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Homology of Equivariant Vector Fields

Gerald W. Schwarz

Abstract: Let K be a compact Lie group. We compute the abelianization of the Lie algebra of equivariant vector fields on a smooth K-manifold X. We also compute the abelianization of the Lie algebra of strata preserving smooth vector fields on the quotient X/K.
Keywords: Equivariant vector fields, homology, diffeomorphisms groups.

1. Introduction

1.1. K. Abe and K. Fukui [AbFu2] have considered the first homology group (abelianization) of the group of equivariant smooth diffeomorphisms of a smooth K-manifold X, where K is finite. They also computed the abelianization for the diffeomorphisms of the quotient orbifold X/K. Our results below are the analogues of their results for vector fields in the case that K is a compact Lie group. The vector fields are, in a sense, the Lie algebras of the relevant diffeomorphism groups, so, hopefully, our results indicate that one should be able to generalize the Abe-Fukui results. There are already generalizations in some cases [AbFu1].

1.2. Let X be a smooth K-manifold where K is compact. Let $\mathcal{X}^\infty(X)$ denote the Lie algebra of smooth vector fields on X and let $\mathcal{X}_c^\infty(X)$ denote the subalgebra of vector fields with compact support. If X is algebraic, then $\mathcal{X}(X)$ will denote the polynomial vector fields on X. By $\mathcal{X}^\infty(X)^K$, etc. we mean the K-invariant

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elements in $\mathcal{X}^\infty(X)$, etc. We will state most of our results for $\mathcal{X}_c^\infty(X)^K$; the corresponding results for $\mathcal{X}^\infty(X)^K$ follow easily from our techniques.

If $\mathfrak{g}$ is a Lie algebra, we denote by $\mathcal{H}(\mathfrak{g})$ the abelianization $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. We denote the Lie algebras of compact Lie groups K, H, etc. by the corresponding gothic letters $\mathfrak{k}$, $\mathfrak{h}$, etc.

1.3. Let $x \in X$. Then we have the isotropy group K_x and its slice representation on $W_x := T_xX/T_x(Kx)$ where Kx denotes the K-orbit through x. We say that the orbit Kx is *isolated* if $W_x^{K_x} = (0)$. It follows from the differentiable slice theorem that Kx is isolated if and only if all isotropy groups K_y of points y near x, $Ky \neq Kx$, are conjugate to a proper subgroup of K_x. There is then a discrete subset $\{x_i\}_{i\in I}$ of X (possibly empty) where we choose one point from each isolated orbit. Let H_i denote K_{x_i} and set $W_i := W_{x_i}$, $i \in I$.

Theorem 1.4. *Let X and the x_i, H_i and W_i be as above Then*

$$\mathcal{H}(\mathcal{X}_c^\infty(X)^K) \simeq \bigoplus_i \mathcal{H}(\mathfrak{k}^{H_i}/\mathfrak{h}_i^{H_i}) \bigoplus_i \mathcal{H}(\operatorname{End}(W_i)^{H_i}).$$

Theorem 1.5. *Let H be a compact Lie group and V an H-module where $V^H = (0)$. Write $V = \oplus_{j=1}^m n_jV_j$ where the V_j are irreducible and pairwise non-isomorphic and n_jV_j denotes the direct sum of n_j copies of V_j. Let l denote the number of V_j such that $\operatorname{End}(V_j)^H \simeq \mathbb{C}$ and let $Z(\operatorname{End}(V)^H)$ denote the center of $\operatorname{End}(V)^H$. Then*

$$\mathcal{H}(\operatorname{End}(V)^H) \simeq Z(\operatorname{End}(V)^H) = \bigoplus_j Z(\operatorname{End}(n_jV_j)^H) \simeq \mathbb{R}^{m-l} \oplus \mathbb{C}^l.$$

Let $\mathcal{X}_c^\infty(X/K)$ denote the Lie algebra of compactly supported smooth strata preserving vector fields on X/K (see §4 for definitions).

