

Spin Cobordism Categories in Low Dimensions

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For Michael Atiyah, in deep gratitude.

Abstract: The Madsen-Tillmann spectra defined by categories of three- and four-dimensional Spin manifolds have a very rich algebraic structure, whose surface is scratched here.

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1. COBORDISM CATEGORIES

1.1 Many variations and generalizations are possible, but to begin, consider the topological two-category $DCobord$ whose objects are oriented smooth closed d -manifolds ($D = d + 1$), with the topological category $DCobord(V, V')$ of morphisms having as objects, D -dimensional cobordisms

$$W : V \rightarrow V'$$

from V to V' ; for our purposes this will mean an identification $\partial W \cong V_{\text{op}} \amalg V'$, extended to a small neighborhood of the boundary. The two-morphisms will be orientation-preserving diffeomorphisms of such cobordisms, which equal the identity near the boundary. The composition functor

$$DCobord(V, V') \times DCobord(V', V'') \rightarrow DCobord(V, V'')$$

is defined by glueing outgoing to incoming boundaries.

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A topological category \mathbf{C} is a kind of simplicial space, and so has a geometric realization $|\mathbf{C}|$; for example,

$$|DCobord(V, V')| = \coprod_{[W:V \rightarrow V']} B\text{Diff}_0^+(W)$$

is the union, indexed by diffeomorphism classes of cobordisms W from V to V' , of the classifying spaces of the groups of orientation-preserving diffeomorphisms of W which equal the identity near the boundary. We'll write $D|\text{Cobord}|$ for the topological category with closed d -manifolds as objects, and the classifying spaces above as morphism objects; it is symmetric monoidal (under disjoint union).

Such categories have an impressive history [3, 16, 17, 21] in topology and physics. This paper applies the recent breakthroughs of [9] which (in great generality) identify the classifying spectra of such categories. The formalism of Galatius, Madsen, Tillmann, and Weiss frames these categories somewhat differently: they work with a category \mathcal{C}_D of manifolds embedded in a high-dimensional Euclidean background, but the description used above is equivalent, and is convenient in physics.

1.2 A topological transformation group $G \times X \rightarrow X$ has an associated homotopy-to-geometric quotient map

$$X//G := EG \times_G X \rightarrow \text{pt} \times_G X = X/G$$

which defines a kind of resolution

$$B\text{Diff} \sim E\text{Diff} \times_{\text{Diff}} \text{Metrics} \rightarrow \text{pt} \times_{\text{Diff}} \text{Metrics}$$

of the moduli space of Riemannian metrics [8] on a manifold. For a closed manifold the action of the diffeomorphism group on the space of metrics is proper; for surfaces of genus > 1 , for example, its isotropy groups are not just compact but finite, making the map a rational homology equivalence.

This resolution defines a monoidal functor

$$D|\text{Cobord}| \rightarrow \text{Gravity}_D$$

to a topological category with moduli spaces of metrics as its morphism objects. The Einstein-Hilbert functional

$$g \mapsto \int_W R(g) \, d\text{vol}_g : \text{Metrics}/\text{Diff} \rightarrow \mathbb{R}$$

is a natural candidate for a Morse function on these objects, so this category models interesting aspects of (Euclidean) general relativity. Witten has suggested that backgrounding the choice of Morse function leads to more general models in which topology change can be treated quite naturally.

1.3.1 This paper is concerned with the cobordism category defined by four-dimensional Spin manifolds. The classifying space $|\mathbf{C}|$ of a **symmetric monoidal** topological category \mathbf{C} is a kind of abelian monoid, or better: a Γ -space in the sense of [20]. Its group completion

$$|\mathbf{C}|^+ := \Omega B|\mathbf{C}|$$

is an infinite loop-space, which is characterized by its associated stable spectrum.

