Pure and Applied Mathematics Quarterly
Volume 6, Number 4
(*Special Issue: In honor of
Joseph J. Kohn, Part 2 of 2*)
1081—1103, 2010

A Remark on the Fefferman-Phong inequality for 2×2 Systems

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Dedicated to Professor J.J.Kohn on the occasion of his 75th birthday

Abstract: We show that the Fefferman-Phong inequality can be extended to certain 2×2 pseudodifferential systems whose symbol is Hermitian non-negative with elliptic matrix-trace.
Keywords: Lower bounds; Systems of pseudodifferential operators; Fefferman-Phong inequality

1. Introduction

Let g be an admissible metric in $\mathbb{R}^n \times \mathbb{R}^n$, and let h be the corresponding Planck function (see Hörmander's book [6] for the background on the Weyl-Hörmander Calculus, and Section 2 below). Fefferman and Phong proved in [3] their celebrated inequality which we state as follows (in the form given in [6], Theorem 18.6.8, page 171).

Theorem 1.1. *Let $a \in S(h^{-2}, g)$ be a (scalar) symbol with $a \geq 0$. Then there exists a constant $C > 0$ such that*

$$(a^{\mathrm{w}}(x, D)u, u) \geq -C\|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \tag{1.1}$$

Received December 6, 2007.
Mathematics Subject Classification. Primary 35S05; Secondary 35B45; 35A30.

Here $(\cdot,\cdot)$ and $\|\cdot\|_0$ denote the $L^2(\mathbb{R}^n)$ inner product and norm, respectively, and a^{w} denotes the Weyl quantization

$$a^{\mathrm{w}}(x,D)u(x) = (2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle}a(\frac{x+y}{2},\xi)u(y)dyd\xi, \quad u\in\mathcal{S}(\mathbb{R}^n).$$

A fundamental step in the proof is the reduction, through microlocalization, to the case of a constant metric (reduction to the "semiclassical case", see [6], Lemma 18.6.10, page 173). One has in fact the following result.

Theorem 1.2. *Let g be a constant metric on $\mathbb{R}^n\times\mathbb{R}^n$, such that $g/g^\sigma \le \lambda^2 \le 1$. Let $0\le a\in C^\infty(\mathbb{R}^{2n})$ be a symbol such that $|a|^g_k(x,\xi)\le\lambda^{-2}$ for all $(x,\xi)\in\mathbb{R}^{2n}$ and for all $k\le N_0$. If N_0 is sufficiently large, one then has*

$$(a^{\mathrm{w}}(x,D)u,u) \ge -C\|u\|_0^2, \quad \forall u\in\mathcal{S}(\mathbb{R}^n),$$

where the constant C is **independent** *of g and of a.*

An inspection of the proof shows that $N_0 = N_0(n)$. However, the proof itself does not give an explicit dependence on the dimension n. In this respect, recent work by Lerner and Morimoto [7] shows that, in the case of the standard pseudo-differential metric $|dx|^2+|d\xi|^2/(1+|\xi|^2)$, one may take $N_0 = 4+2n+1$ (see also the reference to Bony's results and to Boulkhemair's results contained in that paper).

Because of the basic importance of the Fefferman-Phong inequality, there has been a great deal of work, in the scalar case, to extend the Fefferman-Phong inequality (1.1) in various directions (see the bibliography of Parmeggiani [9]; see also [10]).

However, Brummelhuis showed in [1] that in the case of systems the inequality is in general false. He considered the symbol

$$A_B(x,\xi) = \begin{bmatrix} \xi_1^2 & ix_1\xi_1\xi_2 \\ -ix_1\xi_1\xi_2 & x_1^2\xi_2^2 \end{bmatrix},$$

and tested the Fefferman-Phong inequality for $A_B^{\mathrm{w}}(x,D)$ against cut-off functions u_μ, where $\mu>0$ is a parameter, of the kind

$$u_\mu(x_1,x_2) = \begin{bmatrix} u_{\mu,1}(x_1,x_2) \\ u_{\mu,2}(x_1,x_2) \end{bmatrix} = \begin{bmatrix} e^{i\mu x_2}\chi_1(x_1,x_2) \\ \sqrt{\mu}\, e^{i\mu x_2}\chi_2(\mu\, x_1,x_2) \end{bmatrix},$$

where $\chi_1, \chi_2 \in C_0^\infty(|x_1|, |x_2| < 1)$ are real-valued and satisfy

$$\iint \chi_1(0, x_2)\chi_2(x_1, x_2)dx_1dx_2 = 1.$$

Since, as is readily seen, $\|u_\mu\|_0^2 = \|\chi_1\|_0^2 + \|\chi_2\|_0^2$ (from now on, $(\cdot,\cdot)$ and $\|\cdot\|_0$ denote the L^2-inner product and norm, regardless to whether we consider scalar or vector-valued functions), using

$$(A_B^{\mathrm{w}}(x, D)u, u) = \|D_1u_1 + ix_1D_2u_2\|_0^2 - \mathrm{Re}(D_2u_2, u_1)$$

one has

$$(1.2) \qquad (A_B^{\mathrm{w}}(x, D)u_\mu, u_\mu) \sim -\sqrt{\mu}, \quad \text{as} \quad \mu \to +\infty.$$

Hence the Fefferman-Phong inequality cannot hold for the system $A_B^{\mathrm{w}}(x, D)$.

Brummelhuis' counterexample was later generalized to a geometrically characterized class of systems by Parmeggiani in [8], class which is modelled after the example, due to Hörmander [5], of a nonnegative Hermitian matrix whose Weyl-quantization cannot be nonnegative. For A_B^{w} and the isotropic counterexamples of [8] the Sharp Gårding inequality cannot be improved. However, Brummelhuis' counterexample and all the counterexamples given in [8] to inequality (1.1) for systems require at least *two variables*, i.e. $n \geq 2$. As a matter of fact, when $n = 1$, L.-Y.Sung proved in [11] that if $p(x, \xi)$ is an Hermitian $N \times N$ system of *ordinary differential operators* which is *nonnegative* (in the sense of Hermitian matrices) then inequality (1.1) holds for $p^{\mathrm{w}}(x, D)$. His proof is based on the use of Fourier series to reduce the problem to an estimate from below of an infinite-size matrix. We showed in [9] (see also [10]) that Sung's result holds also for systems of *partial differential operators* in $\mathbb{R}^n$ of the kind

$$(1.3) \quad p(x, \xi) = A(x)e(\xi) + B(x, \xi) + C(x) = p(x, \xi)^* \geq -cI, \ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $B(x, \xi) = \sum_{\ell=1}^n B_\ell(x)\xi_\ell$, and e is a *positive homogeneous quadratic form*. Our proof there was in the spirit of the Fefferman-Phong Calderón-Zygmund decomposition of the phase-space $\mathbb{R}^n \times \mathbb{R}^n$ introduced in [4], which allows one to use an *induction on the size* N *of the system*: the microlocalization given by the *Fefferman-Phong metric*, essentially of the form

$$g_{x,\xi} = H(x, \xi)^2|dx|^2 + \frac{|d\xi|^2}{1 + |\xi|^2},$$

where

$$H(x,\xi)^{-1} = \max\Big\{\frac{1}{\sqrt{1+|\xi|^2}}, \sqrt{\mathrm{Tr}(A(x))}\Big\},$$

makes it possible to start the size-reduction of the system, decoupling it, modulo L^2-bounded errors, into a 1×1 block for which we may use the scalar Fefferman-Phong inequality, and into an $(N-1)\times(N-1)$ block that still satisfies the assumptions, that we control by induction.

We note in passing that for $N\times N$ systems of the kind

$$\sum_{j,k=1}^{n} A_{jk}(x)\xi_j\xi_k \geq 0,$$

with $A_{jk} = A_{kj} = A_{jk}^*$ and $\|\partial_x^\alpha A_{jk}\|_{L^\infty} \leq C$, $|\alpha| \leq 2$, and such that

$$\text{(1.4)} \qquad \sum_{j,k=1}^{n} \langle A_{jk}(x)v_j, v_k\rangle_{\mathbb{C}^N} \geq 0, \quad \forall v_1,\dots,v_n \in \mathbb{C}^N,\ \forall x \in \mathbb{R}^n,$$

Brummelhuis proved that inequality (1.1) holds (that proof goes by an elementary integration by parts). Of course, condition (1.4) is too strong. Also, inequality (1.1) is then straightforward for system (1.3) when $B(x,\xi)=0$, and the difficulty when $B(x,\xi)\neq 0$ lies exactly in controlling this first order part.

