Real Embeddings, η-invariant and Chern-Simons Current[*]

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Abstract: We present an alternate proof of the Bismut-Zhang localization formula for η-invariants without using the analytic techniques developed by Bismut-Lebeau. A Riemann-Roch property for Chern-Simons currents, which is of independent interest, is established in due course.

Keywords: Direct image, Chern-Simon current, η-invariant, Riemann-Roch.

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1. Introduction

The η invariant of Atiyah-Patodi-Singer was introduced in [3] as the correction term on the boundary of the index theorem for Dirac operators on manifolds with boundary. Since then it has appeared in many parts of geometry, topology as well as physics. We first recall the definition of this important invariant.

Let M be an odd dimensional oriented closed spin manifold carrying a Riemannian metric g^{TM}. Let $S(TM)$ be the associated Hermitian bundle of spinors. Let μ be a Hermitian vector bundle over M carrying a unitary connection. Let

$$D^\mu : \Gamma(S(TM) \otimes \mu) \longrightarrow \Gamma(S(TM) \otimes \mu) \tag{1.1}$$

denote the corresponding (twisted) Dirac operator, which is formally self-adjoint (cf. [5]).

For any $s \in \mathbf{C}$ with $\mathrm{Re}(s) >> 0$, following [3], set

$$\eta\left(D^{\mu}, s\right) = \sum_{\lambda \in \mathrm{Spec}(D^{\mu}) \backslash \{0\}} \frac{\mathrm{Sgn}(\lambda)}{|\lambda|^{s}}. \tag{1.2}$$

By [3], one knows that $\eta(D^{\mu}, s)$ is a holomorphic function in s when $\mathrm{Re}(s) > \frac{\dim M}{2}$. Moreover, it extends to a meromorphic function over $\mathbf{C}$, which is holomorphic at $s = 0$. The η invariant of D^{μ}, in the sense of Atiyah-Patodi-Singer [3], is defined by

$$\eta\left(D^{\mu}\right) = \eta\left(D^{\mu}, 0\right), \tag{1.3}$$

while the corresponding *reduced* η invariant is defined and denoted by

$$\overline{\eta}\left(D^{\mu}\right) = \frac{\dim\left(\ker D^{\mu}\right) + \eta\left(D^{\mu}\right)}{2}. \tag{1.4}$$

Let $i : Y \hookrightarrow X$ be an embedding between two odd dimensional compact oriented spin Riemannian manifolds. For any Hermitian vector bundle μ over Y carrying a Hermitian connection, Bismut and Zhang [11, Theorem 2.2] established a mod $\mathbf{Z}$ formula, expressing $\overline{\eta}(D^{\mu})$ through the η-invariants associated to certain direct image $i_{!}\mu$ in the sense of Atiyah-Hirzebruch [2], up to some geometric Chern-Simons current. This formula, in some sense, might be thought of as a Riemann-Roch type formula for η-invariants under embeddings.

The proof in [11] relies heavily on the analytic techniques developed in the difficult paper of Bismut and Lebeau [10]. On the other hand, in a special case where X is certain higher dimensional sphere, a more geometric proof of the above Bismut-Zhang formula was given in [16] by making use of the mod k index theorem of Freed-Melrose [12]. As a consequence, one gets a purely geometric formula for the mod $\mathbf{Z}$ part of $\overline{\eta}(D^{\mu})$ (cf. [16, Theorem 2.2]).

It is natural to ask whether the original Bismut-Zhang localization formula for arbitrary X can also be proved without using the techniques developed in [10]. The purpose of this paper is to show that this is indeed the case. More precisely, as indicated in [16, Remark 3.2], we will embed X into a sufficiently high dimensional odd sphere S^{2N-1} and apply the proved case to $Y \hookrightarrow S^{2N-1}$ and $X \hookrightarrow S^{2N-1}$ respectively, to get the final formula. Meanwhile, we also

establish a Riemann-Roch type formula for the involved Chern-Simons currents (cf. Theorem 3.4), which has its own interest.

The rest of this paper is organized as follows. In Section 2, we recall the geometric construction of the direct image $i_!\mu$ and the Bismut-Zhang localization formula. In Section 3, we present our alternate proof of the Bismut-Zhang formula.

2. Direct image and the Bismut-Zhang localization formula for η-invariants

This section is organized as follows. In Section 2.1, we recall some basic notions of super vector bundles. In Section 2.2, we recall the geometric construction of the direct image of a vector bundle under embeddings, as well as the associated Chern-Simons current. In Section 2.3, we recall the statement of the Bismut-Zhang localization formula for η-invariants.

2.1. **Basic notions of super vector bundles.** Let $\xi = \xi_+ \oplus \xi_-$ be a $\mathbf{Z}_2$-graded Hermitian vector bundle in the sense of Quillen [14] over a manifold.

Let $v : \xi_+ \to \xi_-$ be an endomorphism between the vector bundles ξ_+ and ξ_-. Let $v^* : \xi_- \to \xi_+$ be the adjoint of v with respect to the Hermitian metrics on $\xi_\pm$. Then $V = v + v^* : \xi \to \xi$ is an odd self-adjoint endomorphism of the Hermitian super vector bundle ξ. We will denote this set of data by (ξ_+, ξ_-, V).

Let $\mathrm{Supp}(V)$ denote the subset where V is not invertible.

For two super vector bundles with odd endomorphism $((\xi_1)_+, (\xi_1)_-, V_1)$ and $((\xi_2)_+, (\xi_2)_-, V_2)$, we can form their direct sum

$$((\xi_1)_+, (\xi_1)_-, V_1) \oplus ((\xi_2)_+, (\xi_2)_-, V_2) = ((\xi_1)_+ \oplus (\xi_2)_+, (\xi_1)_- \oplus (\xi_2)_-, V_1 \oplus V_2) \tag{2.1}$$

and also the *super* tensor product

$$\begin{aligned}&\left((\xi_1)_+, (\xi_1)_-, V_1\right) \widehat{\otimes} \left((\xi_2)_+, (\xi_2)_-, V_2\right) \\ =&\left((\xi_1)_+ \otimes (\xi_2)_+ \oplus (\xi_1)_- \otimes (\xi_2)_-, (\xi_1)_+ \otimes (\xi_2)_- \oplus (\xi_1)_- \otimes (\xi_2)_+, V_1 \otimes \mathrm{Id}_{\xi_2} + \mathrm{Id}_{\xi_1} \otimes V_2\right),\end{aligned} \tag{2.2}$$

with obvious induced Hermitian metrics.

The following formulas are clear from the definitions,

$$\text{Supp}\,(V_1 \oplus V_2) = \text{Supp}\,(V_1) \cup \text{Supp}\,(V_2)\,, \qquad (2.3)$$

while

$$\text{Supp}\,(V_1 \otimes \text{Id}_{\xi_2} + \text{Id}_{\xi_1} \otimes V_2) = \text{Supp}\,(V_1) \cap \text{Supp}\,(V_2)\,. \qquad (2.4)$$

2.2. Geometric construction of direct images and the associated Chern-Simons current.

For completeness of this paper, we recall the geometric constructions of direct images and the associated Chern-Simons currents from [11] and [16].

Let $i : Y \hookrightarrow X$ be an embedding between two closed oriented spin manifolds. We make the assumption that $\dim X - \dim Y$ is even and that if N denotes the normal bundle to Y in X, then N is orientable, spin and carries an induced orientation as well as a (fixed) spin structure.

Let g^N be a Euclidean metric on N and ∇^N a Euclidean connection on N preserving g^N. Let $S(N)$ be the vector bundle of spinors associated to (N, g^N). Then $S(N) = S_+(N) \oplus S_-(N)$ (resp. its dual $S^*(N) = S_+^*(N) \oplus S_-^*(N)$) is a $\mathbf{Z}_2$-graded complex vector bundle over Y carrying an induced Hermitian metric $g^{S(N)} = g^{S_+(N)} \oplus g^{S_-(N)}$ (resp. $g^{S^*(N)} = g^{S_+^*(N)} \oplus g^{S_-^*(N)}$) from g^N, as well as a Hermitian connection $\nabla^{S(N)} = \nabla^{S_+(N)} \oplus \nabla^{S_-(N)}$ (resp. $\nabla^{S^*(N)} = \nabla^{S_+^*(N)} \oplus \nabla^{S_-^*(N)}$) induced from ∇^N.