GMTW identify $|D|\text{Cobord}|^+$ as the infinite loop space associated to a twisted desuspension

$$MTSO(D) := BSO(D)^{-\mathbf{D}}$$

of the classifying space for the orthogonal group, where \mathbf{D} is the vector bundle associated to the basic representation of $SO(D)$ on \mathbb{R}^D . More generally, a pullback diagram

$$\begin{array}{ccc} G(d) & \longrightarrow & SO(d) \\ \downarrow & & \downarrow \\ G(D) & \longrightarrow & SO(D) \end{array}$$

of groups and homomorphisms defines a topological category $G|\text{Cobord}|$ of manifolds with $G(d)$ -structures on their tangent bundles, up to cobordism through manifolds with $G(D)$ -structures; and the techniques of [9] identify its associated spectrum as

$$MTG(D) := BG(D)^{-\mathbf{D}}$$

where \mathbf{D} is now the D -dimensional representation of $G(D)$ pulled back from the basic representation of the orthogonal group. A further generalization identifies the classifying spectrum for the category of G -manifolds mapped to some parameter space X as

$$X_+ \wedge BG(D)^{-\mathbf{D}} .$$

1.3.2 When $d = 1$, for example, we get the desuspension (alternately denoted $\mathbb{C}P_{-1}^\infty$) of $BSO(2) = \mathbb{C}P^\infty$ by the tautological line bundle. Its homology is free of rank one in even dimensions ≥ -2 , and is otherwise zero. Pinching off its (-2)-dimensional cell defines a cofibration

$$S^{-2} \rightarrow \mathbb{C}P_{-1}^\infty \rightarrow \mathbb{C}P_+^\infty$$

of spectra, with an associated fibration

$$\Omega^\infty S^{-2} \rightarrow |2|\text{Cobord}||^+ \rightarrow Q(\mathbb{C}P^\infty) \times Q(S^0)$$

of loopspaces. Since $\Omega^\infty S^{-2}$ has torsion homotopy, the rational homology of $|2|\text{Cobord}||^+$ is a free bicommutative Hopf algebra generated by $\tilde{H}_* \mathbb{C}P_\infty$ (ie, by the Miller-Morita-Mumford classes κ_i , $i \geq 1$), extended by degree zero classes $\kappa_0^{\pm n}$ coming from the rational cohomology of $Q(S^0)$

1.3.3 The covariant functor [12 Ch III]

$$X \mapsto \text{Spec } H^\pm(\Omega_0^\infty X, \mathbb{F}) := \tilde{\mathbf{H}}_\pm(X, \mathbb{F})$$

(from (Spectra) to unipotent commutative supergroup-schemes over the field \mathbb{F}) is a homotopy-theoretic analog of the ‘big’ quantum cohomology studied in some contexts in physics. For example, the infinite loop-space associated to the suspension spectrum $\Sigma^\infty X$ defined by a connected pointed space splits stably as

$$\Omega^\infty \Sigma^\infty X \sim \coprod_{n \geq 0} E\Sigma_n \wedge_{\Sigma_n} X^{\wedge n}$$

so its rational cohomology is the symmetric algebra on the reduced cohomology of X . In this case $\tilde{\mathbf{H}}_\pm(X, \mathbb{Q})$ can be identified with the affine (super)groupscheme which represents the functor

$$(\mathbb{Q} - \text{algebras}) \ni A \mapsto \tilde{H}_\pm(X; A) \in (\mathbb{Q} - \text{Vect}) .$$

For a general connected spectrum X , $H^\pm(\Omega^\infty X, \mathbb{Q})$ is the universal enveloping Hopf algebra associated to the (super-commutative) Lie algebra $\pi_\pm(X) \otimes \mathbb{Q}$ of primitives. The category of such affine groupschemes is closed and symmetric monoidal, with a product \boxtimes which is not very familiar [12 Ch II]; over \mathbb{Q} , it corresponds to the graded tensor product of spaces of primitives.

If the loop-space associated to a spectrum is not connected, let $\mathbf{H}^0(X, \mathbb{F})$ be the groupscheme represented by the group ring $\mathbb{F}[\pi_0(X)]$, and let $\mathbf{H}_0(X, \mathbb{F})$ be the

spectrum of the ring of finitely-supported \mathbb{F} -valued functions on $\pi_0(X)$; then we can define

$$\mathbf{H}_\pm(X, \mathbb{F}) = \mathbf{H}_0(X, \mathbb{F}) \times \tilde{\mathbf{H}}_\pm(X, \mathbb{F})$$

(and similarly, for cohomology). For example,

$$\mathbf{H}_\pm(S^2 \times MTSO(2), \mathbb{Q})$$

is an analog of the big quantum cohomology related to the Toda lattice [10].