The purpose of this note is to extend the Fefferman-Phong inequality, through the above-mentioned reduction used in [9] (see also [10]), to certain 2×2 systems with (positive) *elliptic* matrix-trace (see Theorem 3.1 and Theorem 3.5 below).

To get an idea why a condition on the trace should work, consider the following "deformation" of Brummelhuis' system A_B:

$$A(x,\xi) = A_B(x,\xi) + \begin{bmatrix} 0 & 0 \\ 0 & \xi_2^2 \end{bmatrix} = A(x,\xi)^* \geq 0,$$

whose trace is $\xi_1^2 + (1+x_1^2)\xi_2^2$, and therefore is elliptic (in the usual $S^2_{1,0}$-calculus). Since

$$\|D_2 u_{\mu,2}\|_0^2 \sim \mu^2, \quad \text{as} \quad \mu \to +\infty,$$

we can no longer say that the Fefferman-Phong inequality does not hold. In fact, (1.1) holds, for by the Cauchy-Schwarz inequality one sees that for any given $u \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^2)$,

$$\begin{aligned}(A^{\mathrm{w}}(x,D)u,u) &= (A_B^{\mathrm{w}}(x,D)u,u) + \|D_2u_2\|_0^2 = \\ &= \|D_1u_1 + ix_1D_2u_2\|_0^2 - \mathrm{Re}\,(D_2u_2,u_1) + \|D_2u_2\|_0^2 \geq\end{aligned}$$

$$\geq -\frac{1}{2}\|D_2u_2\|_0^2 - \frac{1}{2}\|u_1\|_0^2 + \|D_2u_2\|_0^2 \geq -\frac{1}{2}\|u\|_0^2.$$

We shall show that this is in general the case for 2×2 system with elliptic trace, provided an extra assumption is imposed on the off-diagonal terms. The latter condition is automatically fulfilled in the important case of off-diagonal terms that are either always real or always purely imaginary.

The plan of the paper is as follows. In the next section we shall recall, for the sake of completeness, the basic facts about the Weyl-Hörmander Calculus, and in Section 3 we shall state and prove the theorems. In Section 4 we shall show that conditions (3.1) and (3.14) below on the off-diagonal terms are in a sense optimal, by providing an example of system with positive-elliptic trace that does not satisfy either condition and for which the Fefferman-Phong inequality does not hold. In the final Section 5 we shall give some corollaries and concluding remarks.

We wish to thank Francis Nier for many useful discussions.

2. Background on the Weyl-Hörmander calculus

We recall in this section a few basic facts about admissible metrics and weight-functions (see [6], Sections 18.4 and 18.5; see also [5]). We shall denote by $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ the canonical symplectic 2-form in $\mathbb{R}^{2n}_X = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Hence $\sigma(X,Y) = \langle \xi, y\rangle - \langle \eta, x\rangle$, $X=(x,\xi)$, $Y=(y,\eta)$.

Definition 2.1. *An* **admissible metric** *in $\mathbb{R}^{2n}$ is a function $\mathbb{R}^{2n} \ni X \longmapsto g_X$ where g_X is a positive-definite quadratic form on $\mathbb{R}^{2n}$ such that:*

- **Slowness:** *There exists $C_0 > 0$ (the constant of* **slowness***) such that for any given X, $Y \in \mathbb{R}^{2n}$ one has*

$$g_X(Y-X) \leq C_0^{-1} \Longrightarrow C_0^{-1} g_Y \leq g_X \leq C_0 g_Y;$$

- **Uncertainty:** *For any given $X \in \mathbb{R}^{2n}$ one has*

$$g_X \leq g_X^\sigma,$$

where g_X^σ is the **dual** *metric defined by*

$$g_X^\sigma(Y) = \sup_{Z\neq 0} \frac{\sigma(Y,Z)^2}{g_X(Z)};$$

- **Temperateness:** *There exists* $C_1 > 0$ *and* $N_1 \in \mathbb{Z}_+$ *such that for all* X, $Y \in \mathbb{R}^{2n}$ *one has*
$$g_X \leq C_1 g_Y \Big(1 + g_X^\sigma(X - Y)\Big)^{N_1}.$$

The **Planck function** *associated with* g *is by definition*
$$h(X)^2 = \sup_{Z \neq 0} \frac{g_X(Z)}{g_X^\sigma(Z)}.$$

Remark that by the uncertainty property one always has $h \leq 1$.

Definition 2.2. *Given an admissible metric* g, *a* g**-admissible weight** *is a positive function* m *on* $\mathbb{R}^{2n}$ *for which there exist constants* $c, C, C' > 0$ *and* $N' \in \mathbb{Z}_+$ *such that for all* $X, Y \in \mathbb{R}^{2n}$,
$$g_X(X - Y) \leq c \Longrightarrow C^{-1} \leq \frac{m(X)}{m(Y)} \leq C,$$
and
$$\frac{m(X)}{m(Y)} \leq C' \Big(1 + g_X^\sigma(Y - X)\Big)^{N'}.$$

Remark 2.3. *In particular, given an admissible metric* g, *one always has that the Planck function* h *associated with* g *is a* g*-admissible weight.*

Definition 2.4. *Let* g *be an admissible metric and* m *be a* g*-admissible weight. Let* $a \in C^\infty(\mathbb{R}^{2n})$. *Denote by* $a^{(k)}(X; v_1, \ldots, v_k)$ *the* k*-th differential of* a *at* X *in the directions* $v_1, \ldots, v_k$ *of* $\mathbb{R}^{2n}$. *Define*
$$|a|_k^g(X) := \sup_{0 \neq v_1, \ldots, v_k \in \mathbb{R}^{2n}} \frac{|a^{(k)}(X; v_1, \ldots, v_k)|}{\prod_{j=1}^k g_X(v_j)^{1/2}}.$$
We say that $a \in S(m, g)$ *if for any given integer* $k \in \mathbb{Z}_+$ *the following seminorms are finite:*
$$\|a\|_{k, S(m,g)} := \sup_{\ell \leq k,\ X \in \mathbb{R}^{2n}} \frac{|a|_\ell^g(X)}{m(X)} < +\infty. \tag{2.1}$$
With $B_{X_0,r}^g = \{X;\ g_{X_0}(X - X_0) < r^2\}$, *following Bony and Lerner* [2] *we say that* $a \in C^\infty(\mathbb{R}^{2n})$ *is a* **symbol of weight** m **confined** *to the ball* $B_{X_0,r}^g$, *and write* $a \in \mathrm{Conf}(m, g, X_0, r)$, *if for all* $k \in \mathbb{Z}_+$
$$\|a\|_{k, \mathrm{Conf}(m,g,X_0,r)} := \sup_{\ell \leq k,\ X \in \mathbb{R}^{2n}} \frac{|a|_\ell^{g_{X_0}}(X)}{m(X_0)} \big(1 + g_{X_0}^\sigma(X - B_{X_0,r})\big)^{k/2} < +\infty, \tag{2.2}$$

where $g^{\sigma}_{Y}(X-B)=\inf_{Z\in B} g^{\sigma}_{Y}(X-Z)$. *Hence the space of symbols confined to the ball* $B^{g}_{X_0,r}$ *coincides with* $\mathcal{S}(\mathbb{R}^{2n})$ *endowed with the seminorms (2.2). Any given* $\varphi\in C_0^{\infty}(B^{g}_{X_0,r})$ *is automatically confined to the ball* $B^{g}_{X_0,r}$.

As for the composition, one has the following result.

Theorem 2.5. *Given* $a\in S(m_1,g)$, $b\in S(m_2,g)$ *then*

$$a^{\mathrm{w}}(x,D)b^{\mathrm{w}}(x,D)=(a\sharp b)^{\mathrm{w}}(x,D),$$

where for any given $N\in\mathbb{Z}_+$

$$(a\sharp b)(X)=\sum_{j=0}^{N}\frac{1}{j!}\Big(\frac{i}{2}\sigma(D_X,D_Y)\Big)^{j}a(X)b(Y)\Big|_{X=Y}+r_{N+1}(X), \tag{2.3}$$

with $r_{N+1}\in S(h^{N+1}m_1m_2,g)$.

