ of class C^k of the complexified tangent bundle $\mathbb{C}T(M)$ such that the following holds:

- (1) $\dim_{\mathbb{C}} T^{1,0} = n$,
- (2) $T^{1,0} \cap T^{0,1} = \{0\}$, where $T^{0,1} = \overline{T^{1,0}}$,
- (3) $T^{1,0}$ is integrable in the sense that if L_1, L_2 are the C^k sections of $T^{1,0}$, their Lie bracket $[L_1, L_2]$ is a C^{k-1} section of $T^{1,0}$.

We can also use the operator J to define CR structure. Let H be a subbundle of real dimension $2n$ of the real tangent bundle TM and $J : H \rightarrow H, J^2 = -I$. The tensor J is called integrable if for any X and Y in H , so is $[JX, Y] + [X, JY]$ and $J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y]$. The triple (M, H, J) is called a CR manifold of class C^k if J is integrable and of class C^k . One can check easily that these two definitions agree (see e.g. Jacobowitz [Jac]). For simplicity, we only use M to denote the CR manifold $(M, T^{1,0})$ or (M, H, J) . A CR manifold M is called an embedded CR manifold in \mathbb{C}^N if there exists an embedding f from M into \mathbb{C}^N such that the push forward of the CR structure of M is the induced CR structure of $f(M) \subset \mathbb{C}^N$.

On any given CR manifold M of class $C^k, k \geq 1$, we set $\Lambda^{0,q}(M) = (\wedge^q T^{0,1*})$. Sections in $\Lambda^{0,q}$ are called the $(0, q)$ -forms on M . We use $C^m_{(0,q)}(M)$ to denote the space of $(0, q)$ -forms with C^m coefficients, where $0 \leq m \leq k$. Define $\bar{\partial}_b : C^k_{(0,q)}(M) \rightarrow C^{k-1}_{(0,q+1)}(M)$ by the standard derivation formula.

Let $k \geq 1, \phi \in C^m$, where $1 \leq m \leq k + 1$, then $\bar{\partial}_b \phi$ is defined by

$$\langle \bar{\partial}_b \phi, \bar{L} \rangle = \bar{L}(\phi)$$

for all C^k sections \bar{L} of $T^{0,1}(M)$. Then $\bar{\partial}_b$ is extended to $C^k_{(0,q)}(M)$ for $q > 0$ as a derivation. If we let $\theta_{0,q}$ be the projection from $\Lambda^q \mathbb{C}T^*(M)$ onto $\Lambda^{0,q}(M)$, then $\bar{\partial}_b = \theta_{0,q+1} \circ d$, where d is the exterior derivative on M . It is standard to see that $\bar{\partial}_b$ satisfies $\bar{\partial}_b^2 = 0$.

Let $N(M)$ denote the 1-dimensional bundle such that

$$N(M) = \mathbb{C}T(M)/(T^{1,0}(M) + T^{0,1}(M)).$$

We denote the dual bundle of $N(M)$ by $N^*(M)$. Let $\tau \in N^*(M)$, then τ annihilates $T^{1,0} \oplus T^{0,1}$. Thus $N^*(M)$ is called the characteristic bundle.

For a fixed real $\tau \in N^*(M)$, the Levi form Θ_p is defined as the quadratic form

$$\langle \Theta(L, L'), \tau \rangle_p = \sqrt{-1} \langle [L, \bar{L}'], \tau \rangle_p, \quad L, L' \in T^{1,0}(M).$$

Definition 1.2. A CR manifold M of class C^1 is strongly pseudoconvex if the Levi form is positive definite or negative definite.

Following Tanaka, we have (see Proposition 2.1 in [Tan])

Proposition 1.3. Let M be a strongly pseudoconvex CR manifold of class C^k , $k \geq 2$. Then there exists a basic form θ of class C^k such that the hermitian form $-d\theta$ is positive definite on $T^{1,0} + T^{0,1}$ at each $x \in M$.

On a strongly pseudoconvex CR manifold of class C^k , $k \geq 2$, we fix the basic form θ of class C^k , $k \geq 2$ and its corresponding basic field ξ (for definition, see Tanaka [Tan]). Then ξ will be of class C^{k-1} from definition. Set

$$(1.1) \quad -d\theta = \omega.$$

Then ω is a 2-form of class C^{k-1} .

