Pure and Applied Mathematics Quarterly
Volume 6, Number 3
(*Special Issue: In honor of
Joseph J. Kohn*)
647—661, 2010

# Global Regularity of $\bar\partial$ on an Annulus between a Q-pseudoconvex and a P-pseudoconcave Boundary

Heungju Ahn and Giuseppe Zampieri

**Abstract:** We consider the $\bar\partial$ problem over an annulus $\Omega \subset\subset \mathbb{C}^n$ between an internal $p$-pseudoconcave and an external $q$-pseudoconvex hypersuface respectively. We prove that the $C^\infty(\bar\Omega)$-cohomology of $\bar\partial$ on antiholomorphic forms of degree $k$ for $q+1 \le k \le p-1$ is finite-dimensional.
**Keywords:** $\bar\partial$-Neumann problem, $q$ pseudoconvex/concave manifolds.

## 1. Introduction

Let $\Omega_1$ and $\Omega_2$ be two domains of $\mathbb{C}^n$ with $\Omega_2 \subset\subset \Omega_1 \subset\subset \mathbb{C}^n$ and let $\Omega$ be the annulus $\Omega = \Omega_1 \setminus \bar\Omega_2$. We assume that $\Omega$ is $q$-pseudoconvex at $\partial\Omega_1$ and $p$-pseudoconcave at $\partial\Omega_2$ where the indices $q$ and $p$ satisfy $0 \le q+2 \le p \le n-1$. For an antiholomorphic form $f$ in $C^\infty(\bar\Omega)$ of degree $k$ with $q+1 \le k \le p-1$ and which satisfies the compatibility condition $\bar\partial f = 0$, we look for solutions $u$ of degree $k-1$ of the inhomogeneous Cauchy-Riemann equation $\bar\partial u = f$. We prove solvability of this equation in $C^\infty(\bar\Omega)$, modulo harmonic forms $\mathcal{H}$, that is solutions of $(\bar\partial, \bar\partial^*)$. If one strengthens the hypotheses and assumes strong $q$-pseudoconvexity and strong $p$-pseudoconcavity at $\partial\Omega_1$ and $\partial\Omega_2$ respectively, it is classical that local hypoellipticity at the boundary for $(\bar\partial, \bar\partial^*)$ follows: a solution $u$ which is orthogonal to $\ker\bar\partial$ is smooth precisely in the part of $\partial\Omega$ where $f$ is. In particular, when $f$ is in $C^\infty(\bar\Omega)$, then the so called "canonical" solution is also in $C^\infty(\bar\Omega)$. The basic tool of the paper is the $\bar\partial$-Neumann method by Kohn. In

Received September 17, 2007.
MSC: 32F10, 32F20, 32N15, 32T25

Section 3 we establish the $L^2$ estimates for the weighted $\bar\partial$-Neumann problem. For any $s$, by the use of the weight $\varphi_{t_s}=(t_s+c)|z|^2$, they guarantee the existence of a "$\bar\partial$-Neumann operator" $N_{t_s}$, the "quasi" inverse of $\Box_{t_s}=\bar\partial\bar\partial^*_{t_s}+\bar\partial^*_{t_s}\bar\partial$ in the Sobolev space $H^s$ weighted by $\varphi_{t_s}$ as it was proved in [1] and [2]. This permits to solve the $\bar\partial$-problem in the Sobolev spaces $H^s$, modulo $\mathcal{H}$ the finite-dimensional space of harmonic forms. By a procedure of approximation due to Kohn [10] this yields the $C^\infty(\bar\Omega)$ solution modulo $\mathcal{H}$ out of the $H^s$ solutions. In § 4 , we restrict our discussion to the case where $\partial\Omega_1$ and $\partial\Omega_2$ are strongly $q$-pseudoconvex and $p$-pseudoconcave respectively. Though the regularity at the boundary of $\bar\partial$ is classical in this case (cf. e.g. [6]), our appoach carries some novelty and puts the discussion into a unified frame with § 3. Finally, in § 5, we introduce a criterion of decomposition of $\bar\partial$-closed forms. By this, we get, in some cases, a better description of the solution.

We wish to describe in what extent our results are already contained in the literature. First, the estimates of Section 3 and the boundary regularity of $\bar\partial$ were already established by Shaw in [14] in case $\partial\Omega_1$ and $\partial\Omega_2$ are weakly pseudoconvex and pseudoconcave respectively, which corresponds to $q=0$ and $p=n-1$ in our terminology. As for Section 4, we notice that when the assumptions on $q$-pseudoconvexity and $p$-pseudoconcavity are strong, then $\Omega$ satisfies $Z(k)$ for any $k$ s.t. $q+1\le k\le p-1$ and hence the $\bar\partial$-problem satisfies the $\frac{1}{2}$- subelliptic estimates according to Kohn [12], Hörmander [8] and Folland-Kohn [6].

## 2. Q-pseudoconvexity / P-pseudoconcavity

We consider a bounded annulus of $\mathbb{C}^n$ of type $\Omega=\Omega_1\setminus\bar\Omega_2$ where $\Omega_1\subset\subset\Omega_2\subset\subset\mathbb{C}^n$ are domains with $C^\infty$ boundary. We choose two equations $r^1=0,\ r^2=0$ for $M_1,\ M_2$ so that $\Omega$ is defined by $r^1<0,\ r^2>0$. Let $TM_h$ $h=1,2$ be the tangent bundle to $M_h$, $T^{\mathbb{C}}M_h$ the complex tangent bundle, $T^{1,0}M_h$ and $T^{0,1}M_h$ the subbundles of $\mathbb{C}\otimes T^{\mathbb{C}}M$ of forms of type $(1,0)$ and $(0,1)$ respectively. Let $L_{r^h}$ $h=1,2$, resp. $L_{M_h}$ be the Levi forms of $r^h$, resp. $M_h$, which are the hermitian forms represented, in a system of coordinates $z$ of $\mathbb{C}^n$, by the matrices

$$\Big(\partial_{z_i}\partial_{\bar z_j}r^h\Big)_{ij},\quad \text{resp. } \Big(\partial_{z_i}\partial_{\bar z_j}r^h\Big)_{ij}\big|_{T^{\mathbb{C}}M}.$$