For any $r > 0$, set $N_r = \{Z \in N : |Z| < r\}$. Then there is $\varepsilon_0 > 0$ such that $N_{2\varepsilon_0}$ is diffeomorphic to an open neighborhood of Y in X. Without confusion we now view directly $N_{2\varepsilon_0}$ as an open neighborhood of Y in X.

Let $\pi : N \to Y$ denote the projection of the normal bundle N over Y.

If $Z \in N$, let $\tilde{c}(Z) \in \text{End}(S^*(N))$ be the transpose of $c(Z)$ acting on $S(N)$.

Let $\tau^{N*} \in \text{End}(S^*(N))$ be the transpose of τ^N defining the $\mathbf{Z}_2$-grading of $S(N) = S_+(N) \oplus S_-(N)$.

Let $\pi^*(S^*(N))$ be the pull back bundle of $S^*(N)$ over N.

For any $Z \in N$ with $Z \neq 0$, let $\tau^{N*}\tilde{c}(Z) : \pi^*(S_\pm^*(N))|_Z \to \pi^*(S_\mp^*(N))|_Z$ denote the corresponding pull back isomorphisms at Z.

Let (μ, g^μ) be a Hermitian vector bundle over Y carrying a Hermitian connection ∇^μ.

In this paper, by a direct image of μ under the embedding $i : Y \to X$, we always mean a triple (ξ_+, ξ_-, V) described in Section 2.1 with a Hermitian connection $\nabla^\xi = \nabla^{\xi_+} \oplus \nabla^{\xi_-}$ verifying the following fundamental assumptions (cf. [11, (1.10)-(1.12)]):

1) V is invertible on $X \setminus Y$ and $(\ker V)|_Y$ has a constant dimension;

2) the following identification

$$\left(\pi^* (\ker V)|_Y, \pi^* g^{(\ker V)|_Y}, \dot{\partial}_Z V \right) \simeq \left(\pi^* (\mu \otimes S^*(N)), \pi^* g^{\mu \otimes S^*(N)}, \tau^{N*} \tilde{c}(Z) \right) \tag{2.5}$$

holds over N, where the map $\dot{\partial}_Z V$ is defined by

$$\dot{\partial}_Z V = P^{\ker V} (\partial_Z V) P^{\ker V} \tag{2.6}$$

with respect to any smooth trivialization of ξ near $\pi(Z)$ and $P^{\ker V}$ denotes the orthogonal projection from ξ onto $\ker V$;

3) under the identification (2.5) the following connections identification holds,

$$\nabla^{(\ker V)|_Y} = \nabla^{\mu \otimes S^*(N)}, \tag{2.7}$$

where $\nabla^{(\ker V)|_Y}$ is defined by

$$\nabla^{(\ker V)|_Y} = P^{\ker V} \nabla^{\xi|_Y} P^{\ker V}. \tag{2.8}$$

Clearly, $\xi_+ - \xi_- \in \widetilde{K}(X)$ is exactly the Atiyah-Hirzebruch direct image $i_! \mu$ of μ constructed in [2].

We now describe a concrete realization of the direct image of μ for an embedding $i : Y \to X$ which verifies the assumptions 1)–3) (see also [16]).

Let (F, g^F) be a Hermitian vector bundle over Y carrying a Hermitian connection ∇^F such that $S_-(N) \otimes \mu \oplus F$ is a trivial complex vector bundle over Y (cf. [1]). Then

$$\tau^{N*} \tilde{c}(Z) \oplus \pi^* \mathrm{Id}_F : \pi^* \left(S_+^*(N) \otimes \mu \oplus F \right) \to \pi^* \left(S_-^*(N) \otimes \mu \oplus F \right) \tag{2.9}$$

induces an isomorphism between two trivial vector bundles over $N_{2\varepsilon_0} \setminus Y$.

Clearly, $\pi^*(S_\pm^*(N) \otimes \mu \oplus F)|_{\partial N_{2\varepsilon_0}}$ extend smoothly to two trivial complex vector bundles over $X \setminus N_{2\varepsilon_0}$. Moreover, the isomorphism $\tau^{N*}\tilde{c}(Z) \oplus \pi^*\mathrm{Id}_F$ over $\partial N_{2\varepsilon_0}$ extends smoothly to an isomorphism between these two trivial vector bundles over $X \setminus N_{2\varepsilon_0}$.

In summary, what we get is a $\mathbf{Z}_2$-graded Hermitian vector bundle $(\xi = \xi_+ \oplus \xi_-, g^\xi = g^{\xi_+} \oplus g^{\xi_-})$ over X such that

$$\xi_\pm|_{N_{\varepsilon_0}} = \pi^* \left(S_\pm^*(N) \otimes \mu \oplus F \right)\big|_{N_{\varepsilon_0}}, \quad g^{\xi_\pm}|_{N_{\varepsilon_0}} = \pi^* \left(g^{S_\pm^*(N) \otimes \mu} \oplus g^F \right)\Big|_{N_{\varepsilon_0}}, \tag{2.10}$$

where $g^{S_\pm^*(N) \otimes \mu}$ is the tensor product Hermitian metric on $S_\pm^*(N) \otimes \mu$ induced from $g^{S_\pm^*(N)}$ and g^μ.

It is easy to see that there exists an odd self-adjoint automorphism V of ξ such that

$$V|_{N_{\varepsilon_0}} = \tau^{N*}\tilde{c}(Z) \oplus \pi^*\mathrm{Id}_F. \tag{2.11}$$

Moreover, there is a $\mathbf{Z}_2$-graded Hermitian connection $\nabla^\xi = \nabla^{\xi_+} \oplus \nabla^{\xi_-}$ on $\xi = \xi_+ \oplus \xi_-$ over X such that

$$\nabla^{\xi_\pm}|_{N_{\varepsilon_0}} = \pi^* \left(\nabla^{S_\pm^*(N) \otimes \mu} \oplus \nabla^F \right), \tag{2.12}$$

where $\nabla^{S_\pm^*(N) \otimes \mu}$ is the Hermitian connection on $\nabla^{S_\pm^*(N) \otimes \mu}$ defined by $\nabla^{S_\pm^*(N) \otimes \mu} = \nabla^{S_\pm^*(N)} \otimes \mathrm{Id}_\mu + \mathrm{Id}_{S_\pm^*(N)} \otimes \nabla^\mu$.

Clearly, the fundamental assumptions (2.5) and (2.6) hold for this geometric construction. We will call (ξ_+, ξ_-, V) constructed as such a *geometric* direct image of μ.

Before recalling the construction of the associated Chern-Simons current, we give some notation first.

Let $i^{1/2}$ be a fixed square root of $i = \sqrt{-1}$. The objects which will be considered in the sequel do not depend on this square root. Let φ be the map $\alpha \in \Lambda^*(T^*X) \to (2\pi i)^{-\frac{\deg \alpha}{2}} \alpha \in \Lambda^*(T^*X)$.

If E is a real vector bundle over X carrying a connection ∇^E, we denote by $\widehat{A}(E, \nabla^E)$ the Hirzebruch characteristic form defined by

$$\widehat{A}(E, \nabla^E) = \det^{1/2}\left(\frac{\frac{\sqrt{-1}}{4\pi}R^E}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^E\right)}\right) = \varphi \det^{1/2}\left(\frac{\frac{R^E}{2}}{\sinh\left(\frac{R^E}{2}\right)}\right), \qquad (2.13)$$

where $R^E = \nabla^{E,2}$ is the curvature of ∇^E. While if E' is a complex vector bundle carrying a connection $\nabla^{E'}$, we denote by $\mathrm{ch}(E', \nabla^{E'})$ the Chern character form associated to $(E', \nabla^{E'})$ defined by (cf. [15, Section 1])

$$\mathrm{ch}\left(E', \nabla^{E'}\right) = \varphi \operatorname{tr}\left[\exp\left(-R^{E'}\right)\right]. \qquad (2.14)$$

For $T \geq 0$, let C_T be the superconnection on the super vector bundle ξ defined by

$$C_T = \nabla^\xi + \sqrt{T}V. \qquad (2.15)$$

The curvature C_T^2 of C_T is a smooth section of $(\Lambda^*(T^*X)\widehat{\otimes}\mathrm{End}(\xi))^{\mathrm{even}}$.