There exists a pseudo-hermitian metric g of class C^{k-1} corresponding to ω . Notice that g is not a Riemannian metric since $\xi \lrcorner \omega = 0$. Let the volume element on M be given by

$$(1.2) \quad \theta \wedge (d\theta)^{n-1}.$$

The pseudo-hermitian metric together with the volume element induces a metric of class C^{k-1} on M . This metric will be called the (Tanaka-Webster) pseudo-hermitian Levi metric. We can define the formal adjoint ϑ_b of $\bar{\partial}_b$ such that $\vartheta_b : C_{(0,q+1)}^{k-1}(M) \rightarrow C_{(0,q)}^{k-2}(M)$, $k \geq 2$, by requiring

$$(\bar{\partial}_b \phi, \psi) = (\phi, \vartheta_b \psi), \quad \phi \in C_{(0,q)}^k(M), \quad \psi \in C_{(0,q+1)}^k(M).$$

Note that ϑ_b is a first-order differential operator with C^{k-2} coefficients written in local coordinates.

We define the $\bar{\partial}_b$ -Laplacian \square_b (or Kohn's Laplacian) by

$$\square_b = \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b : C_{(0,q)}^{k-1}(M) \rightarrow C_{(0,q)}^{k-3}(M).$$

When $k \geq 3$, the operator \square_b is a system of second-order differential operators with C^{k-3} coefficients in local coordinates.

2. Folland-Stein-Tanaka formula for \square_b on a pseudo-hermitian manifold

Let M be a strongly pseudoconvex CR manifold of class C^3 . We equip M with the pseudohermitian Levi metric of class C^2 . In this section, we will derive the explicit formula for the $\bar{\partial}_b$ -Laplacian

$$(2.1) \quad \square_b = \bar{\partial}_b \vartheta_b + \vartheta_b \bar{\partial}_b.$$

These formula are particular simple under the pseudo-hermitian metric and they agree with the Folland-Stein calculation [FS] for \square_b on the Heisenberg group. Our derivation follows the arguments in Tanaka (see Proposition 5.1 in [Tan]). We include it here only to show that it is valid for CR pseudo-hermitian manifolds of class C^2 . Let ∇ denote the unique canonical connection associated with the pseudo-hermitian metric (For the existence and properties of ∇ , see Chapter 3 in [Tan]).

Proposition 2.1. *Let M be a strongly pseudoconvex CR manifold of class C^2 with the pseudohermitian Levi metric of class C^1 . Let e_1, \dots, e_{n-1} be an orthonormal frame field for $T^{1,0}(M)$ in an open neighborhood in M and let w_1, \dots, w_{n-1} be its dual. Then*

$$(2.2) \quad \bar{\partial}_b = \sum_{j=1}^{n-1} \bar{w}_j \wedge \nabla_{\bar{e}_j}.$$

The equation (2.2) follows from the definition. In fact, this formula holds for any CR manifold without any assumption on the metric. Next we compute the adjoint ϑ_b of $\bar{\partial}_b$ under the pseudoconvex hermitian metric.

Proposition 2.2. *Let M be a strongly pseudoconvex CR manifold of class C^2 with the pseudohermitian Levi metric of class C^1 . Let e_1, \dots, e_{n-1} be an orthonormal frame field for $T^{1,0}(M)$ in an open neighborhood in M and let w_1, \dots, w_{n-1} be its dual. Then*

$$(2.3) \quad \vartheta_b = - \sum_{j=1}^{n-1} \lrcorner(\bar{e}_j) \nabla_{e_j}.$$

Proof. Let ϕ be a $(0, q)$ -form and ψ be a $(0, q + 1)$ -form on M . We have

$$(2.4) \quad \langle \bar{\partial}_b \phi, \psi \rangle = \langle \phi, \vartheta_b \psi \rangle + \delta' \beta$$

where β is a $(0, 1)$ -form defined by $\beta = \beta_j \bar{w}_j$ with $\beta(\bar{e}_j) = \langle \phi, e_j \lrcorner \bar{\psi} \rangle$ and $\delta' \beta$ is the function defined to be

$$\delta' \beta = \sum_j^{n-1} \nabla_{\bar{e}_j} \beta_j.$$

Let $\star \beta$ be a $(2n - 2)$ -form defined by

$$\star \beta = \sum_j^{n-1} \beta_j \bar{e}_j \lrcorner dV.$$

We first prove the following claim:

$$(2.5) \quad d(\star \beta) = \delta' \beta dV.$$

To see this, let $X = \sum_j \beta_j \bar{e}_j$ be the dual of β . Let A_X be the $(1, 1)$ tensor on M defined by

$$A_X = L_X - \nabla_X.$$

Since $\nabla dV = 0$, we have

$$L_X dV = A_X dV.$$

Let ξ_1, \dots, ξ_{2n-1} be a basis for $T_x M$. Since A_X is a derivative which maps every function into zero, we have

$$\begin{aligned} L_X dV(\xi_1, \dots, \xi_{2n-1}) &= A_X dV(\xi_1, \dots, \xi_{2n-1}) \\ &= A_X(dV(\xi_1, \dots, \xi_{2n-1})) - \sum_j dV(\xi_1, \dots, A_X \xi_j, \dots, \xi_{2n-1}) \\ &= - \sum_j dV(\xi_1, \dots, A_X \xi_j, \dots, \xi_{2n-1}) \\ &= -\text{Trace} A_X dV(\xi_1, \dots, \xi_{2n-1}) \end{aligned}$$