We denote by $\lambda_1^h\le\lambda_2^h\le ...,\ h=1,2$, the ordered eigenvalues of $L_{M_h}$. We pass to describe our geometric hypotheses on $\partial\Omega$. We start from the "exterior" boundary defined by $r^1 = 0$, fix a boundary point, consider in a neighborhood of this point an orthonormal basis $\{\omega_j\}_{j=1,...,n}$ with $\omega_n = \partial r^1$, the dual basis $\{\partial_{\omega_j}\}_{j=1,...,n}$ of $(1,0)$ vector fields and denote by $(r^1_{ij}(z))_{i,j=1,...,n}$ the matrix which represents $L_{M_1}$ in this basis. Following [16] and [1] we introduce the following notion. In the neighborhood of a boundary point, we suppose that there exists a $C^2$ smooth bundle $\mathcal{V} \subset T^{1,0}M_1$ of rank $q$, say $\mathcal{V} = \text{Span}\ \{\partial_{\omega_1}, ...\partial_{\omega_q}\}$, such that

$$\sum_{j=1}^{q+1} \lambda^1_j(z) - \sum_{j=1}^{q} r^1_{jj}(z) \geq 0. \tag{2.1}$$

We also consider the situation in which (2.1) holds with strict inequality $> 0$.

It is evident that (2.1) implies $\lambda_{q+1} \geq 0$; hence (2.1) is still true if we replace the first sum $\sum_{j=1}^{q+1} \cdot$ by $\sum_{j=1}^{k} \cdot$ for any $k$ such that $q+1 \leq k \leq n-1$. For ordered multiindices $J = j_1 < ... < j_k$ of length $|J| = k$, let us consider $k$-vectors $w = (w_J)_J$. We assume that $w$ is tangential to $M_1$, that is, $w_J|_{M_1} = 0$ when $n \in J$. If the multiindex is no more ordered, then it is chosen to be alternant: if $J$ decomposes in $jK$, then $w_{jK} := \text{sign}\ \binom{J}{jK} w_J$. Now, (2.1) is equivalent to

$$\sideset{}{'}\sum_{|K|=q} \sum_{ij=1,...,n-1} r^1_{ij} w_{iK} \bar{w}_{jK} - \sideset{}{'}\sum_{|J|=q} \sum_{j \leq q} r^1_{jj} |w_J|^2 \geq 0 \qquad \forall w \text{ tangential of degree } q+1, \tag{2.2}$$

where $\sum'$ denotes the sum over ordered indices. By what we have remarked after (2.1), it follows that (2.2) is in fact true for any $w$ tangential of degree $k \geq q+1$. Sometimes, we also use a variant of (2.2) which is still sufficient for regularity at the boundary:

$$\sideset{}{'}\sum_{|K|=q} \sum_{ij=1,...,n-1} r^1_{ij} w_{iK} \bar{w}_{jK} - \sideset{}{'}\sum_{|K|=q} \sum_{j \leq q} r^1_{jj} |w_{jK}|^2 \geq 0 \qquad \forall w \text{ tangential of degree } q+1\ . \tag{2.3}$$

Again, (2.2) is in fact true for any tangential form of degree $k$ with $q+1 \leq k \leq n-1$.

**Definition 2.1.** $M_1$ is said $q$-pseudoconvex when in a neighborhood of any boundary point either of (2.2) or (2.3) is satisfied. $M_1$ is said strongly $q$-pseudoconvex when either of (2.2) or (2.3) hold with strict inequality.

We pass now to the "interior" boundary $M_2$, fix a point, and choose a local system of smooth forms with $\omega_n = \partial r^2$: this vector points inside $\Omega$ since $r^2 > 0$ on $\Omega$. Let $p$ be an integer $\leq n-1$.

We consider the situation in which, in the neighborhood of a boundary point, there is a $C^2$-bundle $\mathcal{W}$ in $T^{0,1}M_2$ of rank $p$, say $\mathcal{W} = \text{Span}\,\{\partial_{\omega_1}, ..., \partial_{\omega_p}\}$ such that

$$\sum_{j\leq p-1} \lambda_j^2 - \sum_{j\leq p} r_{jj}^2 \geq 0. \tag{2.4}$$

We consider also the case in which (2.4) holds with strict inequality $> 0$.

In the same way as we have seen that (2.1) is equivalent to (2.2), we can see that (2.4) is equivalent to:

$$\sum_{|K|=p-2}{}' \sum_{ij=1,...,n-1} r_{ij}^2 w_{iK}\bar{w}_{jK} - \sum_{|J|=k}{}' \sum_{j\leq p} r_{jj}^2 |w_J|^2 \geq 0 \tag{2.5}$$

for any tangential form $w$ of degree $p-1$.

Similarly as for pseudoconvexity, we also consider the following variant of (2.5)

$$\sum_{|K|=p-2}{}' \sum_{ij=1,...,n-1} r_{ij}^2 w_{iK}\bar{w}_{jK} - \sum_{|J|=k}{}' \sum_{j\leq p} r_{jj}^2 |w_{jK}|^2 \geq 0 \tag{2.6}$$

for any tangential form $w$ of degree $p-1$.

In all (2.4), (2.5) and (2.6) we can replace $p-1$ by any $k \leq p-1$.

**Definition 2.2.** $M_2$ is said $p$-pseudoconcave when in a neighborhood of any boundary point, either of (2.5) or (2.6) are satisfied. $M_2$ is said strongly $p$-pseudoconcave when (2.5) or (2.6) hold with strict inequality.

