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Eta Invariants of Homogeneous Spaces

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Dedicated to Prof. Friedrich Hirzebruch on the occasion of his 80th birthday

Abstract: We derive a formula for the η -invariants of equivariant Dirac operators on quotients of compact Lie groups, and for their infinitesimally equivariant extensions. As an example, we give some computations for spheres.

Keywords: equivariant eta invariants, homogeneous spaces, cubic Dirac operator, Bott localization defect.

Quotients $M = G/H$ of compact Lie groups provide many important examples of Riemannian manifolds with non-negative sectional curvature. The primary characteristic classes and numbers of these spaces have been computed by Borel and Hirzebruch in [5].

The η -invariant has been introduced by Atiyah, Patodi and Singer in [1] as a boundary contribution in an index theorem for manifolds with boundary. It can be used to construct certain secondary invariants of compact manifolds M that were originally defined using zero-bordisms. For example, the Eells-Kuiper and Kreck-Stolz invariants distinguish homeomorphic homogeneous manifolds that are not diffeomorphic. These invariants can be expressed in terms of η -invariants and Chern-Simons numbers, see e.g. [6], [15]. Although the diffeomorphism type of many homogeneous manifolds G/H is well-known, in some cases the explicit

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values of certain η -invariants are needed to complete the diffeomorphism classification. It is therefore worthwhile to have a formula for the η -invariants of equivariant Dirac operators on homogeneous spaces.

First steps in this direction have been made in [8], [9], [10]. There, we computed the equivariant η -invariant of a different operator, called “reductive” or “cubic” Dirac operator, and explained how to recover the η -invariant of the classical Dirac operator. However, one complicated local term remained, called “Bott localisation defect” below. The central result of the present article is a formula for this defect term that is similar to the formula for the equivariant η -invariant of the reductive Dirac operator itself. Thus we get a tractable formula for equivariant η -invariants of homogeneous spaces that is useful for explicit computations. This formula has already been applied in a joint paper [11] with N. Kitchloo and K. Shankar to calculate the Eells-Kuiper invariant of the Berger space $SO(5)/SO(3)$, and to determine its diffeomorphism type. Our formula can be summarised as follows; details and notation will be explained later in the article.

Theorem (see Theorem 2.33 below). *Let $G \supset H$ be compact Lie groups, and let D^κ be the equivariant Dirac operator on $M = G/H$ twisted by the local bundle $V^\kappa M$ associated to an irreducible \mathfrak{h} -representation with highest weight κ . Then $\eta(D^\kappa)$ is a sum of the following terms:*

- (1) *a representation theoretic expression that vanishes if $\text{rk } G \neq \text{rk } H + 1$, that depends on the position of \mathfrak{h} in \mathfrak{g} and on κ , and that takes the form*

$$2 \sum_{w \in W_G} \frac{\text{sign}(w)}{\delta(wX)} \left(\hat{A}(\delta(wX)) e^{-(\alpha - \frac{\delta}{2})(wX)} \prod_{\beta \in \Delta_G^+} \hat{A}(\beta(wX)) \right. \\ \left. - e^{-(\kappa + \rho_H)(wX|_{\mathfrak{s}})} \prod_{\beta \in \Delta_G^+} \hat{A}(\beta(wX|_{\mathfrak{s}})) \right) \prod_{\beta \in \Delta_G^+} \frac{-1}{\beta(X)} \Big|_{X=0}$$

if $\text{rk } G = \text{rk } H + 1$,

- (2) *a local Chern-Simons theoretic contribution*

$$2 \int_M \tilde{\hat{A}}(TM, \nabla^0, \nabla^{\text{LC}}) \text{ch}(V^\kappa M, \nabla^\kappa)$$

given by the integral of a constant multiple of the volume form of M , and

(3) the integer

$$\sum_{\gamma \in \tilde{G}} \dim(V^\gamma) \left(\eta(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right)$$

arising as the spectral flow between two equivariant Dirac-type operators.