Thus we have

$$(2.6) \quad L_X dV = -\text{Trace} A_X dV.$$

Recall the torsion formula (see Proposition 2.5, Chapter 6 in [KoN])

$$(2.7) \quad A_X Y = L_X Y - \nabla_X Y = -\nabla_Y X - T(X, Y).$$

where T is the torsion tensor of ∇ . Now we choose a special frame for $\mathbb{C}T_x(M)$ with $e_j = \xi_j + i\xi_{n+j-1}$, \bar{e}_j , where $j = 1, \dots, n-1$ and $\xi_{2n-1} = \xi$, the basic vector field. Then we have

$$T(e_j, \bar{e}_k) = -\omega(e_i, \bar{e}_k)\xi = \frac{1}{i}\delta_{jk}\nabla\xi$$

and

$$T(Y, \xi) \in T^{1,0} + T^{0,1}$$

for all $Y \in T^{1,0} + T^{0,1}$. Thus the torsion term in (2.7) has no contribution when taking trace of A_X . Also note that $\nabla_X Y$ preserves types for X and Y in $T^{1,0} + T^{0,1}$ and $\nabla\xi = 0$. We have

$$\text{Trace}A_X = \langle \nabla_{\bar{e}_j} X, \bar{e}_j \rangle = \delta'\beta$$

and

$$(2.8) \quad \text{Trace}A_X dV = \delta'\beta dV.$$

For each vector field X , the divergence formula is defined to be

$$(2.9) \quad (\text{div})X dV = L_X dV = d(X \lrcorner dV),$$

we have proved from (2.6)-(2.9) that

$$\delta'\beta dV = L_X dV = d(\star\beta).$$

Integrating with respect to the volume element dV in (2.4), we have proved the proposition.

Theorem 2.3 (Folland-Stein-Tanaka formula for \square_b). *Let $(M, T^{1,0}(M))$ be a strongly pseudoconvex CR manifold of class C^3 with the pseudo-hermitian Levi metric of class C^2 and ξ be a basic field of class C^2 . Let e_1, \dots, e_{n-1} be an*

orthonormal frame field (of class C^2) for $T^{1,0}(M)$ in an open neighborhood in M and let w_1, \dots, w_{n-1} be its dual.

$$(2.10) \quad \square_b = -\sum_j \nabla_{e_j \bar{e}_j}^2 + \frac{q}{i} \nabla_\xi - \sum_{i,j} \bar{w}^k \wedge \lrcorner(\bar{e}_j) R_{e_j \bar{e}_k},$$

$$(2.11) \quad \square_b = -\sum_j \nabla_{\bar{e}_j e_j}^2 - \frac{(n-q-1)}{i} \nabla_\xi,$$

$$(2.12) \quad \begin{aligned} \square_b = & -\frac{n-q-1}{n-1} \sum_j \nabla_{e_j \bar{e}_j}^2 - \frac{q}{n-1} \sum_j \nabla_{\bar{e}_j e_j}^2 \\ & - \frac{n-q-1}{n-1} \sum_{i,j} \bar{w}^k \wedge \lrcorner(\bar{e}_j) R_{e_j \bar{e}_k}, \end{aligned}$$

where

$$\nabla_{XY}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$$

is the second covariant differential and

$$R_{XY} = -\nabla_X \nabla_Y + \nabla_Y \nabla_X - \nabla_{[X,Y]} = \nabla_{YX}^2 - \nabla_{XY}^2$$

is the curvature tensor, both extended \mathbb{C} -linearly to (p, q) -forms.

Proof.

$$\begin{aligned} \vartheta_b \bar{\partial}_b &= -\sum_j \lrcorner(\bar{e}_j) \nabla_{e_j} \left(\sum_k \bar{w}^k \wedge \nabla_{\bar{e}_k} \right) \\ &= -\sum_{j,k} \lrcorner(\bar{e}_j) \bar{w}^k \wedge \nabla_{e_j} \nabla_{\bar{e}_k} \\ &= -\sum_j \nabla_{e_j} \nabla_{\bar{e}_j} + \sum_{j,k} \bar{w}^k \wedge \lrcorner(\bar{e}_j) \nabla_{e_j} \nabla_{\bar{e}_k}. \end{aligned}$$

$$\bar{\partial}_b \vartheta_b = -\sum_k \bar{w}^k \wedge \nabla_{e_k} \left(\sum_j \lrcorner(\bar{e}_j) \nabla_{e_j} \right) = -\sum_{j,k} \bar{w}^k \wedge \lrcorner(\bar{e}_j) (\nabla_{\bar{e}_k} \nabla_{e_j})$$

since $\nabla_{\bar{e}_k} \lrcorner(e_j) = \lrcorner(e_j) \nabla_{\bar{e}_k}$.

