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$\mathcal{L}$-invariants of Tate Curves

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Dedicated to Professor John Tate on the occasion of his eightieth birthday.

Abstract: We compute the Greenberg's $\mathcal{L}$-invariant of the adjoint square of a Hilbert modular Galois representation, assuming a conjecture on the exact form of a certain universal Galois deformation ring (which is known to be valid for almost all cases). If the Galois representation is associated to an elliptic curve E with multiplicative reduction at all places over p, it is equal to $\prod_{\mathfrak{p}|p} \frac{\log_p(Q_\mathfrak{p})}{\mathrm{ord}_p(Q_\mathfrak{p})}$, where $Q_\mathfrak{p}$ is the norm to $\mathbb{Q}_p$ of the Tate period of E at the p-adic place $\mathfrak{p}$. This is an exact analogue of the conjecture of Mazur-Tate-Teitelbaum.
Keywords: exceptional zero, L-invariant, Galois deformation, Tate curve, Tate period.

Contents

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1. Introduction

In Sections 2 to 4 of this paper we describe a computation expressing the $\mathcal{L}$-invariant of Greenberg in an explicit form for the symmetric n-th power for $n \leq 2$ of the Tate module of an elliptic curve over a totally real field F. In the last section, we generalize our computation to the case where $n > 2$. Our result is conjectural if $n > 2$ in the sense that if we have the expected structure theorem of the nearly ordinary Galois deformation ring of the symmetric power, it is likely that the conjecture holds. Indeed, after having written this paper, the author found a proof (under mild assumptions) of this implication: Conjecture 5.4 (on the structure of the deformation ring) implies Conjecture 1.3, which can be found in [H09] and [H07b]. Though the discussion in [H09] and [H07b] of the material treated in Corollary 3.5 and Section 5 of this paper is simpler (and Galois cohomological), we keep our original approach of this paper because (i) it is intriguing to have two essentially different proofs of Corollary 3.5 (one automorphic presented in this paper and the other Galois cohomological given in [H07b] Corollary 1.11) and (ii) our approach in Section 5 to the factor $\mathcal{L}(m)$ might indicate its independence of m, though the discussion in Section 5 is less elaborated than the computation in the proof of [H09] Theorem 1.14. Our conjecture generalizes the conjecture of Mazur-Tate-Teitelbaum [MTT] in the case of $n = 1$ and $F = \mathbb{Q}$, and our computation generalizes the one by Greenberg-Stevens [GS1] again in the case of $n = 1$ and $F = \mathbb{Q}$. A principal point of the conjecture is that the $\mathcal{L}$-invariant of a power of a Tate curve is basically independent of n.

Let p be an odd prime and F be a totally real field of degree $d < \infty$ with integer ring O. Order the prime factors of p in O as $\mathfrak{p}_1, \ldots, \mathfrak{p}_e$. Throughout this paper, we study an elliptic curve $E_{/F}$ with multiplicative reduction at $\mathfrak{p}_j | p > 2$

for $j = 1, 2, \ldots, b$ and ordinary good reduction at $\mathfrak{p}_j|p$ for $j > b$. When $b = 0$, as a convention, we assume that $E_{/F}$ has good ordinary reduction at every p-adic place of F. We assume throughout the paper that E *does not have complex multiplication*. Some cases of complex multiplication are treated in [HMI] Section 5.3.3. Take an algebraic closure $\overline{F}$ of F. Writing $\rho_E : \mathrm{Gal}(\overline{F}/F) \to GL_2(\mathbb{Q}_p)$ for the Galois representation on $T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for the Tate module $T_pE = \varprojlim_n E[p^n]$, at each prime factor $\mathfrak{p}|p$, we have $\rho_E|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \sim \left(\begin{smallmatrix} * & * \\ 0 & \alpha_\mathfrak{p}\end{smallmatrix}\right)$ for an unramified character $\alpha_\mathfrak{p}$. Let S be the set of prime ideals of O prime to p where E has bad reduction. Let $K/\mathbb{Q}_p$ be a finite extension with p-adic integer ring W. We may take $K = \mathbb{Q}_p$, but it is useful to formulate the result allowing other choices of K. We consider the **universal** locally cyclotomic deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_2(R)$ with a pro-Artinian local universal K-algebra R having the residue field $R/\mathfrak{m}_R = K$. Writing $D_\mathfrak{p} = \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ and $I_\mathfrak{p}$ for its inertia group, the couple $(R, \boldsymbol{\rho})$ is universal among the following (p-adically continuous) deformations $\rho_A : \mathrm{Gal}(\overline{F}/F) \to GL_2(A)$ for a local Artinian K-algebra A with $A/\mathfrak{m}_A = K$ for its maximal ideal $\mathfrak{m}_A$ such that

- (K1) unramified outside S, ∞ and p;
- (K2) for each prime $\mathfrak{p}|p$, $\rho_A|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}}\end{smallmatrix}\right)$ for $\alpha_{A,\mathfrak{p}} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_A$ with $\alpha_{A,\mathfrak{p}}|_{I_\mathfrak{p}}$ factoring through $\mathrm{Gal}(F_\mathfrak{p}^{ur}[\mu_{p^\infty}]/F_\mathfrak{p}^{ur})$ for the maximal unramified extension $F_\mathfrak{p}^{ur}/F_\mathfrak{p}$ (the local cyclotomy condition);
- (K3) $\det(\rho_A) = \mathcal{N}$ for the p-adic cyclotomic character $\mathcal{N}$;
- (K4) $\rho_A \equiv (\rho_E \otimes K) \mod \mathfrak{m}_A$.

In other words, for any ρ_A as above, there exists a unique K-algebra homomorphism $\varphi : R \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho_A$. Here we have written $\rho_E \otimes K$ for the K-linear Galois representation on $T_pE \otimes_{\mathbb{Z}_p} K$. The existence of the universal deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{F}/F) \to GL_2(R)$ can be shown in a standard manner.

We write $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_\mathfrak{p}\end{smallmatrix}\right)$ with $\boldsymbol{\delta}_\mathfrak{p} \equiv \alpha_\mathfrak{p} \mod \mathfrak{m}_R$. If $\mathfrak{p} = \mathfrak{p}_j$, we often write $\boldsymbol{\delta}_j$ (resp. α_j) for $\boldsymbol{\delta}_{\mathfrak{p}_j}$ (resp. $\alpha_{\mathfrak{p}_j}$). For simplicity, we write F_j for the $\mathfrak{p}_j$-adic completion $F_{\mathfrak{p}_j}$ of F. Let $\Gamma_{\mathfrak{p}_j} = \Gamma_j \subset \mathcal{N}(I_{\mathfrak{p}_j}) \subset \mathbb{Z}_p^\times$ be the group isomorphic (by $\mathcal{N}$) to the maximal p-profinite subgroup of the inertia subgroup of $\mathrm{Gal}(F_j^{ur}[\mu_{p^\infty}]/F_j^{ur})$. Choose a generator $\gamma_{\mathfrak{p}_j} = \gamma_j \in \mathbb{Z}_p^\times$ of Γ_j and identify $W[[\Gamma_j]]$ with $W[[X_j]]$ for a variable $X_j = X_{\mathfrak{p}_j}$ by $\gamma_j \leftrightarrow 1 + X_j$. Since $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{F}_j/F_j)} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\delta}_j\end{smallmatrix}\right)$ with $\boldsymbol{\delta}_j \equiv \alpha_j \mod \mathfrak{m}_R$, $\boldsymbol{\delta}_j\alpha_j^{-1} : \Gamma_\mathfrak{p} \to R$ induces an algebra structure on R over $W[[X_j]]$. Thus R is an algebra over $K[[X_\mathfrak{p}]]_{\mathfrak{p}|p} = K[[X_j]]_{j=1,\ldots,e}$. If we write

$\varphi : R \to K$ for the morphism with $\varphi \circ \boldsymbol{\rho} \cong (\rho_E \otimes K)$, by our construction, $\mathrm{Ker}(\varphi) \supseteq (X_1, \dots, X_e)$.

Conjecture 1.1. *We have* $R \cong K[[X_1, \dots, X_e]] = K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$.

When $F = \mathbb{Q}$, by a result of Kisin [K] 9.10, [K1] and [K2] 3.4 (generalizing those of Wiles [W] and Taylor-Wiles [TaW]), we always have $R \cong K[[X_p]]$. In general, assuming Hilbert modularity of E and the following condition:

(ai) The $\mathbb{F}_p$-linear Galois representation $\overline{\rho} = (T_pE \mod p)$ is absolutely irreducible over $\mathrm{Gal}(\overline{F}/F[\mu_p])$.

Combining results of Fujiwara (see [F] and [F1]) with that of Lin Chen [C], we can prove $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ (see [HMI] Theorem 3.65 and Proposition 3.78). Instead, by potential modularity proven by R. Taylor and C. Virdol [V], we have $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ under the following condition stronger than (ai) but without assuming modularity over F (see [H08] Proposition 3.2):

(ns) The image $\mathrm{Im}(\overline{\rho}) \subset GL_2(\mathbb{F}_p)$ is non-soluble.

We let $\sigma \in \mathrm{Gal}(\overline{F}/F)$ act on the three dimensional Lie algebra of $SL(2)_{/K}$

$$\mathfrak{sl}_2(K) = \{x \in M_2(K) | \mathrm{Tr}(x) = 0\}$$

by conjugation: $x \mapsto \sigma x = \rho_E(\sigma) x \rho_E(\sigma)^{-1}$. This three dimensional representation is written as $Ad(\rho_E)$ and called the adjoint square representation of ρ_E. We have $Ad(\rho_E) \cong Sym^{\otimes 2}(\rho_E) \otimes \mathcal{N}^{-1} = Sym^{\otimes 2}(\rho_E)(-1)$. By using a canonical isomorphism between the tangent space of $\mathrm{Spf}(R)$ and a certain Selmer group of $Ad(\rho_E)$, we get

Theorem 1.2. *Assume* Conjecture 1.1. *Suppose that the Hilbert-modular elliptic curve E has split multiplicative reduction at $\mathfrak{p}_j$ for $j = 1, 2, \dots, b$ ($b \le e$) with Tate period q_j at $\mathfrak{p}_j$ (in* [T]*) for $j \le b$ and has ordinary good reduction at $\mathfrak{p}_i$ with $i > b$. Then for the local Artin symbol $[p, F_j] = Frob_{\mathfrak{p}_j}$ and the norm $Q_j = N_{F_j/\mathbb{Q}_p}(q_j)$, we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \left(\prod_{j=1}^{b} \frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)} \right) \cdot \det \left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j} \right)_{i>b, j>b} \Big|_{X=0} \prod_{i>b} \frac{\log_p(\gamma_i)}{[F_i : \mathbb{Q}_p] \alpha_i([p, F_i])}$$

for local Artin symbol $[p, F_i]$.

Here $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E))$ is Greenberg's $\mathcal{L}$-invariant defined in [Gr], and all the assumptions in [Gr] made to define the invariant can be verified under Conjecture 1.1. The argument in [HMI] Section 3.4 essentially gives this claim, although in [HMI], we have some seemingly redundant assumptions (to make the book self-contained). The assumption in the theorem that E has split (multiplicative) reduction at $\mathfrak{p}_j$ with $j \leq b$ is inessential, because $Ad(\rho_E) \cong Ad(\rho_E \otimes \chi)$ (for a $K^\times$-valued Galois character χ) and we can bring any elliptic curve with multiplicative reduction at $\mathfrak{p}_j$ to an elliptic curve with split multiplicative reduction at $\mathfrak{p}_j$ by a quadratic twist.

We will prove a refined version of this theorem in Section 5.1 as Theorem 5.3. The following conjecture is a generalization of the above theorem for symmetric powers of ρ_E:

Conjecture 1.3. *Let the notation and the assumption be as in Theorem 1.2. Suppose that the n-th symmetric power motive $Sym^{\otimes n}(H_1(E))(-m)$ with Tate twist by an integer m is critical at 1. Then if $\mathrm{Ind}_F^{\mathbb{Q}}(Sym^{\otimes n}(\rho_E)(-m))$ has an exceptional zero at $s = 1$, we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(Sym^{\otimes n}(\rho_E)(-m))) = \begin{cases} \left(\prod_{i=1}^{b} \frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)}\right) \mathcal{L}(m) & \text{for } \mathcal{L}(m) \in \mathbb{Q}_p^\times \text{ if } n = 2m \text{ with odd } m, \\ \prod_{i=1}^{b} \frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)} & \text{if } n \neq 2m. \end{cases}$$

If $b = e$, we have $\mathcal{L}(m) = 1$, and the value $\mathcal{L}(1)$ is given by

$$\mathcal{L}(1) = \det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}\right)_{i>b, j>b} \Big|_{X_1=X_2=\cdots=X_e=0} \prod_{i>b} \frac{\log_p(\gamma_i)}{[F_i : \mathbb{Q}_p]\alpha_i([p, F_i])}$$

for the local Artin symbol $[p, F_i]$, where γ_i is the generator of Γ_i by which we identify the group algebra $W[[\Gamma_i]]$ with $W[[X_i]]$.

R. Greenberg proved the conjecture for his $\mathcal{L}$-invariant of symmetric powers of Tate curves E over $\mathbb{Q}$ with multiplicative reduction at p. His proof is well hidden in his remark in page 170 of [Gr]. C.-P. Mok [M] has computed the analytic $\mathcal{L}$-invariant of p-adic analytic L-functions (when $n = 1$) of elliptic curves $E_{/F}$ (while this paper was being written), following the method of [GS], and his result confirms the conjecture in some special cases. The expression of the $\mathcal{L}(\rho_{E/F})$ in [M] and $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} \rho_E)$ here appears to be different, but indeed the two formulas are equivalent for the following reasons: Though $L(s, \rho_E) = L(s, \mathrm{Ind}_F^{\mathbb{Q}} \rho_E)$, the modification (vanishing) Euler p-factor computed over $\mathbb{Q}$ following [Gr] (6) and the corresponding Euler $\mathfrak{p}$-factor over F are different if $O/\mathfrak{p} \neq \mathbb{F}_p$. Indeed, the vanishing factor over $\mathbb{Q}_p$ is of degree 1 for each $\mathfrak{p}|p$ (at which E has split multiplicative reduction) and is given by $\prod_{\mathfrak{p}|p,a_{\mathfrak{p}}=1}(1-a_{\mathfrak{p}})$. However over F at $\mathfrak{p}$, it is of degree $f_{\mathfrak{p}} := [O/\mathfrak{p} : \mathbb{F}_p]$, and is given by $(1-a_{\mathfrak{p}}^{f_{\mathfrak{p}}}) = (1-a_{\mathfrak{p}})\prod_{1\neq\zeta\in\mu_{f_{\mathfrak{p}}}}(1-\zeta a_{\mathfrak{p}})$. Thus we have the following identity of the nonvanishing factor

$$\mathcal{E}_+(\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_E) = (\prod_{\mathfrak{p}|p,a_{\mathfrak{p}}=1}\ \prod_{1\neq\zeta\in\mu_{f_{\mathfrak{p}}}}(1-\zeta))\mathcal{E}_+(\rho_{E/F}) = (\prod_{\mathfrak{p}|p,a_{\mathfrak{p}}=1} f_{\mathfrak{p}})\mathcal{E}_+(\rho_{E/F})$$

under the notation in [H07a] Conjecture 0.1, because $f_{\mathfrak{p}} = \prod_{1\neq\zeta\in\mu_{f_{\mathfrak{p}}}}(1-\zeta)$. Thus the expression of the $\mathcal{L}$-invariant $\mathcal{L}(\rho_{E/F})$ over F is slightly different from $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_E)$ over $\mathbb{Q}$, and indeed,

$$\mathcal{L}(\rho_{E/F}) = (\prod_{\mathfrak{p}|p,a_{\mathfrak{p}}=1} f_{\mathfrak{p}})\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_E),$$

since we have $\lim_{s\to1}\frac{L_p(s,\rho)}{(s-1)^e} = \mathcal{L}(\rho)\mathcal{E}^+(\rho)\frac{L(1,\rho)}{c^+(\rho(1))}$ conjecturally for $\rho = \rho_E$ and $\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_E$. Then our formula as above formulated for $\mathcal{L}(\rho_{E/F})$ takes the following shape:

$$\mathcal{L}(\rho_{E/F}) = \prod_{\mathfrak{p}|p}\frac{\log_p(q_{\mathfrak{p}})}{\mathrm{ord}_{\mathfrak{p}}(Q_{\mathfrak{p}})} \quad (\Leftrightarrow \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_E) = \prod_{\mathfrak{p}|p}\frac{\log_p(q_{\mathfrak{p}})}{\mathrm{ord}_p(q_{\mathfrak{p}})}),$$

where $\frac{1}{f_{\mathfrak{p}}}\mathrm{ord}_p(q_{\mathfrak{p}}) = \mathrm{ord}_{\mathfrak{p}}(Q_{\mathfrak{p}})$. C.-P. Mok [M] has computed the analytic $\mathcal{L}$-invariant for some Tate curves E over F following the method of Greenberg–Stevens and got the formula using $\mathrm{ord}_{\mathfrak{p}}(Q_{\mathfrak{p}})$ as above under some assumptions. His formula confirms the expression in Conjecture 1.3 for $n = 1$ if we take $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_E)$ in place of $\mathcal{L}(\rho_{E/F})$ in [M]. For the same reason, the $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho_E)_{/F})$ defined for $Ad(\rho_E) : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_3(K)$ has the following form

$$\mathcal{L}(Ad(\rho_E)_{/F}) = (\prod_{\mathfrak{p}|p} f_{\mathfrak{p}})\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}\,Ad(\rho_E)). \tag{1.1}$$

The motive $Sym^{\otimes n}(H_1(E))(-m)$ is critical at 1 if and only if the following two conditions are satisfied:

- $0 \leq m < n$;
- either n is odd or $n = 2m$ with odd m.

We will make in the text one more conjecture (with less ambitious goal in appearance) similar to Conjecture 1.1 for the deformation ring of $\rho_{n,0} = Sym^{\otimes n}(\rho_E)$

which is closely related to the above conjecture (see Conjecture 5.4). The value $\mathcal{L}(m)$ with $m > 1$ can be conjecturally given by a formula similar to (but more complicated than) the formula of $\mathcal{L}(1)$ in terms of the universal nearly ordinary deformation of $\rho_{m,0}$. There is a wild guess asserting that $\mathcal{L}(m)$ is independent of m and is given by $\mathcal{L}(1)$. We hope to present a general formula for $\mathcal{L}(m)$ and some evidence for this wild guess in our future paper.

Here are some additional remarks about the conjecture:

(1) When $n = 2m$ with even m, the motive associated to $Sym^{\otimes n}(\rho_E)(-m)$ is not critical at $s = 1$; so, the situation is drastically different (and in such a case, we do not make any conjecture; see [H00] Examples 2.7 and 2.8).
(2) The above conjecture is a generalization of the conjecture of Mazur-Tate-Teitelbaum (see [MTT]) and applies to arithmetic and analytic p-adic L-functions.