We also give a reformulation of this result for the η -invariant of the odd signature operator. More generally, we also obtain a formula for the infinitesimally equivariant η -invariant $\eta_{\mathfrak{g}}(D^\kappa)$ of [10]; this invariant is the universal η -form for all families with fibrewise Dirac operator D^κ and compact structure group G . In fact, we will compute the classical η -invariant $\eta(D^\kappa)$ by evaluating $\eta_{\mathfrak{g}}(D^\kappa)$ at $X = 0$. Our main result is stated in Theorem 2.33 for general equivariant Dirac operators, and in Corollary 2.34 for the slightly different special case of the odd signature operator. As an example, we compute the infinitesimally equivariant η -invariants of the untwisted Dirac operator and the odd signature operator for round spheres, and thus we obtain the corresponding η -forms of sphere bundles with compact structure group.

Let us sketch our method. In [8], [9], we presented a formula for the classical equivariant η -invariant of Slebarski's deformed Dirac operator $\tilde{D}^\kappa = D^{\frac{1}{3},\kappa}$ ([16], [14]). This invariant is a function $\eta_G(\tilde{D}^\kappa): G \rightarrow \mathbb{C}$ that is continuous on the subset $G_0 \subset G$ of elements that act freely on M , and its value at the neutral element $e \in G$ is just $\eta(\tilde{D}^\kappa)$. The singularity of $\eta_G(\tilde{D}^\kappa)$ near e has implications on fixpoint sets of G -manifolds N with $\partial N = M$, see [10]. The difference between $\eta_G(D^\kappa)$ and $\eta_G(\tilde{D}^\kappa)$ is comparatively easy to control on G_0 , but since $\eta_G(D^\kappa)$ and $\eta_G(\tilde{D}^\kappa)$ are not continuous at e , we do not obtain the value of $\eta(D^\kappa)$ by this method.

Instead, we will use the infinitesimally equivariant η -invariant $\eta_{\mathfrak{g}}(D^\kappa)$ of [10], which is a power series on \mathfrak{g} with constant term $\eta(D^\kappa)$. For $X \in \mathfrak{g}$ without zeros on M , the infinitesimal η -invariant $\eta_X(D^\kappa)$ differs from the classical equivariant η -invariant $\eta_{e-x}(D^\kappa)$ by the integral of a certain differential form on M . This way, one obtains a formula for the power series $\eta_{\mathfrak{g}}(D^\kappa)$, and the classical η -invariant is just its constant term.

However, the integrand in the formula for the difference $\eta_X(D^\kappa) - \eta_{e-x}(D^\kappa)$ stated in [10] is in general not invariant under the action of G , so that one cannot reduce the problem to a calculation at a single point in M . The new contribution

in the present article is an evaluation of this integral in terms of representation theoretic data of the groups G and H and the relative position of their maximal tori, up to an equivariant Chern-Simons term, see Theorem 2.30. This is an improvement compared with respect to [8], [9], [10] for two reasons. First, our formula for $\eta(D^\kappa)$ involves no more equivariant differential forms. Second, for symmetric spaces, the operators D^κ and \tilde{D}^κ are equal, which means that there will be no Chern-Simons term and no spectral flow.

This paper is organised as follows. In section 1, we recall the results of [8], [9], [10] on equivariant η -invariants. We also calculate the infinitesimally equivariant η -invariants of spheres. In section 2, we give a formula for the Bott localisation defect on homogeneous spaces and present our main result.

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1. EQUIVARIANT η -INVARIANTS

We recall some facts about η -invariants and homogeneous spaces from [8], [9], [10]. In section 1.3, we compute the infinitesimally equivariant η -invariant for the untwisted Dirac operator and the signature operator on round spheres.

1.1. Equivariant η -invariants and their infinitesimal analogues.

Let (M, g^M) be an oriented Riemannian manifold, and let G be a compact group that acts on M by isometries. Let $\mathcal{E} \rightarrow M$ be a G -equivariant Hermitian vector bundle, equipped with a G -equivariant Clifford multiplication. Let ∇^{TM} denote a metric connection on M , and let $\nabla^\mathcal{E}$ be a G -equivariant unitary connection on \mathcal{E} that satisfies the Leibniz rule

$$\nabla^\mathcal{E}(v \cdot s) = \nabla^{TM} v \cdot s + v \cdot \nabla^\mathcal{E} s$$

for all vector fields v on M and all sections $s \in \Gamma(\mathcal{E})$. Then \mathcal{E} is a G -equivariant Clifford module over M , and we have the G -equivariant Dirac operator

$$D: \Gamma(\mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(T^*M \otimes \mathcal{E}) \xrightarrow{g^M} \Gamma(TM \otimes \mathcal{E}) \longrightarrow \Gamma(\mathcal{E}).$$