From the Ricci's formula, we have

$$[\nabla_{e_j}, \nabla_{\bar{e}_k}] = -R_{e_j \bar{e}_k} - \nabla_{T(e_i, \bar{e}_k)}$$

where T is the torsion tensor of ∇ and

$$T(e_j, \bar{e}_k) = -\omega(e_i, \bar{e}_k)\xi = \frac{1}{i} \delta_{jk} \nabla_\xi.$$

Thus we obtain

$$\begin{aligned} \square_b &= -\sum_j \nabla_{e_j} \nabla_{\bar{e}_j} + \sum_{j,k} \bar{w}^k \wedge_{\lrcorner}(\bar{e}_j) [\nabla_{e_j}, \nabla_{\bar{e}_k}] \\ &= -\sum_j \nabla_{e_j} \nabla_{e_j} + \frac{1}{i} q \nabla_\xi - \sum_{i,j} \bar{w}^k \wedge_{\lrcorner}(\bar{e}_j) R_{e_j \bar{e}_k} \end{aligned}$$

which proves (2.10). Equation (2.11) follows from (2.10) and the Ricci formula. Equation (2.12) follows from (2.10) and (2.11).

Notice only C^2 smoothness of the frame fields is required in the derivation.

3. Estimates for \square_b on strongly CR manifolds of class C^3

In this section we first derive Kohn's subelliptic maximal estimates for \square_b in the Hilbert spaces for a strongly pseudoconvex CR manifold M of class C^{k+1} with the pseudo-hermitian Levi metric of class C^k , $k \geq 2$. We denote by $L^2_{(0,q)}(U)$ the space of $(0, q)$ -forms with $L^2(U)$ coefficients. The set $W^s(U)$ denote the usual Sobolev spaces, $0 \leq s \leq k$, i.e., $W^s(U)$ consists of L^2 functions whose sth derivatives in some coordinate system are in L^2 for $1 \leq s \leq k$. This is well defined since different coordinate charts will give equivalent norms. Similarly, $W^s_{(0,q)}(U)$ denote the space of $(0, q)$ -forms with $W^s(U)$ coefficients and we use W^s to denote its norm and omit the subscript $(0, q)$. The norms for noninteger s with $0 < s < k$ can be defined by interpolation norms and denoted by $\| \cdot \|_s$.

We also define the nonisotropic Sobolev spaces $W_*^k(U)$. Let e_1, \dots, e_{n-1} be an orthonormal frame field for $T^{1,0}(M)$ in U and define

$$\|u\|_{\mathcal{L}(U)}^2 = \sum_{i=1}^{n-1} \|e_i u\|^2 + \|u\|^2, \quad \|u\|_{\mathcal{L}(U)}^2 = \sum_{i=1}^{n-1} \|\bar{e}_i u\|^2 + \|u\|^2$$

and

$$\|Xu\|^2 = \sum_{i=1}^{n-1} \|\bar{e}_i u\|^2 + \sum_{i=1}^{n-1} \|e_i u\|^2 = \sum_{i=1}^{2(n-1)} \|X_i u\|^2,$$

where $e_j = X_j + iX_{n-1+j}$. These norms are well defined since different choices of basis will result in equivalent norms. Thus Xu can be viewed as the gradient of u with respect to the “good” directions. We also define inductively for $k \geq 2$,

$$\|X^k u\|^2 = \sum_{i_1, \dots, i_k=1}^{2(n-1)} \|X_{i_1} \cdots X_{i_k} u\|^2,$$

$$\|u\|_{W_*^k(U)}^2 = \sum_{m=1}^k \|X^m u\|^2 + \|u\|^2.$$

The space $W_*^k(U)$ is the completion of $u \in C^k(\bar{U})$ under the $W_*^k(U)$ norm. We also define $W_{0*}^k(U)$ as the completion of $C_0^k(U)$ under the $W_*^k(U)$ norm. Let $W_*^{-1}(U)$ be defined as the dual of $W_{0*}^1(U)$.