Theorem 1.6. *Let X and the x_i, H_i and W_i be as above. Then*

$$\mathcal{H}(\mathcal{X}_c^\infty(X/K)) \simeq \bigoplus_i (Z(\operatorname{End}(W_i)^{H_i})/\mathfrak{s}_i)$$

where each $\mathfrak{s}_i$ is the Lie algebra of a torus S_i lying in $Z(\operatorname{End}(W_i)^{H_i})$.

We will say more about the S_i in §4.

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2. Vanishing of abelianizations

2.1. In the following, let $\mathcal{B}_c^\infty(X)^K$ denote $[\mathcal{X}_c^\infty(X)^K, \mathcal{X}_c^\infty(X)^K]$ and let $\mathcal{C}_c^\infty(X)^K$ denote the compactly supported smooth functions on X. Our first goal is to show that $\mathcal{H}(\mathcal{X}_c^\infty(X \times \mathbb{R})^K)$ is zero.

Lemma 2.2. *Let $A \in \mathcal{X}_c^\infty(X)^K$ and $B \in \mathcal{X}^\infty(X)^K$. Then $[A, B] \in \mathcal{B}_c^\infty(X)^K$.*

Proof. Let $g \in \mathcal{C}_c^\infty(X)^K$ be identically 1 on a neighborhood of $\operatorname{supp} A$. Then $[A, gB] = g[A, B] + A(g)B = [A, B] \in \mathcal{B}_c^\infty(X)^K$. □

Proposition 2.3. *Let K act on $X \times \mathbb{R}$ with the given action on X and the trivial action on $\mathbb{R}$. Then $\mathcal{H}(\mathcal{X}_c^\infty(X \times \mathbb{R})^K) = 0$.*

Proof. Let t denote the usual coordinate function on $\mathbb{R}$ and let $g \in \mathcal{C}_c^\infty(X \times \mathbb{R})^K$. We show that $g\frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$. For $x \in X$ and $s \in \mathbb{R}$ set $h(x, s) = \int_0^s g(x, u)\, du$. Then h is smooth and K-invariant. Let $f \in \mathcal{C}_c^\infty(X \times \mathbb{R})^K$. Then

$$[f\frac{d}{dt}, h\frac{d}{dt}] = f\frac{dh}{dt}\frac{d}{dt} - h\frac{df}{dt}\frac{d}{dt} \text{ and}$$

$$[\frac{d}{dt}, fh\frac{d}{dt}] = f\frac{dh}{dt}\frac{d}{dt} + h\frac{df}{dt}\frac{d}{dt}.$$

Hence $2fg\frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$. If f equals $1/2$ on a neighborhood of $\operatorname{supp} g$, we obtain that $g\frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$.

Now suppose that $A \in \mathcal{X}_c^\infty(X \times \mathbb{R})^K$. By our result above, we can assume that A annihilates t. Set $B(x, s) = \int_0^s A(x, u)\, du$ and let $g \in \mathcal{C}_c^\infty(X \times \mathbb{R})^K$ equal 1 on a neighborhood of $\operatorname{supp} A$. Then $[g\frac{d}{dt}, B] = gA - B(g)\frac{d}{dt}$. We already know that $B(g)\frac{d}{dt} \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$, hence $A \in \mathcal{B}_c^\infty(X \times \mathbb{R})^K$. Thus $\mathcal{H}(\mathcal{X}_c^\infty(X \times \mathbb{R})^K) = 0$. □

2.4. Let H be a closed subgroup of K and W an H-module. Then we have the twisted product $K *^H W$ which is the quotient $(K \times W)/H$ where $h(k, w) = (kh^{-1}, hw)$, $h \in H$, $k \in K$ and $w \in W$. We denote the image of $(k, w) \in K \times W$

in $K *^H W$ by $[k, w]$. Note that $K *^H W$ is naturally a K-vector bundle and a real algebraic K-variety [Schw3].