The most natural choice for ∇^{TM} is the Levi-Civita connection ∇^{LC} , however, on homogeneous spaces it is easier to work with the connection $\nabla^{\frac{1}{3}}$, see section 1.2.

Let $g \in G$, assume $\Re s \geq s_0 > 0$, and define

$$\eta_g(D, s) = \sum_{\lambda} \text{sign}(\lambda) |\lambda|^{-s} \text{tr}(g|_{E_{\lambda}}) = \int_0^{\infty} \frac{t^{\frac{s-1}{2}}}{\Gamma(s + \frac{1}{2})} \text{tr}(gD e^{-tD^2}) dt, \tag{1.1}$$

cf. [7] and [17], where E_{λ} denotes the Eigenspace of the Eigenvalue λ . The function $\eta_g(D, s)$ admits a meromorphic continuation to \mathbb{C} that is finite at $s = 0$, and the *equivariant η -invariant* of D at g is given by

$$\eta_g(D) = \eta_g(D, 0). \tag{1.2}$$

If we have chosen $\nabla^{TM} = \nabla^{LC}$, then the second representation of $\eta_G(D, s)$ actually converges at $s = 0$. The kernel of D does not contribute to (1.1); we define

$$h_g(D) = \text{tr}(g|_{\ker D}).$$

To define the infinitesimal analogue of $\eta_g(D)$, let $X \in \mathfrak{g}$ be an element of the Lie algebra of G , and define the Killing field X_M on M by

$$X_M|_p = \left. \frac{d}{dt} \right|_{t=0} e^{-tX} p. \tag{1.3}$$

Let $\mathcal{L}_X^{\mathcal{E}}$ denote the infinitesimal action of X on sections of \mathcal{E} . We take a Dirac operator D associated with the Levi-Civita connection, and we construct deformations of D and D^2 by

$$D_X = D - \frac{1}{4} X_M \cdot \quad \text{and} \quad H_X = D_{-X}^2 + \mathcal{L}_X^{\mathcal{E}},$$

where X_M acts by Clifford multiplication, see [2]. Then the *infinitesimal \mathfrak{g} -equivariant η -invariant* $\eta_{\mathfrak{g}}(D) \in \mathbb{C}[[\mathfrak{g}^*]]$ of D is defined as

$$\eta_X(D) = \int_0^{\infty} \frac{1}{\sqrt{\pi t}} \text{tr} \left(D_X e^{-tH_X/t} \right) dt, \tag{1.4}$$

see [10], cf. [3], [4]. Note that it is conjecturally possible to define $\eta_{\mathfrak{g}}(D)$ as a function of rX for $X \in \mathfrak{g}$ and $r \in \mathbb{R}$ sufficiently small, rather than as a formal power series. It is also possible to define a mixed equivariant η -invariant $\eta_{g,X}(D)$, where $g \in G$ and $X \in \mathfrak{g}$ commute; this mixed invariant would then occur in an equivariant index theorem for manifolds with boundary combining Theorem 1.6 (1) and (2), and in an equivariant index theorem for families with compact structure group, cf. Remark 1.7. However, this extra generality is not required for the applications we have in mind.

The equivariant η -invariant and its infinitesimal cousin are key ingredients in theorem 1.6 below, which is due to Atiyah, Patodi, Singer [1] and Donnelly [7], and its infinitesimal analogue in [10], which is related to the family index theorem of Bismut and Cheeger in [4]. Suppose that M is the boundary of a compact oriented manifold N , which has a collar isometric to $M \times [0, \varepsilon]$. Suppose that there exists a Clifford module $\bar{\mathcal{E}} = \bar{\mathcal{E}}^+ \oplus \bar{\mathcal{E}}^- \rightarrow N$, such that $\bar{\mathcal{E}}^+|_M \cong \mathcal{E}$ with compatible Clifford multiplications. Then the Dirac operator D is closely related to the Dirac operator \bar{D} acting on sections of $\bar{\mathcal{E}}$.