From the Folland-Stein-Tanaka formula, we immediately arrive at the following results of Kohn’s maximal L^2 estimates for \square_b on a compact strongly pseudoconvex CR manifold of real dimension at least 5 (see Kohn [Ko2]). Notice that Kohn’s original L^2 theory holds for any smooth strongly pseudoconvex CR manifold under any hermitian metric.

Theorem 3.1 (Kohn’s subelliptic $\frac{1}{2}$ estimates for \square_b). *Let M be a compact strongly pseudoconvex CR manifold of real dimension $2n - 1$ and of class C^3 with the pseudo-hermitian metric of class C^2 , $n \geq 3$. Then there exists a constant $C > 0$ such that for any $f \in L^2_{((0,q))}(M) \cap \text{Dom}(\square_b)$, $1 \leq q < n - 1$,*

$$(3.1) \quad (\square_b f, f) + (f, f) \geq C \|f\|_{\frac{1}{2}}^2.$$

Proof. From (2.12), we have using integration by parts,

$$(3.2) \quad (\square_b f, f) = \frac{n - q - 1}{n - 1} \|f\|_{\mathcal{L}}^2 + \frac{q}{n - 1} \|f\|_{\mathcal{L}}^2 + \frac{n - q - 1}{n - 1} (R_* f, f)$$

where R_* is the corresponding curvature operator in (2.12) (see Theorem 5.2 in [Tan]). Theorem 3.1 follows from (3.2) directly.

We also have the following maximal L^2 estimate (see [SW2] for the proof)

Theorem 3.2. *Let M be a compact strongly pseudoconvex CR manifold of real dimension $2n - 1$ and of class C^3 with the pseudo-hermitian metric of class C^2 and $n \geq 3$. Then the following estimates hold: for any $\phi \in L^2_{(0,q)}(M) \cap \text{Dom}(\square_b)$, $1 \leq q < n - 1$,*

$$(3.3) \quad \|\phi\|_{W_*^2}^2 \leq C(\|\square_b \phi\|^2 + \|\phi\|^2).$$

Corollary 3.3. *Let M be the same as in Theorem 3.2. For $1 \leq q \leq n - 2$, $\text{Ker}(\square_b)$ consists of smooth forms and is finite dimensional. There exists compact operators $G_b : L^2_{(0,q)}(M) \rightarrow L^2_{(0,q)}(M)$ and $\mathcal{H}_{(0,q)} : L^2_{(0,q)}(M) \rightarrow L^2_{(0,q)}(M) \cap \text{Ker}(\square_b)$ such that*

- (1) *For any $f \in L^2_{(0,q)}(M)$, $f = \bar{\partial}_b \bar{\partial}_b^* G_b f + \bar{\partial}_b^* \bar{\partial}_b G_b f + \mathcal{H}_{(0,q)} f$.*
- (2) *$G_b \square_b = \square_b G_b = I - \mathcal{H}_{(0,q)}$ on $\text{Dom}(\square_b)$. $G_b \mathcal{H}_{(0,q)} = \mathcal{H}_{(0,q)} G_b = 0$.*
- (3) *$G_b(W_*^s(M)) \subset W_*^{s+2}(M)$, $s = -1, 0$.*

The Szegő projection $\mathcal{S} : L^2(M) \rightarrow L^2(M) \cap \text{Ker}(\bar{\partial}_b)$ on M is given by $\mathcal{S} = I - \bar{\partial}_b^ G_b \bar{\partial}_b$ and satisfies*

$$\|\mathcal{S}f\|_{W_*^1(M)} \leq C\|f\|_{W_*^1(M)}$$

These maximal estimates were obtained in Folland-Stein [FS], Rothschild-Stein [RS] for smooth CR manifolds. From the maximal estimates, we can derive similar estimates in the nonisotropic Hölder spaces (see [SW1] or [SW2]). First we recall some definitions.

Suppose that M is a strongly pseudoconvex CR manifold of class $C^{k,\alpha}$, $k \geq 2$ and $0 < \alpha < 1$. Let x_0 be a point in a neighborhood U in M . Shrinking U if necessary, we choose normal coordinates $(z, t) = (z_1, \dots, z_{n-1}, \bar{z}_1, \dots, \bar{z}_{n-1}, t)$ such that $\partial/\partial z_j = X_j + iX_{n+j-1}$, $1 \leq j \leq n - 1$ at the point $x_0 = 0$. A polynomial in (z, t) is said to be of nonisotropic order m if

$$P(z, t) = \sum_{|I|+|J|+2l \leq m} a_{IJK} z^I \bar{z}^J t^l,$$

where $I = (i_1, \dots, i_{n-1})$, $J = (j_1, \dots, j_{n-1})$ are multiindices and $l \geq 0$. A polynomial of degree 1 is a polynomial of order 1 in the z variables.