2. LOW-DIMENSIONAL SPIN COBORDISMS

2.0 The action

$$u, q \mapsto uqu^{-1} : \mathrm{SU}(2) \times \mathbb{H} \rightarrow \mathbb{H}$$

of the group $\mathrm{SU}(2) = \{u \in \mathbb{H} \mid |u| = 1\}$ of unit quaternions leaves the subspace $\mathbb{R} \subset \mathbb{H}$ invariant, defining a double cover

$$\rho : \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$$

of the rotation group of the subspace orthogonal to it, identifying $\mathrm{SU}(2)$ with $\mathrm{Spin}(3)$. Similarly, the action

$$u_L, u_R, q \mapsto u_L q u_R^{-1} : (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times \mathbb{H} \rightarrow \mathbb{H}$$

factors through the double cover

$$\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4) \rightarrow \mathrm{SO}(4) .$$

It is easy to see that the diagram

$$\begin{array}{ccc} \mathrm{SU}(2) & \xrightarrow{\rho} & \mathrm{SO}(3) \\ \downarrow \Delta & & \downarrow \\ \mathrm{SU}(2) \times \mathrm{SU}(2) & \longrightarrow & \mathrm{SO}(4) \end{array}$$

is a pullback; following §1.3.1, this defines the cobordism category of Spin three-manifolds up to four-dimensional Spin cobordism. Similarly, the $D = 3$ Spin cobordism category is defined by three-dimensional Spin cobordisms between two-dimensional Spin manifolds: in Riemann surface terms [2], the latter structure amounts to a choice of square root for the canonical complex line bundle. [Complex Spin structures are very interesting [22], but they won't be considered here.]

We'll write $\mathbb{H}_{\text{ad}} - 1$ for the three-dimensional representation ρ , and $\mathbb{H} \otimes \mathbb{H}_{\text{op}}$ for the four-dimensional Spin representation, as in [13 §1.4, 15 §1]; then

$$MT\text{Spin}(3) \sim \Sigma BSU(2)^{-\mathbb{H}_{\text{ad}}}$$

and

$$MT\text{Spin}(4) \sim B(SU(2) \times SU(2))^{-\mathbb{H} \otimes \mathbb{H}_{\text{op}}}.$$

These spectra are very nice, with torsion-free integral homology concentrated in degrees $\equiv -1$ (resp. 0) mod four, but they are nontrivial in negative dimensions, starting in degree -3 (resp. -4). It will simplify notation below to introduce their connective suspensions

$$MT(3) := \Sigma^3 MT\text{Spin}(3)$$

and

$$MT(4) := \Sigma^4 MT\text{Spin}(4).$$

2.1 The representation $\mathbb{H} \otimes \mathbb{H}_{\text{op}}$ restricts to \mathbb{H}_{ad} along the diagonal embedding of $SU(2)$ in $SU(2) \times SU(2)$. Since Thom spaces (and spectra) behave nicely under pullback, this defines a morphism

$$\Delta_{\natural} : MT(3) \rightarrow MT(4).$$

The main result of this note asserts that (at least, up to cohomology) this map makes $MT(3)$ a kind of cocommutative and coassociative coalgebra spectrum.

PROPOSITION: The integral cohomology $H^*MT(3)$ can be identified with $H^*BSU(2)$ **as an algebra**, consistent with a splitting

$$\Psi^* : H^*MT(3) \otimes H^*MT(3) \cong H^*MT(4)$$

which identifies $\Delta_{\natural}^* : H^*MT(4) \rightarrow H^*MT(3)$ with the multiplication map.

Proof: If X is a compact connected space, then any $[V] \in \tilde{K}O(X)$ is stably equivalent to a vector bundle V over X , of dimension $v \gg 0$, and Atiyah's Thom spectrum

$$X^{[V]} := \Sigma^{-v} X^V$$

is well-defined up to homotopy. If $[V]$ is orientable (eg if $w_1(V) = 0$, in the case of integral homology), there is a Thom isomorphism

$$\Phi_V : H^*X \rightarrow H^*X^{[V]} .$$

Taking a limit over finite subcomplexes extends such constructions to nice spaces like $BSU(2)$.