By [14], we know that for any $T > 0$,

$$\frac{\partial}{\partial T}\operatorname{Tr}_s\left[\exp\left(-C_T^2\right)\right] = -\frac{d}{2\sqrt{T}}\operatorname{Tr}_s\left[V\exp\left(-C_T^2\right)\right]. \qquad (2.16)$$

By proceeding as in [6], [7] and [11, Definition 1.3], one can construct the Chern-Simons current $\gamma^{\xi,V}$ as

$$\gamma^{\xi,V} = \frac{1}{\sqrt{2\pi i}}\int_0^{+\infty} \varphi\operatorname{Tr}_s\left[V\exp\left(-C_T^2\right)\right]\frac{dT}{2\sqrt{T}}. \qquad (2.17)$$

Let δ_Y denote the current of integration over the oriented submanifold Y of X.

By [11, Theorem 1.4], we have that

$$d\gamma^{\xi,V} = \mathrm{ch}\left(\xi_+, \nabla^{\xi_+}\right) - \mathrm{ch}\left(\xi_-, \nabla^{\xi_-}\right) - \widehat{A}^{-1}\left(N, \nabla^N\right)\mathrm{ch}\left(\mu, \nabla^\mu\right)\delta_Y. \qquad (2.18)$$

Moreover, as indicated in [11, Remark 1.5], by proceeding as in [9, Theorem 3.3], one can prove that $\gamma^{\xi,V}$ is a locally integrable current.

2.3. **The Bismut-Zhang localization formula for η invariants.** We assume in this subsection that $i : Y \hookrightarrow X$ is an embedding between two odd dimensional closed oriented spin manifolds. Then the normal bundle N to Y in X is even dimensional and carries a canonically induced orientation and spin structure.

Let g^{TX} be a Riemannian metric on TX. Let g^{TY} be the restricted Riemannian metric on TY. Let ∇^{TX} (resp. ∇^{TY}) denote the Levi-Civita connection associated to g^{TX} (resp. g^{TY}). Without loss of generality we may and we will make the assumption that the embedding $(Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic.

Let N carry the canonically induced Euclidean metric as well as the Euclidean connection.

The definition of the reduced η invariant for a (twisted) Dirac operator on an odd dimensional spin Riemannian manifold has been recalled in Section 1.

Under our assumptions, we can state the Bismut-Zhang localization formula for η-invariants [11] as follows, of which a special case was proved in [7].

Theorem 2.1. (Bismut-Zhang [11, Theorem 2.2]) *If (ξ_+, ξ_-, V) is a direct image of μ for a totally geodesic embedding $i : Y \hookrightarrow X$, then the following identity holds,*

$$\overline{\eta}\left(D^{\xi_+}\right) - \overline{\eta}\left(D^{\xi_-}\right) \equiv \overline{\eta}\left(D^{\mu}\right) + \int_X \widehat{A}\left(TX, \nabla^{TX}\right) \gamma^{\xi,V} \quad \mod \mathbf{Z}. \qquad (2.19)$$

Remark 2.2. The extra Chern-Simons form in [11, Theorem 2.2] disappears here simply because we have made the simplifying assumption that the isometric embedding $(Y, g^{TY}) \hookrightarrow (X, g^{TX})$ is totally geodesic.

Remark 2.3. The proof of (2.19) in [11] relies heavily on the analytic techniques developed in a difficult paper of Bismut-Lebeau [10]. By using the mod k index theorem of Freed-Melrose [12], Zhang showed in [16] that there exists an embedding $i : Y \hookrightarrow S^{2m-1}$ of Y to some higher dimensional sphere and a geometric direct image (ξ'_+, ξ'_-, V') of μ on S^{2m-1} such that the following special case of (2.19) holds,

$$\overline{\eta}\left(D^{\xi'_+}\right) - \overline{\eta}\left(D^{\xi'_-}\right) \equiv \overline{\eta}\left(D^{\mu}\right) + \int_{S^{2m-1}} \widehat{A}\left(TS^{2m-1}, \nabla^{TS^{2m-1}}\right) \gamma^{\xi',V'} \quad \mod \mathbf{Z},$$
$$(2.20)$$

without using the techniques of Bismut-Lebeau [10].

In the rest of this paper, we will present a proof of (2.19) by using (2.20), which still avoids the use of the techniques in [10].

3. A proof of Theorem 2.1

In this section we will present an alternate proof of Theorem 2.1 by embedding X to a higher dimensional sphere.

Let (ξ_+, ξ_-, V) be a direct image of μ for an embedding $i : Y \hookrightarrow X$ in the sense of Section 2.2.

For the sake of convenience, we denote by $H_X(\xi_+, \xi_-, V) \in \mathbf{R}/\mathbf{Z}$ the quantity defined by

$$H_X\left(\xi_+, \xi_-, V\right) \equiv \overline{\eta}\left(D^{\xi_+}\right) - \overline{\eta}\left(D^{\xi_-}\right) - \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi,V} - \overline{\eta}\left(D^\mu\right) \mod \mathbf{Z}. \tag{3.1}$$

Then (2.19) and (2.20) can be rewritten as

$$H_X\left(\xi_+, \xi_-, V\right) = 0, \tag{3.2}$$

and

$$H_{S^{2m-1}}\left(\xi'_+, \xi'_-, V'\right) = 0 \tag{3.3}$$

respectively.

The rest of this section is organized as follows. In Section 3.1, we prove two basic properties of $H_X(\xi_+, \xi_-, V)$. In Section 3.2, we describe a constructions of direct images under successive embeddings and the associated Chern-Simons current. In Section 3.3, we study the relations between the Chern-Simons currents constructed in Section 3.2 and establish a Riemann-Roch type formula for them. In Section 3.4, we use (3.3) and the Riemann-Roch type formula established in Section 3.3 to give an alternative proof of the Bismut-Zhang localization formula.

3.1. **Basic properties of the H-quantity.** In this subsection, we will prove two properties of the H-quantities defined above, from which one can deduce that the H-quantity depends only on the isotropy class of the embeddings.

Lemma 3.1. *If $i_s : Y_s \hookrightarrow X_s$, $0 \le s \le 1$, is a smooth family of embeddings between odd dimensional compact oriented spin Riemannian manifolds such that $i_0 : Y_0 \hookrightarrow X_1$, $i_1 : Y_1 \hookrightarrow X_1$ are totally geodesic embeddings, and $(\xi_{s,+}, \xi_{s,-}, V_s)$*

is a smooth family of direct images of the complex vector bundle μ_s over Y_s, then the following identity in $\mathbf{R}/\mathbf{Z}$ holds,

$$H_X(\xi_{0,+}, \xi_{0,-}, V_0) = H_X(\xi_{1,+}, \xi_{1,-}, V_1). \tag{3.4}$$

Proof. Without loss of generality, we may and we will assume that the above smooth family is a locally constant family for $s \in [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$.

Set $Y = Y_0$ and $\widehat{Y} = I \times Y$.

We equip $\widehat{Y}$ with the metric $ds^2 \oplus g^{TY_s}$, $0 \leq s \leq 1$. Let $\nabla^{\widehat{Y}}$ denote the associated Levi-Civita connection.

Clearly, $\widehat{Y}$ is an oriented spin even-dimensional Riemannian manifold with boundary $\partial \widehat{Y} = \overline{Y}_1 \bigcup Y_0$, where $\overline{Y}_1$ is a copy of Y_1 but with the reversed orientation.

Let $\widehat{\mu}$ be the canonical complex vector bundle over $\widehat{Y}$ such that $\widehat{\mu}|_{\{s\} \times Y} = \mu_s$, let $g^{\widehat{\mu}}$ (resp. $\nabla^{\widehat{\mu}}$) be the Hermitian metric (resp. connection) on $\widehat{\mu}$ such that $g^{\widehat{\mu}}|_{\mu_s} = g^{\mu_s}$ (resp. $\nabla^{\widehat{\mu}}|_{\mu_s} = \nabla^{\mu_s}$).

By the Atiyah-Patodi-Singer index theorem [3], one has

$$\int_{\widehat{Y}} \widehat{A}\left(T\widehat{Y}, \nabla^{\widehat{Y}}\right) \mathrm{ch}\left(\widehat{\mu}, \nabla^{\widehat{\mu}}\right) - \overline{\eta}\left(D_{\partial\widehat{Y}}^{\widehat{\mu}}\right) \equiv 0 \mod \mathbf{Z}. \tag{3.5}$$

Let $X = X_0$ and $\widehat{X} = I \times X$.