We define the nonisotropic Hölder spaces at a point.

Definition 3.4. A function $u \in C_*^{m,\alpha}(x_0)$ for $m \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha < 1$ and $1 \leq p \leq \infty$, if and only if $u \in C(U)$ for some open neighborhood U of x_0 and there exists a nonisotropic m -th order polynomial P_{x_0} such that

$$(3.4) \quad \sup_{B_\rho(x_0)} |u - P_{x_0}| \leq C\rho^{m+\alpha} \quad \text{for every } B_\rho(x_0) \subset U.$$

A function u is said to be in $C_*^{m,\alpha}(U)$ if $u \in C_*^{m,\alpha}(x_0)$ for every $x_0 \in U$ and the constant C in (3.4) can be chosen independent of x_0 .

We also define the function space $C_*^{-1,\alpha}(x_0)$. A function $u \in C_*^{-1,\alpha}(x_0)$ if it can be written as a finite sum $\sum X_i f_i$ for some $f_i \in C_*^\alpha(x_0)$.

Theorem 3.5. Let M be a compact strongly pseudoconvex CR manifold of class $C^{3,\alpha}$ with real dimension $2n - 1$, $n \geq 3$. Let u and f be locally integrable $(0, q)$ -forms, $1 \leq q \leq n - 2$ which satisfy $\square_b u = f$ on an open set $U \subset M$. If $f \in C_*^{k,\alpha}(x_0)$, then $u \in C_*^{k+2,\alpha}(x_0)$ for $k = -1, 0, 1$ and $0 < \alpha < 1$,

Corollary 3.6. Let M be the same as in Theorem 3.5. The Szegő projection \mathcal{S} is bounded from $C_*^{k,\alpha}(M)$ to itself, where $0 \leq k \leq 2$.

4. Embedding compact strongly pseudoconvex CR manifolds of class $C^{3,\alpha}$

In this section we first extend the embedding theorem of Boutet De Monvel [Bou] to CR manifolds of class $C^{3,\alpha}$.

Theorem 4.1. Let $(M, T^{1,0})$ be a compact strongly pseudoconvex CR manifold with real dimension $2n + 1$, $n \geq 2$, of class $C^{3,\alpha}$, where $0 < \alpha < 1$. Then there exist global CR functions $h_1, \dots, h_N \in C_*^{2,\alpha}(M)$ such that $\bar{\partial}_b h_i = 0$ for every $i = 1, \dots, N$ and $\Phi = (h_1, \dots, h_N) : M \rightarrow \mathbb{C}^N$ is an embedding.

Proof. The arguments are similar to the smooth case used in [Bou]. Since $T^{1,0}(M)$ is of class $C^{3,\alpha}$, we equip M with a metric which is of class $C^{3,\alpha}$. For each point $p \in M$, using a polynomial change of coordinates, we can choose C^3 coordinates $x = (z, t) = (z_1, \dots, z_{n-1}, t)$ near a neighborhood U of $p = 0$ such that a basis for $T^{1,0}$ vector fields is given by

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{i=1}^{n-1} A_{ji} \frac{\partial}{\partial z_i} + (-\sqrt{-1}z_j + B_j) \frac{\partial}{\partial t}, \quad j = 1, \dots, n - 1$$

where both $A_{ji} = \mathcal{O}(|x|^2)$ and $B_j = \mathcal{O}(|x|^2)$. We choose

$$Z(z, t) = Z(x) = (z_1, \dots, z_n),$$

where $z = (z_1, \dots, z_{n-1})$ and $z_n = t + i|z|^2$. Then $Z : U \rightarrow \mathbb{C}^n$. Then we have

$$(4.1) \quad \bar{L}_j z_i = \mathcal{O}(|x|^2), \quad j = 1, \dots, n-1, \quad i = 1, \dots, n.$$

Thus the map Z gives an approximate embedding near p . Extending z_j to be a C^3 function ϕ_j on M , we have functions $\varphi_1, \dots, \varphi_n \in C^3(M)$ such that $\varphi_j(p) = 0$, $d\varphi_1(p), \dots, d\varphi_n(p)$ are linearly independent at p and $\varphi = (\varphi_1, \dots, \varphi_n) : M \rightarrow \mathbb{C}^n$ is a C^3 diffeomorphism of a small neighborhood of p on M into \mathbb{C}^n with $\varphi(p) = 0$ and $\varphi(M)$ is strongly pseudoconvex at the origin.