## 3. Uniform estimates up to the boundary for weakly Q-pseudoconvex/P-pseudoconcave boundaries

For a bounded domain with smooth boundary $\Omega \subset \mathbb{C}^n$, we consider antiholomorphic forms $u = \sum_{|J|=k}' u_J \bar{\omega}_J$ of degree $k$ with coefficients $u_J$ in $C^\infty(\bar{\Omega})$ where $\bar{\omega}_J$ stands for $\bar{\omega}_{j_1} \wedge \bar{\omega}_{j_2} \wedge ....$ We denote by $C^\infty(\bar{\Omega})^k$ the space of such forms and consider the $\bar{\partial}$-complex

$$...C^\infty(\bar{\Omega})^{k-1} \xrightarrow{\bar{\partial}} C^\infty(\bar{\Omega})^k \xrightarrow{\bar{\partial}} C^\infty(\bar{\Omega})^{k+1} \xrightarrow{\bar{\partial}} .... \tag{3.1}$$

We extend the action of $\bar\partial$ to forms with coefficients in $L^2$, possibly endowed with a weight $e^{-\varphi}$, denote by $||\cdot||_{H^0}$ the corresponding norm, and by $\bar\partial^*$ the adjoint operator. We denote by $D_{\bar\partial^*}$ the domain of $\bar\partial^*$. If $\Omega$ is defined, in a neighborhood of a boundary point, by $r<0$ with $\partial r\neq 0$, and $\omega_1,...,\omega_n$ is a basis of orthonormal forms in which $\omega_n=\partial r$, we can check that $u\in D_{\bar\partial^*}$ if and only if $u_J|_{\partial\Omega}\equiv 0$ whenever $n\in J$. We call tangential a form which belongs to $D_{\bar\partial^*}$. We suppose that our domain is an annulus $\Omega=\Omega_1\setminus\bar\Omega_2$, and first define the weight which fits our needs. For a choice of a constant $c$ which will be clarified later, we want $\varphi$ to coincide with $\varphi^1:=c|z|^2$ at $M_1$ and $\varphi^2:=-c|z|^2$ at $M_2$. This can be achieved by taking, e.g., $\chi_1$ and $\chi_2$ smooth, with $\chi_1\equiv 1$ at $M_1$, $\chi_2\equiv 1$ at $M_2$, $\chi_1+\chi_2\equiv 1$, and putting $\varphi=\chi_1\varphi^1+\chi_2\varphi^2$. Under this choice of $\varphi$ we obtain our basic estimates

**Theorem 3.1.** *Let $M_1$ be q-pseudoconvex and $M_2$ p-pseudoconcave for $1\leq q+1\leq p-1\leq n-2$. Then, for suitable $\Omega'\subset\subset\Omega$ and for any $u\in C^\infty(\bar\Omega)^k\cap(D_{\bar\partial^*})^k$, we have*

$$(3.2)\quad ||u||^2_{H^0(\Omega)}+||u||^2_{H^1(\Omega')}\lesssim||\bar\partial u||^2_{H^0(\Omega)}+||\bar\partial^*u||^2_{H^0(\Omega)}+||u||^2_{H^{-1}(\Omega')}\quad \text{if } q+1\leq k\leq p-1.$$

As usual, "$\lesssim$" denotes inequality up to a multiplicative constant (independent of $u$).

*Proof.* We take a local "frame" $\omega_1,...,\omega_n$ with $\omega_n=\partial r$ where $r=r^1$ or $r=r^2$. We denote by $\delta_{\omega_j}$ the adjoint of $-\partial_{\bar\omega_j}$ in the weighted $H^0_\varphi$ inner product; hence $\delta_{\omega_j}=\partial_{\omega_j}-\varphi_j$ where we have put $\varphi_j:=\partial_{\omega_j}\varphi$. By using the adjunction relations between $\partial_{\bar\omega_j}$ and $\delta_{\omega_j}$ as in [9] ch. IV, we get for any $u$ tangential

$$(3.3)\quad ||\bar\partial u||^2_{H^0_\varphi}+||\bar\partial^*_\varphi u||^2_{H^0_\varphi}\gtrsim\sum_{ij=1,...,n}\sum'_{|K|=k-1}\int e^{-\varphi}\left(\delta_{\omega_i}u_{iK}\overline{\delta_{\omega_j}u_{jK}}-\partial_{\bar\omega_j}u_{iK}\overline{\partial_{\bar\omega_i}u_{jK}}\right)dV+\sum_{j=1,...,n}\sum'_{|J|=k}\int e^{-\varphi}|\partial_{\bar\omega_j}u_J|^2dV+...$$

where dots denote an error term in which never occur products of derivatives of $u$. We perform twice integration by parts for all indices $ij$ in the first sum in (3.3). We make also twice integration by parts in some of the terms of the second sum whose choice is different in the two components of the boundary and in the interior. We first remark that by a partition of the unity one can prove

(3.2) separately for forms which have support in neighborhoods of points of $M_1$, $M_2$ and in the interior of $\Omega$. The first is done in [1] or [2] and the third is a consequence of the elliptic estimates in the interior. We will spend later a few words about these two cases, but first point our attention to forms $u$ with support in a neighborhood of a point of $M_2$. We assume that $\mathcal{W} = \text{Span}\{\omega_1, ..., \omega_p\}$ and remember that $\varphi$ is defined as $-c|z|^2$ in a neighborhood of $M_2$. We suppose that $p$-pseudoconcavity holds in the form of (2.5); the variant for (2.6) is obvious. In this case, we apply double integration by parts to the terms of the second sum in which $j \leq p$ and continue the inequality (3.2) by

(3.4)
$$\gtrsim \left( \sum_{ij=1,...,n} \sum_{|K|=k-1}' \int e^{-\varphi} [\delta_{\omega_i}, \partial_{\bar{\omega}_j}] u_{iK} \bar{u}_{jK} dV - \sum_{j \leq p} \sum_{|J|=k}' \int e^{-\varphi} [\delta_{\omega_j} \partial_{\bar{\omega}_j}] |u_J|^2 dV \right)$$
$$+ \left( \sum_{j \leq p} \sum_{|J|=k}' \int e^{-\varphi} |\delta_{\omega_j} u_J|^2 dV + \sum_{j \geq p+1} \sum_{|J|=k}' \int e^{-\varphi} |\partial_{\bar{\omega}_j} u_J|^2 dV \right) + ...$$

We denote by $S$ the second line of (3.4). Now, we can express the commutators as

$$[\delta_{\omega_i}, \partial_{\bar{\omega}_j}] = \varphi_{ij} + r_{ij}(\delta_{\omega_n} - \partial_{\bar{\omega}_n}) + ...$$