We also need equivariant characteristic differential forms. The *equivariant Riemannian curvature* is defined as

$$R_X = (\nabla^{\text{LC}} - 2\pi i \iota_{X_M})^2 + 2\pi i \mathcal{L}_X^{TM} = R + 2\pi i \mu_X^{\text{LC}},$$

where ι_{X_M} denotes interior multiplication of a differential form by X_M , and $\mu_X^{\text{LC}} = \mathcal{L}_X^{TM} - \nabla_{X_M}^{\text{LC}}$ is the *Riemannian moment*. Similarly, define

$$F_X^{\mathcal{E}} = (\nabla^{\mathcal{E}} - 2\pi i \iota_{X_M})^2 + 2\pi i \mathcal{L}_X^{\mathcal{E}} = F^{\mathcal{E}} + 2\pi i \mu_X^{\mathcal{E}}.$$

Following [2], the *equivariant twisting curvature* of \mathcal{E} is defined as

$$F_X^{\mathcal{E}/\mathcal{S}} = F_X^{\mathcal{E}} - \frac{1}{4} \sum_{i,j} \langle R_X e_i, e_j \rangle e_i \cdot e_j, \tag{1.5}$$

where e_1, \dots, e_n is a local orthonormal frame of TM , which acts on \mathcal{E} by Clifford multiplication. Then we have the equivariant characteristic differential forms

$$\hat{A}_X(TM, \nabla^{\text{LC}}) = \det^{\frac{1}{2}} \left(\frac{R_X/4\pi i}{\sinh(R_X/4\pi i)} \right)$$

and

$$\text{ch}_X(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) = \text{tr} \left(e^{-\frac{F_X^{\mathcal{E}/\mathcal{S}}}{2\pi i}} \right).$$

Recall that the equivariant exterior derivative is defined as

$$d_X = d - 2\pi i \iota_{X_M},$$

such that $d_X^2 + 2\pi i \mathcal{L}_X = 0$. Then the forms $\hat{A}_X(TM, \nabla^{\text{LC}})$ and $\text{ch}_X(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}})$ are G -invariant, equivariantly closed, and independent of the choice of an equivariant connection up to equivariantly exact forms. Note that in contrast to [2] and [10], we use the classical convention of including powers of $2\pi i$ in all definitions above. Let N_g denote the fixpoint set of g on N .

Let $\text{ind}_g(\overline{D}) = \text{str}(g|_{\ker D})$ denote the equivariant index of \overline{D} (regarded as the character of a virtual G -representation at $g \in G$) with respect to the APS boundary conditions.

1.6. Theorem ([1], [7], [10]). *Let α_g denote the characteristic form on N_g given by the constant term in the asymptotic development of $\text{str}(g e^{-t\overline{D}^2})$ for $t \rightarrow 0$, then*

$$\text{ind}_g(\overline{D}) = \int_{N_g} \alpha_g(D) - \frac{\eta_g(D) + h_g(D)}{2}. \tag{1}$$

Assume moreover that \overline{D} is associated to the Levi-Civita connection, then so is D , and

$$\text{ind}_{e^{-x}}(\overline{D}) = \int_N \hat{A}_X(TM, \nabla^{\text{LC}}) \text{ch}_X(\mathcal{E}/\mathcal{S}, \nabla \mathcal{E}) - \frac{\eta_X(D) + h_{e^{-x}}(D)}{2}. \tag{2}$$

1.7. Remark. Theorem 1.6 (2) is in fact a special case of the Bismut-Cheeger index theorem for families of manifolds with boundary in [4]. Suppose that $P \rightarrow B$ is a G -principal bundle with curvature form ω and curvature Ω . If we consider the family $P \times_G N \rightarrow B$ with boundary $P \times_G M \rightarrow B$, then Bismut-Cheeger’s theorem is equivalent to (2) above by the Chern-Weil construction. In particular, we can recover the η -form on B of the family of Dirac operators induced by D on M from $\eta_X(D)$.