It is easy to see that there exists a metric $g^{T\widehat{X}}$ on $T\widehat{X}$ such that it equals to $ds^2 \oplus g^{TX_s}$ on $[0, \frac{1}{8}] \times X \cup [\frac{7}{8}, 1] \times X$, and that the canonical embedding $i_{\widehat{Y}} : \widehat{Y} \hookrightarrow \widehat{X}$ is totally geodesic. Let $\nabla^{\widehat{X}}$ denote the associated Levi-Civita connection.

Clearly, $\widehat{X}$ is an oriented spin even-dimensional Riemannian manifold with boundary $\partial \widehat{X} = \overline{X}_1 \bigcup X_0$, where $\overline{X}_1$ is a copy of X_1 but with the reversed orientation.

The smooth family of direct images (ξ_s, V_s) also lifts to $\widehat{X}$ canonically and form a direct image $(\widehat{\xi}, \widehat{V})$ over $\widehat{X}$ of $\widehat{\mu}$ over $\widehat{Y}$, which is of product structure near the boundary. In particular, when restricted to $s \in [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$, $(\widehat{\xi}_s, V_s^{\widehat{\xi}})$ is the (locally constant) geometric direct image of $\mu_s \to Y_s$ for the totally geodesic embedding $i_s : Y_s \to X_s$.

Denote by $\widehat{N}$ the normal bundle to $\widehat{Y}$ in $\widehat{X}$.

By (2.18), one has

$$d\gamma^{\widehat{\xi},\widehat{V}} = \mathrm{ch}\left(\widehat{\xi}_+, \nabla^{\widehat{\xi}_+}\right) - \mathrm{ch}\left(\widehat{\xi}_-, \nabla^{\widehat{\xi}_-}\right) - \widehat{A}^{-1}\left(\widehat{N}, \nabla^{\widehat{N}}\right)\mathrm{ch}\left(\widehat{\mu}, \nabla^{\widehat{\mu}}\right)\delta_{\widehat{Y}}. \qquad (3.6)$$

On the other hand, by the Atiyah-Patodi-Singer index theorem [3], one has

$$\overline{\eta}\left(D_{\partial\widehat{X}}^{\widehat{\xi}}\right) \equiv \int_{\widehat{X}} \widehat{A}\left(T\widehat{X}, \nabla^{\widehat{X}}\right)\mathrm{ch}\left(\widehat{\xi}, \nabla^{\widehat{\xi}}\right) \quad \mathrm{mod}\ \mathbf{Z}. \qquad (3.7)$$

From (3.5), (3.6) and (3.7), one gets

$$\overline{\eta}\left(D^{\xi_{0,+}}\right) - \overline{\eta}\left(D^{\xi_{0,-}}\right) - \left(\overline{\eta}\left(D^{\xi_{1,+}}\right) - \overline{\eta}\left(D^{\xi_{1,-}}\right)\right)$$
$$\equiv \int_{\widehat{X}} \widehat{A}\left(T\widehat{X}, \nabla^{\widehat{X}}\right)\left(\mathrm{ch}\left(\widehat{\xi}_+, \nabla^{\widehat{\xi}_+}\right) - \mathrm{ch}\left(\widehat{\xi}_-, \nabla^{\widehat{\xi}_-}\right)\right)$$
$$\equiv \int_{\widehat{X}} \widehat{A}\left(T\widehat{X}, \nabla^{T\widehat{X}}\right)d\gamma^{\widehat{\xi},\widehat{V}} + \int_{\widehat{X}} \widehat{A}\left(T\widehat{X}, \nabla^{T\widehat{X}}\right)\widehat{A}^{-1}\left(\widehat{N}, \nabla^{\widehat{N}}\right)\mathrm{ch}\left(\widehat{\mu}, \nabla^{\widehat{\mu}}\right)\delta_{\widehat{Y}}$$
$$\equiv \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi_0,V_0} - \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi_1,V_1} + \int_{\widehat{Y}} \widehat{A}\left(T\widehat{Y}, \nabla^{T\widehat{Y}}\right)\mathrm{ch}\left(\widehat{\mu}, \nabla^{\widehat{\mu}}\right)$$
$$\equiv \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi_0,V_0} - \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi_1,V_1} + \overline{\eta}\left(D_{\partial\widehat{Y}}^{\widehat{\mu}}\right)$$
$$\equiv \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi_0,V_0} - \int_X \widehat{A}\left(TX, \nabla^{TX}\right)\gamma^{\xi_1,V_1} + \overline{\eta}\left(D^{\mu_0}\right) - \overline{\eta}\left(D^{\mu_1}\right) \quad \mathrm{mod}\ \mathbf{Z}. \tag{3.8}$$

From (3.1) and (3.8), we get (3.4). Q.E.D.

Lemma 3.2. *Let $i : Y \hookrightarrow X$ be a totally geodesic embedding between the odd dimensional compact oriented spin Riemannian manifolds and μ a Hermitian vector bundle over Y carrying a Hermitian connection. Then for any two direct images $(\xi_{k,+}, \xi_{k,-}, V_k)$, $k = 1, 2$, of μ associated to the embedding i, the following identity in $\mathbf{R}/\mathbf{Z}$ holds,*

$$H_X(\xi_{1,+}, \xi_{1,-}, V_1) = H_X(\xi_{2,+}, \xi_{2,-}, V_2). \tag{3.9}$$

Proof. We first show that any direct image can be deformed smoothly to another one which is of a simpler form.

For any direct image $(\xi_+, \xi_-, V = v + v^*)$ of μ associated to the embedding $i : Y \hookrightarrow X$, let N_ε be a tubular neighborhood of Y which will be identified with

a neighborhood of the zero section in the total space of the normal bundle to Y in X for ε small enough.

From the fundamental assumptions 1)-3) in Section 2.2, over N_ε one has the identification

$$\xi_\pm|_{N_\varepsilon} \simeq \pi^* \left(\mu \otimes S_\pm^*(N) \oplus F_\pm \right), \tag{3.10}$$

where $F_\pm = (\ker V|_Y)^\perp \cap \xi_\pm|_Y$. Moreover, one can write $v : \xi_+ \to \xi_-$ near Y in terms of $Z \in N$ such that

$$v = \begin{pmatrix} \pi^*\mathrm{Id}_\mu \otimes \tilde{c}(Z) + a\ b \\ c \qquad\qquad h \end{pmatrix}, \tag{3.11}$$

where $h : \pi^*F_+ \to \pi^*F_-$ is an isomorphism, and the maps b and c have the infinitesimal order $O(|Z|)$, while a has the order $o(|Z|)$.

Choose a smooth cut-off function $f(Z)$ with support in $N_{\varepsilon/2}$ and $f \equiv 1$ in $N_{\varepsilon/4}$. Set for $0 \le s \le 1$,

$$v_s = \begin{pmatrix} \pi^*\mathrm{Id}_\mu \otimes \tilde{c}(Z) + a - sf(Z)a & b - sf(Z)b \\ c - sf(Z)c & h + sf(Z)(\pi^*(h|_Y) - h) \end{pmatrix}. \tag{3.12}$$

Clearly, for each $s \in [0, 1]$, the map v_s is globally well-defined over X and maps ξ_+ to ξ_-. Moreover, one can choose ε small enough so that v_s is invertible over $X \setminus Y$ for each s.

Note that $v_0 = v$ and over $N_{\varepsilon/4}$

$$v_1 = \begin{pmatrix} \pi^*\mathrm{Id}_\mu \otimes \tilde{c}(Z) & 0 \\ 0 & \pi^*(h|_Y) \end{pmatrix}. \tag{3.13}$$

On the other hand, let $F_\pm$ carry Hermitian metrics $g^{F_\pm}$ and Hermitian connections $\nabla^{F_\pm}$ respectively.