Associated with an admissible metric g one has a partition of unity as follows (see Hörmander [6], and Bony and Lerner [2]).

Lemma 2.6. *Let* g *be an admissible metric, and let* $r^2<C_0^{-1}$. *Then there exists a sequence of centers* $\{X_\nu\}_{\nu\in\mathbb{Z}_+}$, *a covering of* $\mathbb{R}^{2n}$ *made of* g*-balls* $B^{g}_{\nu,r}=\{X; g_{X_\nu}(X-X_\nu)<r^2\}$ *centered at* X_ν *and radius* r, *and a sequence of functions* $\{\varphi_\nu\}$ **uniformly** *in* $S(1,g)$, *with* $\operatorname{supp}\varphi_\nu\subset B^{g}_{\nu,r}$, *such that* $\sum_{\nu\in\mathbb{Z}_+}\varphi_\nu^2=1$. *Moreover, for any given* r_* *such that* $r^2\le r_*^2<C_0^{-1}$, *there exists an integer* N_{r_*} *such that no more than* N_{r_*} *balls* B^{g}_{ν,r_*} *can intersect at each time (i.e. one has an a priori finite number of overlappings of the dilates by* r_*/r *of the* $B^{g}_{\nu,r}$*). In addition, with*

$$g^{\sigma}_{X}(B-B'):=\inf_{Y\in B,\,Y'\in B'} g^{\sigma}_{X}(Y-Y'),\ B,B'\subset\mathbb{R}^{2n},$$

and

$$\Delta_{\mu\nu}(r_*):=\max\big\{1,g^{\sigma}_{X_\mu}(B^{g}_{\mu,r_*}-B^{g}_{\nu,r_*}),g^{\sigma}_{X_\nu}(B^{g}_{\mu,r_*}-B^{g}_{\nu,r_*})\big\}^{1/2},$$

there exist constants $\tilde N$ *and* $\tilde C$ *such that*

$$\sup_{\mu}\sum_{\nu}\Delta_{\mu\nu}(r_*)^{-\tilde N}<\tilde C.$$

Moreover, for all $k\in\mathbb{Z}_+$ *there exist* $C>0$ *and* $\ell\in\mathbb{Z}_+$ *such that for any given* $a\in S(m,g)$ *and* $b\in\mathrm{Conf}(1,g,X,r)$ *one has*

$$\|a\sharp b\|_{k,\mathrm{Conf}(1,g,X,r)}\le Cm(X)\|a\|_{\ell,S(m,g)}\|b\|_{\ell,\mathrm{Conf}(1,g,X,r)}. \tag{2.4}$$

Finally, for all $k, N \in \mathbb{Z}_+$ there exist $C > 0$ and $\ell \in \mathbb{Z}_+$ such that for every $\mu, \nu \in \mathbb{N}$, and every $a \in \mathrm{Conf}(1, g, X_\mu, r)$ and $b \in \mathrm{Conf}(1, g, X_\nu, r)$ one has

$$\|a\sharp b\|_{k,\mathrm{Conf}(1,g,X_\mu,r)} + \|a\sharp b\|_{k,\mathrm{Conf}(1,g,X_\nu,r)} \leq \tag{2.5}$$
$$\leq C\|a\|_{\ell,\mathrm{Conf}(1,g,X_\mu,r)}\|b\|_{\ell,\mathrm{Conf}(1,g,X_\nu,r)}\Delta_{\mu\nu}(r)^{-N}.$$

One has also the following useful lemma, due to Bony and Lerner (see [2]).

Lemma 2.7. *Let g be an admissible metric, and let m be a g-admissible weight. Let B_ν be a g-ball as in Lemma 2.6. Let $g_\nu = g_{X_\nu}$ and $m_\nu = m(X_\nu)$. Let $\{a_\nu\}_{\nu\in\mathbb{Z}_+}$ be a sequence of symbols with $a_\nu \in S(m_\nu, g_\nu)$, such that for any given integer $k \in \mathbb{Z}_+$*

$$\sup_{\nu\in\mathbb{Z}_+} \|a_\nu\|_{k,\mathrm{Conf}(m_\nu,g_\nu,X_\nu,r)} < +\infty.$$

Then $a := \sum_{\nu\in\mathbb{Z}_+} a_\nu$ belongs to $S(m, g)$. The sequence $\{a_\nu\}_{\nu\in\mathbb{Z}_+}$ is said to be **uniformly confined** *in $S(m, g)$. When $m = 1$ we have from the Cotlar-Stein Lemma (see [6], Lemma 18.6.5) that $a^{\mathrm{w}} = \sum_\nu a_\nu^{\mathrm{w}}$ is a bounded operator in L^2.*

In the case of matrix-valued symbols Definitions 2.2 and 2.4, and the composition formula (2.3) hold (being careful with the order of the terms). Upon denoting by M_2 the set of 2×2 complex matrices, we shall write $S(m, g; \mathsf{M}_2)$ for the matrix-valued analogue of the symbol spaces $S(m, g)$ considered above. (Analogous notation will be used for the spaces $S(m, g; \mathbb{C}^2)$ etc.)

In the sequel, given $A, B > 0$, we write $A \lesssim B$ when $A \leq CB$ for some universal constant $C > 0$, and $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

3. The inequality for certain 2×2 system with elliptic matrix-trace

Theorem 3.1. *Let g be an admissible metric. Let*

$$p(X) = \begin{bmatrix} a(X) & \overline{c(X)} \\ c(X) & b(X) \end{bmatrix} = p(X)^* \geq 0, \quad X \in \mathbb{R}^{2n},$$

where $a, b, c \in S(h^{-2}, g)$. Suppose that

$$a\{c, \bar{c}\} - 2i\,\mathrm{Im}(c\{a, \bar{c}\}) \quad \text{and} \quad b\{c, \bar{c}\} - 2i\,\mathrm{Im}(c\{b, \bar{c}\}) \in S(h^{-4}, g), \tag{3.1}$$

and that the matrix-trace of p is (positive) elliptic, that is there exists $c_0 > 0$ such that

$$t(X) := a(X) + b(X) \geq c_0\, h(X)^{-2}, \quad \forall X \in \mathbb{R}^{2n}. \tag{3.2}$$

Then there exists $C > 0$ such that

$$(p^{\mathrm{w}}(x, D)u, u) \geq -C\|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^2). \tag{3.3}$$

Remark 3.2. *Hypothesis (3.1) is clearly simplectically invariant, and is satisfied in the following important cases:*

1. *When $\operatorname{Im} c \in S(h^{-1}, g)$, and hence in particular when c is real;*
2. *When $\operatorname{Re} c \in S(h^{-1}, g)$, and hence in particular when c is purely imaginary;*
3. *In the counterexample given by Brummelhuis in* [1] *and all the counterexamples given in* [8] *(when $N = 2$). Hence, condition (3.1) is* **not** *a decisive restriction for the validity or the failure of the Fefferman-Phong inequality for systems;*
4. *When $p = p(\xi)$, i.e. when p has "constant coefficients" (which is, however, not an invariant condition). Note furthermore that by using the Fourier transform one obtains (3.3) at once just by the sole assumption that $p(\xi) = p(\xi)^* \geq -c\, I$, which shows that condition (3.2) is only sufficient for (3.3) to hold for 2×2 systems.*

Remark 3.3. *The eigenvalues of $p(X)$ are, of course,*

$$\lambda_\pm(X) = \frac{1}{2}\Big(a(X) + b(X) \pm \sqrt{(a(X) - b(X))^2 + 4|c(X)|^2}\Big), \quad X \in \mathbb{R}^{2n}.$$

Hence Theorem 3.1 (as well as Theorem 3.5 and Corollary 3.6 below) allows eigenvalue-crossings: we may have $\lambda_+ = \lambda_-$ at those X for which $a = b$ and $c = 0$. Notice that in this case any diagonalization procedure in general breaks down because of loss of smoothness of the eigenvectors of the symbol.