With this notation, we have a commutative diagram

$$\begin{array}{ccccc} H^*MT(3) \otimes H^*MT(3) & \xrightarrow{\Psi^*} & H^*MT(4) & \xrightarrow{\Delta_{\natural}^*} & H^*MT(3) \\ \Phi_{-\mathbb{H}_{\text{ad}}} \otimes \Phi_{-\mathbb{H}_{\text{ad}}} \uparrow & & \Phi_{-\mathbb{H} \otimes \mathbb{H}_{\text{op}}} \uparrow & & \Phi_{-\mathbb{H}_{\text{ad}}} \uparrow \\ H^*BSU(2)_+ \otimes H^*BSU(2)_+ & \xrightarrow{\cong} & H^*B(SU(2) \times SU(2))_+ & \xrightarrow{\Delta^*} & H^*BSU(2)_+ \end{array}$$

with the composition $\Delta_{\natural}^* \circ \Psi^*$ defining the multiplicative structure.

Verification of associativity amounts to unwinding the collection of Thom isomorphisms which reduce the commutativity (after taking cohomology) of the diagram

$$\begin{array}{ccccccc} MT(3) & \xrightarrow{\Delta_{\natural}} & MT(4) & \xrightarrow{\Psi} & MT(3) \wedge MT(3) & \xrightarrow{1 \wedge \Delta_{\natural}} & MT(3) \wedge MT(4) \\ & & & & \downarrow \Delta_{\natural} \wedge 1 & & \downarrow 1 \wedge \Psi \\ & & & & MT(4) \wedge MT(3) & \xrightarrow{\Psi \wedge 1} & MT(3) \wedge MT(3) \wedge MT(3) \end{array}$$

to a similar diagram expressing the associativity of the usual multiplication on $H^*BSU(2)_+$. Commutativity is a consequence of Δ_{\natural} being essentially a diagonal, and the unit is the composition

$$H^*BSU(2)^{-[\mathbb{H}_{\text{ad}}]} \xrightarrow{\Phi_{\mathbb{H}_{\text{ad}}}^{-1}} H^*BSU(2)_+ \longrightarrow H^*S^0$$

defined by the inclusion of a basepoint into $BSU(2)$. \square

2.2.1 The result above can also be paraphrased in terms of a ring structure on homotopy quantum cohomology, but because

$$\Omega^\infty \text{MT}(3) \sim \mathbb{Z} \times \Omega_0^\infty \text{MT}(3)$$

is not connected, this requires some discussion. According to §1.3.1, $\text{MT}(3)$ is the cobordism spectrum of Spin three-manifolds mapped to the three-sphere. The extra data defined by such a map is (at least, after tensoring with \mathbb{Q}) very close to a framing (in the sense of [4, 15 §2.1]) of a three-dimensional Spin cobordism.

Similarly,

$$\Omega^\infty \text{MTSpin}(4) \sim \mathbb{Z}^2 \times \Omega^\infty \text{MTSpin}(4)$$

with the classes of a K3 surface and the quaternionic projective plane as natural geometric generators for π_0 [11]. The Euler characteristic χ and the signature σ are a basis for the linear functionals on this group, at least over $\mathbb{Z}[1/2]$, and if χ^*, σ^* denote the dual basis elements, then

$$\begin{bmatrix} \mathbb{H}P_2 \\ K3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 16 \end{bmatrix} \begin{bmatrix} \chi^* \\ \sigma^* \end{bmatrix}.$$

Desuspending the isomorphisms in §2.1 yields a splitting

$$\mathbf{H}^\pm(\text{MTSpin}(4), \mathbb{Q}) \cong \boxtimes^2(\mathbf{H}_{KM}^\pm(\Sigma^{-1} \text{MTSpin}(3), \mathbb{Q}))$$

of homotopy-theoretic quantum cohomology: where the subscript on the right indicates an extension of the pro-unipotent object $\tilde{\mathbf{H}}^\pm(\Sigma^{-1} \text{MTSpin}(3), \mathbb{Q})$ by the multiplicative group (represented by the group ring of Kirby-Melvin framings).