We now recall the relation between the two equivariant η -invariants introduced in (1.2) and (1.4) above. From the dual of the Killing field X_M , we construct an equivariant differential form

$$\vartheta_X = \frac{1}{2\pi i} \langle X_M, \cdot \rangle. \tag{1.8}$$

For $X \neq 0$, we have an L_1 -current

$$\frac{\vartheta_X}{d_X \vartheta_X} = - \int_0^\infty \vartheta_X e^{t d_X \vartheta_X} dt$$

on M by Proposition 2.2 below. Note that $\frac{\vartheta_X}{d_X \vartheta_X}$ has a singularity at the zero set of X , but for a smooth compactly supported form α on M , the integral $\int_M \frac{\vartheta_X}{d_X \vartheta_X} \alpha$ is still well-defined, hence the term “current”. The current $\frac{\vartheta_X}{d_X \vartheta_X}$ will be referred to as the *Bott localisation defect* because of (2.1) below.

1.9. **Theorem** ([10], Theorem 0.5). *If the Killing field X_M has no zeros on M , then*

$$\eta_{rX}(D) = \eta_{e^{-rX}}(D) + 2 \int_M \frac{\vartheta_{rX}}{d_{rX} \vartheta_{rX}} \hat{A}_{rX}(M) \operatorname{ch}_{rX}(\mathcal{E}/\mathcal{S}) \in \mathbb{C}[[r]].$$

If there exists a G -manifold manifold N that bounds M equivariantly and if there exists a Dirac operator \bar{D} on N related to D as above, then Theorem 1.9 can be proved by comparing Theorem 1.6 (1) and (2) using Kalkman’s localisation formula for manifolds with boundaries ([13]). A general argument is given in [10].

We will give a formula for the Bott localisation defect in section 2 in the special case that M is a homogeneous space of compact type.

1.2. **Homogeneous spaces.** We recall the formula for the equivariant η -invariants of reductive Dirac operators, and its relation to the η -invariants of Riemannian Dirac operators.

Let $M = G/H$ be a quotient of compact Lie groups with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Recall that all G -equivariant vector bundles over $M = G/H$ are of the form

$$V^\kappa M = G \times_\kappa V^\kappa \rightarrow M,$$

where (κ, V^κ) is a representation of H . We identify sections s of $V^\kappa M$ with H -equivariant functions

$$\hat{s}: G \rightarrow V^\kappa \quad \text{with} \quad \hat{s}(gh) = \kappa_h^{-1} \hat{s}(g)$$

such that $s(gH) = [g, \hat{s}(g)]$.

We fix an Ad_G -invariant metric on \mathfrak{g} and let $\mathfrak{m} = \mathfrak{h}^\perp \subset \mathfrak{g}$. Let $\pi = \operatorname{Ad}|_{H \times \mathfrak{m}}$ denote the isotropy representation of H on \mathfrak{m} . Then the tangent bundle TM is isomorphic to $G \times_\pi \mathfrak{m}$ via

$$[g, v] = \frac{d}{dt} g e^{tv} \tag{1.10}$$

and carries an induced *normally homogeneous* Riemannian metric.

On most bundles $V^\kappa M$, we will mainly use the reductive connection $\nabla^0 = \nabla^{0,\kappa}$, which can be written as

$$\widehat{\nabla_V^0} s = \hat{V}(\hat{s}). \tag{1.11}$$

If κ is a unitary (orthogonal) representation, then $\nabla^{0,\kappa}$ is a unitary (orthogonal) connection. A subscript \mathfrak{h} or \mathfrak{m} will denote orthogonal projection to that subspace

of \mathfrak{g} . One easily calculates the curvature of $\nabla^{0,\kappa}$ as

$$\widehat{F_{V,W}^{0,\kappa}} = -\kappa_*[\hat{V}, \hat{W}]_{\mathfrak{h}} \hat{s} . \tag{1.12}$$

We consider a family of connections on TM given by

$$\widehat{\nabla_V^t W} = \hat{V}(\hat{W}) + t [\hat{V}, \hat{W}]_{\mathfrak{m}} . \tag{1.13}$$

For $t = 0$, we obtain again the reductive connection. Note that because the Lie bracket of vector fields on M is given by

$$[\widehat{V}, \widehat{W}] = \hat{V}(\hat{W}) - \hat{W}(\hat{V}) + [\hat{V}, \hat{W}]_{\mathfrak{m}} ,$$

the Levi-Civita connection on TM with respect to a normal metric is $\nabla^{\text{LC}} = \nabla^{0,\pi} + \frac{1}{2} [\cdot, \cdot]_{\mathfrak{m}}$.