Notice, moreover, that

$$\lambda_-(X) = 0,\ \forall X \Longleftrightarrow |c(X)|^2 = a(X)b(X),\ \forall X \Longleftrightarrow \det p(X) = 0,\ \forall X.$$

In such a case

$$\lambda_+(X) = t(X) > \lambda_-(X) = 0, \quad \forall X.$$

Thus a Taylor-decoupling argument (see Taylor [12]*) is in this case possible, in the hope for exploiting the ellipticity of λ_+. However, we shall give in Section 4* *an example of system with elliptic trace, for which (3.1) (and (3.14) below) is not satisfied, and for which the Fefferman-Phong inequality cannot hold.*

Proof of the theorem. We start by proving the following lemma, which is a consequence of the ellipticity condition (3.2).

Lemma 3.4. *There exist $r > 0$, with $2r < C_0^{-1/2}$, and $c_1 > 0$ such that for any given $Y \in \mathbb{R}^{2n}$ we have*

$$\textit{either}\ \ a(X) \geq c_1\, h(X)^{-2} \ \ \textit{or}\ \ b(X) \geq c_1\, h(X)^{-2},\ \ \forall X \in B^g_{Y,2r}.$$

Proof of the lemma. We put for short $\tilde{r} = C_0^{-1/2}/2$. Let $z_1, \dots, z_{2n}$ be g_Y-orthonormal coordinates centered at Y. Then, with $B_{\tilde{r}}$ the Euclidean ball centered at 0 and radius $\tilde{r}$ (and representing X in z-coordinates),

$$X \in B^g_{Y,\tilde{r}} \iff z \in B_{\tilde{r}}.$$

Consider the nonnegative functions of $z \in B_{\tilde{r}}$,

$$f_1(z) = h(Y)^2 a(X), \quad f_2(z) = h(Y)^2 b(X).$$

If $v_1, \dots, v_{2n}$ are g_Y-orthonormal vectors associated with the z_j, one has

$$\partial_{z_j} f_1(z) = h(Y)^2 a^{(1)}(X; v_j), \quad 1 \leq j \leq 2n,$$

the same holding for f_2. For $X \in B^g_{Y,\tilde{r}}$ we have, with a universal constant $C > 0$,

$$h(Y) \leq C h(X), \ \ \text{and}\ \ g_X(w) \leq C g_Y(w),\ \forall w \in \mathbb{R}^{2n}.$$

Hence there is a universal constant $C' > 0$ such that for all $z \in B_{\tilde{r}}$ and $j = 1, \dots, 2n$,

$$|\partial_{z_j} f_1(z)| = h(Y)^2 \frac{|a^{(1)}(X; v_j)|}{g_Y(v_j)^{1/2}} \leq C^{5/2} h(X)^2 \frac{|a^{(1)}(X; v_j)|}{g_X(v_j)^{1/2}} \leq C', \tag{3.4}$$

for we have that $a \in S(h^{-2}, g)$. The same holds true for f_2.

Now, since $f_1(0) + f_2(0) \geq c_0$ we then have that

$$\text{either}\ \ f_1(0) \geq c_0/2, \ \ \text{or}\ \ f_2(0) \geq c_0/2,$$

whence, by virtue of (3.4), it is straitghforward to see that there is a universal radius $r > 0$, with $2r < \tilde{r}$, such that

$$\text{either}\ \ f_1(z) \geq c_0/4, \ \ \text{or}\ \ f_2(z) \geq c_0/4,\ \forall z \text{ with } |z| < 2r,$$

so that

$$\text{either } \frac{c_0}{4} \le C^2 h(X)^2 a(X), \quad \text{or } \frac{c_0}{4} \le C^2 h(X)^2 b(X), \ \forall X \in B^g_{Y,2r},$$

which concludes the proof of the lemma. □

Hence, using Lemma 3.4 and Lemma 2.6, we may find $0 < r < C_0^{-1/2}/2$ so small that, putting $B_\nu = B^g_{\nu,r}$ and $B^*_\nu = B^g_{\nu,2r}$,

$$\text{either } a(X) \ge c_1 h(X)^{-2} \text{ or } b(X) \ge c_1 h(X)^{-2}, \quad \forall X \in B^*_\nu.$$

Let then $\{\varphi_\nu\}_{\nu\in\mathbb{Z}_+}$ be a partition of unity associated with the B_ν, uniformly in $S(1,g)$, and let $\chi_\nu \in C_0^\infty(B^*_\nu)$, $0 \le \chi_\nu \le 1$, *uniformly* in $S(1,g)$, be such that $\chi_\nu\varphi_\nu = \varphi_\nu$, for all $\nu \in \mathbb{Z}_+$. Then

$$p = \sum_{\nu\in\mathbb{Z}_+} \varphi_\nu(\chi_\nu p)\varphi_\nu,$$

where $\chi_\nu p \in S(h^{-2}, g; \mathsf{M}_2)$ with bounds uniform in $\nu \in \mathbb{Z}_+$. Define next

$$E_\nu(X) = \begin{bmatrix} 1 & -\overline{c(X)}/a(X) \\ 0 & 1 \end{bmatrix}, \quad X \in B^*_\nu,$$

or

$$\tilde{E}_\nu(X) = \begin{bmatrix} 1 & 0 \\ -c(X)/b(X) & 1 \end{bmatrix}, \quad X \in B^*_\nu,$$

according to whether a or b is *elliptic*. Then, according to the cases,

$$E^*_\nu p E_\nu = \begin{bmatrix} a & 0 \\ 0 & b - |c|^2/a \end{bmatrix}, \quad X \in B^*_\nu,$$

or

$$\tilde{E}^*_\nu p \tilde{E}_\nu = \begin{bmatrix} a - |c|^2/b & 0 \\ 0 & b \end{bmatrix}, \quad X \in B^*_\nu.$$

Notice that, according to the case, we have

$$b - \frac{|c|^2}{a} \ge 0, \quad \text{or} \quad a - \frac{|c|^2}{b} \ge 0, \quad \text{on } B^*_\nu.$$

Also,

$$E_\nu^{-1} = \begin{bmatrix} 1 & \bar{c}/a \\ 0 & 1 \end{bmatrix}, \quad \tilde{E}_\nu^{-1} = \begin{bmatrix} 1 & 0 \\ c/b & 1 \end{bmatrix},$$

and *for the first order differentials* we have

$$(E_\nu^{\pm 1})' = \begin{bmatrix} 0 & \mp(\bar{c}/a)' \\ 0 & 0 \end{bmatrix}, \quad (\tilde{E}_\nu^{\pm 1})' = \begin{bmatrix} 0 & 0 \\ \mp(c/b)' & 0 \end{bmatrix}, \tag{3.5}$$

and in general

$$(3.6)\quad (E_\nu^{\pm 1})' E_\nu^j = (E_\nu^{\pm 1})' = E_\nu^j (E_\nu^{\pm 1})',\ \text{for every choice of } j = \pm 1,\ \forall \nu \in \mathbb{Z}_+.$$

Define now, according to the cases,

$$\alpha_\nu = E_\nu^{-1}\varphi_\nu, \quad \text{or} \quad \alpha_\nu = \tilde{E}_\nu^{-1}\varphi_\nu.$$

Then $\alpha_\nu \in S(1, g; \mathsf{M}_2)$ *uniformly* in $\nu \in \mathbb{Z}_+$, with *compact support* in B_ν, and by the Cotlar-Stein Lemma (see Lemma 2.7)

$$(3.7)\qquad \sum_{\nu \in \mathbb{Z}_+} \|\alpha_\nu^{\mathrm{w}} u\|_0^2 \le C\|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^2),$$

for a universal constant $C > 0$.

