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The Atiyah Singer Index Theorem and Chern Weil Forms

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If E is a complex vector bundle over M with a connection A, the holonomy around all closed paths based at a point in M may or may not lie in a compact subgroup of the complex linear automorphisms of the fibre over the base point.

In the compact case one may choose a Hermitian structure on E which is preserved by holonomy and in the noncompact case one may not choose such an invariant Hermitian structure.

The Chern Weil forms $\{1/n!\ \mathrm{trace}(R/2\pi i)^n\,,\ n = 1, 2, 3, ...\}$, where R is the curvature of the connection, are closed forms on M which are real in the compact case and not necessarily so in the noncompact case.

Questions: Which closed forms occur as Chern Weil forms of connections on complex vector bundles over M ? With compact holonomy? With arbitrary holonomy?

Answers can be given using the Atiyah Singer Index Theorem:

Say that a closed total even real cohomology class $C = c + c' + c'' + ...$ on a smooth manifold M satisfies the Atiyah Singer Index Property if given $f : V \to M$ we have $[f^*C \cdot T(V), V]$ is an integer, where V is a closed stably almost complex

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manifold mapping by f into M , $\cdot$ is cup product, $T(V)$ is the Todd genus of V and $[\,,\,]$ is the pairing between homology and cohomology.

Theorem 1: A real even total closed form $w + w' + w'' + ...$ is the total Chern Weil form of a complex bundle with a connection having compact holonomy if and only if the cohomology classes of the real forms satisfy the Atiyah Singer Index Property.

Theorem 2: A complex valued total even closed form $w + w' + w'' + ...$ is the total Chern Weil form of a connection with unrestricted holonomy if and only if the cohomology class of the complex form is real and satisfies the Atiyah Singer Index Property.

Sketch of proof:

(1) It has been known for decades that total cohomology classes that are Chern characters of complex vector bundles are characterized by even dimensionality and the integrality condition imposed by the Atiyah Singer Index Theorem. Here is a sketch. The integrality itself follows by pushing forward to a point the K theory orientation of complex vector bundles defined by the difference between total even and total odd exterior powers. This orientation also defines a natural transformation from complex cobordism to complex K theory from which Conner Floyd in their book on bordism theory derived an amazing isomorphism between bordism tensor Z over the bordism of a point and complex K homology. Here the todd genus makes Z into a module over the bordism of a point. Then one uses the unexpected but easy to prove fact that complex K theory and complex K homology are in the relation called Pontryagin duality. So appropriate homomorphisms on K homology yield elements in K cohomology , which is what we want. This type of argument is discussed at the end of the second author's "1970 MIT Notes on Geometric Topology" which was recently published in book form.

(2) The Theorems can then be deduced from an "onto lemma" asserting the map from connections on products of trivial complex line bundles to the total Chern Simons forms among the total odd forms is onto odd forms mod exact. Here unitary connections are used to realize real odd forms and general connections are used to realize complex odd forms. Using

this lemma [1] we can alter a bundle with connection by a connection on an additional trivial bundle factor to absorb the required exact form.

References

[1] James Simons and Dennis Sullivan, *Structured Bundles and Differential K Theory*, to appear in Publications of the Clay Mathematics Institute, Proceeedings of Alain Connes' 60th Birthday Conference, Paris 2007.

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