An equivariant Clifford module over M is a G -equivariant vector bundle $\mathcal{E} \rightarrow M$ together with a G -equivariant Clifford multiplication $TM \times \mathcal{E} \rightarrow \mathcal{E}$. We can regard the isotropy representation as a homomorphism $\pi: \mathfrak{h} \rightarrow \mathfrak{spin}(\mathfrak{m})$, where $\mathfrak{spin}(\mathfrak{m})$ denotes the spin group associated to the vector space \mathfrak{m} . Let n be the dimension of M , and let $\tilde{\pi}$ denote the pullback of the spin representation of $\mathfrak{spin}(\mathfrak{m})$ to \mathfrak{h} , which acts on a complex $2^{\lfloor \frac{n}{2} \rfloor}$ -dimensional vector space S . If $\tilde{\pi}$ integrates to an H -representation, then $\mathcal{S} = G \times_H S$ is the G -equivariant spinor bundle (unique if H is connected), which is then the most elementary Clifford bundle. In general, every equivariant Clifford module is of the form

$$\mathcal{S}^\kappa M = G \times_H (S \otimes W^\kappa) \rightarrow M , \tag{1.14}$$

where κ is an \mathfrak{h} -representation such that $\tilde{\pi} \otimes \kappa$ integrates to an H -representation ([9], Lemma 3.4). For example, if M is even-dimensional, the complexified bundle of exterior differential forms on M is precisely the G -equivariant Clifford module

$$\Lambda^* TM \otimes \mathbb{C} = \mathcal{S}^{\tilde{\pi}} M = G \times_H (S \otimes S) \rightarrow M .$$

Let $\mathcal{S}^\kappa M \rightarrow M$ be a G -equivariant Clifford module. Following Slebarski [16], we define a family of Dirac operators on $\Gamma(\mathcal{S}^\kappa M)$. For $X \in \mathfrak{g}$, we define $\tilde{\text{ad}}_{\mathfrak{p},X} \in \text{End} S$ by

$$\tilde{\text{ad}}_{\mathfrak{p},X} = \frac{1}{4} \sum_{i,j} \langle [X, e_i], e_j \rangle e_i \cdot e_j \cdot ,$$

where e_i, e_j run through an orthonormal base of \mathfrak{m} . Note that $\widetilde{\text{ad}}_{\mathfrak{p},X} = \widetilde{\pi}_*X$ for $X \in \mathfrak{h}$. Then we set

$$\widehat{D^{t,\kappa}}_{\mathfrak{s}} = \sum_i e_i \cdot (e_i(\hat{s}) + t \widetilde{\text{ad}}_{\mathfrak{p},e_i}).$$

For $t = \frac{1}{2}$, the operator $D^\kappa = D^{\frac{1}{2},\kappa}$ is associated to the Levi-Civita connection on $\mathcal{S}M$ and the reductive connection on $V^\kappa M$ (but these two bundles are in general only locally well defined).

The operator $\widetilde{D}^\kappa = D^{\frac{1}{3},\kappa}$ has distinguished properties. It was shown in [16] and [8], [9] that \widetilde{D}^κ exhibits a very natural behaviour with respect to homogeneous fibrations, hence it was called the “reductive” Dirac operator in [9] (although it is not constructed from the reductive connection on $S^\kappa M$).