Without loss of generality we may clearly restrict summation on those ν such that (say) the 11-entry a of p is *elliptic* on B_ν^*. We now write

$$p = \sum_{\nu \in \mathbb{Z}_+} \alpha_\nu^* \Big(E_\nu^* \chi_\nu p E_\nu \Big) \alpha_\nu,$$

where

$$E_\nu^* \chi_\nu p E_\nu =: p_\nu \in S(h^{-2}, g; \mathsf{M}_2),\ \textit{uniformly} \text{ in } \ \nu \in \mathbb{Z}_+,$$

with

$$(3.8)\qquad p_\nu = \chi_\nu \begin{bmatrix} a & 0 \\ 0 & b - |c|^2/a \end{bmatrix}.$$

By (2.3) we have

$$\alpha_\nu^* \sharp p_\nu \sharp \alpha_\nu = \alpha_\nu^* p_\nu \alpha_\nu - \frac{i}{2}\big(\alpha_\nu^*\{p_\nu, \alpha_\nu\} + \{\alpha_\nu^*, p_\nu \alpha_\nu\}\big) + r_\nu,$$

where, by Lemma 2.7, $\sum_{\nu \in \mathbb{Z}_+} r_\nu^{\mathrm{w}}$ is bounded in $L^2(\mathbb{R}^n; \mathbb{C}^2)$ and

$$\alpha_\nu^*\{p_\nu, \alpha_\nu\},\ \{\alpha_\nu^*, p_\nu \alpha_\nu\} \in S(h^{-1}, g; \mathsf{M}_2),\ \textit{uniformly} \text{ in } \ \nu \in \mathbb{Z}_+.$$

One computes

$$\frac{i}{2}(\alpha_\nu^*\{p_\nu, \alpha_\nu\} + \{\alpha_\nu^*, p_\nu \alpha_\nu\}) = \beta_{1,\nu} + \beta_{2,\nu} + \beta_{3,\nu},$$

where

$$\beta_{1,\nu} = \beta_{1,\nu}^* := \frac{i}{2}\Big(\alpha_\nu^*\{p_\nu, E_\nu^{-1}\}\varphi_\nu + \varphi_\nu\{(E_\nu^{-1})^*, p_\nu\}\alpha_\nu\Big),$$

$$\beta_{2,\nu} = \beta_{2,\nu}^* := \frac{i}{2}\Big(\{(E_\nu^{-1})^*, \varphi_\nu\} p_\nu \alpha_\nu + \alpha_\nu^* p_\nu \{\varphi_\nu, E_\nu^{-1}\}\Big),$$

and

$$\beta_{3,\nu} = \beta^*_{3,\nu} := \frac{i}{2}\sum_{j=1}^{n}\left(\frac{\partial(E_\nu^{-1})^*}{\partial\xi_j}p_\nu\frac{\partial E_\nu^{-1}}{\partial x_j} - \frac{\partial(E_\nu^{-1})^*}{\partial x_j}p_\nu\frac{\partial E_\nu^{-1}}{\partial\xi_j}\right)\varphi_\nu^2.$$

Notice that

$$(\beta_{3,\nu})_{kk'} = \frac{i}{2}\varphi_\nu^2\sum_{k_1,k_2=1}^{2}(p_\nu)_{k_1k_2}\{(E_\nu^{-1})^*_{kk_1},(E_\nu^{-1})_{k_2k'}\} \in S(h^{-1},g),\ \ k,k'=1,2.$$

Therefore

$$\beta_{1,\nu},\beta_{2,\nu},\beta_{3,\nu} \in S(h^{-1},g;\mathsf{M}_2),\quad \textit{uniformly in } \nu\in\mathbb{Z}_+,$$

with *compact support* in B_ν. Moreover, a computation shows that

$$\beta_{3,\nu} = \frac{i}{2}a\varphi_\nu^2\{\frac{c}{a},\frac{\bar c}{a}\}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}. \tag{3.9}$$

By virtue of hypothesis (3.1) we have

$$\varphi_\nu^2\{\frac{c}{a},\frac{\bar c}{a}\} = \varphi_\nu^2\Big(\frac{1}{a^2}\{c,\bar c\} - \frac{2i}{a^3}\mathrm{Im}(c\{a,\bar c\})\Big) \in S(h^2,g),$$

uniformly in $\nu\in\mathbb{Z}_+$, whence (using the fact that $a\approx h^{-2}$ on $\operatorname{supp}\varphi_\nu$, uniformly in $\nu\in\mathbb{Z}_+$)

$$\beta_{3,\nu}\in S(1,g;\mathsf{M}_2),\ \ \textit{uniformly in}\ \ \nu\in\mathbb{Z}_+,$$

so that, by the Cotlar-Stein Lemma, writing $u = \begin{bmatrix}u_1\\ u_2\end{bmatrix}$,

$$\sum_{\nu\in\mathbb{Z}_+}(\beta_{3,\nu}^{\mathrm{w}}u,u) = O(\|u_2\|_0^2).$$

Thus

$$(p^{\mathrm{w}}u,u) = \sum_{\nu\in\mathbb{Z}_+}(p_\nu^{\mathrm{w}}\alpha_\nu^{\mathrm{w}}u,\alpha_\nu^{\mathrm{w}}u) + \sum_{\nu\in\mathbb{Z}_+}(\beta_{1,\nu}^{\mathrm{w}}u,u) + \sum_{\nu\in\mathbb{Z}_+}(\beta_{2,\nu}^{\mathrm{w}}u,u) + O(\|u\|_0^2),$$

and we have to control the terms with $\beta_{1,\nu}$ and $\beta_{2,\nu}$.

We next handle the term $(\beta_{1,\nu}^{\mathrm{w}}u,u)$. It is crucial to note that by (3.5) (see also (3.8))

$$\{p_\nu,E_\nu^{-1}\} = \begin{bmatrix}0 & \{\chi_\nu a,\bar c/a\}\\ 0 & 0\end{bmatrix}, \tag{3.10}$$

with $\{\chi_\nu a, \bar{c}/a\}$ belonging to $S(h^{-1}, g)$ uniformly in $\nu \in \mathbb{Z}_+$. Write then

$$\{\chi_\nu a, \bar{c}/a\}\varphi_\nu = \chi_\nu\sqrt{a}\,\frac{\{\chi_\nu a, \bar{c}/a\}}{\sqrt{a}}\varphi_\nu,$$

where $\frac{i}{2}\varphi_\nu\{\chi_\nu a, \bar{c}/a\}/\sqrt{a} =: \gamma_{1,\nu} \in S(1, g)$ uniformly in $\nu \in \mathbb{Z}_+$, is compactly supported, and $\sum_\nu \gamma_{1,\nu}^{\mathrm{w}}$ is bounded in $L^2(\mathbb{R}^n)$. We now write, with $\alpha_\nu^{\mathrm{w}} u = \begin{bmatrix} (\alpha_\nu^{\mathrm{w}} u)_1 \\ (\alpha_\nu^{\mathrm{w}} u)_2 \end{bmatrix}$,

$$\big((\alpha_\nu^*\{p_\nu, E_\nu^{-1}\}\varphi_\nu)^{\mathrm{w}} u, u\big) = \big((\{\chi_\nu a, \bar{c}/a\}\varphi_\nu)^{\mathrm{w}} u_2, (\alpha_\nu^{\mathrm{w}} u)_1\big) + ({r'}_{1,\nu}^{\mathrm{w}} u, u),$$

and

$$\frac{i}{2}\{\chi_\nu a, \bar{c}/a\}\varphi_\nu = (\chi_\nu\sqrt{a})\sharp\Big(\frac{i}{2}\frac{\{\chi_\nu a, \bar{c}/a\}}{\sqrt{a}}\varphi_\nu\Big) + r_{1,\nu} = (\chi_\nu\sqrt{a})\sharp\gamma_{1,\nu} + r_{1,\nu},$$

where $r_{1,\nu}$ and the entries of $r'_{1,\nu}$ belong to $\mathrm{Conf}(1, g_\nu, X_\nu, 2r)$ *uniformly in* $\nu \in \mathbb{Z}_+$ (recall that X_ν denotes the center of B_ν and $g_\nu = g_{X_\nu}$). Keeping into account that

$$(\{p_\nu, E_\nu^{-1}\})^* = -\{(E_\nu^{-1})^*, p_\nu\},$$

we get

$$(\beta_{1,\nu}^{\mathrm{w}} u, u) = 2\,\mathrm{Re}\,(\gamma_{1,\nu}^{\mathrm{w}} u_2, (\chi_\nu\sqrt{a})^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}} u)_1) + (\tilde{r}_{1,\nu}^{\mathrm{w}} u, u), \tag{3.11}$$

with $\sum_\nu \tilde{r}_{1,\nu}^{\mathrm{w}}$ bounded in L^2. By the Cauchy-Schwarz inequality, with $0 < \varepsilon < 1$ to be picked, we have

$$2\,|\mathrm{Re}\,(\gamma_{1,\nu}^{\mathrm{w}} u_2, (\chi_\nu\sqrt{a})^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}} u)_1)| \le \varepsilon((\chi_\nu^2 a)^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}} u)_1, (\alpha_\nu^{\mathrm{w}} u)_1) +$$
$$+\varepsilon(r_{1,\nu}^{(1)\mathrm{w}} u_1, u_1) + C_\varepsilon(r_{1,\nu}^{(2)\mathrm{w}} u_2, u_2),$$

where $\sum_\nu \mathrm{diag}(r_{1,\nu}^{(1)\mathrm{w}}, r_{1,\nu}^{(2)\mathrm{w}})$ is bounded in $L^2(\mathbb{R}^n, \mathbb{C}^2)$.