We interchange $\delta_{\omega_n}$ and $\partial_{\bar{\omega}_n}$ and get

$$\int_\Omega e^{-\varphi} r_{ij} \delta_{\omega_n} u_J \bar{u}_I dV = \int_{+\partial\Omega} e^{-\varphi} r_{ij} r_n u_J \bar{u}_I dV$$
$$- \int_\Omega e^{-\varphi} r_{ij} u_J \overline{\partial_{\bar{\omega}_n} u_I} dV + ...$$

Thus we continue our estimates by

$$(3.5) \quad \left( \sum_{|K|=k-1}' \sum_{ij=1,...,n-1} \int_{\partial\Omega} e^{-\varphi} r_{ij} u_{iK} \bar{u}_{jK} dS - \sum_{|J|=k}' \sum_{j \leq p} \int_{\partial\Omega} e^{-\varphi} r_{jj} |u_J|^2 dV \right)$$
$$+ \left( \sum_{ij=1,...,n} \sum_{|K|=k-1}' \int_\Omega e^{-\varphi} \varphi_{ij} u_{iK} \bar{u}_{jK} dV - \sum_{j \leq p} \sum_{|J|=k}' \int_\Omega e^{-\varphi} \varphi_{jj} |u_J|^2 dV \right)$$
$$+ S + ...$$

The first line is $\geq 0$ because $k \leq p-1$ and by the assumption of $p$-pseudoconcavity. The second is $\geq (p-k)c||u||^2_{H^0_\varphi} =: c'||u||^2_{H^0_\varphi}$ since $k \leq p-1$. As for the error terms denoted by dots, they can be estimated by $\mathrm{lc}||u||^2_{H^0_\varphi} + \mathrm{sc}\, S$ where lc and sc denote a large and a small constant respectively. Thus, for sc$\leq 1$ and if $c$ is chosen so large that $c' >$lc, (3.5) is $\gtrsim ||u||^2_{H^0_\varphi}$ which completes the proof of (3.2) for forms supported in neighborhoods of points of $M_2$.

At points of $M_1$ we choose our frame $\omega_1, ..., \omega_n$ so that $\omega_n = \partial r^1$ and $\mathcal{V} = \mathrm{Span}\{\omega_1, ..., \omega_q\}$ and the weight $\varphi$ such that $\varphi = c|z|^2$. We suppose that (2.1) or (2.2) hold. We interchange in the second sum of (3.3) the derivatives $\partial_{\bar\omega_j}$ with $\delta_{\omega_j}$ for $1 \leq j \leq q$ and get (3.5) with $p$ replaced by $q$. Since $k \geq q+1$ and $M_1$ is $q$-pseudoconvex, the first line is $\geq 0$. The second is $\geq (k-q)c||u||^2_{H^0_\varphi} =: c'||u||^2_{H^0_\varphi}$ and the error can be estimated as above by $\mathrm{lc}||u||^2_{H^0_\varphi} + \mathrm{sc}\, S$. For sc$\leq 1$ and by choosing $c$ s.t. $c' >$lc, we get (3.2) at $M_1$.

As for the points in the interior of $\Omega$, we observe that no boundary integrals occur. Also, terms involving $\varphi_{ij}$ can be regarded as error terms. We interchange $\frac{1}{2}||\partial_{\bar\omega_j} u_J||^2$ with $\frac{1}{2}||\delta_{\omega_j} u_J||^2$ for all $j = 1, ..., n$ in (3.3). In this case $S$ turns into $\frac{1}{4}|||u|||^2_{H^1}$ where $|||u|||_{H^1}$ is the $L^2$ norm of the derivatives. On the other hand, as a consequence of the Sobolev inequalities, we have

$$||u||^2_{H^0} \leq sc||u||^2_{H^1} + lc||u||^2_{H^{-1}}.$$

Now, the first line of (3.5) is missing, the second is $\gtrsim -||u||^2_{H^0}$ and the third $\gtrsim ||u||^2_{H^1} - ||u||^2_{H^0}$. The proof of Theorem 3.1 is complete.

$\square$

Theorem 3.1 implies $C^\infty(\bar\Omega)$ solvability of $\bar\partial$ as stated in the subsequent Theorem 3.2. For the convenience of the reader we give the outline of the proof for whose full detail we cite [10] and [13]. First, the estimate (3.2) can be transfered from $L^2$ to $H^s$. For this, we remark that the derivatives which are tangential to the boundary, that we denote by $\mathcal{T}_i$, preserve tangentiality of forms: if $u \in D_{\bar\partial^*}$ then $\mathcal{T}_i^j u \in D_{\bar\partial^*}$. For the normal derivative, say $\mathcal{N}$, we have the estimate

$$||\mathcal{N}u||^2_{H^0} \lesssim ||\bar\partial u||^2_{H^0} + ||\bar\partial^* u||^2_{H^0} + \sum_i ||\mathcal{T}_i u||^2_{H^0} + ||u||^2_{H^0},$$

because of the "non-characteristicity" of the boundary for $(\bar\partial, \bar\partial^*)$. So, what is really needed for the control of the $H^s$ norm, is to estimate the $||\mathcal{T}_i^s u||_{H^0}$'s. For this, we apply (3.2) to the $\mathcal{T}_i^s u$'s and consider the commutators $[\bar\partial, \mathcal{T}_i^s]$ and $[\bar\partial^*, \mathcal{T}_i^s]$. Now, the part of order $s$ is independent of $t$ which only enters in the lower order term. By taking $t = t_s$ large enough to compensate the first term, we get

$$||u||^2_{H^s} \lesssim ||\bar\partial u||_{H^s} + ||\bar\partial^* u||_{H^s} + c_s||u||^2_{H^{s-1}}. \tag{3.6}$$

We define now the space of harmonic forms of degree $k$, $\mathcal{H}^s = \mathcal{H}^s_k$, as the space of $u \in (H^s)^k$ such that $\bar\partial u = 0$, $\bar\partial^* u = 0$, denote by $\mathcal{H}^{s\perp}$ the orthogonal complement and by $T : \ H^s \to \mathcal{H}^s$ the orthogonal projection respectively. (Remark that all these definitions depend in fact on the weight $\varphi$ which is omitted in the notations.) If we restrict (3.2) to $\mathcal{H}^s$ we get in particular $||u||_{H^s} \lesssim ||u||_{H^{s-1}}$. Thus, when $\Omega$ is an annulus which satisfies the assumptions of Theorem 3.1 and $k$ has the restraints $q+1 \le k \le p-1$, we obtain, by Rellich's theorem, $\dim(\mathcal{H}^s_k) < \infty$. If, instead, we restrict (3.2) to $\mathcal{H}^{s\perp}_k$, we get

$$||u||^2_{H^s} \lesssim ||\bar\partial u||^2_{H^s} + ||\bar\partial^* u||^2_{H^s} \quad \forall u \in \mathcal{H}^{s\perp}_k. \tag{3.7}$$

