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Formal Meromorphic Functions on Manifolds of Finite Type

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Dedicated to Professor J.J.Kohn on the occasion of his 75th birthday

Abstract: It is shown that a real-valued formal meromorphic function on a formal generic submanifold of finite Kohn-Bloom-Graham type is necessarily constant.

1. Introduction

It is easy to see (and known, see [1]) that if $M \subset \mathbb{C}^N$ is a connected generic real-analytic CR manifold which is of finite type in the sense of Kohn [5] and Bloom-Graham [4] at some point $p \in M$, then any meromorphic map $H\colon U \to \mathbb{C}^m$ defined on a connected neighbourhood of M which satisfies $H(M) \subset E$, where $E \subset \mathbb{C}^m$ is a totally real real-analytic submanifold, is necessarily constant.

Let us give a short proof of this fact. First, we recall the definition of the Segre sets S_p^j. These are defined inductively. First, we define the Segre variety $S_p = S_p^1$

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for $p \in M$. Let $\rho(Z,\bar Z) = (\rho_1(Z,\bar Z),\dots,\rho_d(Z,\bar Z))$ be a (vector-valued) defining function for M defined in a neighbourhood $U \times \bar U$ of $(p,\bar p)$, i.e.

$$M \cap U = \{Z \in U\colon \rho(Z,\bar Z) = 0\}, \quad d\rho_1 \wedge \cdots d\rho_d \neq 0 \text{ on U}, \quad \rho(Z,\bar Z) = \bar\rho(\bar Z, Z).$$

Then if S_q^1 is defined by

$$S_q^1 = \{Z \in U\colon \rho(Z,\bar q) = 0\}, \quad q \in U,$$

the j-th Segre set S_p^j, $j \in \mathbb{N}$, is defined inductively by

$$S_p^j = \bigcup_{q \in S_p^{j-1}} S_q^1.$$

We are using the following Theorem, which characterizes finite type in terms of properties of the Segre sets:

Theorem 1 (Baouendi, Ebenfelt and Rothschild [1]). *Let $M \subset \mathbb{C}^N$ be a generic real-analytic CR manifold. Then M is of finite type at $p \in M$ if and only if there exists an open set $V \subset \mathbb{C}^N$ with $V \subset S_p^{d+1}$.*

Now assume that $H\colon U \to \mathbb{C}^m$ is a meromorphic map which satisfies $H(M) \subset E$, where E is totally real. First note that since M is of finite type at some point p, it is of finite type on the complement of a proper real-analytic subvariety $F \subset M$. So there exists a point $p \in M$ with the property that M is of finite type at p and H is holomorphic in some neighbourhood of p (because M is generic, it is a set of uniqueness for holomorphic functions). We shall prove that in this situation, H is constant on an open set in $\mathbb{C}^N$, and thus constant.

We can find coordinates η in $\mathbb{C}^m$ such that near $H(p)$, E is given by an equation of the form $\eta = \varphi(\bar\eta)$. Thus, $H(Z) = \varphi(\bar H(\bar Z))$, whenever $Z \in M$, and from this we have that $H(Z) = \varphi(\bar H(\zeta))$ whenever $Z \in S_\zeta$ (restricting to a suitable neighbourhood U of p). Thus, $H(Z) = \varphi(\bar H(p))$ for $Z \in S_p$; since $p \in S_p$, $H(Z) = H(p)$ for $Z \in S_p$. Now we consider $Z \in S_p^2$. For each such Z, there is $\zeta \in S_p^1$ with $Z \in S_\zeta^1$. Our equation tells us that $H(Z) = \varphi(\bar H(\zeta)) = \varphi(\bar H(p))$, and again, since $p \in S_p^2$, $H(Z) = H(p)$ for $Z \in S_p^2$.

Continuing the iteration process like this, we see that $H(Z) = H(p)$ for $Z \in S_p^j$ for $j \in \mathbb{N}$. Since S_p^{d+1} contains an open subset of $\mathbb{C}^N$ by Theorem 1, the identity principle implies that $H(Z) = H(p)$ on U. This proves the constancy of such an H.

Our main point in this paper is the extension of this result to the formal category. Here we cannot "move to a good point". In this setting, a *formal meromorphic map* is given by $H = \frac{N}{D}$, where D is a (nonvanishing) formal power series and $N\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$ is a formal holomorphic map. Note that if $E \subset \mathbb{C}^m$ is a formal totally real manifold, then in suitable coordinates $\eta \in \mathbb{C}^m$, E is given by $\operatorname{Im} \eta = 0$. We say that $H = N/D$ maps M into E if for any formal map $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ satisfying $\rho(\gamma_1(t), \gamma_2(t)) = 0$ for every defining function ρ of M we have

$$N_j(\gamma_1(t))\bar{D}(\gamma_2(t)) - \bar{N}_j(\gamma_2(t))D(\gamma_1(t)) = 0$$

for every $j = 1, \ldots, m$. We shall freely use the terminology of formal real submanifolds as explained in e.g. [2]. We show the following:

Theorem 2. *Let $M \subset \mathbb{C}^N$ be a formal generic manifold of finite type, $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^m, 0)$ a formal meromorphic map which satisfies $H(M) \subset E$, where E is a formal totally real manifold. Then H is formal holomorphic, and thus, constant.*

We note that the finite type assumption is necessary. Indeed, every manifold of the form $M = \tilde{M} \times E$ where $\tilde{M}$ is some CR manifold and E is totally real has nonconstant CR maps onto a totally real manifold (the projection onto its second coordinate). On the other hand, here is another example, due to J. Lebl:

Example 1. Let $M \subset \mathbb{C}^3$ be given by

$$w_1 = \overline{w_1} e^{ip|z|^2}, \quad w_2 = \overline{w_2} e^{iq|z|^2},$$

for some integers p and q. Then the function

$$H(z, w_1, w_2) = \frac{w_1^q}{w_2^p}$$

maps M into $\mathbb{R}$ and is not the restriction of a holomorphic function. Also note that this function is not even continuous on M. Our results imply that *no* nonconstant holomorphic choice of projection onto $\mathbb{R}$ can be made.

2. Reflection Identities and Consequences

We shall first show that we can simplify our situation somewhat by choosing "normal" coordinates. Recall that normal coordinates for a formal generic submanifold $(M,0) \subset (\mathbb{C}^N,0)$ means a choice of coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d$ (d being the real codimension of $(M,0)$) together with formal functions $Q_j(z,\chi,\tau) \in \mathbb{C}[[z,\chi,\tau]]$, $j=1,\dots,d$, satisfying

$$Q_j(z,0,\tau) = Q_j(0,\chi,\tau) = \tau_j, \quad j=1,\dots,d,$$

such that $w_j - Q_j(z,\chi,\tau)$ generate the manifold ideal associated to $(M,0)$ in $\mathbb{C}[[z,w,\chi,\tau]]$. We will write $Q=(Q_1,\dots,Q_d)$, and abbreviate the generating set with $w - Q(z,\chi,\tau)$.

We will show that in normal coordinates, a formal meromorphic function H which maps $(M,0)$ into $(\mathbb{R},0)$ actually only depends on the transverse variables w. To do this, we first give a reflection identity which we will use.

Proposition 1. *If $(M,0) \subset (\mathbb{C}^N,0)$ is a formal generic submanifold, and (z,w) are normal coordinates for $(M,0)$ with corresponding generators $w - Q(z,\chi,\tau)$. If $H = \frac{N}{D} \colon (M,0) \to (\mathbb{R},0)$ is formal meromorphic, and N and D do not have any common factors, then there exists a formal holomorphic function $a(z,\chi,z^1,w)$, with $a(0,0,0,0)=1$, such that*

$$\begin{aligned} N\left(z, Q\left(z,\chi,\bar{Q}\left(\chi,z^1,w\right)\right)\right) &= a(z,\chi,z^1,w) N\left(z^1,w\right), \\ D\left(z, Q\left(z,\chi,\bar{Q}\left(\chi,z^1,w\right)\right)\right) &= a(z,\chi,z^1,w) D\left(z^1,w\right). \end{aligned} \tag{1}$$

Proof. The conclusion is clear if N is identically zero, so we assume that this is not the case. By definition, we have

$$\bar{D}(\chi,\tau) N(z,Q(z,\chi,\tau)) = \bar{N}(\chi,\tau) D(z,Q(z,\chi,\tau)). \tag{2}$$

Taking the complex conjugate of the series and replacing χ by z^1, τ by w, and z by χ in this equation, we also have that

$$D(z^1,w)\bar{N}(\chi,\bar{Q}(\chi,z^1,w)) = N(z^1,w)\bar{D}(\chi,\bar{Q}(\chi,z^1,w)). \tag{3}$$

We now substitute $\tau = \bar{Q}(\chi,z^1,w)$ into (2) to obtain

$$\begin{aligned} &\bar{D}(\chi,\bar{Q}(\chi,z^1,w)) N(z,Q(z,\chi,\bar{Q}(\chi,z^1,w))) \\ &\quad = \bar{N}(\chi,\bar{Q}(\chi,z^1,w)) D(z,Q(z,\chi,\bar{Q}(\chi,z^1,w))). \end{aligned} \tag{4}$$

We now multiply the left (and right, respectively) hand sides of (3) and (4) with each other, and after cancelling the (nonvanishing) common factor $\bar{N}(\chi,\bar{Q}(\chi,z^1,w))\bar{D}(\chi,\bar{Q}(\chi,z^1,w))$ we obtain

$$D(z^1,w) N(z,Q(z,\chi,\bar{Q}(\chi,z^1,w))) = D(z,Q(z,\chi,\bar{Q}(\chi,z^1,w))) N(z^1,w). \tag{5}$$