Let $H \to \mathrm{GL}(W)$ be the slice representation at a point $x \in X$. By the differentiable slice theorem, a K-neighborhood of Kx in X is K-diffeomorphic to $K *^H W$. By Proposition 2.3, $\mathcal{H}(\mathcal{X}_c^\infty(K *^H W)^K) = 0$ if $W^H \neq (0)$.

Let F be a closed K-stable subset of X. We say that $\mathcal{H}(\mathcal{X}_c^\infty(X)^K)$ *is supported on* F if $\mathcal{H}(\mathcal{X}_c^\infty(X \setminus F)^K) = 0$. Using a partition of unity argument we can show

Corollary 2.5. *Let $F = \{x \in X \mid W_x^{K_x} = 0\}$. Then $\mathcal{H}(\mathcal{X}_c^\infty(X)^K)$ is supported on F.*

3. Local computations

3.1. Our results above show that there is a discrete set of orbits $\{Kx_i\}$ such that

$$\mathcal{H}(\mathcal{X}_c^\infty(X)^K) \simeq \bigoplus_i \mathcal{H}(\mathcal{X}_c^\infty(K *^{H_i} W_i)^K)$$

where $H_i = K_{x_i}$ and W_i is the slice representation of H_i at x_i. Thus it suffices to compute $\mathcal{H}(\mathcal{X}_c^\infty(K *^H V)^K)$ where H is a closed subgroup of K, V is an H-module and $V^H = (0)$. This computation is the content of the following theorem.

Theorem 3.2. *Let H and V be as above. Then*

$$\mathcal{H}(\mathcal{X}_c^\infty(K *^H V)) \simeq \mathcal{H}(\mathfrak{k}^H/\mathfrak{h}^H) \oplus \mathcal{H}(\mathrm{End}(V)^H).$$

3.3. Our proof of the theorem requires several lemmas. Set $Y := K *^H V$. Then

$$\mathcal{X}(Y)^K \simeq \mathcal{X}(K \times V)^{K \times H} / (\mathcal{O}(K \times V)\mathfrak{h})^{K \times H}$$

(see [Schw2, §4]) where H has the diagonal action (see 2.4) on $K \times V$ (inducing an action of $\mathfrak{h}$) and $\mathcal{O}(K \times V)$ denotes the polynomial functions on $K \times V$. Now

$$\mathcal{X}(K \times V)^{K \times H} \simeq (\mathcal{X}(K) \otimes \mathcal{O}(V) \oplus \mathcal{O}(K) \otimes \mathcal{X}(V))^{K \times H} \simeq (\mathfrak{k} \otimes \mathcal{O}(V))^H \oplus (1 \otimes \mathcal{X}(V)^H)$$

while

$$(\mathcal{O}(K \times V)\mathfrak{h})^{K \times H} \simeq (\mathfrak{h} \otimes \mathcal{O}(V))^H.$$

3.4. We have the Euler operator $E \in \mathcal{X}(V)^H$, where if $x_1, x_2, \dots$ are coordinate functions on V, then $E = \sum_i x_i \frac{\partial}{\partial x_i}$. By the isomorphisms above, E can be considered as a $(K \times H)$-invariant vector field on $K \times V$ and as a K-invariant vector field on Y.

Lemma 3.5. *Let $f \in \mathcal{C}^\infty(Y)^K$. Then $f = E(h)$ for some $h \in \mathcal{C}^\infty(Y)^K$ if and only if $f([e,0]) = 0$.*

Proof. Clearly the condition on f is necessary. Suppose that $f([e,0]) = 0$. Since f is K-invariant, it is determined by its restriction g to $\{[e,v] \mid v \in V\} \simeq V$, where g is H-invariant. Set $h(v) = \int_0^1 (1/t) g(tv)\, dt$. Then $h \in \mathcal{C}^\infty(V)^H$ since $g(0) = 0$. We have

$$E(h)(v) = \int_0^1 \frac{1}{t} \sum_i x_i \frac{\partial g}{\partial x_i}(tv) t\, dt = \int_0^1 \sum_i x_i \frac{\partial g}{\partial x_i}(tv)\, dt$$
$$= \int_0^1 \frac{d}{dt} g(tv)\, dt = g(v) - g(0) = g(v).$$