By using the identification (3.10) one can choose another Hermitian metric g_1^ξ and Hermitian connection ∇_1^ξ on ξ such that when restricted to $N_{\varepsilon/8}$, one has that

$$g_1^{\xi_\pm} = \pi^* g^{\mu \otimes S_+^*(N)} \oplus \pi^* g^{F_\pm}, \quad \nabla_1^{\xi_\pm} = \pi^* \nabla^{\mu \otimes S_+^*(N)} \oplus \pi^* \nabla^{F_\pm}. \tag{3.14}$$

For each $s \in [0, 1]$, set

$$g_s^\xi = s g_1^\xi + (1-s) g^\xi, \quad \nabla_s^\xi = \frac{1}{2} \left(\widetilde{\nabla}_s^\xi + \left(\widetilde{\nabla}_s^\xi \right)^* \right), \tag{3.15}$$

where

$$\widetilde{\nabla}_s^\xi = s \nabla_1^\xi + (1-s) \nabla_0^\xi \tag{3.16}$$

and $(\widetilde{\nabla}_s^\xi)^*$ is the adjoint of $\widetilde{\nabla}_s^\xi$ with respect to the metric g_s^ξ.

Clearly for each $s \in [0, 1]$,

$$\left(\xi, g_s^\xi, \nabla_s^\xi, V_s = v_s + v_s^* \right) \tag{3.17}$$

is a direct image of μ, where v_s^* is the adjoint of v_s with respect to g_s^ξ.

We now come back to the proof of the lemma.

From Lemma 3.1 and the above deformation, it is clear that in order to prove the lemma at hand, one needs only to prove it for direct images $(\xi_{k,+}, \xi_{k,-}, V_k)$ with the properties that

$$\xi_{k,\pm}|_{N_\varepsilon} \simeq \pi^* \left(S_\pm^*(N) \otimes \mu \right) \oplus \pi^* \left(F_{k,\pm} \right), \tag{3.18}$$

$$g^{\xi_{k,\pm}|_{N_\varepsilon}} = \pi^* \left(g^{S_\pm^*(N) \otimes \mu} \oplus g^{F_{k,\pm}} \right) \Big|_{N_\varepsilon}, \nabla^{\xi_{k,\pm}|_{N_\varepsilon}} = \pi^* \left(\nabla^{S_\pm^*(N) \otimes \mu} \oplus \nabla^{F_{k,\pm}} \right) \Big|_{N_\varepsilon} \tag{3.19}$$

and

$$V_k|_{N_\varepsilon} = \tau^{N*} \widetilde{c}(Z) \oplus \pi^* h_{F_k} \tag{3.20}$$

over N_ε for ε small enough, where h_{F_k} are self-adjoint odd automorphisms of F_k, $k = 1, 2$.

Now consider the super vector bundle

$$\left(\widetilde{\xi}_+, \widetilde{\xi}_-, \widetilde{V} \right) = \left(\xi_{1,+} \oplus \xi_{2,-}, \xi_{1,-} \oplus \xi_{2,+}, V_1 \oplus V_2 \right).$$

It is easy to see that both $\widetilde{\xi}_\pm$ contain $\pi^*(S^*(N) \otimes \mu)$.

Clearly over N_ε, one has

$$\widetilde{V} = \begin{pmatrix} \pi^* \mathrm{Id}_\mu \otimes \tau^{N*} \widetilde{c}(Z) & 0 \\ 0 & \pi^* h_{F_1} \oplus \pi^* h_{F_2} \end{pmatrix}. \tag{3.21}$$

Define

$$i_g : \sqrt{-1} g(Z) \mathrm{Id}_{\pi^*(\mu \otimes S^*(N))} \oplus 0_F : \widetilde{\xi}_+ \to \widetilde{\xi}_-, \qquad (3.22)$$

where g is a cut-off function with support in N_ε and $g \equiv 1$ in $N_{\varepsilon/2}$, and 0_F is the zero map from $\pi^*(F_{1,+} \oplus F_{2,-})$ to $\pi^*(F_{1,-} \oplus F_{2,+})$. Then

$$I_g = i_g + i_g^* \qquad (3.23)$$

is a globally defined odd endomorphism of $\widetilde{\xi}_+ \oplus \widetilde{\xi}_-$.

Set

$$\widetilde{V}_g = \widetilde{V} + I_g. \qquad (3.24)$$

One verifies that

$$\left(\widetilde{V}_g \right)^2 \Big|_{N_\varepsilon} = \begin{pmatrix} g(Z)^2 + |Z|^2 & 0 \\ 0 & \pi^* h_{F_1} \oplus \pi^* h_{F_2} \end{pmatrix}. \qquad (3.25)$$

Note that $\widetilde{V}_g$ is invertible over X. By a result of Bismut-Cheeger (cf. [8, Theorem 2.28]), one has

$$\overline{\eta}\left(D^{\widetilde{\xi}_+} \right) - \overline{\eta}\left(D^{\widetilde{\xi}_-} \right) \equiv \int_X \widehat{A}\left(TX, \nabla^{TX} \right) \gamma^{\widetilde{\xi}, \widetilde{V}_g} \mod \mathbf{Z}. \qquad (3.26)$$

Let $F(s, T)$ be the curvature of the smooth family of the superconnections

$$A(s, T) = \nabla^{\widetilde{\xi}} + \sqrt{T}\left(\widetilde{V} + s I_g \right) \qquad (3.27)$$

on $\tilde{\xi}$.

For any closed form ω on X, by the standard double transgression formula for Chern character forms (cf. [13, Proposition 3.1]), one has

$$\int_X \omega \int_0^R \frac{1}{\sqrt{2\pi i}} \varphi \mathrm{tr}_s \left[\widetilde{V}_g e^{-\left(\nabla^{\widetilde{\xi}} + \sqrt{T}\widetilde{V}_g\right)^2} \right] \frac{dT}{2\sqrt{T}}$$

$$- \int_X \omega \int_0^R \frac{1}{\sqrt{2\pi i}} \varphi \mathrm{tr}_s \left[\widetilde{V} e^{-\left(\nabla^{\widetilde{\xi}} + \sqrt{T}\widetilde{V}\right)^2} \right] \frac{dT}{2\sqrt{T}}$$

$$= \int_X \omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \, \mathrm{tr}_s \left[\sqrt{R} I_g e^{-F(s,R)} \right]. \qquad (3.28)$$

Note that when $R \to \infty$, the left hand side of the above equality converges to

$$\int_X \omega \gamma^{\widetilde{\xi}, \widetilde{V}_g} - \int_X \omega \gamma^{\widetilde{\xi}, \widetilde{V}}. \tag{3.29}$$

Now we want to show that the right hand side of (3.28) tends to zero as $R \to \infty$.

Firstly, one sees that in $X \setminus N_{\varepsilon/2}$,

$$\int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \, \mathrm{tr}_s \left[\sqrt{R} I_g e^{-F(s,R)} \right]$$

decays exponentially as $R \to \infty$.

On the other hand, since $g \equiv 1$ in $N_{\varepsilon/2}$, one has

$$\mathrm{tr}_s \left[\sqrt{R} I_g e^{-F(s,R)} \right] = \sqrt{R} \mathrm{tr}_s \left[I_g e^{-(A(0,R)+\sqrt{R}sI_g)^2} \right]$$

$$= \sqrt{R} \mathrm{tr}_s \left[I_g e^{-A(0,R)^2} \right] e^{-Rs^2}. \tag{3.30}$$

Since $A(0,R)^2$ maps ξ_1 to ξ_1 and ξ_2 to ξ_2 respectively, while I_g exchange ξ_1 and ξ_2, one gets in $N_{\varepsilon/2}$ that

$$\mathrm{tr}_s \left[\sqrt{R} I_g e^{-F(s,R)} \right] \equiv 0. \tag{3.31}$$

Hence for any closed form ω, one has

$$\int_X \omega \gamma^{\widetilde{\xi}, \widetilde{V}_g} = \int_X \omega \gamma^{\widetilde{\xi}, \widetilde{V}}. \tag{3.32}$$

By (3.26) and (3.32), one gets

$$\overline{\eta}\left(D^{\widetilde{\xi}_+}\right) - \overline{\eta}\left(D^{\widetilde{\xi}_-}\right) \equiv \int_X \widehat{A}\left(TX, \nabla^{TX}\right) \gamma^{\widetilde{\xi}, \widetilde{V}} \mod \mathbf{Z}, \tag{3.33}$$

which implies (3.9). Q.E.D.