2.2.2 The existence of a multiplication on $H^* \text{MT}(3)$ raises the possibility of the existence of a so-called Hopf algebroid structure on $(H^* \text{MT}(3), H^* \text{MT}(4))$. In fact, two three-dimensional cobordisms mapped to the three-sphere define a fiber product

$$\begin{array}{ccc} W_0 \times_{S^3} W_1 & \dashrightarrow & W_0 \times W_1 \\ \downarrow & & \downarrow \\ S^3 & \xrightarrow{\Delta} & S^3 \times S^3 \end{array}$$

which is generically another such; but whether this can be used to define a geometric product on $MT(3)$ involves subtle questions about framings.

Note that the symplectic pairing (and the associated duality) on the Tate cohomology $t_{\mathbb{T}}H\mathbb{Z}$ studied in connection with $M\mathbb{T}SO(2)$ in [18] has a very nice analog on $t_{SU(2)}H\mathbb{Z}$.

2.2.3 The spectrum

$$MTL(4) := BSU(2)^{-\mathbb{H}}$$

(defined by the obvious action σ of $SU(2)$ by left multiplication on \mathbb{H}) is the Madsen-Tillmann spectrum of the category of three-dimensional Spin manifolds, up to cobordism through four-manifolds with an ‘almost hyperHermitian’ structure (in the sense of [7]). These cobordisms are essentially four-manifolds with $Sl_2(\mathbb{C})$ (ie, Lorentzian Spin) structures; the cohomology of the associated spectrum is concentrated in even dimensions, but it has no very obvious multiplicative structure. It would be interesting to understand better the relations between this spectrum and $M\mathbb{T}Spin(3)$, which has cohomology concentrated in odd degrees: it is tempting to think of $MTL(4)$ as some kind of bosonization of $M\mathbb{T}Spin(3)$.

Behind this lie broader questions about Atiyah-twistings of spectra: the isomorphism

$$H^*BSU(2)^{-\mathbb{H}} \xrightarrow{\Phi_{-\mathbb{H}}^{-1}} H^*BSU(2)_+ \xrightarrow{\Phi_{-\mathbb{H}_{ad}}} H^*BSU(2)^{-\mathbb{H}_{ad}}$$

does not respect Steenrod operations. In general, a vector bundle $V \rightarrow X$ which is oriented with respect to a reasonable multiplicative cohomology theory E^* defines a rank one projective $E^*(X)$ -module $E^*(X^V)$, and thus an element of the Picard group of $E^*(X)$. These groups tend to be trivial, but their equivariant analogs (with respect to the cohomology automorphisms of E) can be more interesting.

The spherical fibration associated to V defines a natural invariant

$$\text{Pic}_{\text{Aut}(E)}(E^*(X)) \rightarrow H^1(\text{Aut}(E), (1 + \tilde{E}^*(X))^\times)$$

which can be pulled back to universal examples involving the J -groups of classifying spaces [6]. Techniques developed for the circle group [15] seem promising for $SU(2)$ as well.

2.2.4 Since this paper was submitted, J. Lurie's important work on topological field theories has become available. We close by drawing attention to some applications of his ideas to the subject of this paper.

Lurie's Theorem 2.5.10 [16] identifies the space of infinite-loop maps to an infinite loop space $X = \{X_n\}$ from $|G|\text{Cobord}|^+$ as the homotopy fixed-point spectrum X^{hG} associated to an action of G on X via the natural action of $\text{SO}(D)$ on suspension coordinates of the stabilization of $\Sigma^D X_{n-D}$. [This action is closely related to the constructions in the preceding paragraph.] The n th space of the fixed-point spectrum X^{hG} is equivalent to the space of maps from the Thom spectrum $MTG(D)$ (in the notation of §1.3.1) to X_{n-D} .

The infinite loop space B_\otimes associated to the monoidal category of real vector spaces under tensor product is an interesting example. A monoidal functor from $|G|\text{Cobord}|$ to (Vect, \otimes) is a generalization of a topological quantum field theory in Atiyah's sense, and it defines an infinite-loop map from $|G|\text{Cobord}|^+$ to B_\otimes , and hence an element of $k_\otimes^{-D}(MTG(D))$. These groups are accessible via the Atiyah-Segal exponential [5, 19].

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