We now handle the term $(\beta_{2,\nu}^{\mathrm{w}} u, u)$. The crucial observation at this point is that, again by (3.5),

$$\{\varphi_\nu, E_\nu^{-1}\} = \begin{bmatrix} 0 & \{\varphi_\nu, \bar{c}/a\} \\ 0 & 0 \end{bmatrix}, \tag{3.12}$$

with $\{\varphi_\nu, \bar{c}/a\} \in S(h, g)$ uniformly in $\nu \in \mathbb{Z}_+$, whence we may write

$$p_\nu\{\varphi_\nu, E_\nu^{-1}\} = \chi_\nu\sqrt{a}\{\varphi_\nu, \bar{c}/a\}\sqrt{a}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where $\frac{i}{2}\{\varphi_\nu, \bar{c}/a\}\sqrt{a} =: \gamma_{2,\nu} \in S(1,g)$, uniformly in $\nu \in \mathbb{Z}_+$, is compactly supported, and $\sum_\nu \gamma_{2,\nu}^{\mathrm{w}}$ is bounded in L^2. Keeping into account that

$$(\{\varphi_\nu, E_\nu^{-1}\})^* = -\{(E_\nu^{-1})^*, \varphi_\nu\},$$

we get (as before in the case of $\beta_{1,\nu}^{\mathrm{w}}$)

$$(\beta_{2,\nu}^{\mathrm{w}}u, u) = 2\,\mathrm{Re}\,(\gamma_{2,\nu}^{\mathrm{w}}u_2, (\chi_\nu\sqrt{a})^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_1) + (r_{2,\nu}^{\mathrm{w}}u, u), \tag{3.13}$$

with $\sum_\nu r_{2,\nu}^{\mathrm{w}}$ bounded in L^2. Again, by the Cauchy-Schwarz inequality, with $0 < \varepsilon < 1$ to be picked, we have

$$2\,|\mathrm{Re}\,(\gamma_{2,\nu}^{\mathrm{w}}u_2, (\chi_\nu\sqrt{a})^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_1)| \le \varepsilon((\chi_\nu^2 a)^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_1, (\alpha_\nu^{\mathrm{w}}u)_1) +$$
$$+\varepsilon(r_{2,\nu}^{(1)\mathrm{w}}u_1, u_1) + C_\varepsilon(r_{2,\nu}^{(2)\mathrm{w}}u_2, u_2),$$

where $\sum_\nu \mathrm{diag}(r_{2,\nu}^{(1)\mathrm{w}}, r_{2,\nu}^{(2)\mathrm{w}})$ is bounded in $L^2(\mathbb{R}^n, \mathbb{C}^2)$.

Hence

$$(p_\nu^{\mathrm{w}}\alpha_\nu^{\mathrm{w}}u, \alpha_\nu^{\mathrm{w}}u) + (\beta_{1,\nu}^{\mathrm{w}}u, u) + (\beta_{2,\nu}^{\mathrm{w}}u, u) \ge$$
$$\ge ((\chi_\nu a - 2\varepsilon\chi_\nu^2 a)^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_1, (\alpha_\nu^{\mathrm{w}}u)_1) + ((\chi_\nu(b - |c|^2/a))^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_2, (\alpha_\nu^{\mathrm{w}}u)_2) +$$
$$+C_\varepsilon(\tilde{r}_{1,\nu}^{\mathrm{w}}u_1, u_1) + C'_\varepsilon(\tilde{r}_{2,\nu}^{\mathrm{w}}u_1, u_1),$$

with $\sum_\nu \mathrm{diag}(\tilde{r}_{1,\nu}^{\mathrm{w}}, \tilde{r}_{2,\nu}^{\mathrm{w}})$ bounded in $L^2(\mathbb{R}^n, \mathbb{C}^2)$. Choose therefore $\varepsilon = 1/4$, so that by the scalar Fefferman-Phong inequality, namely by Theorem 1.2 in the case of the constant metric g_ν, we have that there are *universal* constants $C_1, C_2 > 0$, *independent of* $\nu \in \mathbb{Z}_+$, such that

$$((\chi_\nu a - 2\varepsilon\chi_\nu^2 a)^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_1, (\alpha_\nu^{\mathrm{w}}u)_1) \ge -C_1\|(\alpha_\nu^{\mathrm{w}}u)_1\|_0^2,$$

and

$$((\chi_\nu(b - |c|^2/a))^{\mathrm{w}}(\alpha_\nu^{\mathrm{w}}u)_2, (\alpha_\nu^{\mathrm{w}}u)_2) \ge -C_2\|(\alpha_\nu^{\mathrm{w}}u)_2\|_0^2,$$

which finally yields, by (3.7),

$$(p^{\mathrm{w}}u, u) \ge -C_3 \sum_{\nu\in\mathbb{Z}_+} \|\alpha_\nu^{\mathrm{w}}u\|_0^2 + O(\|u\|_0^2) \ge -C_4\|u\|_0^2,$$

and proves the theorem. □

From Theorem 3.1 we are now in a position to derive the following neat result.

Theorem 3.5. *Let g be an admissible metric. Let*

$$p(X)=\begin{bmatrix} a(X) & \overline{c(X)} \\ c(X) & b(X) \end{bmatrix}=p(X)^*\geq 0,\quad X\in\mathbb{R}^{2n},$$

where $a,b,c\in S(h^{-2},g)$ and where the matrix-trace of p satisfies the ellipticity hypothesis (3.2):

$$a(X)+b(X)\approx h(X)^{-2},\quad \forall X\in\mathbb{R}^{2n}.$$

Suppose that there are constants $\theta_1,\theta_2\in(0,1)$ and a symbol $\omega_1\in S(1,g)$ with $\theta_1\leq\omega_1(X)\leq\theta_2$ for all $X\in\mathbb{R}^{2n}$ such that, writing $c=c_1+ic_2$ and putting $\omega_2:=1-\omega_1$, one has

$$c_j(X)^2\leq\omega_j(X)^2a(X)b(X),\quad \forall X\in\mathbb{R}^{2n},\ j=1,2. \tag{3.14}$$

Then there exists $C>0$ such that

$$(p^{\mathrm{w}}(x,D)u,u)\geq -C\|u\|_0^2,\quad \forall u\in\mathcal{S}(\mathbb{R}^n;\mathbb{C}^2).$$

Proof. We write

$$p(X)=\begin{bmatrix} \omega_1(X)a(X) & c_1(X) \\ c_1(X) & \omega_1(X)b(X) \end{bmatrix}+\begin{bmatrix} \omega_2(X)a(X) & -ic_2(X) \\ ic_2(X) & \omega_2(X)b(X) \end{bmatrix}=$$
$$=: p_1(X)+p_2(X).$$

Then, in view of hypothesis (3.14), each $p_j(X)=p_j(X)^*\geq 0$, $j=1,2$, and they both satisfy the hypotheses of Theorem 3.1, whence the result. □

Using a *scalar* reduction of order, one may prove the following more general version of Theorems 3.1 and 3.5.