Otherwise, there is a sequence $\{u_\nu\}_\nu$ in $\mathcal{H}^{s\perp}_k$ such that

$$\frac{1}{\nu}||u_\nu||_{H^s} \ge ||\bar\partial u_\nu||^2_{H^s} + ||\bar\partial^* u_\nu||^2_{H^s}. \tag{3.8}$$

If we plug (3.8) into (3.2), we get $||u_\nu||^2_{H^s} \lesssim ||u_\nu||^2_{H^{s-1}}$ and hence, by the compactness of the embedding $H^s \hookrightarrow H^{s-1}$, there is a convergent subsequence $\frac{u_{\nu_j}}{||u_{\nu_j}||_{H^{s-1}}} \to u_o$ in $(H^{s-1})^k$. Now, $||u_o||_{H^{s-1}} = 1$, $u_o \in \mathcal{H}^{s\perp}_k$ and also $u_o \in \mathcal{H}^s_k$ by (3.8). This is a contradiction.

Let $\Box_{t_s} + \bar\partial^*_{t_s}\bar\partial + \bar\partial\bar\partial^*_{t_s}$. (3.7) is equivalent to $||u||^2_{H^s} \lesssim \langle \Box_{t_s} u, u\rangle_{H^s}$ for any $u \in \mathcal{H}^{s\perp}_k$ and this implies $||u||_{H^s} \lesssim ||\Box_{t_s} u||_{H^s}$ for any $u \in \mathcal{H}^{s\perp}_k$. Thus $\Box_{t_s}$ has a "quasi" $H^s$-inverse. It fails to be an exact inverse because of the constraint $u \in \mathcal{H}^{s\perp}_k$. In other terms, there is defined an operator $N_{t_s}$ which satisfies $I = \Box_{t_s} N_{t_s} + T$. It satisfies also the commutation relations $\bar\partial N_{t_s} = N_{t_s}\bar\partial$ and $\bar\partial^*_{t_s} N_{t_s} = N_{t_s}\bar\partial^*_{t_s}$. By means of $N_{t_s}$, we get that if $\bar\partial f \in \mathcal{H}^{s\perp}_k$, then $u_{t_s} := \bar\partial^*_{t_s} N_{t_s} f$ is a solution of the equation $\bar\partial u = f$ modulo $\mathcal{H}^s_k$. By a procedure of approximation (cf. [10]), we get a $C^\infty(\bar\Omega)^k$ solution $u$ from the $(H^s)^k$ solutions $u_{t_s}$, though it is not clear whether it is canonical. In particular it is not clear whether it is orthogonal to harmonic forms. Let $\mathcal{H} = \mathcal{H}^0$; what we have obtained is the content of the following

**Theorem 3.2.** *Let $\Omega = \Omega_1 \setminus \bar\Omega_2$ be an annulus $q$-pseudoconvex at $M_1$ and $p$-pseudoconcave at $M_2$. Then, for any $k$ satisfying $q+1 \le k \le p-1$, the following* *holds. For any $f \in C^\infty(\bar\Omega)^k$ which is orthogonal to $\mathcal{H}_k$ and with $\bar\partial f \in \mathcal{H}_{k-1}$, there is $u \in C^\infty(\bar\Omega)^{k-1}$ such that $\bar\partial u = f$ modulo $\mathcal{H}_k$.*

Since $\dim(\mathcal{H}_k) < \infty$, Theorem 3.2 implies that the cohomology of $\bar\partial$ over $C^\infty(\bar\Omega)$ forms is finite-dimensional for any $k$ such that $q+1 \le k \le p-1$.

We turn now our attention to the domain $\Omega_1$ with $q$-pseudoconvex boundary. If our interest is confined to the $\bar\partial$-problem over $\Omega_1$, the estimate (3.2) can be improved. For this purpose, we have to adapt the weight and choose a single positive function $\varphi = (t+c)|z|^2$ where $c$ is used as before to control error terms, and $t$ is a big parameter which raises the left side (3.2). We get in this case

**Theorem 3.3.** *Let $\Omega_1$ be $q$-pseudoconvex. Then, for any $u \in C^\infty(\bar\Omega)^k \cap (D_{\bar\partial^*})^k$, we have*

$$t||u||^2_{H^0} \le ||\bar\partial u||^2_{H^0} + ||\bar\partial^* u||_{H^0} \text{ if } k \ge q+1. \tag{3.9}$$

*Proof.* In (3.3), we interchange $\partial_{\bar\omega_j}$ with $\delta_{\omega_j}$ for $1 \le j \le q$. The first line of (3.5) is still $\ge 0$ by $q$-pseudoconvexity. The second is $\ge (k-q)(c+t)||u||^2_{H^0}$. Errors are controlled by $c||u||^2_{H^2} + Su$. This yields (3.9) at the boundary $\partial\Omega_1$. At the interior points we do not perform any commutation of $\partial_{\bar\omega_j}$ with $\delta_{\omega_j}$, and just notice that $\sum'_{|K|=k-1} \sum_{ij} \varphi_{ij} u_{iK} \bar u_{jK} \ge k(c+t)|u|^2$.

□

We then follow the same argument as the one which led to Theorem 3.2 but without the constraint $u \in \mathcal{H}_k^{s\perp}$ in our estimates and obtain

**Theorem 3.4.** *([1], [16]) Let $\Omega_1$ be $q$-pseudoconvex and let $k \ge q+1$. Then for any $f \in C^\infty(\bar\Omega)^k$ with $\bar\partial f = 0$, we can find $u \in C^\infty(\bar\Omega)^{k-1}$ such that $\bar\partial u = f$.*

Thus the cohomology of $\bar\partial$ over $C^\infty(\bar\Omega)$ forms for $k \ge q+1$ is 0 in the present case.