□

Corollary 3.6. *Let $g \in \mathcal{C}_c^\infty(Y)^K$ such that $g([e,0]) = 0$. Then $gE \in \mathcal{B}_c^\infty(Y)^K$.*

Proof. By Lemma 3.5, $g = E(h)$ for some $h \in \mathcal{C}^\infty(Y)^K$. Let $f \in \mathcal{C}_c^\infty(Y)^K$ such that f is $1/2$ in a neighborhood of $\operatorname{supp} g$. Then, as in Proposition 2.3,

$$[E, fhE] + [fE, hE] = 2fE(h)E = 2fgE,$$

so that $gE \in \mathcal{B}_c^\infty(Y)^K$. □

3.7. Since Y is real algebraic, the results in [Schw1, §6] show that $\mathcal{X}^\infty(Y) \simeq \mathcal{C}^\infty(Y) \otimes_{\mathcal{O}(Y)} \mathcal{X}(Y)$. For compactly supported sections we clearly have that $\mathcal{X}_c^\infty(Y) = \mathcal{C}_c^\infty(Y)\mathcal{X}(Y)$.

3.8. We have an E-eigenspace decomposition

$$\mathcal{X}(K \times V)^{K \times H} \simeq \bigoplus_{m \geq 0} (\mathfrak{k} \otimes \mathcal{O}(V)_m)^H \oplus (1 \otimes \mathcal{X}(V)_m^H)$$

and similarly for $(\mathfrak{h} \otimes \mathcal{O}(V))^H$. The weights that occur in $\mathcal{X}(V)^H$ are all positive since $V^H = (0)$. We have an induced decomposition

$$\mathcal{X}(Y)^K = \bigoplus_{m \geq 0} \mathcal{X}(Y)_m^K.$$

Remark 3.9. Since the sum only contains terms for $m \geq 0$, an element of $\mathcal{X}(Y)^K$ applied to an element of $\mathcal{C}^\infty(Y)^K \simeq \mathcal{C}^\infty(V)^H$ always vanishes at $[e,0]$.

Lemma 3.10. *Let $A \in \mathcal{X}(Y)^K_m$ and let $f \in \mathcal{C}^\infty_c(Y)^K$. Then $fA \in \mathcal{B}^\infty_c(Y)^K$ if*

(1) *$m > 0$ or*
(2) *$f([e,0]) = 0$.*

Proof. Suppose that $m > 0$. Then $[(1/m)fE, A] = fA - (1/m)A(f)E$ where $A(f)E \in \mathcal{B}^\infty_c(Y)^K$ by Corollary 3.6. Hence $fA \in \mathcal{B}^\infty_c(Y)^K$. If $m = 0$ and $f([e,0]) = 0$, then let $h \in \mathcal{C}^\infty(Y)^K$ be such that $E(h) = f$, and let $g \in \mathcal{C}^\infty_c(Y)^K$. Then

$$[gE, hA] = gE(h)A - hA(g)E = gfA - hA(g)E,$$

where $hA(g)E \in \mathcal{B}^\infty_c(Y)^K$ by Corollary 3.6. We may arrange that $gfA = fA$, so $fA \in \mathcal{B}^\infty_c(Y)^K$. □