In order to state the formula for $\eta_G(\widetilde{D}^\kappa)$, we need some more notation and conventions. We choose maximal tori $S \subset T$ of $H \subset G$ with Lie algebras $\mathfrak{s} \subset \mathfrak{t}$, and fix Weyl chambers $P_G \subset i\mathfrak{t}^*$ and $P_H \subset i\mathfrak{s}^*$. Let Δ_G^+ and Δ_H^+ denote the corresponding sets of positive roots, and let ρ_G and ρ_H be their half sums. The choices of P_G and P_H also determine orientations on $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{h}/\mathfrak{s}$ as follows. If $\beta_1, \dots, \beta_l \in i\mathfrak{t}^*$ are the positive roots of \mathfrak{g} with respect to P_G , we can choose a complex structure on $\mathfrak{g}/\mathfrak{t}$ and a complex basis z_1, \dots, z_n such that $\text{ad}|_{\mathfrak{t} \times (\mathfrak{g}/\mathfrak{t})}$ takes the form

$$\text{ad}_X = \begin{pmatrix} \beta_1(X) & & \\ & \ddots & \\ & & \beta_l(X) \end{pmatrix} \quad \text{for all } X \in \mathfrak{t}. \tag{1.15}$$

Then we declare the basis $z_1, iz_1, z_2, \dots, iz_l$ of $\mathfrak{g}/\mathfrak{t}$ as a real vector space to be positive oriented. The orientation of H/S is constructed similarly.

If we fix an orientation on $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ and choose orientations on $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{h}/\mathfrak{s}$ as above, there is a unique orientation on $\mathfrak{t}/\mathfrak{s}$ such that the two induced orientations on

$$\mathfrak{g}/\mathfrak{s} \cong \mathfrak{m} \oplus (\mathfrak{h}/\mathfrak{s}) \cong (\mathfrak{g}/\mathfrak{t}) \oplus (\mathfrak{t}/\mathfrak{s}) \tag{1.16}$$

agree. We assume for the moment that $\text{rk} G = \text{rk} H + 1$, so $\mathfrak{s}^\perp \subset \mathfrak{t}$ is one-dimensional. Let $E \in \mathfrak{t}/\mathfrak{s} \cong \mathfrak{s}^\perp$ be the positive unit vector, and let $\delta \in i\mathfrak{t}^*$ be the unique weight such that

$$-i\delta(E) > 0 \quad \text{and} \quad \delta(X) \in 2\pi i \mathbb{Z} \iff e^X \in S \tag{1.17}$$

for all $X \in \mathfrak{t}$.

By abuse of notation, let $\kappa \in i\mathfrak{s}^*$ denote the highest weight of the \mathfrak{h} -representation κ used to construct $\mathcal{S}^\kappa M$. Then there is a unique weight $\alpha \in i\mathfrak{t}^*$ of \mathfrak{g} such that

$$\alpha|_{\mathfrak{s}} = \kappa + \rho_H \quad \text{and} \quad -i(\alpha - \delta)(E) < 0 \leq -i\alpha(E). \quad (1.18)$$

Let A_G denote the alternating sum over the Weyl group of G acting on \mathfrak{t} , and write $A_G(\rho_G)$ shorthand for $A_G(e^{\rho_G(\cdot)})$. Recall that for $X \in \mathfrak{t}$, $e^X \in T$ is regular iff $A_G(\rho_G)(X) \neq 0$. We can now compute the classical equivariant eta-invariant of \tilde{D}^κ .

1.19. Theorem ([8], [9]). *If $\text{rk } G \neq \text{rk } H + 1$, then $\eta_G(\tilde{D}^\kappa) = 0$ identically. If $\text{rk } G = \text{rk } H + 1$, then $\eta_G(\tilde{D}^\kappa)$ is continuous on the set $T_0 \subset T$ that acts freely on M . Moreover, if $\alpha, \delta \in i\mathfrak{s}^*$ and $E \in \mathfrak{t}$ are given as above, then for all $X \in \mathfrak{t}$ such that $e^X \in T_0$ and e^X is regular,*

$$(\eta + h)_{e^X}(\tilde{D}_M^\kappa) = \frac{A_G\left(\frac{1}{\sinh(\frac{\delta}{2}(\cdot))} e^{(\alpha - \frac{\delta}{2})(\cdot)}\right)}{A_G(\rho_G)}(X).$$

1.3. Round Spheres. As an example for our main result, we compute the infinitesimally equivariant η -invariant of the untwisted Dirac operator and the odd signature operator for odd-dimensional spheres. Note that for a symmetric space, the last two terms in Theorem 2.33 (1) and (2) vanish, so in particular, we do not have to integrate equivariant Chern-Simons classes.