Let g be a polynomial $g(z, t) = -iz_n + z_n^2$. Then g satisfies $dg(p) \neq 0$ and $\bar{\partial}_b g(p) = \mathcal{O}(|x|^2)$. Choosing U small, we have

$$(4.2) \quad \text{Reg} = |z|^2 + t^2 - (|z|^2)^2 \geq c|x|^2,$$

where c is a positive constant. Extend g to M such that $g \in C^3(M)$ and g satisfies $\text{Reg} > 0$ on $M \setminus p$.

Let η be a cut-off function such that $\eta = 1$ near a neighborhood $V \subset\subset U$ of p and η is supported in U . To construct CR embedding functions, we set $z_j^\lambda = \eta\varphi_j e^{-\lambda g}$ for sufficiently large $\lambda > 0$. Notice that Theorems 3.1, 3.2 and Corollary 3.3 hold for any metric of class C^3 , not necessarily the pseudo-hermitian Levi metric. Let G_b be the inverse of \square_b . Set

$$h^j = \mathcal{S}(z_j^\lambda) = z_j^\lambda - \vartheta_b G_b \bar{\partial}_b z_j^\lambda \quad \text{for } j = 1, \dots, n,$$

where \mathcal{S} is the Szegö projection on M . We have $\bar{\partial}_b h^j = 0$ in M for $j = 1, \dots, n$. Since z_j^λ is in $C^3(M) \subset C_*^{2,\alpha}(M)$, from Corollary 3.6 we have $h^j \in C_*^{2,\alpha}(M)$ for any $0 < \alpha < 1$.

Using (4.1) and (4.2), it follows that in a neighborhood U near p

$$\begin{aligned} |\bar{\partial}_b z_j^\lambda| &= |\bar{\partial}_b(\eta\varphi_j)e^{-\lambda g} - \lambda\eta\varphi_j e^{-\lambda g}(\bar{\partial}_b g)| \\ &\leq C(|x|^2 + \lambda|x|^3)e^{-\lambda|x|^2} \\ &\leq C(\lambda^{-1} + \lambda^{-\frac{1}{2}})\sup_{v>0}(|v|^1 e^{-v} + |v|^{1+\frac{1}{2}} e^{-v}) \leq C\lambda^{-\frac{1}{2}} \rightarrow 0, \quad \lambda \rightarrow \infty. \end{aligned}$$

This gives that $\|\bar{\partial}_b z_j^\lambda\| \rightarrow 0$ as $\lambda \rightarrow \infty$. Since $\bar{\partial}_b z_j^\lambda \rightarrow 0$ in $C^{1,\alpha}(0)$ as $\lambda \rightarrow \infty$, we have $\vartheta_b G_b \bar{\partial}_b z_j^\lambda$ in $C_*^{2,\alpha}(0)$ can be made arbitrarily small as $\lambda \rightarrow \infty$. This gives $dh^1(0), \dots, dh^n(0)$ are linearly independent for large λ . The map $\Phi = (h^1, \dots, h^n)$ forms a local embedding of M into \mathbb{C}^n by global $C_*^{2,\alpha}$ CR functions. From the same arguments as in [Bou], we also have that global CR functions separating points. From the compactness of M and a partition of unity, there exists a global embedding map consisting of CR functions. This proves the theorem.

Theorem 4.2. *Let (M, g) be a complete Kähler manifold of complex dimension $n \geq 3$ with nonpositive sectional curvature. Suppose that the boundary ∂M admits a $C^{3,\alpha}$ CR structures. Then there exist bounded holomorphic functions on M .*

The proof follows the same arguments used in Bland (see Theorem 2.7 in [Bla]) and Theorem 4.1. If we use Hölder regularity for elliptic systems, then we can obtain the following version of the Newlander-Nirenberg theorem (see [NW, We2, NW]).

Theorem (Newlander-Nirenberg). *Let $(M, T^{1,0})$ be an almost complex manifold of class $C^k(U) \cap C^{k,\alpha}(x_0)$ in a neighborhood U of a point x_0 in M where $k \geq 1$ and $0 < \alpha < 1$. Assume that the almost complex structure $T^{1,0}$ is (formally) integrable. Then there exist a neighborhood $V \subset U$ of x_0 and holomorphic coordinates of class $C^k(V) \cap C^{k+1,\alpha}(x_0)$ in V which embed V into \mathbb{C}^n .*

Proof. The proof is exactly as in [Ko1] combined with the pointwise interior Hölder regularity for elliptic operators.