Proof of Theorem 3.2. We first define a map of Lie algebras $\varphi\colon \mathcal{X}^\infty_c(Y)^K \to \mathcal{X}(Y)^K_0$. Let $B = \sum_{i=1}^m f_i B_i \in \mathcal{X}^\infty_c(Y)^K$ where $f_i \in \mathcal{C}^\infty_c(Y)^K$ and $B_i \in \mathcal{X}(Y)^K_{m_i}$, $i = 1, \dots, m$. Define $\varphi(B) := \sum_{m_i=0} f_i([e,0])B_i \in \mathcal{X}(Y)^K_0$. It is obvious that φ is surjective. Suppose that C, $D \in \mathcal{X}(Y)^K$ are eigenvectors for E and that f, $g \in \mathcal{C}^\infty_c(Y)^K$. Then $[fC, gD] = fC(g)D - gD(f)C + fg[C,D]$ where $C(g)$ and $D(f)$ vanish at $[e,0]$. Thus $\varphi([fC,gD]) = (fg)(0)\varphi([C,D]) = (fg)(0)[\varphi(C),\varphi(D)] = [\varphi(fC),\varphi(gD)]$. Now φ induces $\tilde{\varphi}\colon \mathcal{H}(\mathcal{X}^\infty_c(Y)^K) \to \mathcal{H}(\mathcal{X}(Y)^K_0)$, which is again surjective. Suppose that $B = \sum_i f_i B_i \in \operatorname{Ker}(\tilde{\varphi})$ where the B_i are in $\mathcal{X}(Y)^K_0$. Then $\varphi(B) = \sum_j [C_j, D_j]$ where C_j, $D_j \in \mathcal{X}(Y)^K_0$ for all j. Let $f \in \mathcal{C}^\infty_c(Y)^K$ such that f is 1 on a neighborhood of $[e,0]$. Then $B - \sum_j [fC_j, fD_j] \in \mathcal{B}^\infty_c(Y)^K$. Hence $\tilde{\varphi}$ is an isomorphism. From our equations in 3.3 it follows that $\mathcal{H}(\mathcal{X}(Y)^K_0) \simeq \mathcal{H}(\mathfrak{k}^H/\mathfrak{h}^H) \oplus \mathcal{H}(\operatorname{End}(V)^H)$. □

Proof of Theorem 1.4. The theorem is immediate from 3.1 and Theorem 3.2 □

Proof of Theorem 1.5. Let $V = \oplus_{j=1}^m n_j V_j$ and H be as in 1.5. Then $\operatorname{End}(V)^H \simeq \oplus_j \operatorname{End}(n_j V_j)^H$. There are three cases to consider.

Case 1: $\operatorname{End}(V_j)^H \simeq \mathbb{R}$. Then $\operatorname{End}(n_j V_j)^H \simeq \mathfrak{gl}(n_j, \mathbb{R})$ and $\mathcal{H}(\mathfrak{gl}(n_j,\mathbb{R})) \simeq Z(\mathfrak{gl}(n_j,\mathbb{R})) \simeq \mathbb{R}$.

Case 2: $\operatorname{End}(V_j)^H \simeq \mathbb{C}$. Then $\operatorname{End}(n_j V_j)^H \simeq \mathfrak{gl}(n_j, \mathbb{C})$ and $\mathcal{H}(\mathfrak{gl}(n_j,\mathbb{C})) \simeq Z(\mathfrak{gl}(n_j,\mathbb{C})) \simeq \mathbb{C}$.

Case 3: $\mathrm{End}(V_j)^H \simeq \mathbb{H}$, the quaternions. Then $\mathrm{End}(n_j V_j)^H \simeq \mathfrak{gl}(n_j, \mathbb{H})$ and we have that $\mathcal{H}(\mathfrak{gl}(n_j, \mathbb{H})) \simeq Z(\mathfrak{gl}(n_j, \mathbb{H})) \simeq \mathbb{R}$. The theorem follows. □