Corollary 3.6. *Let g be an admissible metric and m be a g-admissible weight. Let*

$$p(X)=\begin{bmatrix} a(X) & \overline{c(X)} \\ c(X) & b(X) \end{bmatrix}=p(X)^*\geq 0,\quad X\in\mathbb{R}^{2n},$$

where $a,b,c\in S(m,g)$ and where the matrix-trace of p satisfies the ellipticity hypothesis:

$$a(X)+b(X)\approx m(X),\quad \forall X\in\mathbb{R}^{2n}.$$

Suppose that either

- $a\{c,\bar c\}-2i\,\mathrm{Im}(c\{a,\bar c\})$ *and* $b\{c,\bar c\}-2i\,\mathrm{Im}(c\{b,\bar c\})\in S(h^2m^3,g)$, *or that there are constants $\theta_1, \theta_2 \in (0,1)$ and a symbol $\omega_1 \in S(1,g)$ with $\theta_1 \leq \omega_1(X) \leq \theta_2$ for all $X \in \mathbb{R}^{2n}$ such that, writing $c = c_1 + ic_2$ and putting $\omega_2 := 1 - \omega_1$, one has*

- $$c_j(X)^2 \leq \omega_j(X)^2 a(X) b(X), \quad \forall X \in \mathbb{R}^{2n},\ j = 1, 2.$$

Then there exists a symbol $q = q^* \in S(h^2 m, g; \mathsf{M}_2)$ such that

$$p^{\mathrm{w}}(x,D) \geq q^{\mathrm{w}}(x,D).$$

4. On conditions (3.1) and (3.14)

In this section we show an example of 2×2 system with positive-elliptic trace, for which conditions (3.1) and (3.14) are not satisfied, and for which the Fefferman-Phong inequality does not hold.

Let

$$L(x,\xi) = \begin{bmatrix} \xi_1 \\ (1 - ix_1)\xi_2 \end{bmatrix},$$

and let

$$A(x,\xi) = L(x,\xi)^* \otimes L(x,\xi) = L(x,\xi)\, {}^t\overline{L(x,\xi)}$$

("column-times-row"). Hence

$$A(x,\xi) = \begin{bmatrix} \xi_1^2 & (1 + ix_1)\xi_1\xi_2 \\ (1 - ix_1)\xi_1\xi_2 & (1 + x_1^2)\xi_2^2 \end{bmatrix},$$

and

$$t(x,\xi) = \lambda_+(x,\xi) = |L(x,\xi)|^2 = \xi_1^2 + (1 + x_1^2)\xi_2^2,$$

is thus elliptic, with $\lambda_- \equiv 0$. It is also readily seen that both (3.1) and (3.14) do not hold.

It is convenient here to refer to the following (localized) version of the Fefferman-Phong inequality: *For any given compact $K \subset \mathbb{R}^n$ there exists $C_K > 0$ such that*

$$(A^{\mathrm{w}}(x,D)u, u) \geq -C_K \|u\|_0^2, \quad \forall u \in C_0^\infty(K; \mathbb{C}^2). \tag{4.1}$$

Lemma 4.1. *For the system $A^{\mathrm{w}}(x,D)$ the Fefferman-Phong inequality (4.1) cannot hold.*

Proof. Consider

$$w_-(x,\xi)=\begin{bmatrix}-(1+ix_1)\xi_2\\ \xi_1\end{bmatrix}\in \operatorname{Ker} A(x,\xi),$$

and

$$w_+(x,\xi)=L(x,\xi)=\begin{bmatrix}\xi_1\\ (1-ix_1)\xi_2\end{bmatrix}\in \operatorname{Ker} A(x,\xi)^{\perp}.$$

It is well known that we may find classical (properly supported) pseudodifferential operators $B_-^{\mathrm{w}}(x,D)$ and $\Lambda_-^{\mathrm{w}}(x,D)$, for which

$$A^{\mathrm{w}}(x,D)B_-^{\mathrm{w}}(x,D)=\Lambda_-^{\mathrm{w}}(x,D)B_-^{\mathrm{w}}(x,D)+\text{smoothing operator},$$

where

$$B_-(x,\xi)\sim v_-^{(0)}(x,\xi)+v_-^{(-1)}(x,\xi)+\dots,\ \text{ has order } 0,$$

and

$$\Lambda_-(x,\xi)\sim \lambda_-(x,\xi)+\lambda_-^{(1)}(x,\xi)+\dots,\ \text{ has thus order } 1.$$

We find out from $Av_-^{(0)}=0$ that we may take $v_-^{(0)}(x,\xi)=w_-(x,\xi)/|\xi|$, and from

$$-\frac{i}{2}\{A,v_-^{(0)}\}+Av_-^{(-1)}=\lambda_-^{(1)}v_-^{(0)}$$

that $\lambda_-^{(1)}$ is chosen so as to have

$$\lambda_-^{(1)}v_-^{(0)}+\frac{i}{2}\{A,v_-^{(0)}\}\in \operatorname{Ker} A^{\perp}.$$

We have

$$\{A,v_-^{(0)}\}=\{A,\frac{1}{|\xi|}\}w_-+\frac{1}{|\xi|}\{A,w_-\},$$

and we see that on the one hand

$$\frac{i}{2}\{A,\frac{1}{|\xi|}\}w_-=\frac{i}{2}\frac{\xi_1}{|\xi|^3}\begin{bmatrix}0 & i\xi_1\xi_2\\ -i\xi_1\xi_2 & 2x_1\xi_2^2\end{bmatrix}\begin{bmatrix}-(1+ix_1)\xi_2\\ \xi_1\end{bmatrix}=$$

$$=-\frac{\xi_1^2\xi_2}{2|\xi|^3}\begin{bmatrix}\xi_1\\ (1-ix_1)\xi_2\end{bmatrix}=-\frac{\xi_1^2\xi_2}{2|\xi|^3}w_+,$$

and that on the other

$$\frac{i}{2}\frac{1}{|\xi|}\{A,w_-\}=\frac{i}{2}\frac{1}{|\xi|}\Big(L\{{}^t\bar{L},w_-\}+\frac{\partial L}{\partial\xi_1}\,{}^t\bar{L}\,\frac{\partial w_-}{\partial x_1}-\frac{\partial L}{\partial x_1}\,{}^t\bar{L}\,\frac{\partial w_-}{\partial \xi_1}\Big)=$$

$$=\frac{\xi_2}{|\xi|}\Big(w_++\frac{1}{2}\begin{bmatrix}\xi_1\\ -(1+ix_1)\xi_2\end{bmatrix}\Big).$$

Hence

$$\lambda_-^{(1)}\frac{w_-}{|\xi|}+\frac{i}{2}\{A,\frac{w_-}{|\xi|}\}=\lambda_-^{(1)}\frac{w_-}{|\xi|}+\Big(\frac{\xi_2}{|\xi|}-\frac{\xi_1^2\xi_2}{2|\xi|^3}\Big)w_+ +\frac{\xi_2}{2|\xi|}\begin{bmatrix}\xi_1\\ -(1+ix_1)\xi_2\end{bmatrix},$$

and imposing

$$\Big\langle\lambda_-^{(1)}\frac{w_-}{|\xi|}+\frac{i}{2}\{A,\frac{w_-}{|\xi|}\},\frac{w_-}{|\xi|}\Big\rangle_{\mathbb{C}^2}=0$$

yields the equation

$$\lambda_-^{(1)}\frac{|w_-|^2}{|\xi|^2}+\frac{\xi_2}{2|\xi|^2}\Big\langle\begin{bmatrix}\xi_1\\ -(1+ix_1)\xi_2\end{bmatrix},\begin{bmatrix}-(1+ix_1)\xi_2\\ \xi_1\end{bmatrix}\Big\rangle_{\mathbb{C}^2}=0,$$

that is, finally,

$$\lambda_-^{(1)}(x,\xi)=\frac{\xi_1\xi_2^2}{\lambda_+(x,\xi)},\quad \xi\neq 0. \tag{4.2}$$