2. Galois deformation and $\mathcal{L}$-invariant

Here is a theorem proved in [HMI] as Theorem 3.73:

Theorem 2.1. *Suppose $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then, if $\varphi \circ \boldsymbol{\rho} \cong \rho_E$, for the local Artin symbol $[p, F_{\mathfrak{p}}] = Frob_{\mathfrak{p}}$, we have the following formula of Greenberg's $\mathcal{L}$-invariant:*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}{\partial X_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{X=0} \prod_{\mathfrak{p}} \frac{\log_p(\gamma_{\mathfrak{p}})}{[F_{\mathfrak{p}} : \mathbb{Q}_p]\alpha_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}.$$

The statement in [HMI] Theorem 3.73 is

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \det\left(\frac{\partial \boldsymbol{\delta}_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}{\partial x_{\mathfrak{p}'}}\right)_{\mathfrak{p},\mathfrak{p}'}\Big|_{x=0} \prod_{\mathfrak{p}} \frac{\log_p(\gamma_{\mathfrak{p}})}{\alpha_{\mathfrak{p}}([p, F_{\mathfrak{p}}])}.$$

The variable $x_{\mathfrak{p}}$ is defined by $\boldsymbol{\delta}_{\mathfrak{p}}([\gamma_{\mathfrak{p}}, F_{\mathfrak{p}}]) = (1 + x_{\mathfrak{p}})$ and has relation $(1 + x_{\mathfrak{p}}) = (1 + X_{\mathfrak{p}})^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}$, and thus we get $[F_{\mathfrak{p}} : \mathbb{Q}_p]^{-1} \frac{\partial}{\partial X_{\mathfrak{p}}}\Big|_{X=0} = \frac{\partial}{\partial x_{\mathfrak{p}}}\Big|_{X=0}$. Indeed, by local class field theory, $[\gamma_{\mathfrak{p}}, F_{\mathfrak{p}}]|_{\mathbb{Q}^{ab}} = [N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(\gamma_{\mathfrak{p}}), \mathbb{Q}_p] = [\gamma_{\mathfrak{p}}, \mathbb{Q}_p]^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}$ and for an element $\sigma \in I_{\mathfrak{p}}$ with $\sigma|_{F_{\mathfrak{p}}^{ab}} = [u, F_{\mathfrak{p}}]$ and $N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u) = \gamma_{\mathfrak{p}}$, we have $[u, F_{\mathfrak{p}}]|_{\mathbb{Q}_p^{ab}} = [\gamma_{\mathfrak{p}}, \mathbb{Q}_p]$. Thus $\boldsymbol{\delta}_{\mathfrak{p}}([u, F_{\mathfrak{p}}]^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}) = \boldsymbol{\delta}_{\mathfrak{p}}([\gamma_{\mathfrak{p}}, F_{\mathfrak{p}}])$, and on the other hand, $\boldsymbol{\delta}_{\mathfrak{p}}([u, F_{\mathfrak{p}}]) = (1 + X_{\mathfrak{p}})$ by definition. In other words, we have $\boldsymbol{\delta}_{\mathfrak{p}}([\gamma_{\mathfrak{p}}, F_{\mathfrak{p}}]) = \boldsymbol{\delta}_{\mathfrak{p}}(\gamma_{\mathfrak{p}}^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}) = (1 + X_{\mathfrak{p}})^{[F_{\mathfrak{p}}:\mathbb{Q}_p]}$. This explains the equivalence of the two formulas. Also some more conditions are assumed in [HMI] Theorem 3.73 to assure

$R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$, but the proof given there is valid only assuming $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Indeed, the (seemingly redundant) condition (vsl) in [HMI] Theorem 3.73 is equivalent to $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ by [HMI] Proposition 3.78.

We recall briefly an F-version (given in [HMI] Definition 3.85) of Greenberg's formula of the $\mathcal{L}$–invariant for a general p-adic totally p-ordinary Galois representation V (of $\mathrm{Gal}(\overline{F}/F)$) with an exceptional zero. This definition is equivalent to the one in [Gr] if we apply it to $\mathrm{Ind}_F^{\mathbb{Q}} V$ as proved in [HMI] (in Definition 3.85). When $V = Ad(\rho_E)$, the definition can be outlined as follows. Under some hypothesis, he found a unique subspace $\mathbf{H} \subset H^1(\mathbb{Q}, \mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E))$ of dimension $e = |\{\mathfrak{p}|p\}|$. By Shapiro's lemma, $H^1(\mathbb{Q}, \mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \cong H^1(F, Ad(\rho_E))$, and one can give a definition of the image $\mathbf{H}_F$ of $\mathbf{H}$ in $H^1(F, Ad(\rho_E))$ without reference to the induction $\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)$ ([HMI] Definition 3.85) as we recall the precise definition later. The space $\mathbf{H}_F$ is represented by cocycles $c : \mathrm{Gal}(\overline{F}/F) \to Ad(\rho_E)$ such that

(1) c is unramified outside p;
(2) c restricted to the decomposition subgroup $D_{\mathfrak{p}} \subset \mathrm{Gal}(\overline{F}/F)$ at $\mathfrak{p}|p$ is upper triangular after conjugation, and $c|_{D_{\mathfrak{p}}}$ modulo nilpotent matrices becomes unramified over $F_{\mathfrak{p}}[\mu_{p^\infty}]$ for all $\mathfrak{p}|p$.

This subspace $\mathbf{H}_F$ coincides with the locally cyclotomic Selmer group $\mathrm{Sel}_F^{cyc}(Ad(\rho_E))$ whose precise definition will be given in the following subsection. By the condition (2), $c|_{D_{\mathfrak{p}'}}$ with a prime $\mathfrak{p}'|p$ (expressed in a matrix form taking a basis so that $\rho_E = \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{\mathfrak{p}'} \end{smallmatrix}\right)$) modulo upper nilpotent matrices factors through the cyclotomic Galois group $\mathrm{Gal}(F_{\mathfrak{p}'}[\mu_{p^\infty}]/F_{\mathfrak{p}'})$, and hence $c|_{D_{\mathfrak{p}'}}$ modulo upper nilpotent matrices becomes unramified everywhere over the cyclotomic $\mathbb{Z}_p$-extension F_∞/F. In other words, the cohomology class $[c]$ is in $\mathrm{Sel}_{F_\infty}(Ad(\rho_E))$ but not in $\mathrm{Sel}_F(Ad(\rho_E))$.

Take a basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$ of $\mathbf{H}_F$ over K. Write

$$c_{\mathfrak{p}}(\sigma) \sim \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \quad \text{for } \sigma \in D_{\mathfrak{p}'}.$$

Then $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \to K$ is a homomorphism. We now have two $e \times e$ matrices with coefficients in K: $A = \left(a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}$ and $B = \left(\log_p(\gamma_{\mathfrak{p}'})^{-1} a_{\mathfrak{p}}([\gamma_{\mathfrak{p}'}, F_{\mathfrak{p}'}])\right)_{\mathfrak{p},\mathfrak{p}'|p}$.

Under Conjecture 1.1, we can show that B is invertible. Then Greenberg's $\mathcal{L}$-invariant is defined by

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) = \det(AB^{-1}). \tag{2.1}$$

The determinant $\det(AB^{-1})$ is independent of the choice of the basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$.

Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, and put F_∞/F for the composite of F and $\mathbb{Q}_\infty$. Choose a generator γ of $\mathcal{N}(\mathrm{Gal}(F_\infty/F)) \subset \mathbb{Z}_p^\times$ for the p-adic cyclotomic character $\mathcal{N}$, and identify $\Lambda = W[[\mathrm{Gal}(F_\infty/F)]]$ with $W[[T]]$ by $\gamma \mapsto 1+T$. The adjoint square Selmer group $\mathrm{Sel}_{F_\infty}(Ad(T_pE) \otimes (\mathbb{Q}_p/\mathbb{Z}_p))$ has its Pontryagin dual which is a Λ-module of finite type. Choose a characteristic power series $\Phi^{arith}(T) \in \Lambda$ of the Pontryagin dual. Put $L_p^{arith}(s, Ad(\rho_E)) = \Phi^{arith}(\gamma^{1-s} - 1)$. This p-adic L-function corresponds to the Selmer group $\mathrm{Sel}_{F_\infty}(Ad(\rho_E)^*)$, and hence we need to use the $\mathcal{L}(Ad(\rho_E)_{/F})$ in (1.1) in place of $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(Ad(\rho_E))$ to describe its property at $s = 1$. The following conjecture for the arithmetic L-function $L_p^{arith}(s, Ad(\rho_E))$ is a theorem (except for the nonvanishing $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \neq 0$) essentially under (mild but possibly restrictive) conditions (see [Gr] Proposition 4 and Theorem 5.3 in the text for a precise statement):

Conjecture 2.2 (Greenberg)**.** *Suppose* (ds) *and that $\overline{\rho}$ is absolutely irreducible. Then $L_p^{arith}(s, Ad(\rho_E))$ has zero of order equal to $e = \left|\{\mathfrak{p}|p\}\right|$ and for the constant $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \in K^\times$ specified by the determinant as in the theorem, we have*

$$\lim_{s\to 1} \frac{L_p^{arith}(s, Ad(\rho_E))}{(s-1)^d} = \mathcal{L}(Ad(\rho_E)_{/F})\left|\left|\mathrm{Sel}_F(Ad(\rho_E)^*)\right|\right|_p^{-1/[K:\mathbb{Q}_p]}$$

up to units.

Note here that $\mathrm{Ind}_F^{\mathbb{Q}} \rho_E$ is not ordinary in the sense of [Gr] if p ramifies in $F/\mathbb{Q}$; so, Greenberg made the above conjecture under the extra hypothesis of unramifiedness of p in $F/\mathbb{Q}$. In the above conjecture, the modifying Euler factor at the p-adic places $\mathfrak{p}_j$ of good reduction ($j > b$):

$$\mathcal{E}^+(Ad(\rho_E)) = \prod_{j>b}(1 - \alpha_j^{-2}(Frob_{\mathfrak{p}})N(\mathfrak{p}_j))(1 - \alpha_j^{-2}(Frob_{\mathfrak{p}}))$$

does not appear. However, taking the Bloch-Kato Selmer group $S_F(Ad(\rho_E)^*)$ over F (crystalline at $\mathfrak{p}_j$ for $j > b$) in place of Greenberg's Selmer group $\mathrm{Sel}_F(Ad(\rho_E)^*) \cong \mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_E^{\mathbb{Q}} Ad(\rho_E)^*)$, we have the relation

$$\left|\left|\mathrm{Sel}_F(Ad(\rho_E)^*)\right|\right|_p^{-1/[K:\mathbb{Q}_p]} = \mathcal{E}^+(Ad(\rho_E))\left|\left|S_F(Ad(\rho_E)^*)\right|\right|_p^{-1/[K:\mathbb{Q}_p]}$$

up to p-adic units (as described in [MFG] page 284). Thus if one uses the formulation of Bloch-Kato, we do have the modifying Euler factor in the formula, and the size of the Bloch-Kato Selmer group is expected to be equal to the primitive archimedean L-values (divided by a suitable period; see Greenberg's Conjecture 0.1 in [H07a]). Though $\mathrm{Sel}_F(Ad(\rho_E)^*) \cong \mathrm{Sel}_{\mathbb{Q}}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)^*)$ by [HMI] Proposition 3.80, the nonvanishing modification factor $\mathcal{E}^+(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E))$ and $\mathcal{E}^+(Ad(\rho_E))$ could be different as explained above (1.1); so, we need to use $\mathcal{L}(Ad(\rho_E)_{/F})$ in place of $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)_{/F})$ in the above conjecture.

2.1. **Selmer Groups.** First we recall Greenberg's Selmer groups. Write $F^{(S)}/F$ for the maximal extension unramified outside S, p and ∞. Put $\mathfrak{G} = \mathrm{Gal}(F^{(S)}/F)$ and $\mathfrak{G}_M = \mathrm{Gal}(F^{(S)}/M)$. Let V be a potentially ordinary representation of $\mathfrak{G}$ on a K-vector space V. Thus V has decreasing filtration $\mathcal{F}_{\mathfrak{p}}^i V$ such that an open subgroup of $I_{\mathfrak{p}}$ (for each prime factor $\mathfrak{p}|p$) acts on $\mathcal{F}_{\mathfrak{p}}^i V/\mathcal{F}_{\mathfrak{p}}^{i+1} V$ by the i-th power $\mathcal{N}^i$ of the p-adic cyclotomic character $\mathcal{N}$. We fix a W-lattice T in V stable under $\mathfrak{G}$.

Write $D = D_{\mathfrak{p}} \subset \mathfrak{G}$ for the decomposition group of each prime factor $\mathfrak{p}|p$. Put $\mathcal{F}_{\mathfrak{p}}^+ V = \mathcal{F}_{\mathfrak{p}}^1 V$ and $\mathcal{F}_{\mathfrak{p}}^- V = \mathcal{F}_{\mathfrak{p}}^0 V$. We have a 3-step filtration:

$$\text{(ord)} \qquad V \supset \mathcal{F}_{\mathfrak{p}}^- V \supset \mathcal{F}_{\mathfrak{p}}^+ V \supset \{0\}.$$

Its dual $V^*(1) = \mathrm{Hom}_K(V, K) \otimes \mathcal{N}$ again satisfies (ord).

Let M/F be a subfield of $F^{(S)}$, and put $\mathfrak{G}_M = \mathrm{Gal}(F^{(S)}/M)$. We write $\mathfrak{p}$ for a prime of M over p and $\mathfrak{q}$ for general primes outside p of M. We write $I_{\mathfrak{p}}$ and $I_{\mathfrak{q}}$ for the inertia subgroup in $\mathfrak{G}_M$ at $\mathfrak{p}$ and $\mathfrak{q}$, respectively. We put

$$L_{\mathfrak{p}}(A) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{p}}, A) \to H^1(I_{\mathfrak{p}}, \frac{A}{\mathcal{F}_{\mathfrak{p}}^+(A)})),$$

and

$$L_{\mathfrak{q}}(A) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{q}}, A) \to H^1(I_{\mathfrak{q}}, A)).$$

Then we define the Selmer submodule in $H^1(M_{\mathfrak{l}}, V/T)$ by

$$\text{(2.2)} \qquad \mathrm{Sel}_M(A) = \mathrm{Ker}(H^1(\mathfrak{G}_M, A) \to \prod_{\mathfrak{q}} \frac{H^1(M_{\mathfrak{q}}, A)}{L_{\mathfrak{q}}(A)}) \times \prod_{\mathfrak{p}} \frac{H^1(M_{\mathfrak{p}}, A)}{L_{\mathfrak{p}}(A)})$$

for $A = V, V/T$. The classical Selmer group of V is given by $\mathrm{Sel}_M(V/T)$, equipped with discrete topology. We define the "minus", the "locally cyclotomic" and the "strict" Selmer groups $\mathrm{Sel}_M^-(A)$, $\mathrm{Sel}_M^{cyc}(A)$ and $\mathrm{Sel}_M^{st}(A)$, respectively, replacing $L_{\mathfrak{p}}(A)$ by

$$L_{\mathfrak{p}}^-(A) = \mathrm{Ker}(\mathrm{Res} : H^1(M_{\mathfrak{p}}, V) \to H^1(I_{\mathfrak{p}}, \frac{V}{\mathcal{F}_{\mathfrak{p}}^-(A)})) \supset L_{\mathfrak{p}}(A)$$
$$L_{\mathfrak{p}}^{cyc}(A) = \mathrm{Ker}(\mathrm{Res} : L_{\mathfrak{p}}^-(A) \to H^1(I_{\mathfrak{p},\infty}, \frac{V}{\mathcal{F}_{\mathfrak{p}}^+(A)})) \subset L_{\mathfrak{p}}^-(A)$$
$$L_{\mathfrak{p}}^{st}(A) = \mathrm{Ker}(\mathrm{Res} : L_{\mathfrak{p}}^-(A) \to H^1(M_{\mathfrak{p}}, \frac{V}{\mathcal{F}_{\mathfrak{p}}^+(A)})) \supset L_{\mathfrak{p}}(A),$$

where $I_{\mathfrak{p},\infty}$ is the inertia group of $\mathrm{Gal}(\overline{M}_{\mathfrak{p}}/M_{\mathfrak{p}}[\mu_{p^\infty}])$. Then we have

$$\mathrm{Sel}_F^{cyc}(A) = \mathrm{Res}_{F_\infty/F}^{-1}(\mathrm{Sel}_{F_\infty}(A)).$$

Lemma 2.3. *Suppose* $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. *Then we have* $\mathrm{Sel}_F(Ad(\rho_E)) = 0$ *and* $\mathrm{Sel}_F^{cyc}(Ad(\rho_E)) \cong \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$.

Proof. Let $V = Ad(\rho_E)$. Then we have the filtration:

$$V \supset \mathcal{F}_{\mathfrak{p}}^- V \supset \mathcal{F}_{\mathfrak{p}}^+ V \supset \{0\},$$

where taking a basis so that $\rho_E|_{D_{\mathfrak{p}}} = \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{\mathfrak{p}} \end{smallmatrix}\right)$, $\mathcal{F}_{\mathfrak{p}}^- V$ is made up of upper triangular matrices and $\mathcal{F}_{\mathfrak{p}}^+ V$ is made up of upper nilpotent matrices, and on $\mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$, $D_{\mathfrak{p}}$ acts trivially (getting eigenvalue 1 for $Frob_{\mathfrak{p}}$). We consider the space $Der_K(R, K)$ of continuous K-derivations. Let $K[\varepsilon] = K[t]/(t^2)$ for the dual number $\varepsilon = (t \bmod t^2)$. Then writing each K-algebra homomorphism $\phi : R \to K[\varepsilon]$ as $\phi(r) = \phi_0(r) + \partial_\phi(r)\varepsilon$ and sending ϕ to $\partial_\phi \in Der_K(R, K)$, we have $\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong Der_K(R, K) = \mathrm{Hom}_K(\mathfrak{m}_R/\mathfrak{m}_R^2, K)$. By the universality of $(R, \boldsymbol{\rho})$, we have

$$\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \cong \frac{\{\rho : \mathrm{Gal}(\overline{F}/F) \to GL_2(K[\varepsilon])|\rho \text{ satisfies (K1–4)}\}}{\cong}$$

by $\mathrm{Hom}_{K\text{-alg}}(R, K[\varepsilon]) \ni \phi \mapsto \rho_\phi = \phi \circ \boldsymbol{\rho} = \rho_E + \varepsilon\partial_\phi\boldsymbol{\rho}$. Pick $\rho = \rho_\phi$ as above. Write $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$ with $\rho_1(\sigma) = \frac{\partial\rho}{\partial t} = \partial_\phi\boldsymbol{\rho}(\sigma)$. Then $c_\rho = (\partial_\phi\boldsymbol{\rho})\rho_E^{-1}$ can be easily checked to be a 1-cocycle having values in $M_2(K) \supset V$. Since $\det(\rho) = \det(\rho_E) \Rightarrow \mathrm{Tr}(c_\rho) = 0$, c_ρ has values in $V = \mathfrak{sl}_2(K)$. By the reducibility condition (K2), $[c_\rho]$ vanishes in $\frac{H^1(M_{\mathfrak{p}},A)}{L_{\mathfrak{p}}^-(A)}$. By the local cyclotomy condition in (K2), $[c_\rho]$ is unramified over F_∞. For $\mathfrak{q} \in S$, first suppose that E has potentially good reduction at $\mathfrak{q}$. Then $\rho_E(I_{\mathfrak{q}})$ is a finite group. Take a finite Galois extension $L/F_{\mathfrak{q}}$ over which E has good reduction. Then the inertia group I of $\mathrm{Gal}(\overline{F}_{\mathfrak{q}}/L)$ acts trivially on T_pE (see [ST]), and $H^1(I_{\mathfrak{q}}/I, V) = 0$ because $I_{\mathfrak{q}}/I$ is a finite group (and V is a $\mathbb{Q}_p$-vector space). We have the inflation-restriction exact sequence (made of $Frob_{\mathfrak{q}}$-linear maps):

$$0 = H^1(I_{\mathfrak{q}}/I, V) \to H^1(I_{\mathfrak{q}}, V) \xrightarrow{\text{Res}} H^1(I, V) = \text{Hom}(\mathbb{Z}_p(1), V).$$

The Frobenius $Frob_{\mathfrak{q}}$ acts on I and its p-profinite quotient $\mathbb{Z}_p(1)$ by conjugation. Restricting c_ρ to I, we get a homomorphism of the p-profinite Tame inertia group $c_\rho : \mathbb{Z}_p(1) \to V$ compatible with the action of $Frob_{\mathfrak{q}}$. Since E has good reduction over L, after restricting V to I, the action of $Frob_{\mathfrak{q}}$ on V and on $\mathbb{Z}_p(1)$ does not match; so, $\text{Res}(c_\rho) = 0$, which shows that c_ρ is unramified at $\mathfrak{q}$. If E has potentially multiplicative reduction at $\mathfrak{q} \in S$, the unramifiedness of c_ρ follows from the following lemma. Thus the cohomology class $[c_\rho]$ of c_ρ is in $\text{Sel}_F^{cyc}(V)$. We see easily that $\rho \cong \rho' \Leftrightarrow [c_\rho] = [c_{\rho'}]$.