## 4. Subelliptic estimates for a strongly $q$-pseudoconvex and $p$-pseudoconcave annulus

When the boundary of the annulus satisfies conditions of strong convexity/concavity, then we have $\frac{1}{2}$-subelliptic estimates for $(\bar\partial, \bar\partial^*)$.

**Theorem 4.1.** *Let $\Omega = \Omega_1 \setminus \bar{\Omega}_2$ be an annulus such that $M_1$ is strongly $q$-pseudoconvex and $M_2$ is strongly $p$-pseudoconcave for $1 \leq q+1 \leq p-1 \leq n-2$. Then, for suitable $\Omega' \subset\subset \Omega$ and for any $u \in C^\infty(\bar{\Omega})^k \cap (D_{\bar{\partial}^*})^k$, we have*

$$(4.1)\quad ||u||^2_{H^{\frac{1}{2}}(\Omega)} \lesssim ||\bar{\partial}u||^2_{H^0(\Omega)} + ||u||^2_{H^0(\Omega)} + ||u||^2_{H^{-\frac{1}{2}}(\Omega')}, \qquad \text{if } q+1 \leq k \leq p-1.$$

*Proof.* We need to modify the proof of Theorem 3.1. We still use a weight $\varphi$ which is $c|z|^2$ at $M_1$ and $-c|z|^2$ at $M_2$ and first turn our attention to a neighborhood of $M_2$: thus, we consider forms $u$ with support in a neighborhood of a point of $M_2$. By double integration by parts, we interchange the terms $||\partial_{\bar{\omega}_j}u||^2$ in the following manner

$$(4.2)\quad \begin{cases} ||\partial_{\bar{\omega}_j}u||^2 = (1-\epsilon)||\delta_{\omega_j}u||^2 + \epsilon||\partial_{\bar{\omega}_j}u||^2 - (1-\epsilon)\displaystyle\int_\Omega e^{-\varphi}[\partial_{\omega_j}, \partial_{\bar{\omega}_j}]|u|^2 dV + ... \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 \leq j \leq p, \\ ||\partial_{\bar{\omega}_j}u||^2 = \epsilon||\delta_{\omega_j}u||^2 + (1-\epsilon)||\partial_{\bar{\omega}_j}u||^2 - \epsilon\displaystyle\int_\Omega e^{-\varphi}[\partial_{\omega_j}, \partial_{\bar{\omega}_j}]|u|^2 dV + ... \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } p+1 \leq j \leq n-1. \end{cases}$$

Hence, the estimate (3.5) turns into

$$(4.3)\quad \begin{aligned} ||\bar{\partial}u||^2_{H^0} + ||\bar{\partial}^*u||^2_{H^0} \geq & \left(\sum_{ij \leq n-1} \int_{\partial\Omega_2} \cdot - (1-\epsilon)\sum_{j \leq p} \int_{\partial\Omega} \cdot - \epsilon \sum_{p+1 \leq j \leq n-1} \int_{\partial\Omega} \cdot \right) \\ & + \left(\sum_{ij=1,\dots,n} \int_\Omega \cdot - (1-\epsilon)\sum_{j \leq p} \int_\Omega \cdot - \epsilon \sum_{p+1 \leq j \leq n-1} \int_\Omega \cdot \right) \\ & + \left(\sum_{ij=1,\dots,n-1} ||\partial_{\omega_j}u||^2_{H^0} + \sum_{ij=1,\dots,n} ||\partial_{\bar{\omega}_j}u||^2_{H^0}\right) + ... \end{aligned}$$

The right side of the first line is $\geq 0$ for $\epsilon$ small and $k \leq p-1$ by the strong $p$-pseudoconvexity of $M_2$. As for the second line, recall that $\varphi = -c|z|^2$ at $M_2$; thus, it is $\geq (p-(n-1)\epsilon-k)c||u||^2_{H^0}$ which is positive, for small $\epsilon$, since $k \leq p-1$; we denote by $c'||u||^2_{H^0}$ this positive quantity. We finally denote by $Su$ the term between parentheses in the third line. In conclusion, the right side of (4.3) is $\geq Su + c'||u||^2_{H^0}$. Again, by the strong $p$-pseudoconcavity, the Levi form of $M_2$ has finite type 2: commutators of the $\partial_{\bar{\omega}_j}$'s and $\partial_{\omega_j}$'s for all $j \leq p-1$, generate a vector field $\mathcal{N} \in \mathbb{C} \otimes TM_2$ which is transversal to $\mathbb{C} \otimes T^{\mathbb{C}} M_2$. It is readily seen that this implies (cf. [11])

$$||\mathcal{N}u||^2_{H^{\frac{1}{2}}} \underset{\sim}{<} Su.$$

On the other hand, the systems $\{\partial_{\bar\omega_j}\}_{j\leq n}$ and $\{\partial_{\omega_j}\}_{j\leq n-1}$, supplemented by $\mathcal{N}$, span all the tangent vector fields. This implies

$$\begin{aligned} ||u||^2_{H^{\frac{1}{2}}} &\underset{\sim}{<} Su + ||\mathcal{N}u||^2_{H^0} \\ &\underset{\sim}{<} Su + ||u||^2_{H^0}. \end{aligned}$$

Also, the error terms can be estimated by $\text{sc}Su + \text{lc}||u||^2_{H^0}$. Thus, if $\text{sc}< 1$ and $c$ and $\epsilon$ are chosen so that $c' >$lc, the estimate (4.1) follows at $M_2$.