4. Computations on the quotient

We now consider the abelianization of the strata preserving vector fields on the quotient X/K. We recall a few facts about X/K from [Schw1]. Let $\pi\colon X \to X/K$ denote the canonical map, where X/K is given the quotient topology. Then X/K has a differentiable structure where for U an open subset of X/K, $\mathcal{C}^\infty(U) = \mathcal{C}^\infty(\pi^{-1}(U))^K$. Let H be a closed subgroup of K. Then we have the corresponding stratum $X^{(H)} := \{x \in X \mid K_x \text{ is conjugate to } H\}$ and its image $(X/K)^{(H)} \subset X/K$. The *isotropy strata* $(X/K)^{(H)} \subset X/K$ and $X^{(H)} \subset X$ are smooth and locally closed submanifolds and $\pi\colon X^{(H)} \to (X/K)^{(H)}$ is naturally a smooth fiber bundle (with structure group $N_K(H)/H$). The number of isotropy strata is locally finite on X and X/K. Let $\mathrm{Der}(\mathcal{C}^\infty(X/K))$ denote the derivations of $\mathcal{C}^\infty(X/K)$ and let $\mathcal{X}^\infty(X/K)$ denote those derivations that preserve the ideals of functions I_{H_i} vanishing on the isotropy strata $(X/K)^{(H_i)}$ of X/K. Each element of $\mathcal{X}^\infty(X)^K$ restricts to a derivation of $\mathcal{C}^\infty(X/K)$, so there is a canonical map $\pi_*\colon \mathcal{X}^\infty(X)^K \to \mathrm{Der}(\mathcal{C}^\infty(X/K))$. The main theorem of [Schw1] is that $\mathrm{Im}\,\pi_* \subset \mathcal{X}^\infty(X/K)$ and that π_* is surjective. Clearly π_* is a homomorphism of Lie algebras so we have an induced surjection $\mathcal{H}(\mathcal{X}^\infty(X)^K) \to \mathcal{H}(\mathcal{X}^\infty(X/K))$. We only need to compute what happens in the case of $X = K *^H V$ where H is a closed subgroup of K and V is an H-module such that $V^H = (0)$. Let $V = \oplus_{j=1}^m n_j V_j$ as in Theorem 1.5. The following has Theorem 1.6 as a corollary.

Theorem 4.1. *Assume that* $\mathrm{End}(V_j)^H \simeq \mathbb{C}$ *if and only if* $j \le l$ *where* $l \le m$. *Let* T *be the corresponding torus* $(S^1)^l \subset \prod_{j=1}^l Z(\mathrm{End}(V_j)^H)$. *Then* T *acts on* V *commuting with the action of* H, *and we have an induced map* $T \to \mathrm{Aut}(V/H)$. *Let* S *denote the kernel where* $\dim S = k$. *Then*

$$\mathcal{H}(\mathcal{X}_c^\infty((K *^H V)/K)) \simeq \mathcal{H}(\mathcal{X}(V/H)) \simeq \mathbb{R}^{m-l+k} \oplus \mathbb{C}^{l-k}.$$

Proof. We have the canonical surjection of Lie algebras $\pi_*\colon \mathrm{End}(V)^H \to \mathcal{X}_0(V/H)$ and π_* induces a surjection of $\mathcal{H}(\mathrm{End}(V)^H)$ onto $\mathcal{H}(\mathcal{X}(V/H))$. For every j we have the identity $\mathrm{Id}_j \in \mathrm{End}(n_j V_j)^H$ and clearly these elements give linearly independent derivations of $\mathcal{O}(V)^H$. Now consider the action of T on V/H and its kernel S. Then $\mathfrak{s}$ is the kernel of the restriction of π_* to the center of $\mathrm{End}(V)^H$, so that $\mathfrak{s}$ is the kernel on homology. □

Example 4.2. Suppose that H is a torus acting faithfully on V and $V = \sum_{j=1}^m n_j V_j$ where $V^H = (0)$ as in Theorem 1.5. Then $\mathfrak{s} \simeq \mathfrak{h}$ and $\mathcal{H}(\mathcal{X}(V/H)) \simeq \mathbb{R}^k \oplus \mathbb{C}^{m-k}$ where $k = \dim H$.

Example 4.3. Let $V = \mathbb{C}^n \oplus \wedge^2\mathbb{C}^n$ with the canonical action of $\mathrm{SU}(n,\mathbb{C})$, $n \geq 3$. Then T has dimension 2 and S has dimension 1. See [Schw1, Table I].

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Gerald W. Schwarz
Department of Mathematics
Brandeis University
PO Box 549110
Waltham, MA 02454-9110
E-mail: schwarz@brandeis.edu