We can reverse the above argument starting with a cocycle c giving an element of $\text{Sel}_F^{cyc}(V)$ to construct a deformation $\rho_c = \rho_E + \varepsilon(c\rho_E)$ with values in $GL_2(K[\varepsilon])$. Thus we have

$$\frac{\{\rho : \text{Gal}(\overline{F}/F) \to GL_2(K[\varepsilon])|\rho \text{ satisfies the conditions (K1–4)}\}}{\cong} \cong \text{Sel}_F^{cyc}(V).$$

The isomorphism $Der_K(R, K) \cong \text{Sel}_F^{cyc}(V)$ is given by $Der_K(R, K) \ni \partial \mapsto [c_\partial] \in \text{Sel}_F^{cyc}(V)$ for the cocycle $c_\partial = c_\rho = (\partial\boldsymbol{\rho})\rho_E^{-1}$, where $\rho = \rho_E + \varepsilon(\partial\boldsymbol{\rho})$. Since the algebra structure of R over $W[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ is given by $\boldsymbol{\delta}_{\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1}$, the K-derivation $\partial = \partial_\phi : R \to K$ corresponding to a $K[\varepsilon]$-deformation ρ is a $W[[X_{\mathfrak{p}}]]$-derivation if and only if $\partial\boldsymbol{\rho}|_{\text{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \sim \left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$, which is equivalent to $[c_\partial] \in \text{Sel}_F(V)$, because we have already shown that $\text{Tr}(c_\partial) = \text{Tr}(c_\rho) = 0$. Thus we have $\text{Sel}_F(V) \cong Der_{W[[X_{\mathfrak{p}}]]}(R, K) = 0$. □

Lemma 2.4. *Let $\mathfrak{q}$ be a prime outside p at which E has potentially multiplicative reduction. Then for a deformation ρ of ρ_E satisfying* (K1–4)*, the cocycle c_ρ (defined in the above proof) is unramified at $\mathfrak{q}$.*

Proof. By our assumption, $\rho_E|_{\text{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}})} \cong \begin{pmatrix} \eta\mathcal{N} & * \\ 0 & \eta \end{pmatrix}$ for a finite order character η. Since $Ad(\rho_E \otimes \eta^{-1}) \cong Ad(\rho_E)$, twisting by a character, we may assume that the restriction of ρ_E to the inertia group $I_{\mathfrak{q}}$ has values in the upper unipotent subgroup; so, it factors through the tame inertia group $\cong \widehat{\mathbb{Z}}^{(q)}(1)$. By the theory of Tate curves, ρ_E ramifies at $\mathfrak{q}$. The p-factor of $\widehat{\mathbb{Z}}^{(q)}$ is of rank 1 isomorphic to $\mathbb{Z}_p(1)$. Then $\rho(I_{\mathfrak{q}})$ is cyclic, and therefore $\dim_K \rho(I_{\mathfrak{q}}) = 1 = \dim_K \rho_E(I_{\mathfrak{q}})$. Thus the deformation ρ is constant over the inertia subgroup, and hence c_ρ restricted to $I_\mathfrak{q}$ is trivial. □

Let $\rho_{n,m} = Sym^{\otimes n}(\rho_E)(-m)$, and write V for the representation space of $\rho_{n,m}$. For each prime $\mathfrak{q} \in S \cup \{\mathfrak{p}|p\}$, we put
(2.3)

$$\overline{L}_\mathfrak{q}(V) = \begin{cases} \mathrm{Ker}(H^1(F_j, V) \to H^1(F_j, \frac{V}{\mathcal{F}^+_{\mathfrak{p}_j}(V)})) \subset L_{\mathfrak{p}_j}(V) & \text{if } \mathfrak{q} = \mathfrak{p}_j \text{ and } j \le b, \\ L_\mathfrak{q}(V) & \text{otherwise} \end{cases}$$

Once $\overline{L}_\mathfrak{q}(V)$ is defined, we define $\overline{L}_\mathfrak{q}(V^*(1)) = \overline{L}_\mathfrak{q}(V)^\perp$ under the local Tate duality between $H^1(F_\mathfrak{q}, V)$ and $H^1(F_\mathfrak{q}, V^*(1))$, where $V^*(1) = \mathrm{Hom}_K(V, \mathbb{Q}_p(1))$ as Galois modules. Then we define the balanced Selmer group $\overline{\mathrm{Sel}}_F(V)$ (resp. $\overline{\mathrm{Sel}}_F(V^*(1))$) by the same formula as in (2.2) replacing $L_\mathfrak{p}(V)$ (resp. $L_\mathfrak{p}(V^*(1))$) by $\overline{L}_\mathfrak{p}(V)$ (resp. $\overline{L}_\mathfrak{p}(V^*(1))$). By definition, $\overline{\mathrm{Sel}}_F(V) \subset \mathrm{Sel}_F(V)$. We have

Lemma 2.5. *If $V = \rho_{n,m}$ is motivic and critical at $s = 1$,*

$$\text{(V)} \qquad \mathrm{Sel}_F(V) = 0 \Rightarrow H^1(\mathfrak{G}, V) \cong \prod_{\mathfrak{q}\in S} \frac{H^1(F_\mathfrak{q}, V)}{L_\mathfrak{q}(V)} \times \prod_{\mathfrak{p}|p} \frac{H^1(F_\mathfrak{p}, V)}{\overline{L}_\mathfrak{p}(V)}.$$

Proof. Since $\overline{\mathrm{Sel}}_F(V) \subset \mathrm{Sel}_F(V)$, the assumption implies $\overline{\mathrm{Sel}}_F(V) = 0$. Then the Poitou-Tate exact sequence tells us the exactness of the following sequence:

$$\overline{\mathrm{Sel}}_F(V) \to H^1(\mathfrak{G}, V) \to \prod_{\mathfrak{l}\in S\sqcup\{\mathfrak{p}|p\}} \frac{H^1(F_\mathfrak{l}, V)}{\overline{L}_\mathfrak{l}(V)} \to \overline{\mathrm{Sel}}_F(V^*(1))^*.$$

It is an old theorem of Greenberg (which assumes criticality at $s = 1$) that

$$\dim \overline{\mathrm{Sel}}_F(V) = \dim \overline{\mathrm{Sel}}_F(V^*(1))^*$$

(see [Gr] Proposition 2 or [HMI] Proposition 3.82); so, we have the assertion (V). In [HMI], Proposition 3.82 is formulated in terms of $\overline{\mathrm{Sel}}_\mathbb{Q}(\mathrm{Ind}_F^\mathbb{Q} V)$ and $\overline{\mathrm{Sel}}_\mathbb{Q}(\mathrm{Ind}_F^\mathbb{Q} V^*(1))$ defined in [HMI] (3.4.11), but this does not matter because we can easily verify $\overline{\mathrm{Sel}}_\mathbb{Q}(\mathrm{Ind}_F^\mathbb{Q} ?) \cong \overline{\mathrm{Sel}}_F(?)$ (similarly to [HMI] Corollary 3.81). □

Here we note that there is an error in the definition of $\overline{U}_p(\mathrm{Ind}_F^\mathbb{Q} V)$ in [HMI] page 265 at line 8 from the bottom. It has to be the pull back image of $\mathcal{F}^+H^1(\mathbb{Q}_p, \mathcal{Y}) \times \mathrm{Hom}(D_p/I_p, K)^{t_0} \times H^1_{fl}(\mathbb{Q}_p, K(1))^{t_1} \subset H^1(\mathbb{Q}_p, Y)$ decomposing $Y = \mathcal{Y} \oplus K^{t_0} \oplus K(1)^{t_1}$ for the product $\mathcal{Y}$ of nontrivial extensions of K by $K(1)$ (see the errata list of [HMI] posted at the author's web page for more details).

2.2. **Greenberg's $\mathcal{L}$–invariant.** In this subsection, we let $V = Ad(\rho_E)$. Assuming for simplicity that $F_\mathfrak{p}/\mathbb{Q}_p$ is a Galois extension for all prime factors $\mathfrak{p}|p$, we recall a little more detail of the F-version of Greenberg's definition of $\mathcal{L}(\mathrm{Ind}_F^\mathbb{Q} V)$ (which is equivalent to the one given in [Gr] if we apply Greenberg's definition to $\mathrm{Ind}_F^\mathbb{Q} V$ as explained in [HMI] 3.4.4 without assuming the simplifying condition). Write $\mathfrak{G}_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $\mathfrak{G}_\mathfrak{p} = \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$; so, $\mathfrak{G}_\mathfrak{p}$ surjects down to $D_\mathfrak{p}$. The long exact sequence associated to the short one $\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V \hookrightarrow V/\mathcal{F}_\mathfrak{p}^+ V \twoheadrightarrow V/\mathcal{F}_\mathfrak{p}^- V$ gives a homomorphism

$$H^1(F_\mathfrak{p}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})^{\mathfrak{G}_p} = \mathrm{Hom}(\mathfrak{G}_\mathfrak{p}^{ab}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})^{\mathfrak{G}_p} \xrightarrow{\iota_\mathfrak{p}} H^1(F_\mathfrak{p}, V)/\overline{L}_\mathfrak{p}(V),$$

where $\mathfrak{G}_p$ acts on $H^1(F_\mathfrak{p}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})$ regarding $\frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V}$ as the trivial $\mathfrak{G}_p$-module; so, its action on $\phi \in \mathrm{Hom}(\mathfrak{G}_\mathfrak{p}^{ab}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})$ is given by $\phi \mapsto \tau \cdot \phi(\sigma) = \phi(\tau\sigma\tau^{-1})$. Note that canonically

$$H^1(F_\mathfrak{p}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})^{\mathfrak{G}_p} \xleftarrow[\mathrm{Res}]{\sim} \mathrm{Hom}(\mathfrak{G}_p^{ab}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V}) \cong \mathrm{Hom}(\mathbb{Q}_p^\times, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V}) \cong K^2$$

by $\phi \mapsto (\frac{\phi([\gamma, F_\mathfrak{p}])}{\log_p(\gamma)}, \phi([p, F_\mathfrak{p}]))$. Here $[x, F_\mathfrak{p}]$ is the local Artin symbol. Identifying $H^1(F_\mathfrak{p}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})^{\mathfrak{G}_p}$ with $\mathrm{Hom}(\mathfrak{G}_p^{ab}, \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V})$, a homomorphism $\phi : \mathfrak{G}_p^{ab} \to \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V}$ in $\mathrm{Ker}(\iota_\mathfrak{p})$ is unramified; so, the image of $\iota_\mathfrak{p}$ is one-dimensional (those ramified classes modulo unramified ones). In other words, the image of $\iota_\mathfrak{p}$ is isomorphic to $\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V \cong K$.

Suppose $R \cong K[[X_\mathfrak{p}]]_{\mathfrak{p}|p}$. Then by (V) in Lemma 2.5 (and Lemma 2.3), we have a unique subspace $\mathbf{H}_F$ of $H^1(\mathfrak{G}, V)$ projecting down onto

$$\prod_\mathfrak{p} \mathrm{Im}(\iota_\mathfrak{p}) \hookrightarrow \prod_\mathfrak{p} \frac{H^1(F_\mathfrak{p}, V)}{\overline{L}_\mathfrak{p}(V)}.$$

Then by the restriction, $\mathbf{H}_F$ gives rise to a subspace L of

$$\prod_\mathfrak{p} \mathrm{Hom}(\mathfrak{G}_\mathfrak{p}^{ab}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V)^{\mathfrak{G}_p} \cong \prod_\mathfrak{p} \mathrm{Hom}(\mathfrak{G}_p^{ab}, \mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V) \cong \prod_\mathfrak{p} \left(\frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V}\right)^2$$

isomorphic to $\prod_\mathfrak{p}(\mathcal{F}_\mathfrak{p}^- V/\mathcal{F}_\mathfrak{p}^+ V)$. If a cocycle c representing an element in $\mathbf{H}_F$ is unramified, it gives rise to an element in $\mathrm{Sel}_F(V)$. By the vanishing of $\mathrm{Sel}_F(V)$ (Lemma 2.3), this implies $c = 0$; so, the projection of L to the first factor $\prod_\mathfrak{p} \frac{\mathcal{F}_\mathfrak{p}^- V}{\mathcal{F}_\mathfrak{p}^+ V}$ (via $\phi \mapsto (\phi([\gamma, F_\mathfrak{p}])/\log_p(\gamma))_\mathfrak{p}$) is surjective. Thus this subspace L is a graph of a

K–linear map $\mathcal{L} : \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V \to \prod_{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}}^- V/\mathcal{F}_{\mathfrak{p}}^+ V$. We then define $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} V) = \det(\mathcal{L}) \in K$. This is a brief description of the direct construction of $\mathbf{H}_F$ assuming normality of $F_{\mathfrak{p}}/\mathbb{Q}_p$ for all $\mathfrak{p}|p$. The general non-Galois case is treated in pages 273–274 of [HMI] (or Section 1.2 of [H07b]).

Now we return to the general setting allowing prime factors $\mathfrak{p}|p$ with non-Galois extension $F_{\mathfrak{p}}/\mathbb{Q}_p$. By $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$, we have $\dim_K \mathrm{Sel}_F^{cyc}(V) = e$ by Lemma 2.3. For cocycles c in $\mathbf{H}_F$, $(c|_{D_{\mathfrak{p}}} \mod \mathcal{F}_{\mathfrak{p}}^+ V)$ is a $\mathfrak{G}_p$-invariant homomorphism of $\mathfrak{G}_{\mathfrak{p}}$ into K; so, it extends to $\mathfrak{G}_p$, and hence it factors through $\mathfrak{G}_p^{ab} \cong \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p)$. Thus c is locally cyclotomic, and we have $\mathbf{H}_F = \mathrm{Sel}_F^{cyc}(V)$.

Let $\boldsymbol{\rho} : \mathfrak{G}_F \to GL_2(R)$ be the universal nearly ordinary deformation with $\boldsymbol{\rho}|_{D_{\mathfrak{p}'}} = \begin{pmatrix} * & * \\ 0 & \boldsymbol{\delta}_{\mathfrak{p}'} \end{pmatrix}$. Then $c_{\mathfrak{p}} = \frac{\partial \boldsymbol{\rho}}{\partial X_{\mathfrak{p}}}|_{X=0} \rho_E^{-1}$ is a 1-cocycle (by the argument proving Lemma 2.3) giving rise to a class of $\mathbf{H}_F$ (by (K2)). The cocycles $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ give a basis of $\mathbf{H}_F$ over K (by Lemma 2.3). We have

$$\boldsymbol{\delta}_{\mathfrak{p}'}([u, F_{\mathfrak{p}'}]) = (1 + X_{\mathfrak{p}'})^{-\log_p(N_{F_{\mathfrak{p}'}/\mathbb{Q}_p}(u))/\log_p(\gamma_{\mathfrak{p}'})}$$

for $u \in O_{\mathfrak{p}'}^\times$ (because $\mathcal{N}([u, F_{\mathfrak{p}'}]) = N_{F_{\mathfrak{p}'}/\mathbb{Q}_p}(u)^{-1}$). Writing

$$c_{\mathfrak{p}} = \begin{pmatrix} -a_{\mathfrak{p}} & * \\ 0 & a_{\mathfrak{p}} \end{pmatrix} \rho_E^{-1} \quad \text{over } D_{\mathfrak{p}'},$$

we have $a_{\mathfrak{p}} = \boldsymbol{\delta}_{\mathfrak{p}'}^{-1} \frac{d\boldsymbol{\delta}_{\mathfrak{p}'}}{dX_{\mathfrak{p}}}|_{X=0}$ over $D_{\mathfrak{p}'}$, and from this and (2.1) we get the desired formula of $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E))$ in Theorem 2.1 (see the proof of Theorem 3.73 in [HMI] for computational details).

If one restricts $c \in \mathbf{H}_F$ to $\mathfrak{G}_\infty = \mathrm{Gal}(F^{(S)}/F_\infty)$, its ramification is exhausted by $\Gamma = \mathrm{Gal}(F_\infty/F)$ (because of the definition of $\mathrm{Sel}_F^{cyc}(Ad(\rho_E))$ and $\mathbf{H}_F$) giving rise to a class $[c] \in \mathrm{Sel}_{F_\infty}(V)$. The kernel of the restriction map: $H^1(\mathfrak{G}, V) \to H^1(\mathfrak{G}_\infty, V)$ is given by $H^1(\Gamma, H^0(\mathfrak{G}_\infty, V)) = 0$ because $H^0(\mathfrak{G}_\infty, V) = 0$. Thus the image of $\mathbf{H}_F$ in $\mathrm{Sel}_{F_\infty}(V/T)$ gives rise to the order e exceptional zero of $L^{arith}(s, Ad(\rho_E))$ at $s = 1$. We have proved a weaker version of [Gr] Proposition 3:

Proposition 2.6. *Suppose $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. Then for the number e of prime factors of p in F, we have*

$$\mathrm{ord}_{s=1} L_p^{arith}(s, Ad(\rho_E)) \geq e.$$

3. Hecke algebras for quaternion algebras

We make some preparation for the proof of Theorem 1.2, gathering known facts. We assume that $F \neq \mathbb{Q}$ (otherwise the theorem is known by [H04] and by Greenberg-Stevens [GS] and [GS1]). Take first a quaternion algebra $D_{0/F}$ central over F unramified everywhere such that $D_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{d-r}$ with $0 \leq r \leq 1$ (so $r \equiv d \mod 2$). Let S be an open compact subgroup of $GL_2(\widehat{O})$ with $S = S_p \times S^{(p)}$ such that $S_p = GL_2(O_p)$. For an ideal $N \subset O$, we make the following specific choice of $S^{(p)}$:

$$S^{(p)} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\widehat{O}^{(p)}) \middle| c \equiv 0 \mod N \right\} \tag{3.1}$$

for $\widehat{O}^{(p)} = \prod_{\mathfrak{l} \nmid p} O_{\mathfrak{l}}$. Then we consider the automorphic variety (either a Shimura curve ($r = 1$) or a 0-dimensional point set ($r = 0$)) given by

$$X_{11}(p^n) = D_0^\times \backslash D_{0,\mathbb{A}}^\times / S_{11}(p^n) Z_{\mathbb{A}} C_\infty,$$

where $Z_{\mathbb{A}} \cong F_{\mathbb{A}}^\times$ is the center of $D_{\mathbb{A}}^\times$, C_∞ is a maximal compact subgroup of the identity component of $D_{0,\infty}^\times$ and identifying $D_{0,\mathfrak{l}}^{(\infty)} = D_0 \otimes_{\mathbb{Q}} F_{\mathfrak{l}}$ with $M_2(F_{\mathfrak{l}})$ for all primes $\mathfrak{l}$, and $S_{11}(p^n)$ is given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod p^n \text{ with } N_{F_j/\mathbb{Q}_p}(a/d) \equiv 1 \mod p^n \right\}$$

for $\widehat{O} = \prod_{\mathfrak{l}} O_{\mathfrak{l}}$. Let B (resp. Z) be the upper triangular Borel subgroup $B \subset GL(2)$ and the center $Z \subset GL(2)$. Write $S_{11}(p^\infty) = \bigcap_{n=1}^\infty S_{11}(p^n)$. Then

$$B(O_{\mathfrak{p}})/S_{11}(p^\infty)_{\mathfrak{p}} \cong \mathrm{Im}(\mathcal{N}(\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}}))) \subset \mathbb{Z}_p^\times$$

by sending $\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right)$ to $N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(ad^{-1}) \in \mathbb{Z}_p^\times$. Consider $M_n \cong H_r(X_{11}(p^n), \mathbb{Z}_p)$ which is the Pontryagin dual of $H^r(X_{11}(p^n), \mathbb{Q}_p/\mathbb{Z}_p)$. The $\mathbb{Z}_p$-module M_n is a free module (if $n \gg 0$) having the action of the following three type of linear operators:

- Hecke operator $T(\mathfrak{n})$ for each integral ideals $\mathfrak{n}$ outside p,
- the $U(p)$-operator given by the double coset $S_{11}(p^n) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) S_{11}(p^n) \subset D_{0,\mathbb{A}}^\times$ (along with $U(p_{\mathfrak{p}})$ for the $\mathfrak{p}$-component $p_{\mathfrak{p}} \in O_{\mathfrak{p}}$ of p associated with $S_{11}(p^n) \left(\begin{smallmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right) S_{11}(p^n) \subset D_{0,\mathbb{A}}^\times$ and $U(\mathfrak{p})$ associated with $S_{11}(p^n) \left(\begin{smallmatrix} \varpi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right) S_{11}(p^n)$ for (a choice of) a uniformizer $\varpi_{\mathfrak{p}} \in O_{\mathfrak{p}}$),
- the diamond operator action $\langle z \rangle$ coming from $\left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $z \in O_p^\times$.