As for $M_1$, we make the similar commutations as in (4.2) but, this time, for $j \leq q$ in the first line and for $q + 1 \leq j \leq n - 1$ in the second. We get the similar conclusion as in (4.3) but with the first and second terms in the right replaced by

$$\text{(4.4)} \quad \left( \sum_{ij=1,\dots,n-1} \int_{\partial\Omega} \cdot - (1-\epsilon)\sum_{j\leq q} \int \partial\Omega \cdot - \epsilon \sum_{q+1\leq j\leq n-1} \int_{\partial\Omega} \cdot \right) \\ + \left( \sum_{ij=1,\dots,n} \int_{\Omega} \cdot - (1-\epsilon)\sum_{j\leq q} \int_{\Omega} \cdot - \epsilon \sum_{q+1\leq j\leq n-1} \int_{\Omega} \cdot \right),$$

respectively. The first line of (4.4) is $\geq 0$ for $\epsilon$ small and $k \geq q+1$ by the strong $q$-pseudoconvexity of $M_1$. For the second line, recall that $\varphi = c|z|^2$ at $M_1$; thus this is $\geq (k-(n-1)\epsilon - q)c||u||^2_{H^0}$. Again, this is positive, say $\geq c'||u||^2_{H^0}$, for small $\epsilon$, because $k \geq q+1$. The rest of the proof of the estimate at $M_1$ goes through by the same argument as for $M_2$.

Finally, for the points in the interior, the estimate is the same as in Theorem 3.1.

□

*Remark* 4.2. If, instead of the annulus $\Omega$, we consider only the domain $\Omega_1$ with strongly $q$-pseudoconvex boundary, then we use a single weight $\varphi = c|z|^2$ all over $\bar\Omega_1$ and get a much better conclusion. The first part of the above proof yields, for any $u \in C^\infty(\bar\Omega)^k \cap (D_{\bar\partial^*})^k$,

$$||u||^2_{H^{\frac{1}{2}}} \underset{\sim}{<} ||\bar\partial u||^2_{H^0} + ||\bar\partial^* u||^2_{H^0} \quad \text{if } k \geq q+1.$$

We derive now the main consequences of (4.1). Notice that it implies readily, (cf. [11]), the hypoellipticity of the system $(\bar\partial,\ \bar\partial^*)$: $u$ is smooth if $\bar\partial u$ and $\bar\partial^* u$ are smooth. In particular, for the harmonic forms $\mathcal{H}_k$, we have $\mathcal{H}_k \subset C^\infty(\bar\Omega)^k$. We can also restrict (4.1) to $u \in \mathcal{H}_k^\perp$, as we have already seen in Section 3, and obtain a $\bar\partial$-Neumann operator $N$ in $H^0_k$ which satisfies $I = \Box N + T$ where $T : H^0_k \to \mathcal{H}_k$ is the orthogonal projection. By the above remarks, if $f \in C^\infty(\bar\Omega)^k$ and $\bar\partial f \in \mathcal{H}_{k+1}$, then the canonical solution $u := \bar\partial^* N f$, which is a priori only $H^0$, is in fact $C^\infty(\bar\Omega)$. This follows from the fact that $\bar\partial u = f + Tf \in C^\infty(\bar\Omega)$ and $\bar\partial^* u = 0$. Thus $u$ is the $C^\infty(\bar\Omega)$ solution of $\bar\partial u = f$, modulo $\mathcal{H}$, orthogonal to $\ker \bar\partial$. We have thus showed

**Corollary 4.3.** *Let $\Omega = \Omega_1 \setminus \Omega_2$ be an annulus such that $M_1$ is strongly $q$-pseudoconvex and $M_2$ strongly $p$-pseudoconcave respectively and let $k$ satisfy $q + 1 \le k \le p - 1$. Then for any $f \in C^\infty(\bar\Omega)^k$ with $\bar\partial f = 0$, we have that $u = \bar\partial^* N f$ is the $C^\infty(\bar\Omega)^{k-1}$ solution, modulo $\mathcal{H}$, of $\bar\partial u = f$ which is orthogonal to $\mathcal{H}_{k-1}$.*

In particular, the cohomology of $\bar\partial$ is finite-dimensional in the specified degrees. If we go back to Remark 4.2, we see that if $\Omega_1$ is strongly $q$-pseudoconvex, then harmonic forms on $\Omega_1$ are 0 and therefore the $C^\infty(\bar\Omega)$ cohomology of $\bar\partial$ is 0 for $k \ge q + 1$ which we already know from Theorem 3.4 for a more general weakly $q$-pseudoconvex domain. However, what we have more, is that we know now that the canonical solution is smooth. More generally, we know that $(\bar\partial,\ \bar\partial^*)$ is hypoelliptic in degree $k \ge q + 1$.

## 5. An example

We first state a general result on decomposition of $\bar\partial$-closed forms.

**Theorem 5.1.** *Let $A_1$ and $A_2$ be domains of $\mathbb{C}^n$, and define $B := A_1 \cap A_2$, $C := A_1 \cup A_2$. Assume*

(5.1) the $C^\infty(\bar C)$ cohomology of $\bar\partial$ is 0 in degree $k+1$

(5.2) $$(\bar A_1 \setminus B) \cap (\bar A_2 \setminus B) = \emptyset.$$

*Then, for any $f \in C^\infty(\bar B)^k$ satisfying $\bar\partial f = 0$, we can find $f_h \in C^\infty(\bar A_h)^k$, $h = 1, 2$, satisfying $\bar\partial f_h = 0$ and such that*

(5.3) $$f = f_1 + f_2 \text{ in } B.$$

*Proof.* We take a pair of functions $\chi_1$ and $\chi_2$ in $C^\infty(\bar C)$ such that

$$\begin{cases} \chi_1 + \chi_2 \equiv 1 \text{ in } C, \\ \chi_h \equiv 0 \text{ in } A_h \setminus B \text{ for } h = 1, 2. \end{cases}$$

It follows, for $h = 1, 2$

$$\begin{cases} \chi_h f \in C^\infty(\bar A_h), \\ \bar\partial(\chi_h f) \in C^\infty(\bar C), \\ \bar\partial(\chi_h f)|_{C\setminus B} \equiv 0. \end{cases} \tag{5.4}$$

Because of (5.1) we can find $u_h$ such that

$$\bar\partial u_h = \bar\partial(\chi_h f),\ u_h \in C^\infty(\bar C)^k.$$

From the third of (5.4) we get

$$\bar\partial(u_1 + u_2)|_{C\setminus B} \equiv 0.$$

On the other hand

$$\begin{aligned} \bar\partial(u_1 + u_2)|_B &= \bar\partial(\chi_1 f + \chi_2 f)|_B \\ &= \bar\partial f|_B = 0. \end{aligned}$$

Thus, in conclusion, $\bar\partial(u_1 + u_2) \equiv 0$ in $C$. We then put

$$f_1 := (\chi_1 f - u_1), \quad f_2 := (\chi_2 f - u_2) + (u_1 + u_2),$$

and get the desired decomposition (5.3).