From the two lemmas above, one deduces easily that the H-quantity of a direct image on X of μ depends only on μ as well as the isotropy class of the embedding. From now on, we may denote this quantity by $H(Y, X, \mu)$, that is

$$H_X(\xi_+, \xi_-, V) = H(Y, X, \mu). \tag{3.34}$$

Moreover, one sees that this quantity now does not depend on the metrics and connections define it and is indeed a smooth invariant!

3.2. Direct images and Chern-Simons currents associated with successive embeddings.

Let (ξ_+, ξ_-, V) be a geometric direct image of μ for $i : Y \hookrightarrow X$ constructed in Section 2.2.

Let $j : X \hookrightarrow M$ be a totally geodesic embedding of X into an odd-dimensional closed oriented spin Riemannian manifold M. Since the embedding $i : Y \hookrightarrow X$ is totally geodesic, the induced embedding $j \circ i : Y \hookrightarrow M$ is also totally geodesic.

Let N_X be the normal bundle to X in M. One can take $\varepsilon_0 > 0$ appearing in Section 2.2 small enough so that $N_{X,2\varepsilon_0} = \{u \in N_X : |u| < 2\varepsilon_0\}$ is diffeomorphic to an open neighborhood of X in M. Without confusion we now view directly $N_{X,2\varepsilon_0}$ as an open neighborhood of X in M.

Let N_Y be the normal bundle to Y in M. Clearly,

$$N_Y = N \oplus i^* N_X. \tag{3.35}$$

Then one can choose $\varepsilon_0 > 0$ small enough so that $N_{2\varepsilon_0} \oplus i^* N_{X,2\varepsilon_0}$ is diffeomorphic to an open neighborhood of Y in M. Without confusion we now view directly $N_{2\varepsilon_0} \oplus i^* N_{X,2\varepsilon_0}$ as an open neighborhood of Y in M.

Let $(\zeta_{+,+}, \zeta_{+,-}, W_+)$ (resp. $(\zeta_{-,+}, \zeta_{-,-}, W_-)$) be the geometric direct image of ξ_+ (resp. ξ_-) in the sense of Section 2.2.

Let $\zeta = \zeta_+ \oplus \zeta_-$ be the $\mathbf{Z}_2$-graded Hermitian vector bundle over M such that

$$\zeta_+ = \zeta_{+,+} \oplus \zeta_{-,-}, \quad \zeta_- = \zeta_{+,-} \oplus \zeta_{-,+}. \tag{3.36}$$

Then $\zeta_+ - \zeta_- \in \widetilde{K}(M)$ is a representative of $(j \circ i)_! \mu$ in the sense of [2].

Let

$$W = W_+ \oplus W_- \tag{3.37}$$

be the induced odd endomorphism on ζ. Then $\mathrm{Supp}(W) = X$.

Let $f \in C^\infty(M)$ be such that $\mathrm{Supp}(f) \subset N_{X,2\varepsilon_0}$ and $f \equiv 1$ on N_{X,ε_0}.

Let $\pi_X : N_X \to X$ denote the canonical projection.

Recall that by the construction of geometric direct images in Section 2.2, one has

$$\pi_X^* \left(S^*(N_X) \widehat{\otimes} \xi \right) |_{N_{X,2\varepsilon_0}} \subset \zeta|_{N_{X,2\varepsilon_0}}. \tag{3.38}$$

For any $Z \in N_{X,2\varepsilon_0}$, set

$$V_f(Z) = f(Z)\pi_X^* \left(\mathrm{Id}_{S^*(N_X)}\widehat{\otimes}V\right) : \pi_X^* \left(S^*(N_X)\widehat{\otimes}\xi\right) \to \pi_X^* \left(S^*(N_X)\widehat{\otimes}\xi\right). \quad (3.39)$$

By (3.38) it can be viewed as an endomorphism of $\zeta|_{N_{X,2\varepsilon_0}}$, which vanishes on the orthogonal complement of $\pi_X^*(S^*(N_X)\widehat{\otimes}\xi)$ in $\zeta|_{N_{X,2\varepsilon_0}}$.

Also, since $\mathrm{Supp}(f) \subset N_{X,2\varepsilon_0}$, one can extend V_f as zero endomorphism to $M \setminus N_{X,2\varepsilon_0}$.

Thus, we may view V_f as an odd self-adjoint endomorphism on ζ.

Let W_f be the odd self adjoint endomorphism of ζ defined by

$$W_f = W + V_f. \quad (3.40)$$

By (2.4), (3.37), (3.39) and (3.40), one verifies easily that

$$\mathrm{Supp}\left(W_f\right) = Y. \quad (3.41)$$

Moreover, by (3.35), one sees that the obvious analogues of the conditions [11, (1.10) and (1.12)] verify for (ζ_+, ζ_-, W_f) near $Y \subset M$.

Thus, one gets a well-defined Chern-Simons current γ^{ζ,W_f} over M as in (2.17).

Proposition 3.3. *If g is another cut-off function verifying the same condition as f, then up to smooth exact forms on M, $\gamma^{\zeta,W_f} = \gamma^{\zeta,W_g}$.*

Proof. Take $f_s = f + s(g - f)$, $0 \leq s \leq 1$. Then f_s is a family of cut-off functions verifying the same condition as f. Let

$$A(s,T) = \nabla^\zeta + \sqrt{T}W_{f_s} \quad (3.42)$$

be the corresponding smooth family of superconnections on ζ. Let $F(s,T)$ be the curvature of $A(s,T)$.

By the double transgression formula for Chern character forms (cf. [13, Proposition 3.1]), one has

$$\frac{d}{ds}\mathrm{tr}_s\left[\frac{dA(s,T)}{dT}e^{-F(s,T)}\right] - \frac{d}{dT}\mathrm{tr}_s\left[\frac{dA(s,T)}{ds}e^{-F(s,T)}\right] = dv(s,T), \quad (3.43)$$

where

$$v(s,T) = -\int_0^1 \mathrm{tr}_s\left[\frac{dA(s,T)}{ds}e^{-uF(s,T)}\frac{dA(s,T)}{dT}e^{-(1-u)F(s,T)}\right]du$$

$$= -\int_0^1 \mathrm{tr}_s\left[\left(\sqrt{T}(g-f)\mathrm{Id}_{\pi_X^*S^*(N_X)\widehat{\otimes}\pi_X^*(V)}\right)e^{-uF(s,T)}\frac{dA(s,T)}{dT}e^{-(1-u)F(s,T)}du\right]$$

$$= -(g-f)\int_0^1 \mathrm{tr}_s\left[\left(\sqrt{T}\mathrm{Id}_{\pi_X^*S^*(N_X)\widehat{\otimes}\pi_X^*(V)}\right)e^{-uF(s,T)}\frac{dA(s,T)}{dT}e^{-(1-u)F(s,T)}du\right].$$

(3.44)

Since

$$\int_0^1 ds \int_0^{+\infty} dT \int_0^1 \mathrm{tr}_s\left[\sqrt{T}\pi_X^*\left(\mathrm{Id}_{S^*(N_X)}\widehat{\otimes}V\right)e^{-uF(s,T)}\frac{dA(s,T)}{dT}e^{-(1-u)F(s,T)}du\right]$$

is a smooth form on $N_{X,\varepsilon_0}\setminus X$ (cf. [13, Proposition 3.8]) and $g-f$ vanishes near X, one sees that

$$\int_0^1 ds \int_0^{+\infty} dT\, v(s,T)$$

is a smooth form. Thus, one has

$$\gamma^{\zeta,W_g} - \gamma^{\zeta,W_f} = \lim_{R\to+\infty}\int_0^1 ds\frac{d}{ds}\left(\int_0^R \frac{1}{\sqrt{2\pi i}}\varphi\,\mathrm{tr}_s\left[\frac{dA(s,T)}{dT}e^{-F(s,T)}\right]dT\right)$$

$$= \lim_{R\to+\infty}\int_0^1 ds\int_0^R \frac{1}{\sqrt{2\pi i}}\frac{d}{dT}\varphi\,\mathrm{tr}_s\left[\frac{dA(s,T)}{ds}e^{-F(s,T)}\right]dT$$

$$+ \frac{1}{\sqrt{2\pi i}}\varphi\int_0^1 ds\int_0^{+\infty} dv(s,T)$$

$$= \int_0^1 ds \lim_{T\to+\infty}\frac{1}{\sqrt{2\pi i}}\varphi\,\mathrm{tr}_s\left[\left(\sqrt{T}(g-f)\pi_X^*\left(\mathrm{Id}_{S^*(N_X)}\widehat{\otimes}V\right)\right)e^{-F(s,T)}\right]$$

$$+ \frac{1}{\sqrt{2\pi i}}\varphi\,d\int_0^1 ds\int_0^{+\infty} v(s,T)$$

$$= \frac{1}{\sqrt{2\pi i}}\varphi\,d\int_0^1 ds\int_0^{+\infty} v(s,T), \quad (3.45)$$

from which Proposition 3.3 follows. Q.E.D.