By the above remark on $B(O_{\mathfrak{p}})/S_{11}(p^\infty)_{\mathfrak{p}}$, the action of $\langle z \rangle$ for $z \in 1 + \mathfrak{p}O_{\mathfrak{p}}^\times$ factor through $\Gamma_{\mathfrak{p}} \subset \mathbb{Z}_p^\times$. Let $e = \lim_{n \to \infty} U(p)^{n!}$ as an operator acting on M_n ($U(p) =$

$\prod_{\mathfrak{p}} U(p_{\mathfrak{p}})$). Let M_n^{ord} be the direct summand eM_n. We have natural trace map $M_m \twoheadrightarrow M_n$ for $m > n$ compatible with all Hecke operators and all diamond operators. By the diamond operator action, $M_\infty^{ord} = \varprojlim_n M_n^{ord}$ naturally become a $W[[\Gamma_F]]$-module. Here is an old theorem of mine (see [HMI] 3.2.9):

Theorem 3.1. *The $W[[\Gamma_F]]$-module M_∞^{ord} is free of finite rank over $W[[\Gamma_F]]$.*

Let $\mathbf{h}$ be the $W[[\Gamma_F]]$-subalgebra of $\mathrm{End}_{W[[\Gamma_F]]}(M_\infty^{ord})$ generated over $W[[\Gamma_F]]$ by $T(\mathfrak{n})$ for all $\mathfrak{n}$ prime to p and $U(\mathfrak{p})$ for all prime factors $\mathfrak{p}$ of p. Then we have

Corollary 3.2. *The algebra $\mathbf{h}$ is torsion free of finite type over $W[[\Gamma_F]]$ with $\mathbf{h}_F/(X_{\mathfrak{p}})_{\mathfrak{p}|p}\mathbf{h}_F$ pseudo isomorphic to the Hecke algebra of $H_r(X_{11}(p), W)$.*

Actually if $p \geq 5$, $\mathbf{h}$ is known to be free over $W[[\Gamma_F]]$, and the pseudo isomorphism as above is actually an isomorphism (see [HMI] 4.3.9).

We take N to be the prime-to–p part of the conductor of E. Let $\mathbb{T}$ be the local ring of the universal nearly ordinary Hecke algebra $\mathbf{h}$ acting nontrivially on the Hecke eigenform associated to E. Let $P \in \mathrm{Spec}(\mathbb{T})(K)$ corresponding to ρ_E, that is, $\rho_{\mathbb{T}} \mod P \sim (\rho_E \otimes K)$. Let $\widehat{\mathbb{T}}_P = \varprojlim_n \mathbb{T}_P/P^n\mathbb{T}_P$ for the localization $\mathbb{T}_P$ of $\mathbb{T}$ at P. Since ρ_E is absolutely irreducible, by the technique of pseudo representation, we can construct the modular deformation $\rho_{\mathbb{T}} : \mathfrak{G} \to GL_2(\widehat{\mathbb{T}}_P)$ which satisfies (K1–4); in particular, $\det \rho_{\mathbb{T}} = \mathcal{N}$, because the central character is trivial. Since E is modular over F, we have the surjective K-algebra homomorphism $R \to \widehat{\mathbb{T}}_P$ for the localization-completion $\widehat{\mathbb{T}}_P$. Since $\widehat{\mathbb{T}}_P$ is integral and of dimension e, we have

Corollary 3.3. *If $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$, then $R \cong \widehat{\mathbb{T}}_P$.*

The isomorphism $R \cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ is often proven by showing $R \cong \widehat{\mathbb{T}}_P$ first (see [HMI] Theorem 3.65 and Proposition 3.78).

Take a quaternion algebra $D_{1/F}$ such that $D_{1,\infty} := D_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^q \times \mathbb{H}^{d-q}$ with $q \leq 1$ and D_1 is ramified only at $\mathfrak{p}_1$ (among finite places). At $\mathfrak{p}_1$, we have a unique maximal order R_1 in $D_{\mathfrak{p}_1}$. Identify $D_{1,\mathbb{A}}^{(\mathfrak{p}_1\infty)}$ with $M_2(F_{\mathbb{A}}^{(\mathfrak{p}_1\infty)})$. Then we define $S'_{11}(p^n)$ to be the product of $S_{11}(p^n)^{(\mathfrak{p}_1)}$ and $R_1^\times$ and define

$$Y_{11}(p^n) = D_1^\times \backslash D_{1,\mathbb{A}}^\times / S'_{11}(p^n) Z_{\mathbb{A}} C_\infty,$$

where C_∞ is again the maximal compact subgroup of the identity component of $D_{1,\infty}^\times$. Let $U(p^{(\mathfrak{p}_1)}) = \prod_{j>1} U(p_{\mathfrak{p}_j})$ for the Hecke operator $U(p_{\mathfrak{p}})$ associated to $S'_{11}(p^n)\left(\begin{smallmatrix} p_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right) S'_{11}(p^n)$, and define $e_1 = \lim_{n\to\infty} U(p^{(\mathfrak{p}_1)})^{n!}$ acting on the dual $N_n = H_q(Y_{11}(p^n), \mathbb{Z}_p)$ of the cohomology group $H^q(Y_{11}(p^n), \mathbb{Q}_p/\mathbb{Z}_p)$. Put $N_\infty = \varprojlim_n N_n$. Then e_1 and $U(p^{(\mathfrak{p}_1)})$ act on N_∞. Let $\Gamma_1 = \prod_{\mathfrak{p}\neq\mathfrak{p}_1} \Gamma_{\mathfrak{p}}$. We go through all the above process and define $\mathbf{h}_1 \subset \mathrm{End}_{W[[\Gamma_1]]}(\varprojlim_n e_1 N_n)$ by the $W[[\Gamma_1]]$-subalgebra generated by $T(\mathfrak{n})$ for all $\mathfrak{n}$ prime to p and $U(\mathfrak{p})$ for all prime factors $\mathfrak{p}$ of p. Here $U(\mathfrak{p}_1)$ is associated to $S'_{11}(p^n)\varpi_1 S'_{11}(p^n) \subset D^\times_{1,\mathbb{A}}$ for (a choice of) $\varpi_1 \in R_1$ whose reduced norm is equal to $\varpi_{\mathfrak{p}_1}$. Since ρ_E (or more precisely, the corresponding Hilbert-modular automorphic representation π_E with $L(s,\pi_E) = L(s,E)$) is Steinberg at $\mathfrak{p}_1$, by the Jacquet-Langlands correspondence (e.g., [HMI] 2.3.6) combined with the Eichler–Shimura isomorphism (e.g., [PAF] 4.3.4), we have a Hecke eigenvector f_1 in $H^q(Y_{11}(p), \mathbb{Z}_p)$ giving rise to E. Then we define $\mathbb{T}_1$ to be the local ring of $\mathbf{h}_1$ acting nontrivially on f_1. Let $P_1 \in \mathrm{Spec}(\mathbb{T}_1)(W)$ be the point associated to ρ_E. We then have a deformation $\rho_{\mathbb{T}_1} : \mathfrak{G} \to GL_2(\widehat{\mathbb{T}}_{1,P})$ of ρ_E. Since the central character is trivial, we have $\det \rho_{\mathbb{T}_1} = \mathcal{N}$.

Theorem 3.4. *We have*

(1) $\mathbf{h}_1$ *is torsion-free finite over* $W[[\Gamma_1]]$, *and* $\widehat{\mathbb{T}}_{1,P_1} \cong K[[X_2,\dots,X_e]]$*;*
(2) $\rho_{\mathbb{T}_1}$ *restricted to* $\mathrm{Gal}(\overline{F}_1/F_1)$ *is isomorphic to* $\left(\begin{smallmatrix} \varepsilon\mathcal{N} & * \\ 0 & \varepsilon \end{smallmatrix}\right)$, *where* $\varepsilon = \pm 1$ *is the eigenvalue of* $Frob_{\mathfrak{p}_1}$ *on the étale quotient of* T_pE*;*
(3) *There is a surjective algebra homomorphism* $\mathbb{T}/X_1\mathbb{T} \twoheadrightarrow \mathbb{T}_1$ *inducing an isomorphism* $\widehat{\mathbb{T}}_P/X_1\widehat{\mathbb{T}}_P \cong \widehat{\mathbb{T}}_{1,P_1}$*;*
(4) *There is a surjective algebra homomorphism* $\mathbb{T}/(U(\mathfrak{p}_1)-\varepsilon)\mathbb{T} \twoheadrightarrow \mathbb{T}_1$ *sending* $T(\mathfrak{n})$ *to* $T(\mathfrak{n})$, *where* $U(\mathfrak{p}_1)$ *is given by the action of the double coset* $S_{11}(p^n)\left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right) S_{11}(p^n)$ *for a prime element* ϖ *of* $\mathfrak{p}_1 O_{\mathfrak{p}_1}$.

Here is a sketch of proof (see [H00] Proposition 7.1 and [HMI] Corollary 3.57 for a detailed proof). The first assertion follows from construction; in other words, it can be proven by the same way as the proof of Corollary 3.2. By the Jacquet-Langlands correspondence, $\mathbb{T}$ covers $\mathbb{T}_1$. Any Hilbert-modular automorphic representation π corresponding to a point of $\mathrm{Spec}(\mathbb{T}_1)(\overline{\mathbb{Q}}_p) \subset \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ is Steinberg at $\mathfrak{p}_1$ because D_1 ramifies at $\mathfrak{p}_1$. Since points corresponding classical automorphic representations are Zariski dense in $\mathrm{Spec}(\mathbb{T}_1)$, the Galois representation has to have the form as in (2) (see [HMI] Proposition 2.44 (2)). Thus the eigenvalue of $U(\mathfrak{p}_1)$ of π is ± 1 and the corresponding Galois representation has the form as in (2). The assertion (1) implies (3). By (2), $U(\mathfrak{p}_1)$ is either

± 1. Since $U(\mathfrak{p}_1)$ is a formal function on the connected $\mathrm{Spf}(\mathbb{T}_1)$, $U(\mathfrak{p}_1) = \varepsilon$ is a constant, which implies (4). □

Note that $[\varpi, F_1]^{e_1} = [p, F_1]$ on any unramified abelian extension of F_1 for the ramification index e_1 of $\mathfrak{p}_1$ over $\mathbb{Q}_p$, and hence $U(p_{\mathfrak{p}_1}) = U(\mathfrak{p}_1)^{e_1}$. By Theorem 3.4 (3–4) (and Theorem 2.43 in [HMI]), we find $U(\mathfrak{p}_1) = \boldsymbol{\delta}_1([\varpi, F_1]) = \varepsilon + X_1\Phi(X_1, \dots, X_e)$ for $\Phi \in W[[X_1, \dots, X_e]]$, and hence, we get $\frac{\partial U(p_{\mathfrak{p}_1})}{\partial X_j}|_{X_1=0} = \frac{\partial U(\mathfrak{p}_1)}{\partial X_j}|_{X_1=0} = 0$ for all $j \geq 2$. In other words,

$$\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}(0)\right)_{i,j} = \begin{pmatrix} \frac{\partial \boldsymbol{\delta}_1([p,F_1])}{\partial X_1}(0) & 0 \\ * & * \end{pmatrix}.$$

Thus we have

$$\det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}(0)\right) = \frac{\partial \boldsymbol{\delta}_1([p, F_1])}{\partial X_1}(0) \times \det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}(0)\right)_{i \geq 2, j \geq 2}.$$

Inductively, we can continue choosing a quaternion algebra D_i exactly ramifying at i p-adic places $\mathfrak{p}_1, \dots, \mathfrak{p}_i$ and $D_i \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{d-r}$ with $0 \leq r \leq 1$ and prove the theorem similar to Theorem 3.4. See [H00] Proposition 7.1 and [HMI] Corollary 3.57 for the exact statement, which yield the following result given in [HMI] as Proposition 3.91:

Corollary 3.5. *Suppose the Hilbert modular elliptic curve $E_{/F}$ has multiplicative reduction at $\mathfrak{p}_k$ for $k = 1, \dots, b$ and ordinary good reduction at $\mathfrak{p}_j$ for $j > b$. Then we have*

$$\det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}(0)\right) = \prod_{k=1}^{b} \frac{\partial \boldsymbol{\delta}_k([p, F_k])}{\partial X_k}(0) \times \det\left(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}(0)\right)_{i > b, j > b}.$$

Although we have given an automorphic proof of the above factorization formula (over p-adic places) of the $\mathcal{L}$-invariant, there is another Galois cohomological proof, which is discussed in [H09] Section 1.3 and [H07b] Corollary 1.11.

4. Extensions of $\mathbb{Q}_p$ by its Tate twist

Let L be a p-adic field with p-adic integer ring W. We start with an extension of local Galois modules

$$0 \to K(1) \to T \to K \to 0$$

over $\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ for a finite extension $L/\mathbb{Q}_p$. This type of extensions (for $K = \mathbb{Q}_p$) can be obtained by the p-adic Tate module $T = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of a Tate elliptic curve $E_{/L}$ with multiplicative reduction.

We prepare some general facts. The following is a slight generalization of [GS1] Section 2: Let L and K be a finite extension of $\mathbb{Q}_p$ inside a fixed algebraic closure $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ and T be a two dimensional vector space over K on which $D := \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ acts. We write $H^i(?)$ for $H^i(D, ?)$. By definition, $H^1(M) = \mathrm{Ext}^1_{K[D]}(K, M)$ for a D-module M, and hence, there is a one-to-one correspondence:

$$\left\{\begin{matrix}\text{nontrivial extensions}\\ \text{of } K \text{ by } M\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{1-dimensional subspaces}\\ \text{of } H^1(M)\end{matrix}\right\}.$$

From the left to the right, the map is given by $(M \hookrightarrow X \twoheadrightarrow K) \mapsto \delta_X(1)$ for the connecting map $K = H^0(K) \xrightarrow{\delta_X} H^1(M)$ of the long exact sequence attached to $(M \hookrightarrow X \twoheadrightarrow K)$. Out of a 1-cocycle $c : D \to M$, one can easily construct an extension $(M \hookrightarrow X \twoheadrightarrow K)$ taking $X = M \oplus K$ and letting D acts on X by $g(v, t) = (gv + t \cdot c(g), t)$, and $[c] \mapsto (M \hookrightarrow X \twoheadrightarrow K)$ gives the inverse map.

By Kummer's theory, we have a canonical isomorphism:

$$H^1(K(1)) \cong \left(\varprojlim_n L^\times/(L^\times)^{p^n}\right) \otimes_{\mathbb{Z}_p} K.$$

We write $\gamma_q \in H^1(K(1))$ for the cohomology class associated to $q \otimes 1$ for $q \in L^\times$. The class γ_q is called the Kummer class of q. A canonical cocycle ξ_q in the class γ_q is given as follows. Define $\xi_n : D \to \mu_{p^n}$ by $\xi_n(\sigma) = (q^{1/p^n})^{\sigma-1}$, which is a 1-cocycle. Then $\xi_q = \varprojlim_n \xi_n$ having values in $\mathbb{Z}_p(1) \subset K(1)$.

Suppose we have a non-splitting exact sequence of D-modules $K(1) \hookrightarrow T \twoheadrightarrow K$ with the splitting field $\bigcup_n L[\mu_{p^n}, q^{1/p^n}]$ for $q \in L$ with $0 < |q|_p < 1$. We have proven

Proposition 4.1. *If T is isomorphic to the representation $\sigma \mapsto \begin{pmatrix} \mathcal{N}(\sigma) & \xi_q(\sigma) \\ 0 & 1 \end{pmatrix}$, then for the extension class of $[T] \in H^1(K(1))$, we have $K[T] = K\gamma_q$. In particular, $K\gamma_q$ is in the image of the connecting homomorphism $H^0(K) \xrightarrow{\delta_0} H^1(K(1))$ coming from the extension $K(1) \hookrightarrow T \twoheadrightarrow K$.*

Corollary 4.2. *Let $E_{/L}$ be an elliptic curve. If E has split multiplicative reduction over W, the extension class of $[T]$ for the p-adic Tate module T is in $\mathbb{Q}_p\gamma_{q_E}$ for the Tate period $q_E \in L^\times$.*

Write $\mathcal{D} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \supset D$. We consider $\mathcal{V} = \mathrm{Ind}_L^{\mathbb{Q}_p} T := \mathrm{Ind}_D^{\mathcal{D}} T$. Then we have a D-stable exact sequence $0 \to \mathcal{F}^+T \to T \to T/\mathcal{F}^+T \to 0$ such that D acts by $\mathcal{N}$ on $\mathcal{F}^+T$. Thus $\mathcal{F}^+T$ is one dimensional. We then have the exact sequence of the induced modules:

$$0 \to \mathrm{Ind}_L^{\mathbb{Q}_p} \mathcal{F}^+T \to \mathrm{Ind}_L^{\mathbb{Q}_p} T \to \mathrm{Ind}_L^{\mathbb{Q}_p}(T/\mathcal{F}^+T) \to 0.$$

We put $\mathcal{F}^+\mathcal{V} := \mathrm{Ind}_L^{\mathbb{Q}_p} \mathcal{F}^+T$, and define $\mathcal{F}^{00}\mathcal{V}$ by the maximal subspace of $\mathcal{V}$ stable under $\mathcal{D}$ such that $\mathcal{D}$ acts on $\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V}$ trivially. In other words, we have

$$H^0(\mathcal{D}, \mathcal{V}/\mathcal{F}^+\mathcal{V}) = \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V}.$$

Similarly, we define $\mathcal{F}^{11}\mathcal{V} \subset \mathcal{V}$ to be the smallest subspace stable under $\mathcal{D}$ such that $\mathcal{D}$ acts on $\mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V}$ by $\mathcal{N}$; so, we have

$$H_0(\mathcal{D}, \mathcal{F}^+\mathcal{V}(-1)) = (\mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V})(-1).$$

Since we have $\mathrm{Ind}_L^{\mathbb{Q}_p}(T/\mathcal{F}^+T) \cong \mathrm{Ind}_L^{\mathbb{Q}_p} \mathbf{1}$ and $\mathrm{Ind}_L^{\mathbb{Q}_p} \mathcal{F}^+T \cong \mathrm{Ind}_L^{\mathbb{Q}_p} \mathcal{F}^+\mathcal{N} \cong (\mathrm{Ind}_L^{\mathbb{Q}_p} \mathbf{1}) \otimes \mathcal{N}$, we find $\dim_K(\mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V}) = \dim_K(\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V}) = 1$, because $H^0(\mathcal{D}, \mathrm{Ind}_L^{\mathbb{Q}_p} \mathbf{1}) \cong H_0(\mathcal{D}, \mathrm{Ind}_L^{\mathbb{Q}_p} \mathbf{1}) \cong K$. Thus we get an extension

$$0 \to \mathcal{F}^+\mathcal{V}/\mathcal{F}^{11}\mathcal{V} \to \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V} \to \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^+\mathcal{V} \to 0 \tag{4.1}$$

of $K[\mathcal{D}]$-modules.

Let $\widetilde{K} := K[\varepsilon] = K[t]/(t^2)$ with $\varepsilon \leftrightarrow (t \mod t^2)$. A $\widetilde{K}[D]$-module $\widetilde{T}$ is called an infinitesimal deformation of T if $\widetilde{T}$ is $\widetilde{K}$-free of rank 2 and $\widetilde{T}/\varepsilon\widetilde{T} \cong T$ as $K[D]$-modules. Since the map $\varepsilon : \widetilde{T} \twoheadrightarrow T \subset \widetilde{T}$ given by $v \mapsto \varepsilon v$ is Galois equivariant, we have an exact sequence of D-modules

$$0 \to T \to \widetilde{T} \to T \to 0$$

if $\widetilde{T}$ is an infinitesimal deformation of T. Pick an infinitesimal character $\psi : D \to \widetilde{K}^\times$ with $\psi \mod (\varepsilon) = 1$. Define $\widetilde{K}(\psi)$ for the space of the character ψ. Obviously, $\frac{d\psi}{d\varepsilon} : D \to K$ is a homomorphism; so, $\frac{d\psi}{d\varepsilon} \in \mathrm{Hom}(D, K) = H^1(K)$. Since the extension $\widetilde{T}$ is split if and only if $\frac{d\psi}{d\varepsilon} = 0$, we get

Proposition 4.3. *The correspondence $\widetilde{K}(\psi) \leftrightarrow \frac{d\psi}{d\varepsilon} \in H^1(K)$ gives a one-to-one correspondence:*

$$\left\{\begin{matrix}\text{Nontrivial infinitesimal}\\ \text{deformations of } K\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{1-dimensional}\\ \text{subspaces of } H^1(K)\end{matrix}\right\},$$

and we have $K[\widetilde{T}(\psi)] = K\frac{d\psi}{d\varepsilon}$ in $H^1(K)$.