Now consider a 0th-order classical (properly supported) pseudodifferential operator $B_+^{\mathrm{w}}(x,D)$ with principal symbol $w_+/|\xi|$. Then

$$B^{\mathrm{w}}(x,D)\begin{bmatrix}f_1\\ f_2\end{bmatrix}:=B_-^{\mathrm{w}}(x,D)f_1+B_+^{\mathrm{w}}(x,D)f_2=$$

$$=\Big[B_-^{\mathrm{w}}(x,D)\Big|B_+^{\mathrm{w}}(x,D)\Big]\begin{bmatrix}f_1\\ f_2\end{bmatrix},\quad f_1,f_2\in C_0^\infty,$$

is *elliptic*, and one has

$$(B^{\mathrm{w}})^*A^{\mathrm{w}}B^{\mathrm{w}}=\begin{bmatrix}(B_-^{\mathrm{w}})^*\Lambda_-^{\mathrm{w}}B_-^{\mathrm{w}} & (B_-^{\mathrm{w}})^*A^{\mathrm{w}}B_+^{\mathrm{w}}\\ (B_+^{\mathrm{w}})^*A^{\mathrm{w}}B_-^{\mathrm{w}} & (B_+^{\mathrm{w}})^*A^{\mathrm{w}}B_+^{\mathrm{w}}\end{bmatrix}+\text{smoothing operator},$$

where the principal symbol of $(B_+^{\mathrm{w}})^*A^{\mathrm{w}}B_+^{\mathrm{w}}$ is $\lambda_+|w_+|^2/|\xi|^2$, and that of $(B_+^{\mathrm{w}})^*A^{\mathrm{w}}B_-^{\mathrm{w}}=\big((B_-^{\mathrm{w}})^*A^{\mathrm{w}}B_+^{\mathrm{w}}\big)^*$ is $\lambda_-^{(1)}\langle w_-,w_+\rangle/|\xi|^2$. Since $\langle w_-,w_+\rangle=0$, one therefore has that $(B_+^{\mathrm{w}})^*A^{\mathrm{w}}B_-^{\mathrm{w}}$ and $(B_-^{\mathrm{w}})^*A^{\mathrm{w}}B_+^{\mathrm{w}}$ are 0th-order operators, and thus they are bounded in L^2. Hence, using the ellipticity of $(B_+^{\mathrm{w}})^*A^{\mathrm{w}}B_+^{\mathrm{w}}$, one sees that the Fefferman-Phong inequality holds for A^{w} iff the Sharp-Gårding inequality holds for $(B_-^{\mathrm{w}})^*\Lambda_-^{\mathrm{w}}B_-^{\mathrm{w}}$, that is iff the principal symbol $\lambda_-^{(1)}|w_-|^2/|\xi|^2$ of $(B_-^{\mathrm{w}})^*\Lambda_-^{\mathrm{w}}B_-^{\mathrm{w}}$ is nonnegative. But by (4.2) $\lambda_-^{(1)}$ is not nonnegative, whence the Fefferman-Phong inequality cannot hold for A^{w}. □

5. Final remarks

5.1. **Why taking $N = 2$?** The reduction procedure cannot in general be iterated, for it may destroy the ellipticity assumption on the trace, which is the reason why we have to take $N = 2$. This is easily seen by considering the system

$$A(x,\xi) = \begin{bmatrix} \xi_1^2 & ix_1\xi_1\xi_2 & 0 \\ -ix_1\xi_1\xi_2 & x_1^2\xi_2^2 & 0 \\ 0 & 0 & \xi_2^2 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$A_B(x,\xi)$}} & 0 \\ & & 0 \\ \hline 0 & 0 & \xi_2^2 \end{array}\right], \ (x,\xi) \in \mathbb{R}^{2n}.$$

The trace of $A(x,\xi)$ is (positive) elliptic, and the off-diagonal entries of $A(x,\xi)$ fulfill hypothesis (3.1). However, $A^{\mathrm{w}}(x,D)$ fails to satisfy the Fefferman-Phong inequality on functions $u \in C_0^\infty(\mathbb{R}^{2n};\mathbb{C}^3)$ of the kind $u = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$ where $u_1, u_2 \in C_0^\infty(|x_1|, |x_2| \leq 1)$, because of the 2×2 block $A_B(x,\xi)$ (see (1.2)).

5.2. **Optimality of the ellipticity condition.** The assumption that the trace be elliptic is by no means optimal. Let $a, b \in S(h^{-2}, g)$, real-valued and such that $a(X) \geq |b(X)|$ for all $X \in \mathbb{R}^{2n}$. Consider then the system

$$A(X) = a(X)\, I + b(X) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A(X)^* \geq 0.$$

Using a *constant* unitary transformation $U\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that

$$U^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we may write

$$U^* A(X) U = \begin{bmatrix} a(X) + b(X) & 0 \\ 0 & a(X) - b(X) \end{bmatrix}.$$

By the hypothesis we have $a \pm b \geq 0$, so that we may use the scalar Fefferman-Phong inequality on the diagonal. Hence the Fefferman-Phong inequality holds for $A^{\mathrm{w}}(x,D)$.

5.3. $N \times N$ **examples.** Following the construction given in [8], it is now an easy matter to construct systems of size $N \times N$ (that are, however, 2×2 "in disguise") with determinant identically zero, for which the Fefferman-Phong inequality holds. We have the following corollary of Theorem 3.1.

Corollary 5.1. *Let v_1, v_2 be orthonormal vectors of $\mathbb{C}^N$, $N \geq 3$. Let g be an admissible metric. Let $q_1, q_2 \in S(h^{-1}, g)$ be real-valued with*

$$q_1(X)^2 + q_2(X)^2 \approx h(X)^{-2}, \ \forall X \in \mathbb{R}^{2n},$$

and let

$$L\colon \mathbb{R}^{2n} \ni X \longmapsto L(X) = q_1(X)v_1 + q_2(X)v_2 \in \mathbb{C}^N.$$

Consider the symbol

$$0 \leq p = L^* \otimes L = p^* \in S(h^{-2}, g; \mathsf{M}_N). \tag{5.1}$$

Then there exists $C > 0$ such that

$$(p^{\mathrm{w}}(x, D)u, u) \geq -C\|u\|_0^2, \quad \forall u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^2).$$

Recall that $(v^* \otimes v)w = \langle w, v\rangle_{\mathbb{C}^N} v$.

Proof. After having completed the set $\{v_1, v_2\}$ into a unitary basis of $\mathbb{C}^N$, take a unitary *constant* matrix $U\colon \mathbb{C}^N \to \mathbb{C}^N$ such that $Uv_j = e_j$, $j = 1, \dots, N$, where $\{e_1, \dots, e_N\}$ is the canonical basis of $\mathbb{C}^N$. Hence, since $U(v^* \otimes v)U^* = (Uv)^* \otimes (Uv)$, we have that

$$Up(X)U^* = \left[\begin{array}{cc|c} q_1(X)^2 & q_1(X)q_2(X) & \\ q_1(X)q_2(X) & q_2(X)^2 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right], \quad \forall X \in \mathbb{R}^{2n}.$$

Since

$$q_1(X)^2 + q_2(X)^2 \approx h(X)^{-2},$$

and $q_1 q_2$ is real-valued, we are in a position to use Theorem 3.1, which yields the desired conclusion. □

Notice that in (5.1) the system p has constant rank 1 and positive elliptic trace.

Of course, it is now straightforward to prove also the following corollary.

Corollary 5.2. *Given real-valued symbols* $q_{2j-1}, q_{2j} \in S(h^{-1}, g)$*,* $1 \le j \le d$*, and orthonormal vectors* v_{2j-1}, v_{2j}*,* $1 \le j \le d$*, of* $\mathbb{C}^N$*, where* $N \ge 2d$*, define*

$$L_j \colon \mathbb{R}^{2n} \ni X \longmapsto L_j(X) = q_{2j-1}(X)v_{2j-1} + q_{2j}(X)v_{2j},\ 1 \le j \le d,$$

and consider the $N \times N$ *system*

$$0 \le p = \sum_{j=1}^{d} L_j^* \otimes L_j = p^* \in S(h^{-2}, g; \mathsf{M}_N). \tag{5.2}$$

Suppose that for every $j = 1, \ldots, d$*,*

$$q_{2j-1}(X)^2 + q_{2j}(X)^2 \approx h(X)^{-2},\ \forall X \in \mathbb{R}^{2n}. \tag{5.3}$$

Then $p^{\mathrm{w}}(x, D)$ *satisfies the Fefferman-Phong inequality.*

Notice that in (5.2) the system p has constant rank d and positive elliptic trace. Notice also that as soon as condition (5.3) fails for some j, then Corollary 5.2 cannot hold in general.

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