3.3. **Real embeddings and Chern-Simons current: a Riemann-Roch formula.** In this section, we prove the following result, which might be thought of as a Riemann-Roch type formula for the Chern-Simons currents γ^{ζ,W_f} and $\gamma^{\zeta,W} = \gamma^{\zeta_+,W_+} - \gamma^{\zeta_-,W_-}$.

Theorem 3.4. *For any closed form ω on M, one has*

$$\int_M \omega \gamma^{\zeta,W_f} - \int_M \omega \gamma^{\zeta,W} = \int_X (j^*\omega)\, \widehat{A}^{-1}(N_X)\gamma^{\xi,V}. \tag{3.46}$$

In other words, $\gamma^{\zeta,W_f} - \gamma^{\zeta,W} - \widehat{A}^{-1}(N_X)\gamma^{\xi,V}\delta_X$ is a current on M eliminating closed forms.

Proof. By using formula (3.43) for $f_s = sf$, one sees that for any closed form ω on M, one has

$$\int_M \omega \int_0^R \frac{1}{\sqrt{2\pi i}}\varphi \operatorname{tr}_s\left[W_f e^{-(\nabla^\zeta+\sqrt{T}W_f)^2}\right]\frac{dT}{2\sqrt{T}}$$
$$- \int_M \omega \int_0^R \frac{1}{\sqrt{2\pi i}}\varphi \operatorname{tr}_s\left[W e^{-(\nabla^\zeta+\sqrt{T}W)^2}\right]\frac{dT}{2\sqrt{T}}$$
$$= \int_M \omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}}\varphi \operatorname{tr}_s\left[\left(\sqrt{R}f\pi_X^*\left(\operatorname{Id}_{S^*(N_X)}\widehat{\otimes}V\right)\right)e^{-F(s,R)}\right]. \tag{3.47}$$

It is clear that in order to prove (3.46), one needs to evaluate the limit as $R \to +\infty$ of the right hand side of (3.47).

In the current case

$$A(s,T) = \nabla^\zeta + \sqrt{T}W_{sf} = \nabla^\zeta + \sqrt{T}\left(W + sV_f\right). \tag{3.48}$$

From (3.39), (3.48), the construction of (ζ_+,ζ_-,W) and the fact that $f \equiv 1$ on N_{X,ε_0}, one deduces that for any $T \geq 0$, $0 \leq s \leq 1$, one has on N_{X,ε_0} that

$$\operatorname{tr}_s\left[\left(\sqrt{T}f\pi_X^*\left(\operatorname{Id}_{S^*(N_X)}\widehat{\otimes}V\right)\right)e^{-F(s,T)}\right]$$
$$= \pi_X^*\operatorname{tr}_s\left[\sqrt{T}V e^{-(\nabla^\xi+s\sqrt{T}V)^2}\right] \cdot \operatorname{tr}_s\left[e^{-\left(\nabla^{\pi_X^* S(N_X)}+\sqrt{T}\tau^{N_X*}\widetilde{c}(Z)\right)^2}\right]. \tag{3.49}$$

Let $\psi \geq 0$ be a smooth function on M such that $\operatorname{Supp}(\psi) \subset N_{X,\varepsilon_0}$, $\psi \equiv 1$ on $N_{X,\frac{1}{2}\varepsilon_0}$.

By (3.49), one has

$$
\int_M \omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right]
$$

$$
= \int_{N_{X,\varepsilon_0}} \psi\omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right]
$$

$$
+ \int_M (1-\psi)\omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right], \quad (3.50)
$$

with

$$
\int_{N_{X,\varepsilon_0}} \psi\omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right]
$$

$$
= \int_X \int_0^1 \frac{ds}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\sqrt{T} V e^{-(\nabla^\xi + s\sqrt{T}V)^2} \right] \int_{N_X/X} \psi\omega\varphi \operatorname{tr}_s \left[e^{-\left(\nabla^{\pi_X^* S(N_X)} + \sqrt{T}\tau^{N_X*}\widetilde{c}(Z) \right)^2} \right]
$$

$$
= \int_X \int_0^T \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[V e^{-(\nabla^\xi + \sqrt{s}V)^2} \right] \frac{ds}{2\sqrt{s}} \int_{N_X/X} \psi\omega\varphi \operatorname{tr}_s \left[e^{-\left(\nabla^{\pi_X^* S(N_X)} + \sqrt{T}\tau^{N_X*}\widetilde{c}(Z) \right)^2} \right],
$$

$$
(3.51)
$$

where $\int_{N_X/X}$ is the integration of differential forms along the fibre of N_X over X.

By proceeding as in [11, Theorem 1.2], one sees that there exists $C > 0$ such that as $T > 0$ is large enough,

$$
\left\| \int_{N_X/X} \psi\omega\varphi \operatorname{tr}_s \left[e^{-\left(\nabla^{\pi_X^* S(N_X)} + \sqrt{T}\tau^{N_X*}\widetilde{c}(Z) \right)^2} \right] - (j^*\omega) \widehat{A}^{-1} \left(N_X, \nabla^{N_X} \right) \right\|_{C^1(X)}
$$

$$
\leq \frac{C}{\sqrt{T}} \|\omega\|_{C^2(M)}. \quad (3.52)
$$

By (3.52) and [11, Theorem 1.2] again, one sees that as $T \to +\infty$,

$$
\left| \int_X \left(\int_{N_X/X} \psi\omega\varphi \operatorname{tr}_s \left[e^{-\left(\nabla^{\pi_X^* S(N_X)} + \sqrt{T}\tau^{N_X*}\widetilde{c}(Z) \right)^2} \right] - (j^*\omega) \widehat{A}^{-1} \left(N_X, \nabla^{N_X} \right) \right) \right.
$$

$$
\left. \cdot \int_0^T \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[V e^{-(\nabla^\xi + \sqrt{s}V)^2} \right] \frac{ds}{2\sqrt{s}} \right| \to 0. \quad (3.53)
$$

From (2.12), (3.51) and (3.53), one finds that as $T \to +\infty$,

$$\int_{N_{X,\varepsilon_0}} \psi\omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right]$$

$$\to \int_X (j^*\omega) \widehat{A}^{-1} \left(N_X, \nabla^{N_X} \right) \gamma^{\xi,V}. \quad (3.54)$$

On the other hand, since $\operatorname{Supp}(f) \subset N_{X,2\varepsilon_0}$, if we write

$$\int_M (1-\psi)\omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right]$$

$$= \int_X \int_0^T \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[V e^{-(\nabla^\xi + \sqrt{s}V)^2} \right] \frac{ds}{2\sqrt{s}}$$

$$\cdot \int_{N_X/X} (1-\psi) f\omega\varphi \operatorname{tr}_s \left[e^{-\left(\nabla^{\pi_X^* S(N_X)} + \sqrt{T}\tau^{N_X*}\widetilde{c}(Z) \right)^2} \right], \quad (3.55)$$

then by noting that as $T \to +\infty$,

$$\int_0^T \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[V e^{-(\nabla^\xi + \sqrt{s}V)^2} \right] \frac{ds}{2\sqrt{s}}$$

grows polynomially in T (compare with [13, Lemma 6.1]), while since $1 - \psi \equiv 0$ on $N_{X,\frac{1}{2}\varepsilon}$,

$$\int_{N_X/X} (1-\psi) f\omega\varphi \operatorname{tr}_s \left[e^{-\left(\nabla^{\pi_X^* S(N_X)} + \sqrt{T}\tau^{N_X*}\widetilde{c}(Z) \right)^2} \right]$$

decays exponentially in T, one sees that as $T \to +\infty$,

$$\int_M (1-\psi)\omega \int_0^1 ds \frac{1}{\sqrt{2\pi i}} \varphi \operatorname{tr}_s \left[\left(\sqrt{T} f \pi_X^* \left(\operatorname{Id}_{S^*(N_X)} \widehat{\otimes} V \right) \right) e^{-F(s,T)} \right] \to 0. \quad (3.56)$$

From (3.47), (3.50), (3.54) and (3.56), one gets (3.46), which completes the proof of Theorem 3.4. Q.E.D.