We have the restriction map Res : $H^1(\mathcal{D}, K(m)) \to H^1(K(m))$ and the transfer map Tr : $H^1(K(m)) \to H^1(\mathcal{D}, K(m))$. They are adjoint each other under the Tate duality. Thus we have the cup product pairing giving Tate duality and the following commutative diagram:

$$\begin{array}{cccccccc} \langle\cdot,\cdot\rangle : & H^1(K(1)) & \times & H^1(K) & \to & H^2(K(1)) & \cong K \\ & \mathrm{Tr}\ \downarrow & & \uparrow\ \mathrm{Res} & & \| & \\ \langle\cdot,\cdot\rangle : & H^1(\mathcal{D}, K(1)) & \times & H^1(\mathcal{D}, K) & \to & H^2(\mathcal{D}, K(1)) & \cong K. \end{array}$$

By Shapiro's lemma (and the Frobenius reciprocity; cf., [HMI] Section 3.4.4), we get

Lemma 4.4. *We have* $\mathrm{Tr}([T]) = [\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}] \in H^1(\mathcal{D}, K(1))$ *for the class* $[T] \in H^1(K(1))$ *of the extension* $K(1) \hookrightarrow T \twoheadrightarrow K$*.*

Proof. Decompose $\mathcal{D} = \bigsqcup_{\sigma\in\Sigma} D\sigma$; so, $\Sigma \cong \mathrm{Hom}_{\mathrm{field}}(L, \overline{\mathbb{Q}}_p)$. Then for $\tau \in \mathcal{D}$, we have $\sigma\tau = \tau_\sigma\sigma'$ for $\sigma' \in \Sigma$ and $\tau_\sigma \in D$. We look at the matrix form of the induced representation. If the matrix form of T is given by $\left(\begin{smallmatrix} \mathcal{N} & \xi \\ 0 & 1 \end{smallmatrix}\right)$ for a 1-cocycle $\xi : D \to K(1)$, the cocycle giving the extension $K(1) \hookrightarrow \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V} \twoheadrightarrow K$ is given by $\tau \mapsto \sum_{\sigma\in\Sigma} \xi(\tau_\sigma)^\sigma$, which represents the class of $\mathrm{Tr}([\xi])$. Here $\mathcal{D}$ acts on the right on $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ following the tradition of right Galois action on roots of unity $\zeta \mapsto \zeta^\sigma$. □

Corollary 4.5. *Let $E_{/L}$ be an elliptic curve with p-adic Tate module T. If E has split multiplicative reduction over W, the extension class of* $[\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}]$ *for* $\mathcal{V} = \mathrm{Ind}_L^{\mathbb{Q}_p} T$ *is in* $\mathbb{Q}_p\gamma_{N_{L/\mathbb{Q}_p}(q_E)}$ *for the Tate period* $q_E \in L^\times$*.*

Proof. We keep the notation introduced in the proof of the above lemma. Consider the cocycle $\xi_n(\tau) = (q_E^{1/p^n})^{\tau-1}$ of D with values in μ_{p^n}. Then we have

$$\mathrm{Tr}(\xi_n)(\sigma) = \prod_{\sigma\in\Sigma} (q_E^{1/p^n})^{(\tau_\sigma-1)\sigma} = \prod_{\sigma\in\Sigma} (q_E^{1/p^n})^{\sigma(\tau-1)} = (N_{L/\mathbb{Q}_p}(q_E)^{1/p^n})^{\tau-1}.$$

Thus $\mathrm{Tr}([T]) = [\mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}]$ is represented by the cocycle ξ given by $\lim_n \mathrm{Tr}(\xi_n)$ for $\mathrm{Tr}(\xi_n)(\tau) = (N_{L/\mathbb{Q}_p}(q_E)^{1/p^n})^{\tau-1}$, which implies the identity $\mathrm{Tr}([T]) = \gamma_{N_{L/\mathbb{Q}_p}(q_E)}$. □

Note that

$$H^1(\mathcal{D}, K) \cong \mathrm{Hom}(\mathcal{D}, K) = \mathrm{Hom}(\mathcal{D}^{ab}, K) \cong K^2,$$

where the last isomorphism is given by

$$\mathrm{Hom}(\mathcal{D}^{ab}, K) \ni \phi \mapsto (\frac{\phi([\gamma, \mathbb{Q}_p])}{\log_p(\gamma)}, \phi([p, \mathbb{Q}_p])) \in K^2$$

for $\gamma \in \mathbb{Z}_p^\times$ of infinite order. This follows from class field theory. Since the Tate duality $\langle \cdot, \cdot \rangle$ is perfect, for any line ℓ in $H^1(\mathcal{D}, K)$, one can assign its orthogonal complement $\ell^\perp$ in $H^1(\mathcal{D}, K(1))$. Thus we have

Proposition 4.6. *Suppose $L = \mathbb{Q}_p$. The correspondence of a line in $H^1(\mathcal{D}, K)$ and its orthogonal complement in $H^1(\mathcal{D}, K(1))$ gives a one-to-one correspondence:*

$$\left\{ \begin{matrix} \textit{Nontrivial extensions} \\ \textit{of } K \textit{ by } K(1) \textit{ as } K[\mathcal{D}]\textit{-modules} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{nontrivial infinitesimal} \\ \textit{deformations of } K \textit{ over } \mathcal{D} \end{matrix} \right\}.$$

Normalize the Artin symbol $[x, \mathbb{Q}_p]$ so that

- $\mathcal{N}([u, \mathbb{Q}_p]) = u^{-1}$ for $u \in \mathbb{Z}_p^\times$,
- $[p, \mathbb{Q}_p]$ is the arithmetic Frobenius element.

Let $\sigma_q = [q, \mathbb{Q}_p]^{-1}$. Then we have $\langle \gamma_q, \xi \rangle = \xi(\sigma_q)$ for $\gamma_q \in H^1(\mathcal{D}, \mathbb{Q}_p(1))$ and $\xi \in \mathrm{Hom}(\mathcal{D}, \mathbb{Q}_p) = H^1(\mathcal{D}, \mathbb{Q}_p)$. Now we are ready to prove the following version of a theorem of Greenberg-Stevens (cf. [GS1] 2.3.4):

Theorem 4.7. *Let $E_{/L}$ be an elliptic curve with split multiplicative reduction, and let $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widetilde{\mathbb{Q}_p}^\times$ be a nontrivial character which is congruent to 1 modulo ε. Let $T = T_pE \otimes \mathbb{Q}_p$ for the p-adic Tate module T_pE of E, $\mathcal{V}$ be the induced Galois representation $\mathrm{Ind}_L^{\mathbb{Q}_p} T$ and $q_E \in L^\times$ be the Tate period of E. Then the following statements are equivalent:*

(a) $\frac{d\psi}{d\varepsilon}(\sigma_{N_{L/\mathbb{Q}_p}(q_E)}) = 0$;
(b) $\mathcal{W} := \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}$ *corresponds to* $\widetilde{\mathbb{Q}_p}(\psi)$ *under the correspondence of* Proposition 4.6;
(c) *There is an infinitesimal deformation $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ and a commutative diagram:*

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(1) & \overset{\hookrightarrow}{\longrightarrow} & \widetilde{\mathcal{W}} & \overset{\twoheadrightarrow}{\longrightarrow} & \widetilde{\mathbb{Q}_p}(\psi) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \underset{\hookrightarrow}{\longrightarrow} & \mathcal{W} & \underset{\twoheadrightarrow}{\longrightarrow} & \mathbb{Q}_p, \end{array}$$

in which the top row is an exact sequence of $\widetilde{\mathbb{Q}_p}[\mathcal{D}]$-modules and the vertical map is the reduction modulo ε.

Proof. Since $\langle\gamma_q,\xi\rangle=\xi(\sigma_q)$ for $\xi\in H^1(\mathcal{D},\mathbb{Q}_p)$ and $\gamma_q\in H^1(\mathcal{D},\mathbb{Q}_p(1))$, applying these formulas to $\xi=\frac{d\psi}{d\varepsilon}$, we get (a) $\Leftrightarrow$ (b) by the definition of the correspondence in Proposition 4.6.

The equivalence (b) $\Leftrightarrow$ (c) can be proven in exactly the same manner as in the proof of [GS1] 2.3.4. Here is the argument proving (b) $\Rightarrow$ (c). Let c be a 1-cocycle representing γ_Q for $Q=N_{L/\mathbb{Q}_p}(q_E)$. Then $\mathcal{D}\times\mathcal{D}\ni(\sigma,\tau)\mapsto c(\sigma)\frac{d\psi}{d\varepsilon}(\tau)\in\mathbb{Q}_p(1)$ is the 2-cocycle representing the cup product $\gamma_Q\cup[\widetilde{\mathbb{Q}_p}(\psi)]$, which vanishes by (b). Thus it is a 2-coboundary:

$$c(\sigma)\frac{d\psi}{d\varepsilon}(\tau)=\partial\xi(\sigma,\tau)=\xi(\sigma\tau)-\mathcal{N}(\sigma)\xi(\tau)-\xi(\sigma)$$

for a 1-chain $\xi:\mathcal{D}\to\mathbb{Q}_p(1)$. Then defining an action of $\sigma\in\mathcal{D}$ on $\widetilde{\mathbb{Q}_p}^2$ via the matrix multiplication by $\begin{pmatrix}\mathcal{N}(\sigma) & c(\sigma)+\xi(\sigma)\varepsilon\\ 0 & \psi(\sigma)\end{pmatrix}$, the resulting $\widetilde{\mathbb{Q}_p}[\mathcal{D}]$-module $\widetilde{\mathcal{W}}$ fits well in the diagram in (c).

Conversely suppose we have the commutative diagram as in (c), which can be written as the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{\mathbb{Q}_p}(1) & \longrightarrow & \widetilde{\mathcal{W}} & \longrightarrow & \widetilde{\mathbb{Q}_p} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The connecting homomorphism $d:H^1(\mathcal{D},\mathbb{Q}_p(1))\to H^2(\mathcal{D},\mathbb{Q}_p(1))$ vanishes because the leftmost vertical sequence splits. On the other hand, letting $\delta_\psi: H^0(\mathcal{D},\mathbb{Q}_p)\to H^1(\mathcal{D},\mathbb{Q}_p)$ stand for the connecting homomorphism of degree 0 coming from the rightmost vertical sequence, and letting $\delta_i:H^i(\mathcal{D},\mathbb{Q}_p)\to H^{i+1}(\mathcal{D},\mathbb{Q}_p(1))$ be the connecting homomorphism of degree i associated to the bottom row (and also to the top row), by the commutativity of the diagram, we get the following commutative square:

$$\begin{array}{ccc} H^0(\mathcal{D},\mathbb{Q}_p)=\mathbb{Q}_p & \xrightarrow{\delta_0} & H^1(\mathcal{D},\mathbb{Q}_p(1)) \\ \delta_\psi\downarrow & & \downarrow d=0 \\ H^1(\mathcal{D},\mathbb{Q}_p) & \xrightarrow[\delta_1]{} & H^2(\mathcal{D},\mathbb{Q}_p(1)). \end{array}$$

Since $\delta_\psi(1)=\frac{d\psi}{d\varepsilon}$, we confirm $\frac{d\psi}{d\varepsilon}\in\mathrm{Ker}(\delta_1)$. By Proposition 4.1, γ_Q is the in the image of δ_0. Thus the assertion (b) follows if we can show that $\mathrm{Ker}(\delta_1)$ is orthogonal to $\mathrm{Im}(\delta_0)$.

Since $\mathcal{V}=\mathrm{Ind}_L^{\mathbb{Q}}T$ is the p-adic Tate module of the principally polarized abelian variety $A=\mathrm{Res}_{L/\mathbb{Q}_p}E_{/L}$ (the Weil restriction), $\mathcal{V}$ is self dual under the polarization pairing, which induces a self duality of $\mathcal{W}$ and also the self (Cartier) duality of the exact sequence $0\to\mathbb{Q}_p(1)\to\mathcal{W}\to\mathbb{Q}_p\to 1$. In particular the inclusion $\iota:\mathbb{Q}_p(1)\to\mathcal{W}$ and the projection $\pi:\mathcal{W}\twoheadrightarrow\mathbb{Q}_p$ are mutually adjoint under the pairing. Thus the connecting maps $\delta_0:H^0(\mathcal{D},\mathbb{Q}_p)\to H^1(\mathcal{D},\mathbb{Q}_p(1))$ and $\delta_1:H^1(\mathcal{D},\mathbb{Q}_p)\to H^2(\mathcal{D},\mathbb{Q}_p(1))$ are mutually adjoint each other under the Tate duality pairing. In particular, $\mathrm{Im}(\delta_0)$ is orthogonal to $\mathrm{Ker}(\delta_1)$. □

Take a prime $\mathfrak{p}|p$ in F, and let $D=\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ ($L=F_\mathfrak{p}$) and $\mathcal{D}=\mathrm{Gal}(\overline{F}_\mathfrak{p}/\mathbb{Q}_p)$. We write $\mathcal{I}$ (resp. I) for the inertia group of $\mathcal{D}$ (resp. D).

Lemma 4.8. *Let $\rho_A:\mathrm{Gal}(\overline{F}/F)\to GL_2(A)$ be a deformation of ρ_E for an artinian local K-algebra A with residue field K. Write $\rho_A|_D=\begin{pmatrix} * & * \\ 0 & \delta_A\end{pmatrix}$ with $\delta_A\equiv\alpha_\mathfrak{p} \mod \mathfrak{m}_A$. Suppose that $\alpha_\mathfrak{p}$ can be extended to a character $\widetilde{\alpha}_\mathfrak{p}:\mathcal{D}\to K^\times$. If $\delta_A|_I$ factors through $\mathrm{Gal}(F_\mathfrak{p}[\mu_{p^\infty}]/F_\mathfrak{p})$, the character δ_A extends to a unique character $\widetilde{\delta}_A$ of $\mathcal{D}$ with values in $A^\times$ such that $\widetilde{\delta}_A\equiv\widetilde{\alpha}_\mathfrak{p} \mod \mathfrak{m}_A$.*

Proof. Let $F_\mathfrak{p}^{ab}$ (resp. $F_\mathfrak{p}^{ur}$) be the maximal abelian extension of $F_\mathfrak{p}$ (resp. the maximal unramified extension of $F_\mathfrak{p}$). Then we have

$$F_\mathfrak{p}[\mu_{p^\infty}]\subset F_\mathfrak{p}^{ur}[\mu_{p^\infty}]=F_\mathfrak{p}\mathbb{Q}_p^{ur}[\mu_{p^\infty}]=F_\mathfrak{p}\mathbb{Q}_p^{ab}.$$

So $\mathrm{Gal}(F_\mathfrak{p}\mathbb{Q}_p^{ur}[\mu_{p^\infty}]/F_\mathfrak{p})$ is identified with the subgroup $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{ab}\cap F_\mathfrak{p})$ of $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ of finite index. Since δ_A is a character of $\mathrm{Gal}(F_\mathfrak{p}\mathbb{Q}_p^{ur}[\mu_{p^\infty}]/F_\mathfrak{p})$, regarding it as a character of $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p^{ab}\cap F_\mathfrak{p})$, we only need to extend it to $\mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$. Since $F_\mathfrak{p}\cap\mathbb{Q}_p^{ab}/\mathbb{Q}_p$ is a finite Galois extension with an abelian Galois group Δ, by the theory of the Schur multiplier, the obstruction of extending character lies in $H^2(\Delta,A^\times)$ (see [MFG] Section 3.3.5). Since $\delta_A\equiv\alpha_\mathfrak{p} \mod \mathfrak{m}_A$, the obstruction class $Ob(\delta_A) \equiv Ob(\alpha_{\mathfrak{p}}) = 0 \mod \mathfrak{m}_A$. Thus $Ob(\delta_A) \in H^2(\Delta, 1+\mathfrak{m}_A)$. Since $1+\mathfrak{m}_A$ is uniquely divisible (by $\log : 1+\mathfrak{m}_A \cong \mathfrak{m}_A$ as K-vector spaces), we get the vanishing $H^2(\Delta, 1+\mathfrak{m}_A) = 0$ for the finite group Δ. Then we can extend δ_A to $\widetilde{\delta}_A$ with $\widetilde{\delta}_A \equiv \widetilde{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_A$ as proven in [MFG] Section 5.4. If δ' is another extension with $\delta' \equiv \widetilde{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_A$, we find $\widetilde{\delta}_A^{-1}\delta'$ is a character of Δ, which has to be trivial by the condition $\widetilde{\delta}_A \cong \widetilde{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_A$. Thus the extension is unique. □

5. Symmetric power of Tate curves

In this section, we state a conjectural formula of the $\mathcal{L}$-invariant of the L-function of symmetric power p-adic L-functions of elliptic curves with semi-stable ordinary reduction at p. We prove the conjecture for the adjoint square arithmetic L-function under mild assumptions (Theorem 5.3).

5.1. **A general conjecture and a proof of the theorem.** Take an elliptic curve E with multiplicative reduction over the finite extension $L_{/\mathbb{Q}_p}$. Let $T = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Let $T(n,m)$ be the symmetric n-th power of T with $(-m)$-th Tate twist:

$$T(n,m) = (Sym^{\otimes n}T)(-m).$$

We then put $\mathcal{V}(n,m) = \mathrm{Ind}_L^{\mathbb{Q}_p} T(n,m)$. Suppose $0 \le m < n$. We have a decreasing filtration $\mathcal{F}^kX$ of $X = T(n,k)$ and $\mathcal{V}(n,k)$ stable under $D = \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ so that an open subgroup of the inertia group I of D acts on $\mathcal{F}^kX/\mathcal{F}^{k+1}X$ by the k-th power of the p-adic cyclotomic character. We have $\mathcal{F}^j\mathcal{V}(n,m) = \mathrm{Ind}_L^{\mathbb{Q}_p} \mathcal{F}^jT(n,m)$. We put $\mathcal{F}^+X = \mathcal{F}^1X$ and $\mathcal{F}^-X = \mathcal{F}^0X$. We define $\mathcal{F}^{00}\mathcal{V}(n,m) \subset \mathcal{V}(n,m)$ so that $\mathcal{F}^{00}\mathcal{V}(n,m) \supset \mathcal{F}^+\mathcal{V}(n,m)$ and $\frac{\mathcal{F}^{00}\mathcal{V}(n,m)}{\mathcal{F}^+\mathcal{V}(n,m)} = H^0(\mathbb{Q}_p, \frac{\mathcal{V}(n,m)}{\mathcal{F}^+\mathcal{V}(n,m)})$. Similarly, we define $\mathcal{F}^{11}\mathcal{V}(n,m) \subset \mathcal{F}^+\mathcal{V}(n,m)$ so that

$$\mathcal{F}^+\mathcal{V}(n,m)/\mathcal{F}^{11}\mathcal{V}(n,m) = H_0(\mathbb{Q}_p, \mathcal{F}^+\mathcal{V}(n,m)(-1)).$$

Put $\mathcal{D} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Lemma 5.1. *We have*

$$\dim_{\mathbb{Q}_p} \mathcal{F}^+\mathcal{V}(n,m)/\mathcal{F}^{11}\mathcal{V}(n,m) = \dim_{\mathbb{Q}_p} \mathcal{F}^{00}\mathcal{V}(n,m)/\mathcal{F}^+\mathcal{V}(n,m) = 1$$

and a Tate extension of $\mathcal{D}$ modules

$$0 \to \mathbb{Q}_p(1) \to \mathcal{F}^{00}\mathcal{V}(n,m)/\mathcal{F}^{11}\mathcal{V}(n,m) \to \mathbb{Q}_p \to 0.$$

Moreover, writing q for the Tate period of E, we have $\left[\frac{\mathcal{F}^{00}\mathcal{V}(n,m)}{\mathcal{F}^{11}\mathcal{V}(n,m)}\right] \in \mathbb{Q}_p\gamma_Q$ *with* $Q = N_{L/\mathbb{Q}_p}(q)$ *and* $\left[\frac{\mathcal{F}^{00}\mathcal{V}(n,m)}{\mathcal{F}^{11}\mathcal{V}(n,m)}\right] = \mathrm{Tr}([T]) \in H^1(\mathcal{D},\mathbb{Q}_p(1))$ *for the transfer map* $\mathrm{Tr} : H^1(D,\mathbb{Q}_p(1)) \to H^1(\mathcal{D},\mathbb{Q}_p(1))$.