□

*Remark* 5.2. For forms which are smooth in the interior but not at the boundary, the conclusion of the theorem is still true, even if we release the assumption (5.2). This is the so called “Mayer-Vietoris” decomposition. However, condition (5.2) is needed in our Theorem 5.1. In fact, at any point of $(\bar A_1 \setminus B) \cap (\bar A_2 \setminus B)$, the function $\chi_1 + \chi_2$ would be forced to take both the value 0 and 1.

We want to end our discussion by setting up in a different way our $\bar\partial$-problem for an annulus $\Omega = \Omega_1 \setminus \bar\Omega_2$ which is (weakly) $q$-pseudoconvex at $M_1$ and strongly $p$-pseudoconcave at $M_2$. We choose a ball $\mathbb{B}^n \supset\supset \Omega_1$, define $\Omega_3 = \mathbb{B}^n \setminus \Omega_2$ and remark that $\Omega = \Omega_1 \cap \Omega_3$ and $\mathbb{B}^n = \Omega_1 \cup \Omega_3$. Since $\partial\Omega_1 \cap \partial\Omega_3 = \emptyset$, then (5.2) follows; also, the $C^\infty(\bar{\mathbb{B}})$ cohomology of $\bar\partial$ is 0 in any degree $k \geq 1$ as it is classical (and is also ensured, e.g., by Theorem 3.4). Let us consider the equation

$$\bar\partial u = f \text{ for } f \in C^\infty(\bar\Omega)^k \text{ satisfying } \bar\partial f = 0. \tag{5.5}$$

We first decompose $f = f_1 + f_3$ with $\bar\partial f_h = 0$ for $h = 1, 3$. We denote by $u_1$ the solution in $C^\infty(\bar\Omega_1)^{k-1}$ of $\bar\partial u_1 = f_1$ whose existence follows from Theorem 3.4. We denote by $N_3$ the $\bar\partial$-Neumann operator on $H^0(\Omega_3)$ whose existence follows, for instance, from Theorem 3.1, and also denote by $T_3$ the orthogonal projection $T_3 : H^0(\Omega_3) \to \mathcal{H}(\Omega_3)$. (We do not need to use here the full strength of Theorem 4.1.) We get

$$\begin{aligned} f &= f_1 + f_3 \\ &= \bar\partial u_1 + \bar\partial(\bar\partial^* N_3 f_3) + T_3 f_3. \end{aligned}$$

So far, we did not use the advantage of replacing $\Omega_1 \setminus \bar\Omega_2$ by $\Omega_3 \setminus \bar\Omega_2$ which is strongly $q$-pseudoconvex at the "exterior" boundary. We do it now and remark that this implies, by Theorem 4.1, that $\bar\partial^* N_3 f \in C^\infty(\bar\Omega_3)^{k-1}$. Thus, we have obtained a $C^\infty(\bar\Omega)^{k-1}$ solution of $\bar\partial u = f$ modulo $\mathcal{H}$ in the more explicit form $u = u_1 + \bar\partial^* N_3 f$.

## References


[1] H. Ahn—Global boundary regularity of the $\bar\partial$-equation on $q$-pseudoconvex domains, *Math. Reich.* **280** (2007), 343-350

[2] L. Baracco, G. Zampieri—Regularity at the boundary for $\bar\partial$ on $q$-pseudoconvex domains, *J. d'Analyse Math.* **95** (2005), 45-61

[3] A. Boggess—CR manifolds and the tangential Cauchy-Riemann complex, *CRC Press* Boca Raton FL. (1991)

[4] S.C. Chen, M.C. Shaw—Partial differential equations in several complex variables, *Studies in Adv. Math. - AMS Int. Press* **19** (2001)

[5] M. Derridj—Regularité pour $\bar\partial$ dans quelques domaines faiblement pseudo-convexes, *J. Differential Geometry* **13** (1978), 559-576

[6] G.B. Folland, J.J. Kohn—The Neumann problem for the Cauchy-Riemann complex, *Ann. Math. Studies, Princeton Univ. Press*, Princeton, N.J., 1972

[7] G.M. Henkin—H. Lewy's equation and analysis on pseudoconvex manifolds (Russian), I, *Uspehi Mat. Nauk.* **32**-(3) (1977), 57–118

[8] L. Hörmander—$L^2$ estimates and existence theorems for the $\bar\partial$ operator, *Acta Math.* **113** (1965), 89-152

[9] L. Hörmander—An introduction to complex analysis in several complex variables, Van Nostrand, Princeton N.J., 1966

[10] J.J. Kohn—Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds, em Transactions of the A.M.S. **181** (1973), 273–292

[11] J.J. Kohn—Methods of partial differential equations in complex analysis, *Proceedings of Symposia in pure Mathematics* **30** (1977),215–237

[12] J.J. Kohn—Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions, *Acta Math.* **142** (1979), 79–122

[13] J.J. Kohn, L. Nirenberg—Non-coercive boundary value problems, *Comm. on Pure and Appl. Math.*, **18** (1965), 443–492

[14] M.C. Shaw—Global solvability and regularity for $\bar{\partial}$ on an annulus between two weakly pseudo-convex domains, *Trans. of the A.M.S.* **291** (1) (1985), 255-267

[15] M.C. Shaw—Local existence theorems with estimates for $\bar{\partial}_b$ on weakly pseudoconvex boundaries, *Mat. Ann.* **294** (1992), 677-700

[16] G. Zampieri—Q-Pseudoconvexity and regularity at the boundary for solutions of the $\bar{\partial}$-problem, *Compositio Math.* **121** (2000), 155–162

Heungju Ahn
Department of Mathematics, Seoul Nat. University,
San 56-1, Shinrimdong Kwanak-gu Seoul, 151-747,
Korea
E-mail: hjahn@math.snu.ac.kr

Giuseppe Zampieri
Dipartimento di Matematica,
Università di Padova,
via Belzoni 7, 35131 Padova,
Italy
E-mail: zampieri@math.unipd.it