Let L_1, \dots, L_n be a local basis for smooth sections of $T^{1,0}(M)$. Let x_1, \dots, x_{2n} be the real coordinates for M and we write $z_j = x_j + ix_{n+j}$. We can using compatibility condition, after a quadratic change of coordinates, assume that

$$L_i = \frac{\partial}{\partial z_i} + \sum_{j=1}^n a_{ij} \frac{\partial}{\partial \bar{z}_j}, \quad i = 1, \dots, n,$$

where the a_{ij} 's are C^k functions and $a_{ij}(0) = 0$ for all $i, j = 1, \dots, n$. At the origin, L_i is the constant coefficient operator $\partial/\partial z_i$. Let $L_i^\epsilon = \frac{\partial}{\partial z_i} + \sum_{j=1}^n a_{ij}(\epsilon x) \frac{\partial}{\partial \bar{z}_j}$, $i =$

$1, \dots, n$, where $\epsilon > 0$ is small. Then $T_\epsilon^{0,1} = \langle L_1^\epsilon, \dots, L_n^\epsilon \rangle$ defines an almost complex structure that is integrable for each $\epsilon < \epsilon_0$ for some sufficiently small $\epsilon_0 > 0$.

Let $\bar{\partial}_\epsilon$ denote the Cauchy-Riemann complex associated with the almost complex structure $T_\epsilon^{0,1}$ equipped with a Hermitian metric. Then the existence and regularity theory developed for $\bar{\partial}$ in the previous section on any complex manifold can be applied to M with $\bar{\partial}$ substituted by $\bar{\partial}_\epsilon$. Let $\phi = \sum_{i=1}^n |z_i|^2 = |x|^2$, ϕ is a strictly plurisubharmonic function near 0. Choosing ϵ_0 sufficiently small, we may assume that $\Omega = \{x \in \mathbb{R}^{2n} \mid |x|^2 < 1\} = B_1$ and Ω is strongly pseudoconvex with respect to the almost complex structure $T_\epsilon^{0,1}(M)$. Using the L^2 existence results for $\bar{\partial}$ (see [Ko1] or [Hö2]), the $\bar{\partial}$ -Neumann operator N_ϵ exists on Ω and there exists a solution $u_i^\epsilon = \bar{\partial}_\epsilon^* N_\epsilon \bar{\partial}_\epsilon z_i$ on Ω such that $\bar{\partial}_\epsilon u_i^\epsilon = \bar{\partial}_\epsilon z_i$ and

$$(4.3) \quad \|u_i^\epsilon\|_\Omega \leq C \|\bar{\partial}_\epsilon z_i\|_\Omega$$

where C can be chosen uniformly for $\epsilon < \epsilon_0$. Since

$$\bar{L}_i^\epsilon z_j = \bar{a}_{ij}(\epsilon x),$$

we have

$$D^I \bar{\partial}_\epsilon z_i = O(\epsilon)$$

for any $D^I = (\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_{2n})^{i_{2n}}$, where the i_ν 's are nonnegative integers with $|I| = i_1 + \cdots + i_{2n} \leq k$. It is easy to check that

$$|\bar{\partial}_\epsilon z_i|_{C^{k,\alpha}(0)} \rightarrow 0 \quad \text{if } \epsilon \rightarrow 0.$$

Let $\zeta_i^\epsilon = z_i - u_i^\epsilon$. From the interior Hölder regularity for N_ϵ and (4.3), we have

$$\|u_i^\epsilon\|_{C^{k+1,\alpha}(0)} \leq C(|\bar{\partial}_\epsilon z_i|_{C^{k,\alpha}(0)} + \|u_i^\epsilon\|) \rightarrow 0 \quad \text{if } \epsilon \rightarrow 0.$$

We have that $\bar{\partial}_\epsilon \zeta_i^\epsilon = 0$ in Ω and also $d\zeta_i^\epsilon(0) = dz_i - du_i^\epsilon(0)$ are linearly independent if ϵ is sufficiently small. If we pull back ζ_i^ϵ to $\epsilon\Omega$ by setting $\zeta_i = \zeta_i^\epsilon(x/\epsilon)$, then we have that $\bar{\partial}\zeta_i = 0$ and $d\zeta_i$ are linearly independent in $\epsilon\Omega$ provided we choose ϵ sufficiently small. This proves the theorem.

Remarks (1) In the Newlander-Nirenberg Theorem above, we only need the assumption on Hölder condition $C^{k,\alpha}$ to be just at one point x_0 .

(2) If $(M, T^{1,0})$ is of real dimension 2, then we can relax the assumption by requiring only $k \geq 0$ since there is no compatibility condition to be satisfied. This is the result obtained in Bers and Chern (see [Ber, Che]). Again, the assumption on Hölder condition C^α is only required at one point.

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Mei-Chi Shaw
 Department of Mathematics
 University of Notre Dame
 244 Hayes-Healy Notre Dame, IN 46556
 E-mail: shaw.1@nd.edu