Proof. Write the representation ρ_E on D as $\left(\begin{smallmatrix}\mathcal{N} & \xi_q\\ 0 & 1\end{smallmatrix}\right)$ (with respect to the basis (x,y) and the Tate period $q \in L$). Then the matrix expression of $\rho_{n,0}$ on D with respect the basis $(x^n, x^{n-1}y, \dots, y^n)$ of $T(n,m)$ is given by

$$\begin{pmatrix} \mathcal{N}^n & n\mathcal{N}^{n-1}\xi_q & * & \cdots & * \\ 0 & \mathcal{N}^{n-1} & (n-1)\mathcal{N}^{n-2}\xi_q & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{N} & \xi_q \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Thus we get an extension $\mathbb{Q}_p(m+1) \hookrightarrow \mathcal{F}^mT(n,0)/\mathcal{F}^{m+2}T(n,0) \twoheadrightarrow \mathbb{Q}_p(m)$ of D-modules on which D acts by $\left(\begin{smallmatrix}\mathcal{N}^{m+1} & (m+1)\mathcal{N}^m\xi_q\\ 0 & \mathcal{N}^m\end{smallmatrix}\right)$. The extension class of the twist: $\mathbb{Q}_p(1) \hookrightarrow \frac{\mathcal{F}^mT(n,0)}{\mathcal{F}^{m+2}T(n,0)}(-m) \twoheadrightarrow \mathbb{Q}_p$ is described by the 1-cocycle $(m+1)\xi_q : D \to \mathbb{Q}_p$. Since $\left(\begin{smallmatrix}m+1 & 0\\ 0 & 1\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}\mathcal{N} & (m+1)\xi_q\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}m+1 & 0\\ 0 & 1\end{smallmatrix}\right) \cong \left(\begin{smallmatrix}\mathcal{N} & \xi_q\\ 0 & 1\end{smallmatrix}\right)$, we have an isomorphism of D-modules

$$\mathcal{F}^-T(n,m)/\mathcal{F}^+T(n,m) = (\mathcal{F}^mT(n,0)/\mathcal{F}^{m+2}T(n,0))(-m) \cong T.$$

This proves $\mathcal{F}^-T(n,m)/\mathcal{F}^+T(n,m) \cong \mathbb{Q}_p$ as D-modules. By induction from L to $\mathbb{Q}_p$, we get

$$\mathcal{F}^{00}\mathcal{V}(n,m)/\mathcal{F}^{11}\mathcal{V}(n,m) \cong \mathcal{F}^{00}\mathcal{V}/\mathcal{F}^{11}\mathcal{V}$$

for $\mathcal{V} = \mathrm{Ind}_L^{\mathbb{Q}_p} T$. Then all the assertions follow from Lemma 4.4 and Corollary 4.5. □

We restate Conjecture 1.3 in a slightly different fashion: Order the prime factors of p in F as $\mathfrak{p}_1, \dots, \mathfrak{p}_e$.

Conjecture 5.2. *For an elliptic curve $E_{/F}$, suppose that E is split multiplicative at $\mathfrak{p}_j$ for $j = 1, 2, \dots, b$ ($0 \le b \le e$) with Tate period $q_j \in F_j^\times$ and has ordinary good reduction at $\mathfrak{p}_i$ with $i > b$. Suppose that the motive $Sym^{\otimes n}(H_1(E))(-m)$ for an integer m with $0 \le m < n$ is critical at 1 ($\Leftrightarrow$ either n is odd or $n = 2m$ with m odd). Then if $\mathrm{Ind}_F^{\mathbb{Q}}(Sym^{\otimes n}(\rho_E)(-m))$ has an exceptional zero at $s = 1$,* *we have*

$$\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}(Sym^{\otimes n}(\rho_E)(-m))) = \begin{cases} \left(\prod_{j=1}^b \frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)}\right)\mathcal{L}(m) & \text{for } \mathcal{L}(m)\in\mathbb{Q}_p^\times \text{ if } n=2m \text{ with odd } m,\\ \prod_{j=1}^b \frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)} & \text{if } n\neq 2m,\end{cases}$$

where $Q_j = N_{F_j/\mathbb{Q}_p}(q_j)$. *We have* $\mathcal{L}(m)=1$ *if* $b=e$, *and when* $m=1$, *assuming that* $R\cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$, *we conjecture that* $\mathcal{L}(1)$ *is given by*

$$\mathcal{L}(1) = \det\left(\frac{\partial\boldsymbol{\delta}_i([p,F_i])}{\partial X_j}\right)_{i>b,j>b}\Big|_{X_1=X_2=\cdots=X_e=0}\prod_{i>b}\frac{\log_p(\gamma_i)}{[F_i:\mathbb{Q}_p]\alpha_i([p,F_i])}$$

for the local Artin symbol $[p,F_i]$.

Here are some more remarks about the conjecture:

(1) As we have said, the above conjecture applies to the (often hypothetical) analytic p-adic L-function of $L_p(s,T(n,m))$ and the arithmetic p-adic L-function $L_p^{arith}(s,T(n,m))$. The analytic p-adic L-function interpolates the complex L-values of $L(1,T(n,m)\otimes\varepsilon)$ (up to a power of the Néron period of E and a power of the Gauss sum $G(\varepsilon)$) over finite order characters $\varepsilon : \mathrm{Gal}(F[\mu_{p^\infty}]/F)\to\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$.

(2) Write $\gamma\in\mathbb{Z}_p^\times$ for the image under the p-adic cyclotomic character of a generator of $\Gamma=\mathrm{Gal}(F_\infty/F)$, and identify $\mathbb{Z}_p[[\Gamma]]$ with $\mathbb{Z}_p[[T]]$ by $\gamma\mapsto 1+T$. Then the arithmetic p-adic L-function is defined by

$$L_p^{arith}(s,T(n,m)) = \Phi(\gamma^{1-s}-1)$$

choosing a characteristic power series $\Phi(T)\in\mathbb{Z}_p[[T]]$ of the Pontryagin dual Iwasawa module of $\mathrm{Sel}_{F_\infty}((Sym^{\otimes n}T_pE)(-m)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$. Thus in the arithmetic case, the $\mathcal{L}$-invariant is the one defined by Greenberg in [Gr]. We will heuristically show in the following section the prerequisites to have well-defined Greenberg's $\mathcal{L}$-invariant for $T(2m,m)$ with m odd.

(3) As already remarked, if $F=\mathbb{Q}$, the assumption $R\cong K[[X_p]]$ is shown, without any other assumptions, by Kisin. If $F\neq\mathbb{Q}$, by Fujiwara, this assertion $R\cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$ is shown under (ds) and (ai).

To prove a stronger version (with additional information) of the theorem in the introduction, we prepare some notation. Let $W=\mathbb{Z}_p$, and write $\overline{\rho}_n : \mathrm{Gal}(\overline{F}/F_n)\to GL_2(\mathbb{F}_p)$ (resp. $\overline{\alpha}_{\mathfrak{p}} : \mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})\to\mathbb{F}_p^\times$) for ($\rho_n \mod p$) (resp. ($\alpha_{\mathfrak{p}} \mod p$)). We consider the representation $\rho_n : \mathrm{Gal}(\overline{F}/F_n) \to GL_2(\mathbb{Z}_p) = GL(T_pE)$. Here F_n/F is the n-th layer in F_∞; so, $[F_n : F] = p^n$ (we exclusively use n for the index of the n-th layer F_n to avoid confusion with the $\mathfrak{p}_j$-adic completion F_j, and the completion is either denoted by F_j or F_i). Write $(\mathcal{R}_n, \varrho_n)$ for the couple universal among the following couples (A, ρ_A) of a p-adically continuous representations $\rho_A : \mathrm{Gal}(\overline{F}/F_n) \to GL_2(A)$ and a local pro-artinian $\mathbb{Z}_p$-algebra A with $A/\mathfrak{m}_A = \mathbb{F}_p$ such that

(W1) unramified outside S, ∞ and p;
(W2) for each prime factor $\mathfrak{p}|p$ of F_n, $\rho_A|_{\mathrm{Gal}(\overline{F}_{n,\mathfrak{p}}/F_{n,\mathfrak{p}})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \alpha_{A,\mathfrak{p}} \end{smallmatrix}\right)$ for $\alpha_{A,\mathfrak{p}} \equiv \overline{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_A$ with $\alpha_{A,\mathfrak{p}}|_{I_\mathfrak{p}}$ factoring through $\mathrm{Gal}(F^{ur}_{n,\mathfrak{p}}[\mu_{p^\infty}]/F^{ur}_{n,\mathfrak{p}})$ for the maximal unramified extension $F^{ur}_{n,\mathfrak{p}}/F_{n,\mathfrak{p}}$ (the local cyclotomy condition);
(W3) $\det(\rho_A) = \mathcal{N}$ for the p-adic cyclotomic character $\mathcal{N}$;
(W4) $\rho_A \equiv \overline{\rho}_n \mod \mathfrak{m}_A$.

Then under (ai) and (ds), the universal couple $(\mathcal{R}_n, \varrho_n)$ exists. Writing

$$\varrho_n|_{\mathrm{Gal}(\overline{F}_{n,\mathfrak{p}}/F_{n,\mathfrak{p}})} \cong \left(\begin{smallmatrix} * & * \\ 0 & \boldsymbol{\alpha}_{n,\mathfrak{p}} \end{smallmatrix}\right)$$

with $\boldsymbol{\alpha}_{n,\mathfrak{p}} \equiv \overline{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_{\mathcal{R}_n}$, we confirm that the character $\boldsymbol{\alpha}_{0,\mathfrak{p}}\alpha_{\mathfrak{p}}^{-1} : \Gamma_\mathfrak{p} \to \mathcal{R}_n^\times$ induces $W[[X_\mathfrak{p}]]$-algebra structure on $\mathcal{R}_0$. Since $\boldsymbol{\alpha}_{n,\mathfrak{p}}|_{I_\mathfrak{p}}$ factors through the inertia group $\mathrm{Gal}(F^{ur}_{n,\mathfrak{p}}[\mu_{p^\infty}]/F^{ur}_{n,\mathfrak{p}})$, it factors through $\mathrm{Gal}(F^{ab}_\mathfrak{p}/F_\mathfrak{p})$. Thus we may evaluate $\boldsymbol{\alpha}_{n,\mathfrak{p}}$ at $[p, F_\mathfrak{p}]$. In particular, we may think of the closed subalgebra $\boldsymbol{\Lambda}_n$ of $\mathcal{R}_n$ generated topologically over W by $\boldsymbol{\alpha}_{n,\mathfrak{p}}([p, F_\mathfrak{p}])$ for all $\mathfrak{p}|p$. Since $\boldsymbol{\delta}_\mathfrak{p}$ factors through $\boldsymbol{\alpha}_{0,\mathfrak{p}}$, $\boldsymbol{\Lambda}_0$ covers the closed subring $\overline{\boldsymbol{\Lambda}}_0$ of R topologically generated over W by $\boldsymbol{\delta}_\mathfrak{p}([p, F_\mathfrak{p}])$ for all $\mathfrak{p}|p$. If $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \neq 0$, we have $\det(\frac{\partial \boldsymbol{\delta}_\mathfrak{p}([p,F_\mathfrak{p}])}{\partial X_{\mathfrak{p}'}})_{\mathfrak{p},\mathfrak{p}'} \neq 0$, and $\overline{\boldsymbol{\Lambda}}_0$ is a power series ring of e variables over W; so, $\boldsymbol{\Lambda}_0 \cong \overline{\boldsymbol{\Lambda}}_0$. Since $\varrho_0|_{\mathrm{Gal}(\overline{F}/F_n)}$ is a deformation classified by $(\mathcal{R}_n, \varrho_n)$, we have a local W-algebra homomorphism $\pi_n : \mathcal{R}_n \to \mathcal{R}_0$. Since $\mathcal{R}_0$ is generated by trace of ϱ_0, π_n is surjective. Then π_n induces a surjective morphism: $\boldsymbol{\Lambda}_n \twoheadrightarrow \boldsymbol{\Lambda}_0$; so, if $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \neq 0$, we have $\boldsymbol{\Lambda}_n \cong \boldsymbol{\Lambda}_0 \cong W[[t_\mathfrak{p}]]_{\mathfrak{p}|p}$ for variables $t_\mathfrak{p}$. Let $P \in \mathrm{Spec}(\mathcal{R}_0)$ be the point corresponding to ρ_E; so, $\varrho_n \mod P \cong \rho_E$. Via π_n, we may regard $P \in \mathrm{Spec}(\mathcal{R}_n)$ for $n = 1, 2, \ldots, \infty$. We consider the module of continuous 1-differentials $\Omega_{\mathcal{R}_n/B} \otimes_{\mathcal{R}_n} A$ for $A = \mathcal{R}_n/P \cong W$, which will be written as $M^A_{n/B}$ hereafter (for simplicity). Here the continuity of 1-differentials on $\mathcal{R}_n$ over B is under the profinite topology. As shown in [HMI] proposition 3.87, $M^A_{0/W}$ is canonically isomorphic to the Pontryagin dual of $\mathrm{Sel}_F^{cyc}(Ad(\rho_0)\otimes(\mathbb{Q}_p/\mathbb{Z}_p))$. Similarly, $M^A_{\infty/W}$ is isomorphic to the Pontryagin dual of $\mathrm{Sel}_{F_\infty}(Ad(\rho_0)\otimes(\mathbb{Q}_p/\mathbb{Z}_p))$ (because there is no difference between the locally cyclotomic Selmer group and the standard Selmer group over F_∞. If p is unramified in $F/\mathbb{Q}$, $M^A_{\infty/\mathbf{\Lambda}_\infty}$ is isomorphic to the Pontryagin dual of $\mathrm{Sel}^{st}_{F_\infty}(Ad(\rho_0)\otimes(\mathbb{Q}_p/\mathbb{Z}_p))$ by the same argument which proves [HMI] proposition 3.87.

We state a stronger version (with additional information) of the theorem in the introduction. Note here that $T(2,1)=Ad(\rho_E)$.

Theorem 5.3. *Suppose $n=2$ and $m=1$. Suppose that the Hilbert-modular elliptic curve E has split multiplicative reduction at $\mathfrak{p}_j$ for $j=1,2,\dots,b$ ($b\le e$) for $j\le b$ and has ordinary good reduction at $\mathfrak{p}_i$ with $i>b$. If* Conjecture 1.1 *holds for ρ_E, then $\mathrm{Sel}_F(T(2,1))=0$ and the formula in* Conjecture 5.2 *gives the $\mathcal{L}$-invariant of Greenberg defined in* [Gr]. *If we assume further the following six conditions:*

(0) *each prime factor $\mathfrak{p}|p$ is unramified in $F/\mathbb{Q}$ (so, $\mathfrak{p}$ fully ramifies in F_∞/F);*
(1) $\overline{\rho}_0=(\rho_0 \mod p\mathbb{Z}_p)$ *is absolutely irreducible over* $\mathrm{Gal}(\overline{F}/F[\mu_p])$*;*
(2) *the semisimplification of $\overline{\rho}_0|_{\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})}$ is the sum of two distinct characters for each prime factor $\mathfrak{p}$ of p;*
(3) *if E has multiplicative reduction at a prime $\mathfrak{q}$ outside p, $\overline{\rho}_0$ restricted to the inertia group $I_\mathfrak{q}$ at $\mathfrak{q}$ is indecomposable;*
(4) *E is semi-stable over O,*
(5) *One of the following equivalent conditions:*
    (a) *The Pontryagin dual of* $\mathrm{Sel}^{st}_{F_\infty}(Ad(T_pE)\otimes(\mathbb{Q}_p/\mathbb{Z}_p))$ *has no nontrivial pseudo-null submodule non-null;*
    (b) $H^0(\Gamma, M^A_{\infty/\mathbf{\Lambda}_\infty})=0$ *for* $\Gamma=\mathrm{Gal}(F_\infty/F)$.

we have $\Phi^{arith}(T)=T^e\Psi(T)$ in $\mathbb{Z}_p[[T]]$ and

$$\Psi(0)=\mathcal{L}(Ad(\rho_E)_{/F})\left(\prod_{j=1}^{e}\log_p(\gamma_j)\right)^{-1}|\mathrm{Sel}_F(Ad(T_pE)\otimes\mathbb{Q}_p/\mathbb{Z}_p)|$$

up to units for the characteristic power series $\Phi^{arith}(T)$ of the Pontryagin dual Iwasawa module of $\mathrm{Sel}_{F_\infty}(Ad(T_pE)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p)$. *Without assuming the assumption (5), the number $\Psi(0)$ is a factor of the right-hand-side of the above formula.*

A formula almost identical to the ones in the above theorem (covering more general cases of nonadjoint type) have been proven (with a different set of assumptions) by Greenberg as [Gr] Proposition 4, but our method of proof is different via Galois deformation theory (and infinitesimal p-adic calculus).

Here are some more remarks on the theorem:

- We use in place of $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E))$ the $\mathcal{L}$-invariant $\mathcal{L}(Ad(\rho_E)_{/F})$ in (1.1) for the reason explained after stating Conjecture 2.2.
- The unramifiedness in the condition (0) is probably inessential (we can presumably remove it, though we need to replace $[p, F_{\mathfrak{p}}]$ in the definition of $\mathbf{\Lambda}_n$ by $[\varpi_{\mathfrak{p}}, F_{\mathfrak{p}}]$ for a prime element $\varpi_{\mathfrak{p}} \in F_{\mathfrak{p}}^{\times}$ which is the universal norm from $F_{\infty,\mathfrak{p}}^{\times}$ if $\mathfrak{p}$ ramifies in $F/\mathbb{Q}$ wildly). We hope to treat general cases in our subsequent paper.
- The condition (3) can be removed if we assume the full level lowering statement at $\mathfrak{q}$ (in other words, if we assume to have a Hilbert modular Hecke eigenform of weight 2 of level $N/\mathfrak{q}$ whose Galois representation is congruent to ρ_E modulo $\mathfrak{m}_W$ if (3) fails at $\mathfrak{q}$, where N is the conductor of E).
- In the condition (5), (a)$\Rightarrow$(b) is easy because $H^0(\Gamma, M^A_{\infty/\mathbf{\Lambda}_\infty})$ is pseudo-null $W[[\Gamma]]$-module (as we will see below) and $M^A_{\infty/\mathbf{\Lambda}_\infty}$ is isomorphic to the Pontryagin dual of $\mathrm{Sel}^{st}_{F_\infty}(Ad(\rho_0) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ as already remarked. The reverse direction is [HMI] Lemma 5.24 (3).
- In [H00] Theorem 6.3 (4), the second assertion is claimed without assuming the condition (5), but the proof there also requires this condition (so omission of the condition (5) there is an error).
- The last assertion of the theorem is just the restatement of [H00] Theorem 6.3 (3).

Proof. At the beginning, to give heuristics for the conjecture, we do not suppose $n = 2$ and $m = 1$. Let $T = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for the p-adic Tate module T_pE of E, and put $T(n, m) = (Sym^{\otimes n}T)(-m)$ for $0 \le m < n$. The global representation $\mathbb{V}(n, m) = \mathrm{Ind}_F^{\mathbb{Q}} T(n, m)$ has decreasing filtration $\mathcal{F}^i\mathbb{V}(n, m)$ such that an open subgroup of the inertia group I_p at p acts on $\mathcal{F}^i\mathbb{V}(n, m)/\mathcal{F}^{i+1}\mathbb{V}(n, m)$ by the i-th power of the cyclotomic character $\mathcal{N}$ and $\mathcal{F}^1\mathbb{V}(n, m) \subsetneq \mathbb{V}(n, m)$. Put $\mathcal{F}^+\mathbb{V}(n, m) = \mathcal{F}^1\mathbb{V}(n, m)$. Recall $\mathcal{D} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $\mathcal{F}^{00}\mathbb{V}(n, m)$ be the maximal $\mathcal{D}$-stable subspace of $\mathbb{V}(n, m)$ containing $\mathcal{F}^+\mathbb{V}(n, m)$ such that any vector in $\mathcal{F}^{00}\mathbb{V}(n,m)/\mathcal{F}^{+}\mathbb{V}(n,m)$ is fixed by $\mathcal{D}$. Similarly, let $\mathcal{F}^{11}\mathbb{V}(n,m)$ be the minimal $\mathcal{D}$-stable subspace of $\mathbb{V}(n,m)$ contained in $\mathcal{F}^{+}\mathbb{V}(n,m)$ such that $\mathcal{D}$ acts on $\mathcal{F}^{+}\mathbb{V}(n,m)/\mathcal{F}^{11}\mathbb{V}(n,m)$ by $\mathcal{N}$. We may regard T as a $\mathrm{Gal}(\overline{F}_j/F_j)$-module, and consider $\mathcal{V}_j = \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} T$. Then again we have $\mathcal{F}^{00}\mathcal{V}_j \supset \mathcal{F}^{11}\mathcal{V}_j$ as defined above (4.1) for $L = F_j$. From [HMI] (3.4.4) in page 263 and the fact that the eigenvalues of $[p, F_j]$ on $\mathbb{V}(n,m)$ can be 1 or p only for $j \le b$ if $n \ne 2m$, we see easily that

$$\mathcal{F}^{00}\mathbb{V}(n,m)/\mathcal{F}^{11}\mathbb{V}(n,m) \cong \begin{cases} \bigoplus_{j=1}^{b} \frac{\mathcal{F}^{00}\mathcal{V}_j}{\mathcal{F}^{11}\mathcal{V}_j} & \text{if } n \ne 2m, \\ \bigoplus_{j=1}^{e} \frac{\mathcal{F}^{00}\mathcal{V}_j}{\mathcal{F}^{11}\mathcal{V}_j} & \text{if } n = 2m \end{cases} \tag{5.1}$$

as $\mathcal{D}$-modules. Fix an index j, and write $D = \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ for $L = F_j$. We consider the universal couple $(R, \boldsymbol{\rho})$ of ρ_E under the conditions (K1–4). Put $\mathfrak{m}_j := (X_1, \ldots, X_{j-1}, X_j^2, X_{j+1}, \ldots, X_e) \subset R = K[[X_j]]_{j=1,\ldots,e}$. Consider $\widetilde{T}_j(n,m) = (Sym^{\otimes n}(\boldsymbol{\rho})(-m) \mod \mathfrak{m}_j)$. Write

$$\mathbf{V}(n,m) = (Sym^{\otimes n}\boldsymbol{\rho}) \otimes \mathcal{N}^{-m}.$$

Then taking the filtration $\mathcal{F}^i\mathbf{V}(n,m)$ stable under D such that D acts on the j-th graded piece $\mathcal{F}^i\mathbf{V}(n,m)/\mathcal{F}^{i+1}\mathbf{V}(n,m)$ by $\boldsymbol{\delta}_j^{n-2i-2m}\mathcal{N}^i$.