Remark 3.5. Formula (3.56) also follows from the fact that $W_{sf}^2 > C$, for some positive constant $C > 0$, uniformly for $s \in [0,1]$. We thank the referee for pointing out this.

Remark 3.6. From (2.18) and (3.46), one formally gets

$$d\gamma^{\zeta,W_f} = \mathrm{ch}\left(\zeta,\nabla^\zeta\right) + \widehat{A}^{-1}\left(N_X,\nabla^{N_X}\right)\left(d\gamma^{\xi,V} - \mathrm{ch}\left(\xi,\nabla^\xi\right)\right)\delta_X$$

$$= \mathrm{ch}\left(\zeta,\nabla^\zeta\right) - \widehat{A}^{-1}\left(N_Y,\nabla^{N_Y}\right)\mathrm{ch}\left(\mu,\nabla^\mu\right)\delta_Y, \quad (3.57)$$

which fits with (2.13) again.

Corollary 3.7. *Under the assumptions and notations above, the following identity in* $\mathbf{R}/\mathbf{Z}$ *holds,*

$$H(Y,M,\mu) = H(Y,X,\mu) + H(X,M,\xi_+) - H(X,M,\xi_-). \qquad (3.58)$$

Proof. By (3.1) and (3.46), one has

$$H_M\left(\zeta_+,\zeta_-,W_f\right) \equiv \overline{\eta}\left(D^{\zeta_+}\right) - \overline{\eta}\left(D^{\zeta_-}\right) - \int_M \widehat{A}\left(TM,\nabla^{TM}\right)\gamma^{\zeta,W_f} - \overline{\eta}\left(D^\mu\right) \mod \mathbf{Z}$$

$$\equiv \overline{\eta}\left(D^{\zeta_{++}}\right) + \overline{\eta}\left(D^{\zeta_{+-}}\right) - \overline{\eta}\left(D^{\zeta_{-+}}\right) - \overline{\eta}\left(D^{\zeta_{+-}}\right) - \int_M \widehat{A}\left(TM,\nabla^{TM}\right)\gamma^{\zeta,W}$$

$$- \int_X \widehat{A}\left(TX,\nabla^{TX}\right)\gamma^{\xi,V} - \overline{\eta}\left(D^\mu\right) \mod \mathbf{Z}$$

$$\equiv \overline{\eta}\left(D^{\zeta_{++}}\right) - \overline{\eta}\left(D^{\zeta_{+-}}\right) - \int_M \widehat{A}\left(TM,\nabla^{TM}\right)\gamma^{\zeta_+,W_+} - \overline{\eta}\left(D^{\xi_+}\right)$$

$$- \left(\overline{\eta}\left(D^{\zeta_{-+}}\right) - \overline{\eta}\left(D^{\zeta_{--}}\right) - \int_M \widehat{A}\left(TM,\nabla^{TM}\right)\gamma^{\zeta_-,W_-} - \overline{\eta}\left(D^{\xi_-}\right)\right)$$

$$+ \overline{\eta}\left(D^{\xi_+}\right) - \overline{\eta}\left(D^{\xi_-}\right) - \int_X \widehat{A}(TX,\nabla^{TX})\gamma^{\xi,V} - \overline{\eta}\left(D^\mu\right) \mod \mathbf{Z}$$

$$= H_M\left(\zeta_{++},\zeta_{+-},W_+\right) - H_M\left(\zeta_{-+},\zeta_{--},W_-\right) + H_X\left(\xi_+,\xi_-,V\right). \quad (3.59)$$

By (3.34) and (3.59), one gets (3.58), which completes the proof of Corollary 3.7. Q.E.D.

3.4. **Proof of the Bismut-Zhang localization formula.** Recall that by Remark 2.3 one knows that there exists a totally geodesic embedding $i_0 : Y \hookrightarrow S^{2m-1}$ such that

$$H\left(Y,S^{2m-1},\mu\right) = 0. \qquad (3.60)$$

We first show that this holds for any embedding of Y to an arbitrary sphere.

Lemma 3.8. *Let μ be a complex vector bundle over an odd dimensional closed oriented spin manifold Y. Then for any embedding $i : Y \hookrightarrow S^{2n-1}$, the following identity in $\mathbf{R}/\mathbf{Z}$ holds,*

$$H\left(Y, S^{2n-1}, \mu\right) = 0. \tag{3.61}$$

Proof. For the two embeddings $i_0 : Y \hookrightarrow S^{2m-1}$ and $i : Y \hookrightarrow S^{2n-1}$, we first consider the associated embeddings $i_0' : Y \hookrightarrow S^{2m-1}(1) \subset \mathbf{R}^{2m}$ and $i' : Y \hookrightarrow S^{2n-1}(1) \subset \mathbf{R}^{2n}$ to the standard unit spheres.

By using a trick in [4, Page 498], we construct a smooth family of embeddings $j_s : Y \hookrightarrow S^{2m+2n-1}(1)$, $0 \le s \le 1$, obtained by

$$y \in Y \mapsto \frac{1}{\sqrt{s^2 + (1-s)^2}} \left((1-s)i_0'y, si'y\right) \in S^{2m+2n-1}(1) \subset \mathbf{R}^{2m} \oplus \mathbf{R}^{2n}.$$
$$\tag{3.62}$$

Now since j_0 and j_1 are isotropic to each other by (3.62), by Lemma 3.1 one has

$$H_{j_0}\left(Y, S^{2m+2n-1}, \mu\right) = H_{j_1}\left(Y, S^{2m+2n-1}, \mu\right). \tag{3.63}$$

On the other hand, by the Bott periodicity, any complex vector bundle over an odd dimensional sphere can be expressed as a difference of two trivial vector bundles so that can be extended to a bounding ball. By using the simple arguments in [7] (see also [16]) which apply to this situation directly, one sees that

$$H\left(S^{2k-1}, S^{2k'-1}, \nu\right) = 0 \tag{3.64}$$

for any standard embedding between spheres and ν any complex vector bundle over S^{2k-1}.

Now by applying Corollary 3.7 and (3.64) to the successive embedding $j_0 : Y \hookrightarrow S^{2m-1} \hookrightarrow S^{2m+2n-1}$ and $j_1 : Y \hookrightarrow S^{2n-1} \hookrightarrow S^{2m+2n-1}$ respectively, one gets

$$H_{j_0}\left(Y, S^{2m+2n-1}, \mu\right) = H\left(Y, S^{2m-1}, \mu\right) = 0 \tag{3.65}$$

and

$$H_{j_1}\left(Y, S^{2m+2n-1}, \mu\right) = H\left(Y, S^{2n-1}, \mu\right) \tag{3.66}$$

respectively.

From (3.63), (3.65) and (3.66), one gets (3.61).   Q.E.D.

We now come to the proof of the Bismut-Zhang localization Theorem 2.1 which is equivalent to saying that for any embedding $i : Y \hookrightarrow X$ between two odd dimensional closed oriented spin manifolds and a complex vector bundle μ over Y, one has

$$H(Y, X, \mu) = 0. \tag{3.67}$$

Indeed, let $j : X \hookrightarrow S^{2N-1}$ be a further embedding of X into a higher odd dimensional sphere, let $\xi = \xi_+ \oplus \xi_-$ be a $\mathbf{Z}_2$-graded vector bundle over X so that by giving suitable metrics, connections and odd endomorphism V of ξ, (ξ_+, ξ_-, V) realizes a direct image of μ in the sense of Section 2.2.

By Corollary 3.7 and Lemma 3.8, one deduces that

$$H(Y, X, \mu) = H\left(Y, S^{2N-1}, \mu\right) - H\left(X, S^{2N-1}, \xi_+\right) + H\left(X, S^{2N-1}, \xi_-\right) = 0, \tag{3.68}$$

which via (3.67) completes the proof of Theorem 2.1.   Q.E.D.

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