Now, using the fact that N and D do not have any common factors, unique factorization in the ring $\mathbb{C}[[z,\chi,z^1,w]]$ implies that there exists a unit $a(z,\chi,z^1,w)$ such that (1) holds. By evaluating (1) at $z=z^1$, and using the reality property $Q(z,\chi,\bar{Q}(\chi,z,w))=w$, we have that $a(z,\chi,z,w)=1$, so in particular, $a(0,0,0,0)=1$. □

Lemma 2. *Let $(M,0)\subset(\mathbb{C}^N,0)$ be a formal generic submanifold. Assume that $H(Z)=\frac{N(Z)}{D(Z)}$ is a formal meromorphic map sending $(M,0)$ into $(\mathbb{R},0)$. Then for any choice of normal coordinates (z,w) for $(M,0)$, we have that $H(z,w)=H(0,w)$; i.e., there exist formal functions $\tilde{N}(w)$ and $\tilde{D}(w)$ such that $H(z,w)=\frac{\tilde{N}(w)}{\tilde{D}(w)}$.*

Proof. We use Proposition 1. Setting $\chi=z^1=0$, we see that

$$N(z,w)=a(z,0,0,w)N(0,w),\quad D(z,w)=a(z,0,0,w)D(0,w).$$

The Lemma follows. □

3. Prolongation of the reflection along Segre maps and proof of Theorem 2

We will denote by

$$v^1(z,\chi,z^1;w)=Q(z,\chi,\bar{Q}(\chi,z^1,w));$$

in the usual Segre-map terminology, $v^1(z,\chi,z^1;0)$ is the transversal component of the second Segre map of $(M,0)$. We define $S^{(0)}=z$, and for $j\geq 1$

$$S^{(j)}=(z,\chi,z^1,\chi^1,\ldots,z^j),$$

and write $S_k^{(j)}=(z^k,\chi^k,\ldots,z^j)$ for $k\leq j$. With that notation and our simplification from Lemma 2, our reflection identity (1) now reads

$$\begin{aligned} N\left(v^1(S^{(1)};w)\right)&=a(S^{(1)},w)N\left(w\right),\\ D\left(v^1(S^{(1)};w)\right)&=a(S^{(1)},w)D\left(w\right). \end{aligned}\tag{6}$$

For $j\geq 2$, we define inductively

$$v^j\left(S^{(j)};w\right)=v^1(z,\chi,z^1;v^{j-1}(S_1^{(j)};w)).$$

We can now state the finite type criterion of Baouendi, Ebenfelt and Rothschild [3], for later reference, as follows:

Theorem 3. *If $(M,0)$ is of finite type in the sense of Kohn-Bloom-Graham, then there exists a $j \geq 1$ such that*

$$S^{(j)} \mapsto v^j\left(S^{(j)};0\right), \quad (\mathbb{C}^{(2j-1)n},0) \to (\mathbb{C}^d,0),$$

is of generic full rank d.

Thus, if we for $j \geq 2$ replace w by $v^{j-1}(S_1^{(j)};w)$ in (6), we obtain

$$\begin{aligned} N\left(v^j(S^{(j)};w)\right) &= N\left(v^1(S^{(1)};v^{j-1}(S_1^{(j)};w))\right) \\ &= a(S^{(1)};v^{j-1}(S_1^{(j)};w))N\left(v^{j-1}(S_1^{(j)};w)\right). \end{aligned}$$

Applying induction, we see that the following holds:

Lemma 3. *For every $j \geq 1$, there exists a unit $a_j(S^{(j)},w)$ such that*

$$N\left(v^j(S^{(j)};w)\right) = a_j(S^{(j)},w)N(w), \quad D\left(v^j(S^{(j)};w)\right) = a_j(S^{(j)},w)D(w). \tag{7}$$

We can now prove Theorem 2: By Theorem 3, there exists a j such that $v^j(S^{(j)};0)$ is of generic full rank. Assuming that $D(0) = 0$, we see that $D(v^j(S^{(j)};0)) = 0$. Since v^j is of generic full rank, this implies that $D(w) = 0$; this contradiction shows that $D(0) \neq 0$. Hence, we can assume that $H(w) = N(w)$ is holomorphic, and without loss of generality, $N(0) = 0$. Now the same argument as before shows that $N(w) = 0$, and so, H is constant.

Remark 1. More generally, if we do not assume that $(M,0)$ is of finite type, then we can define the formal variety

$$V_j = \overline{\text{image}(v^j(S^{(j)};0))} \cong \{f \in \mathbb{C}[[w]] \colon f \circ v^j(S^{(j)};0) = 0\},$$

and $V = \cup_j V_j$ (which is again a formal variety). The same arguments as above show that D, as well as N, are constant on V. This corresponds to the statement that a real-valued CR meromorphic function is constant along the CR-orbits of M.

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