Pick $j \le b$. Again we consider $\widetilde{\mathcal{V}}_j(n,m) := \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \widetilde{T}_j(n,m)$. We put $\mathcal{F}^{+}\widetilde{\mathcal{V}}_j(n,m) = \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathcal{F}^{+}\widetilde{T}_j(n,m)$. We have a D-stable filtration $\mathcal{F}^i\widetilde{T}_j(n,m) \subset \widetilde{T}_j(n,m)$ such that D acts on the i-th graded piece $\mathcal{F}^i\widetilde{T}_j(n,m)/\mathcal{F}^{i+1}\widetilde{T}_j(n,m)$ by $\delta_j^{n-2i-2m}\mathcal{N}^i$ for the nearly ordinary character

$$\delta_j := (\boldsymbol{\delta}_j \mod (X_1, \ldots, X_{j-1}, X_j^2, X_{j+1}, \ldots, X_e)).$$

The character δ_j satisfies $\delta_j \equiv \alpha_j = \mathbf{1} \mod (X_j)$ for the trivial character $\mathbf{1}$ of D. Since α_j can be extended to $\mathbf{1} : \mathcal{D} \to \mathbb{Q}_p^{\times}$, by Lemma 4.8, δ_j has a unique extension $\widetilde{\delta}_j : \mathcal{D} \to \widetilde{\mathbb{Q}_p}^{\times}$ with $\widetilde{\delta}_j \equiv \mathbf{1} \mod (X_j)$ (identifying $\widetilde{\mathbb{Q}_p}$ with $\mathbb{Q}_p[X_j]/(X_j)^2$). Thus we have

$$\mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \frac{\mathcal{F}^{-}\widetilde{T}_j(n,m)}{\mathcal{F}^{+}\widetilde{T}_j(n,m)} = \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \delta_j^{n-2m} \cong \widetilde{\delta}_j^{n-2m} \otimes \mathrm{Ind}_{F_j}^{\mathbb{Q}_p} \mathbf{1},$$

and we have a unique subspace $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m) \subset \widetilde{\mathcal{V}}_j(n,m)$ such that

$$\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)/\mathcal{F}^{+}\widetilde{\mathcal{V}}_j(n,m) = H^0(\mathcal{D}, \widetilde{\mathcal{V}}_j(n,m)/\mathcal{F}^{+}\widetilde{\mathcal{V}}_j(n,m)(\widetilde{\delta}_j^{-n+2m})).$$

The $\widetilde{\mathbb{Q}_p}$-module $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)/\operatorname{Ind}_{F_j}^{\mathbb{Q}_p}\mathcal{F}^+\widetilde{T}_j(n,m)$ is free of rank 1 over $\widetilde{\mathbb{Q}_p}$. Similarly we have a unique subspace $\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m) \subset \mathcal{F}^+\widetilde{\mathcal{V}}_j(n,m)$ such that

$$H_0(\mathcal{D}, \mathcal{F}^+\widetilde{\mathcal{V}}_j(n,m)(\widetilde{\delta}_j^{2+2m-n}\mathcal{N}^{-1})) = \mathcal{F}^+\widetilde{\mathcal{V}}_j(n,m)/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m).$$

Again $\mathcal{F}^+\widetilde{\mathcal{V}}_j(n,m)/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m)$ is $\widetilde{\mathbb{Q}_p}$-free of rank 1.

Since by the fixed determinant condition (K3), we have the D-equivariant duality pairing $\widetilde{T}_j(n,m) \times \widetilde{T}_j(n,m) \to \widetilde{\mathbb{Q}_p}(n-2m)$, the duality extends to a $\mathcal{D}$-equivariant duality pairing $\widetilde{\mathcal{V}}_j(n,m) \times \widetilde{\mathcal{V}}_j(n,m) \to \widetilde{\mathbb{Q}_p}(n-2m)$, and we have $\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m) \subset \operatorname{Ind}_{F_j}^{\mathbb{Q}_p}\mathcal{F}^+\widetilde{T}_j(n,m)$ given by $(\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m))^\perp$. The matrix form of the $\mathcal{D}$-representation $\frac{\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)}{\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m)}$ is $\begin{pmatrix} \widetilde{\delta}_j^{n-2-2m}\mathcal{N} & * \\ 0 & \widetilde{\delta}_j^{n-2m} \end{pmatrix}$. Twist $\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m)$ by $\chi = \widetilde{\delta}_j^{2+2m-n}$; then, the quotient $\frac{\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)}{\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m)}(\chi)$ has the matrix form $\begin{pmatrix} \mathcal{N} & * \\ 0 & \psi_j \end{pmatrix}$ for $\psi_j = \widetilde{\delta}_j^2$. Then $\frac{\mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)}{\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m)}(\chi)$ is an infinitesimal extension of $\frac{\mathcal{F}^{00}\mathcal{V}_j(n,m)}{\mathcal{F}^{11}\mathcal{V}_j(n,m)}$ making the following diagram commutative:

$$\begin{array}{ccccc} \widetilde{\mathbb{Q}_p}(1) & \hookrightarrow & \mathcal{F}^{00}\widetilde{\mathcal{V}}_j(n,m)/\mathcal{F}^{11}\widetilde{\mathcal{V}}_j(n,m)(\chi) & \twoheadrightarrow & \widetilde{\mathbb{Q}_p}(\psi_j) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Q}_p(1) & \hookrightarrow & \mathcal{F}^{00}\mathcal{V}_j(n,m)/\mathcal{F}^{11}\mathcal{V}_j(n,m) & \twoheadrightarrow & \mathbb{Q}_p. \end{array}$$

This diagram satisfies the condition (c) of Theorem 4.7, and by Lemma 5.1

$$\frac{\partial\psi_j([Q_j,\mathbb{Q}_p])}{\partial X_j}\Big|_{X_j=0} = 2\widetilde{\delta}_j\frac{\partial\widetilde{\delta}_j([Q_j,\mathbb{Q}_p])}{\partial X_j}\Big|_{X_j=0} = 0 \Rightarrow \frac{\partial\widetilde{\delta}_j([Q_j,\mathbb{Q}_p])}{\partial X_j}\Big|_{X_j=0} = 0.$$

Write $Q_j = p^a u$ for $a = \operatorname{ord}_p(Q_j)$ and $u \in \mathbb{Z}_p^\times$. Then $\log_p(u) = \log_p(Q_j)$. Write $d_j = [F_j : \mathbb{Q}_p]$ and $N_j = N_{F_j/\mathbb{Q}_p} : F_j^\times \to \mathbb{Q}_p^\times$ for the norm map. Since $[p,\mathbb{Q}_p]^{d_j} = [N_j(p),\mathbb{Q}_p] = [p,F_j]|_{\mathbb{Q}_p^{ab}}$ and $[u,\mathbb{Q}_p]^{d_j} = [N_j(u),\mathbb{Q}_p] = [u,F_j]|_{\mathbb{Q}_p^{ab}}$, we have

$$\begin{aligned}\widetilde{\delta}_j([N(q_j),\mathbb{Q}_p]^{d_j}) &= \delta_j([p,F_j])^a\delta_j([u,F_j]) \\ &= \delta_j([p,F_j])^a(1+X_j)^{-\log_p(\mathcal{N}([u,F_j]))/\log_p(\gamma_j)} \\ &= \delta_j([p,F_j])^a(1+X_j)^{-d_j\log_p(u)/\log_p(\gamma_j)}\end{aligned}$$

(because $\mathcal{N}([u,F_j]) = u^{-d_j}$). Differentiating this identity with respect to X_j, we get from $\delta_j([u,F_j])|_{X_j=0} = \delta_j([p,F_j])|_{X_j=0} = \alpha_j([p,F_j]) = 1$

$$a\frac{\partial\boldsymbol{\delta}_j}{\partial X_j}\Big|_{X_j=0}([p,F_j]) - \frac{d_j\log_p(u)}{\log_p(\gamma_j)} = 0.$$

From this we conclude

$$(5.2)\qquad \frac{\partial\boldsymbol{\delta}_j([p,F_j])}{\partial X_j}\Big|_{X_j=0} d_j^{-1}\log_p(\gamma_j)\alpha_j([p,F_j])^{-1}=\frac{\log_p(Q_j)}{\mathrm{ord}_p(Q_j)},$$

since $\alpha_j([p,F_j])=1$ (by split multiplicative reduction of E at $\mathfrak{p}_j$ with $j\le b$).

We now assume that $n=2$ and $m=1$. Then $T(2,1)\cong Ad(\rho_E)$, and by Lemma 2.3, $\mathrm{Sel}_F(Ad(\rho_E))=0$, assuming that $R\cong K[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. By the formulas in Theorem 2.1 and Corollary 3.5 combined, we get

$$(5.3)\quad \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E))=\prod_{i=1}^{b}\frac{\partial\boldsymbol{\delta}_i([p,F_i])}{\partial X_i}\Big|_{X_i=0} d_i^{-1}\log_p(\gamma_i)\alpha_i([p,F_i])^{-1}$$
$$\times\det\left(\frac{\partial\boldsymbol{\delta}_i([p,F_i])}{\partial X_j}\right)_{i>b,j>b}\Big|_{X=0}\prod_{j>b} d_j^{-1}\log_p(\gamma_j)\alpha_j([p,F_j])^{-1}.$$

From this and (5.2), the desired formula follows.

The second assertion of the theorem follows from [Gr] Proposition 3 by Proposition 2.6 if the $\mathcal{L}$-invariant as above vanishes. If the $\mathcal{L}$-invariant does not vanish, it follows from [H00] Theorem 6.3 (4) (adding the assumption (5) here). We shall give a proof of the formula different from that of [H00] Theorem 6.3 (4) based on our more recent work [HMI] Theorem 5.27 (because in [H00] some redundant conditions are assumed). The Galois representation $\rho_0:\mathrm{Gal}(\overline{F}/F)\to GL_2(\mathbb{Z}_p)$ satisfies the assumptions (h1–3) and (sf) of [HMI] Theorem 5.27 by the semi-stability of E over O, satisfies the assumption (h4) of the theorem by (3) and satisfies the assumption ($\mathrm{ai}_{F[\mu_p]}$) by (1); so, we can apply [HMI] Theorem 5.27 to the present setting. Here we have $\rho_E=\rho_0\otimes\mathbb{Q}_p$. Let us recall some notation of [HMI]. The couple $(\mathcal{R}_n,\varrho_n)=(\mathcal{R}_{F_n},\varrho_{F_n})$ (resp. $(\mathcal{R}_\infty,\varrho_\infty)$) is the locally-cyclotomic universal couple over W deforming ρ_0 over $\mathrm{Gal}(\overline{F}/F_n)$ (resp. $\mathrm{Gal}(\overline{F}/F_\infty)$ studied in [HMI] 3.2.8 (resp. [HMI] Proposition 5.1). Then by [HMI] Theorem 3.50, if n is finite, $\mathcal{R}_n$ is free of finite rank over $W[[X_{\mathfrak{p}}]]_{\mathfrak{p}|p}$. The deformation ring R (over K) is isomorphic to the P-adic localization-completion of $\mathcal{R}_0$ ([HMI] Theorem 3.65) and $PR=(X_{\mathfrak{p}})_{\mathfrak{p}|p}$.

We identify $W[[\Gamma]]=W[[T]]$ via $\gamma\mapsto 1+T$ by choosing a generator γ of $\Gamma:=\mathrm{Gal}(F_\infty/F)$. Thus under the notation in Theorem 5.27 of [HMI], taking $W=\mathbb{Z}_p$, we have $A=\mathcal{R}_0/P\cong W$ and

$$\Phi^{arith}(T):=\mathrm{char}_{W[[T]]}(M^A_{\infty/W})=\mathrm{char}_{W[[T]]}(L^A_\infty)\,\mathrm{char}_{W[[T]]}(M^A_{0,W}),$$

where $L_n^A = (\mathrm{Ker}(\pi_n)/\mathrm{Ker}(\pi_n)^2) \otimes_{\mathcal{R}_n} A$ and $M_{n/B}^A = \Omega_{\mathcal{R}_n/B} \otimes_{\mathcal{R}_n} \mathcal{R}_n/P$ for a closed subalgebra B of $\mathcal{R}_n$. Recall that the module $M_{n/W}^A$ is isomorphic to the Pontryagin dual of $\mathrm{Sel}_{F_n}^{cyc}(Ad(\rho_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ (cf. Lemma 2.3 and [HMI] Proposition 3.87). Thus, by Lemma 2.3, the $W[[T]]$-module $M_{0/W}^A$ is pseudo-isomorphic to $\mathbb{Z}_p^e$ whose characteristic power series is T^e. When $n = \infty$, we have $\mathrm{Sel}_{F_\infty}^{cyc}(Ad(\rho_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Sel}_{F_\infty}(Ad(\rho_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$; so, $L_p^{arith}(s, Ad(\rho_E)) = \Phi^{arith}(\gamma^{1-s} - 1)$. Thus, writing $\Psi(T) = \mathrm{char}_{W[[T]]}(L_{\infty/W}^A)$, we have $\Phi^{arith}(T) = \Psi(T)T^e$. Again by [HMI] Theorem 5.27, we have $\Psi(0) \neq 0$.

As shown in [HMI] Definition 5.22, we have a canonical $W[[T]]$-linear surjective map $M_{n/\mathbf{\Lambda}_n}^A \twoheadrightarrow L_n^A$ with kernel $X_{n/\mathbf{\Lambda}_n}^A$. Indeed, we have $L_n^A = T(M_{n/\mathbf{\Lambda}_n}^A)$ by Definition 5.22 in [HMI], and the map $M_{n/\mathbf{\Lambda}_n}^A \twoheadrightarrow L_n^A$ is given by $x \mapsto Tx$. Since $T = \gamma - 1$, we have

$$X_{n/\mathbf{\Lambda}_n}^A = H^0(\Gamma, M_{n/\mathbf{\Lambda}_n}^A) \ \text{ for } n = 1, 2, \ldots, \infty. \tag{5.4}$$

By the universality of $\mathcal{R}_m$, we have the canonical morphism $\pi_{m,n} : \mathcal{R}_m \to \mathcal{R}_n$ $(m > n)$ inducing $\varrho_n|_{\mathrm{Gal}(\overline{F}/F_m)} \cong \pi_{m,n} \circ \varrho_m$. This morphism in turn induces $\pi_{m,n,*} : X_{m/\mathbf{\Lambda}_m}^A \to X_{n/\mathbf{\Lambda}_n}^A$. By [HMI] (5.2.7) (valid for all finite n), if $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \neq 0$, $|X_{n/\mathbf{\Lambda}_n}^A|$ is a non-zero constant independent of finite n; so, $X_{\infty/\mathbf{\Lambda}_\infty}^A = \varprojlim_n X_{n/\mathbf{\Lambda}_n}^A$ is a pseudo-null $W[[T]]$-module; so, $\Psi(T) = \mathrm{char}_{W[[T]]}(M_{\infty/\mathbf{\Lambda}_\infty}^A)$. This in particular shows that $H^0(\Gamma, M_{\infty/\mathbf{\Lambda}_\infty}^A)$ is pseudo-null as we claimed already, because $\Psi(0) \neq 0$ by [HMI] Theorem 5.27. By [HMI] Lemma 5.24 (3), under $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_E)) \neq 0$ and the assumption (5), $M_{\infty/\mathbf{\Lambda}_\infty}^A$ has no nontrivial pseudo-null $W[[T]]$-module non-null. Moreover, by [HMI] Proposition 5.6, we have $M_{\infty/\mathbf{\Lambda}_\infty}^A / TM_{\infty/\mathbf{\Lambda}_\infty}^A \cong M_{0/\mathbf{\Lambda}_0}^A$. Thus, up to units,

$$\Psi(0) = |M_{0/\mathbf{\Lambda}_0}^A| = \eta \det\Big(\frac{\partial \boldsymbol{\delta}_i([p, F_i])}{\partial X_j}\Big)\Big|_{X=0} \tag{5.5}$$

for $\eta = |\mathrm{Sel}_F(Ad(\rho_0 \otimes \mathbb{Q}_p/\mathbb{Z}_p))| = |\Omega_{\mathcal{R}_0/W[[X_\mathfrak{p}]]_{\mathfrak{p}|p}} \otimes_{W[[X_\mathfrak{p}]]} A|$ (cf. [HMI] Proposition 3.87 and (5.2.6)). Indeed, if $W[[x_\mathfrak{p}]]_{\mathfrak{p}|p}$ and $W[[t_\mathfrak{p}]]_{\mathfrak{p}|p}$ are two subrings of $\mathcal{R}_0$ isomorphic to a power series ring of e variables, we have

$$|\Omega_{\mathcal{R}_0/W[[X_\mathfrak{p}]]_{\mathfrak{p}|p}} \otimes_{\mathcal{R}_0} A| \cdot \det\Big(\frac{\partial t_\mathfrak{p}}{\partial X_{\mathfrak{p}'}}\Big)\Big|_{X=0} = |\Omega_{\mathcal{R}_0/W[[t_\mathfrak{p}]]_{\mathfrak{p}|p}} \otimes_{\mathcal{R}_0} A|,$$

and applying this to $t_\mathfrak{p} = \boldsymbol{\alpha}_\mathfrak{p}\alpha_\mathfrak{p}^{-1}([p, F_\mathfrak{p}]) - 1$ and $X_\mathfrak{p} = \boldsymbol{\alpha}_\mathfrak{p}\alpha_\mathfrak{p}^{-1}(\gamma_\mathfrak{p}) - 1$ for our chosen generator $\gamma_\mathfrak{p}$ of $\Gamma_\mathfrak{p}$, we get $|M_{0/\mathbf{\Lambda}_0}^A| = \eta \det(\frac{\partial \boldsymbol{\delta}_i([p,F_i])}{\partial X_j})\Big|_{X=0}$. The formula

(5.5) is equivalent to the desired formula by Theorem 2.1, because by (1.1),

$$\det(\frac{\partial \boldsymbol{\delta}_i([p,F_i])}{\partial X_j})\Big|_{X=0} = \mathcal{L}(Ad(\rho_F))(\prod_{j=1}^{e} \log_p(\gamma_j)^{-1}\frac{d_j}{f_j})$$

for $f_j = f_{\mathfrak{p}_j}$ and the ramification index $\frac{d_j}{f_j}$ of $\mathfrak{p}_j/p$ is equal to 1 under the unramifiedness condition.

In the above proof of the evaluation formula of $\Psi(0)$, the assumption (5) is used only where we relate the value $\Psi(0)$ with $|M^A_{0/\boldsymbol{\Lambda}_0}|$. Without assuming (5), we can replace $M^A_{\infty/\boldsymbol{\Lambda}_\infty}$ by its maximal quotient N modulo the maximal pseudo-null submodule of $M^A_{\infty/\boldsymbol{\Lambda}_\infty}$. Then $\Psi(0) = |N/XN|$ up to units, and $M^A_{0/\boldsymbol{\Lambda}_0} = M^A_{\infty/\boldsymbol{\Lambda}_\infty}/TM^A_{\infty/\boldsymbol{\Lambda}_\infty}$ surjects down to N/TN; so, $\Psi(0)$ is a factor of

$$|M^A_{0/\boldsymbol{\Lambda}_0}| = \eta \det(\frac{\partial \boldsymbol{\delta}_i([p,F_i])}{\partial X_j})\Big|_{X=0},$$

proving the last assertion. □

By the above proof, heuristics for the conjecture in the cases $n \neq 2m$ and $n = 2m$ with $m = 1$ are clear.

5.2. **The global invariant $\mathcal{L}(m)$.** At the end of this section after some preparation, we will describe our heuristics for the factorization of the $\mathcal{L}$-invariant (in the formula of Conjecture 1.3) into the product of the local terms involving the Tate periods q_i and the global factor $\mathcal{L}(m)$. Thus we study $Ad(\rho_{m,0})$ and $T(2m,m)$ for odd m in this section.

Consider $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We then define $J_n = Sym^{\otimes n}(J_1)$. Since ${}^t\alpha J_1 \alpha = \det(\alpha) J_1$ for $\alpha \in GL(2)$, we have ${}^t\rho_{n,0}(\sigma) J_n \rho_{n,0}(\sigma) = \mathcal{N}^n(\sigma) J_n$, where $\rho_{n,0} = Sym^{\otimes n}(\rho_E)$. Define an algebraic group G_n over $\mathbb{Z}_p$ by

$$G_n(A) = \left\{\alpha \in GL_{n+1}(A) \middle| {}^t\alpha J_n \alpha = \nu(\alpha) J_n\right\}$$

with the similitude homomorphism $\nu : G_n \to \mathbb{G}_m$. Then G_n is a quasi-split orthogonal or symplectic group according as n is even or odd. The representation $\rho_{n,0}$ of $\mathrm{Gal}(\overline{F}/F)$ has values in $G_n(\mathbb{Z}_p)$. Let S_n be the derived group of G_n, and consider the Lie algebra $\mathfrak{s}_n$ of S_n. Then $\sigma \in \mathrm{Gal}(\overline{F}/F)$ acts on $\mathfrak{s}_n$ by $X \mapsto \rho_{n,0}(\sigma) X \rho_{n,0}(\sigma)^{-1}$. Write this Galois module as $Ad(\rho_{n,0})$. Then we have (cf.

[H00] Examples 2.8 and 6.2)

$$Ad(\rho_{n,0}) \cong \bigoplus_{j:\text{odd},\ 1\le j\le n} T(2j,j). \tag{5.6}$$

We study as before the universal Galois deformation ring of deformations of $\rho_{n,0}$ with values in G_n (note here that $G_1 = GL(2)$). For simplicity, we assume that

(st) *E is semi-stable over O.*

Write $\rho_E|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \sim \left(\begin{smallmatrix}\beta_{\mathfrak{p}} & * \\ 0 & \alpha_{\mathfrak{p}}\end{smallmatrix}\right)$ with unramified $\alpha_{\mathfrak{p}}$. Start with $\rho_{n,0}$ and consider the deformation ring $(R_n, \boldsymbol{\rho}_n)$ which is universal among the following deformations: Galois representations $\rho_A : \mathrm{Gal}(\overline{F}/F) \to G_n(A)$ for Artinian local K-algebras A, such that

(K$_n$1) unramified outside S, ∞ and p;

(K$_n$2) $\rho_A|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \cong \begin{pmatrix} \alpha_{0,A,\mathfrak{p}} & * & \cdots & * \\ 0 & \alpha_{1,A,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,A,\mathfrak{p}} \end{pmatrix}$ $(\alpha_{i,A,\mathfrak{p}} \equiv \beta_{\mathfrak{p}}^{n-i}\alpha_{\mathfrak{p}}^{i} \mod \mathfrak{m}_A)$ with $\alpha_{i,A,\mathfrak{p}}|_{I_{\mathfrak{p}}}$ $(i = 0,1,\dots,n)$ factoring through $\mathrm{Gal}(F_{\mathfrak{p}}^{ur}[\mu_{p^\infty}]/F_{\mathfrak{p}}^{ur})$ for the maximal unramified extension $F_{\mathfrak{p}}^{ur}/F_{\mathfrak{p}}$ for all prime factors $\mathfrak{p}$ of p;

(K$_n$3) $\nu \circ \rho_A = \mathcal{N}^n$ for the p-adic cyclotomic character $\mathcal{N}$;

(K$_n$4) $\rho_A \equiv \rho_{n,0} \mod \mathfrak{m}_A$.

Since $\rho_{n,0}$ is absolutely irreducible as long as E does not have complex multiplication (because $\mathrm{Im}(\rho_E)$ is open in $GL_2(\mathbb{Z}_p)$ by a result of Serre) and all $\alpha_{\mathfrak{p}}^{i}\beta_{\mathfrak{p}}^{n-i}$ for $i = 0,1,\dots,n$ are distinct, the deformation problem specified by (K$_n$1–4) is representable by a universal couple $(R_n, \boldsymbol{\rho}_n)$. Write now

$$\boldsymbol{\rho}_n|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \cong \begin{pmatrix} \boldsymbol{\delta}_{0,\mathfrak{p}} & * & \cdots & * \\ 0 & \boldsymbol{\delta}_{1,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\delta}_{n,\mathfrak{p}} \end{pmatrix}$$

with $\boldsymbol{\delta}_{i,\mathfrak{p}} \equiv \beta_{\mathfrak{p}}^{n-i}\alpha_{\mathfrak{p}}^{i} \mod \mathfrak{m}_A$. Note that $\beta_{\mathfrak{p}}^{n-i}\alpha_{\mathfrak{p}}^{i} = \mathcal{N}^{n-i}\alpha_{\mathfrak{p}}^{2i-n}$, because $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = \mathcal{N}$. Write $n = 2m-1$ if n id odd and $n = 2m$ if n is even. Since we have a relation $\boldsymbol{\delta}_{i,\mathfrak{p}}\boldsymbol{\delta}_{n-i,\mathfrak{p}} = \mathcal{N}^n$, $\boldsymbol{\delta}_{i,\mathfrak{p}} : \Gamma_{\mathfrak{p}} \to R_n^{\times}$ for $i = 0,1,\cdots,m-1$ could induce an independent algebra structure over $W[[\Gamma_{\mathfrak{p}}]] \cong W[[X_{\mathfrak{p}}]]$. Thus the number of variables coming from the inertia character $\boldsymbol{\delta}_{i,\mathfrak{p}}$ of R_n is at most the number of odd integers j in the interval $[0,n]$.

Conjecture 5.4. *We have*

$$R_n \cong K[[X_{j,\mathfrak{p}}]]_{\mathfrak{p}|p,\ j:odd,\ 1\le j\le n}$$

for variables $X_{j,\mathfrak{p}}$*; in particular,* $\dim R_n = e\cdot \operatorname{rank} S_n = e\left[\frac{n+1}{2}\right]$.

As we explain later, we have a good reason for indexing the variable $X_{j,\mathfrak{p}}$ by odd integers $j \in [0, n]$ if n is odd. Since $G_1 = GL(2)$ is the spin cover of $G_2 = GO(1, 2)$, Conjecture 5.4 is known for $1 \le n \le 2$ in almost all cases by the result of Kisin and Fujiwara already quoted. An exposition of this type of results is given in [HMI] as Theorems 3.50, 3.65 and Proposition 3.78. In [HMI], to make the book self-contained, some redundant assumptions are made (for example, the condition (sf) in page 183), but the general case follows from [F] and [F1], or we can reduce the general case to the cases treated in [HMI] by base-change to a soluble totally real extension of F. Since $G_3 \cong GSp(4)$ is the spin cover of $G_4 = GO(2, 3)$ and some progress has been made in [GeT] and [Ti] towards the identification of Galois deformation rings and $GSp(4)$-Hecke algebras (for $F = \mathbb{Q}$), there is a good prospect to get a proof of Conjecture 5.4 when $n = 3$ and 4. More generally, letting n denote an odd integer, the *Gpin*-cover $\widetilde{G}_{n+1}$ of G_{n+1} and the symplectic group $G_n = GSp(n+1)$ are Langlands dual each other. The symplectic group G_n has associated Siegel–Shimura varieties; so, if $\rho_{n+1,0}$ is modular with respect to G_n (and each discrete series automorphic representation of $G_n(F_{\mathbb{A}})$ has the associated Galois representation into G_{n+1}), we know the dimension of the universal locally cyclotomic Hecke algebra to be equal to $e\cdot(\frac{n+1}{2})$ as expected. Taylor et al ([CHT] Theorem B and [Ta] Theorem A) have proved potential automorphy of $\rho_{n,0}$ when $F = \mathbb{Q}$ with respect to $\widetilde{G}_{n+1}$.

Let us write $\mathfrak{m}_n$ for the maximal ideal of R_n. Then in the same manner as in the proof of Lemma 2.3, we get

Lemma 5.5. *Suppose* Conjecture 5.4. *Then we have*

$$\begin{aligned}\mathrm{Sel}_F^{cyc}(Ad(\rho_{n,0})) &\cong \mathrm{Hom}_K(\mathfrak{m}_n/\mathfrak{m}_n^2, K) = Der_K(R,K)\\ &= \bigoplus_{j:odd,1\le j\le n}\ \bigoplus_{\mathfrak{p}|p} K\cdot\frac{\partial}{\partial X_{j,\mathfrak{p}}} \cong \bigoplus_{j:odd,1\le j\le n} \mathrm{Sel}_F^{cyc}(T(2j,j))\end{aligned}$$

and $\mathrm{Sel}_F(T(2j,j)) = 0$ *for odd* j *with* $1\le j\le n$.

By Lemma 2.5, we also have

Lemma 5.6. *For an odd integer $m \geq 1$, we have for primes $\mathfrak{l}$ of F*

$$\mathrm{Sel}_F(T(2m,m)) = 0 \Rightarrow H^1(\mathfrak{G}, T(2m,m)) \cong \prod_{\mathfrak{l}} \frac{H^1(F_{\mathfrak{l}}, T(2m,m))}{\overline{L}_{\mathfrak{l}}(T(2m,m))}.$$

The two lemmas combined verifies the essential prerequisite ([Gr] Conjecture 1) for Greenberg's $\mathcal{L}$-invariant to be well-defined for $T(2m,m)$ (under Conjecture 5.4), and the rest of the hypothesis in [Gr] (Hypothesis S, T and U in [Gr] obviously holds in our setting of elliptic curves).

Conjecture 1.1 implies $\dim_{\mathbb{Q}_p} \mathrm{Sel}_F^{cyc}(T(2,1)) = e$. Then the two lemmas implies (by induction on n) that $\dim_{\mathbb{Q}_p} \mathrm{Sel}_F^{cyc}(T(2j,j)) = e$ for all odd j and that we can normalize the variable $X_{j,\mathfrak{p}}$ so that the cocycles $\left\{\frac{\partial \boldsymbol{\rho}_n}{\partial X_{j,\mathfrak{p}}} \rho_{n,0}^{-1}\right\}_{\mathfrak{p}|p}$ span $\mathrm{Sel}_F^{cyc}(T(2j,j))$.

If n is odd (given by $n = 2m-1$), we can make another natural choice of the variables $\{X'_{j,\mathfrak{p}}\}_{j,\mathfrak{p}}$ of R_n. Recall the upper triangular shape of

$$\boldsymbol{\rho}_n|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})} \cong \begin{pmatrix} \boldsymbol{\delta}_{0,\mathfrak{p}} & * & \cdots & * \\ 0 & \boldsymbol{\delta}_{1,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\delta}_{n,\mathfrak{p}} \end{pmatrix}$$

with $\boldsymbol{\delta}_{i,\mathfrak{p}} \equiv \beta_{\mathfrak{p}}^{n-i}\alpha_{\mathfrak{p}}^{i} \mod \mathfrak{m}_A$. Note that $\beta_{\mathfrak{p}}^{n-i}\alpha_{\mathfrak{p}}^{i} = \mathcal{N}^{n-i}\alpha_{\mathfrak{p}}^{2i-n}$, because $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = \mathcal{N}$. Thus the characters $\beta_{\mathfrak{p}}^{n-i}\alpha_{\mathfrak{p}}^{i}$ are all distinct, and hence $\boldsymbol{\delta}_{i,\mathfrak{p}}$ $(i = 1, 2, \ldots, n)$ are all distinct. We also have the relation $\boldsymbol{\delta}_{i,\mathfrak{p}}\boldsymbol{\delta}_{n-i,\mathfrak{p}} = \mathcal{N}^n$ for $i = 0, 1, \ldots, n$. The set of variables $\{X'_{j,\mathfrak{p}}\}_{j:\mathrm{odd},\mathfrak{p}}$ is induced by $\boldsymbol{\delta}_{j,\mathfrak{p}}$. In other words, $\boldsymbol{\delta}_{j,\mathfrak{p}}(\beta_{\mathfrak{p}}^{n-j}\alpha_{\mathfrak{p}}^{j})^{-1}(\gamma_{\mathfrak{p}}) = 1 + X'_{j,\mathfrak{p}}$.

Here is the heuristic supporting Conjecture 1.3. Let $m > 1$ be an odd integer, and write $\widetilde{G}_{m+1}$ for the *Gpin* cover of the orthogonal group G_{m+1}. Note that $\widetilde{G}_{m+1}(\mathbb{C})$ is the L-group of G_m. The Greenberg's cocycles in $\mathbf{H}_F = \mathrm{Sel}_F^{cyc}(T(2m,m))$ for $T(2m,m)$ (with odd m) can be described in terms of $\left.\frac{\partial \boldsymbol{\delta}_{m,\mathfrak{p}}}{\partial X_{m,\mathfrak{p}}}\right|_{X=0}$, and using the derivatives $\left.\frac{\partial \boldsymbol{\delta}_{i,\mathfrak{p}}}{\partial X'_{j,\mathfrak{p}}}\right|_{X'=0}$, a formula similar to the one in Theorem 2.1 holds for $Ad(\rho_{m,0})$ (by the same reason behind the proof of Theorem 3.73 in [HMI]). The exact formula is discussed in [H09] Section 1.3. Once an analogue of Theorem 2.1 is established for $Ad(\rho_{m,0})$, a $\widetilde{G}_{m+1}$-analogue of the result proven for the Hecke algebras on $GL(2)$ in Section 3 should be also true for $\widetilde{G}_{m+1}$, because for primes $\mathfrak{p}_j$ with $j \leq b$ (at which E has multiplicative reduction), the automorphic representation of the Langlands dual $\widetilde{G}_{m+1}$ of G_m (conjecturally) associated to $\rho_{m,0}$ is Steinberg at $\mathfrak{p}_j$. Then, combined with the argument given in the proof of Theorem 4.7, we should be able to factorize $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_{m,0}))$ into the product of a global determinant factor similar to $\mathcal{L}(1)$ involving $\frac{\partial \boldsymbol{\delta}_{i,\mathfrak{p}}}{\partial X'_{j,\mathfrak{p}_k}}\Big|_{X'=0}$ for $k > b$ and the local term coming from $\mathfrak{p}_g$ with $g \leq b$ which is a product of $\{\frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)}\}_{i \leq b}$ with multiplicity. Since $Ad(\rho_{m,0}) = \bigoplus_{j:\mathrm{odd},0<j\leq m} T(2j,j)$, we can factor $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_{m,0}))$ into the product of $\mathcal{L}(T(2j,j))$. By the argument in the proof of Theorem 4.7, the local term (for $\mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}} Ad(\rho_{m,0}))$ and $\mathcal{L}(T(2m,m)))$ is the product of $\frac{\log_p(Q_i)}{\mathrm{ord}_p(Q_i)}$ as in Conjecture 1.3. If the choice of the variables $\{X_{j,\mathfrak{p}}\}$ adjusted to the decomposition in Lemma 5.5 coincides modulo $\mathfrak{m}^2$ with the variables $\{X'_{j,\mathfrak{p}}\}$ induced by $\boldsymbol{\delta}_{i,\mathfrak{p}}$, we might be able to prove that $\mathcal{L}(m) = \mathcal{L}(1)$. We hope to discuss more details of these points in our future papers (see [H09] and [H07b]).

References

[C] Lin Chen, Framed deformation of Galois representation, preprint, 2008, arXiv:0804.0226

[CHT] L. Clozel, M. Harris and R. Taylor, Automorphy for some l-adic lifts of automorphic mod l Galois representations, preprint, 2005 (preprint available at `www.math.harvard.edu/~rtaylor/`)

[F] K. Fujiwara, Deformation rings and Hecke algebras in totally real case, preprint, 1999 (arXiv.math.NT/0602606)

[F1] K. Fujiwara, Galois deformations and arithmetic geometry of Shimura varieties, ICM proceedings, II, 347–371, Eur. Math. Soc., 2006

[GeT] A. Genestier and J. Tilouine, Systémes de Taylor-Wiles pour GSp_4, Astérisque **302** (2005), 177–290

[Gr] R. Greenberg, Trivial zeros of p–adic L–functions, Contemporary Math. **165** (1994), 149–174

[GS] R. Greenberg and G. Stevens, p-adic L–functions and p–adic periods of modular forms, Inventiones Math. **111** (1993), 407–447

[GS1] R. Greenberg and G. Stevens, On the conjecture of Mazur, Tate, and Teitelbaum, Contemporary Math. **165** (1994), 183–211

[H89] H. Hida, On nearly ordinary Hecke algebras for $GL(2)$ over totally real fields, Adv. Studies in Pure Math. **17** (1989), 139–169.

[H00] H. Hida, Adjoint Selmer groups as Iwasawa modules, Israel Journal of Math. **120** (2000), 361–427

[H02] H. Hida, Control theorems for coherent sheaves on Shimura varieties of PEL–type, Journal of the Inst. of Math. Jussieu **1** (2002), 1–76

[H04] H. Hida, Greenberg's $\mathcal{L}$-invariants of adjoint square Galois representations, IMRN. **59** (2004), 3177–3189 (a preprint version available at `www.math.ucla.edu/~hida`)

[H07a] H. Hida, $\mathcal{L}$-invariant of p-adic L-functions, in "The conference on L-functions" pp.17–53, 2007, World Scientific (a preprint version available at `www.math.ucla.edu/~hida`)

[H07b] H. Hida, On a generalization of the conjecture of Mazur–Tate–Teitelbaum, International Mathematics Research Notices, vol. **2007**, Article ID rnm102, 49 pages. `doi:10.1093/imrn/rnm102` (a preprint version available at `www.math.ucla.edu/~hida`)

[H08] H. Hida, $\mathcal{L}$-invariant of tensor power of Hilbert modular forms, Hokkaido University Technical Report Series in Mathematics, **137** (2008), 150–172 (a preprint version available at `www.math.ucla.edu/~hida` and the entire volume at `http://eprints3.math.sci.hokudai.ac.jp/1922/`)

[H09] H. Hida, $\mathcal{L}$-invariant of symmetric powers of Tate curves, Publications of RIMS, **45** No.1 (2009),
(a preprint version available at `www.math.ucla.edu/~hida`)

[HMI] H. Hida, *Hilbert modular forms and Iwasawa theory*, Oxford University Press, 2006 (an errata list available at `www.math.ucla.edu/~hida`)

[K] M. Kisin, Overconvergent modular forms and the Fontaine-Mazur conjecture, Inventiones Math. **153** (2003), 373–454

[K1] M. Kisin, Geometric deformations of modular Galois representations, Inventiones Math. **157** (2004), 275–328

[K2] M. Kisin, Moduli of finite flat group schemes, and modularity, to appear in Ann. of Math.
(preprint available at `www.math.uchicago.edu/~kisin/preprints.html`)

[MFG] H. Hida, *Modular Forms and Galois Cohomology*, Cambridge Studies in Advanced Mathematics **69**, 2000, Cambridge University Press

[M] C.-P. Mok, Exceptional zero conjectures for Hilbert modular forms, preprint, 2007, to appear in Compositio Mathematica (preprint available at `http://math.berkeley.edu/~mok/`)

[MTT] B. Mazur, J. Tate and J. Teitelbaum, On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer, Inventiones Math. **84** (1986), 1–48.

[PAF] H. Hida, *p-Adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics, 2004, Springer

[ST] J.-P. Serre and J. Tate, Good reduction of abelian varieties, Ann. of Math. **88** (1968), 452–517 (Serre's Œuvres II, 472–497, No. 79).

[T] J. Tate, A review of nonarchimedean elliptic functions, in "Elliptic Curves, Modular Forms, & Fermat's Last Theorem" edited by J. Coates and S. T. Yau, International Press, 1995, 162–184

[Ta] R. Taylor, Automorphy for some l-adic lifts of automorphic mod l Galois representations. II, preprint, 2006
(preprint available at `www.math.harvard.edu/~rtaylor/`)

[TaW] R. Taylor and A. Wiles, Ring theoretic properties of certain Hecke algebras, Ann. of Math. **141** (1995), 553–572.

[Ti] J. Tilouine, Nearly ordinary rank four Galois representations and p-adic Siegel modular forms, 2006, Compositio Math. **142** (2006), 1122–1156

[V] C. Virdol, Non-solvable base change for Hilbert modular representations and zeta functions of twisted quaternionic Shimura varieties, preprint 2008 (available at `http://www.math.columbia.edu/~virdol/research.html`)

[W] A. Wiles, Modular elliptic curves and Fermat's last theorem, Ann. of Math. **141** (1995), 443–551

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