Fix n and embed $G = \text{Spin}(2n) \subset \text{Cl}(2n)$. We let $H = \text{Spin}(2n - 1)$ denote the subgroup belonging to the Clifford subalgebra spanned by the vectors e_1, \dots, e_{2n-1} of an orthonormal base of \mathbb{R}^{2n} . Then we can write $S^{2n-1} = G/H$.

We choose the maximal tori $S \subset H$ and $T \subset G$ with Lie algebras

$$\begin{aligned} \mathfrak{t} &= \left\{ X = \frac{x_1}{2} e_1 e_2 + \dots + \frac{x_n}{2} e_{2n-1} e_{2n} \mid (x_1, \dots, x_n) \in \mathbb{R}^n \right\} \\ \text{and} \quad \mathfrak{s} &= \{ X \in \mathfrak{t} \mid x_n = 0 \}. \end{aligned}$$

We also choose Weyl chambers

$$\begin{aligned} P_G &= \{ i(\gamma_1 x^1 + \dots + \gamma_n x^n) \mid \gamma_1 \geq \dots \geq \gamma_{n-1} \geq |\gamma_n| \} \\ \text{and} \quad P_H &= \{ i(\gamma_1 x^1 + \dots + \gamma_{n-1} x^{n-1}) \mid \gamma_1 \geq \dots \geq \gamma_{n-1} \geq 0 \}. \end{aligned}$$

Then we have

$$\begin{aligned} \Delta_G^+ &= \{ i(x_j \pm x_k) \mid 1 \leq j < k \leq n \} \\ \text{and } \Delta_H^+ &= \{ i(x_j \pm x_k) \mid 1 \leq j < k \leq n-1 \} \cup \{ ix_j \mid 1 \leq j \leq n-1 \}. \end{aligned} \tag{1.20}$$

We fix $\delta(X) = ix_n$ in (1.17), so that S^{2n-1} is now oriented by (1.15) and (1.16).

1.21. Theorem. *For $X = (x_1, \dots, x_n) \in \mathfrak{t}$ as above, the infinitesimally equivariant η -invariant of the untwisted Dirac operator D and the odd signature operator $B = D^{\tilde{\pi}}$ of the odd spheres are given by*

$$\eta_X(D_{S^{2n-1}}) = \frac{i^n}{2^{n-1} \sin \frac{x_1}{2} \cdots \sin \frac{x_n}{2}} \left(1 - \sum_{j=1}^n \frac{2}{x_j} \sin \frac{x_j}{2} \prod_{k \neq j} \frac{x_k^2}{x_k^2 - x_j^2} \right), \tag{1}$$

$$\eta_X(B_{S^{2n-1}}) = i^n \cot \frac{x_1}{2} \cdots \cot \frac{x_n}{2} \left(1 - \sum_{j=1}^n \frac{2}{x_j} \tan \frac{x_j}{2} \prod_{k \neq j} \frac{x_k^2}{x_k^2 - x_j^2} \right). \tag{2}$$

Note that these formulas have been proved by Zhang for S^1 in [18] and for S^3 by the author in [10]. For other symmetric spaces, one can similarly compute $\eta_X(D)$ and $\eta_X(B)$ using the formulas for the equivariant η -invariants in [8], [9].

Proof. The classical equivariant η -invariants of B and D have been calculated in [1] and [12]. Let us start with the untwisted Dirac operator. We take $X \in \mathfrak{t}$ as above. By [12], we have

$$\eta_{e^{-X}}(D_{S^{2n-1}}) = \frac{i^n}{2^{n-1} \sin \frac{x_1}{2} \cdots \sin \frac{x_n}{2}}.$$

Let W_π^+ denote the positive weights of the isotropy representation π , which are all of multiplicity one in our case. With (1.20) and $\delta(X) = ix_n$, we calculate

$$\frac{1}{\delta(-X)} \prod_{\beta \in \Delta_G^+} \frac{\beta(-X|_s)}{\beta(-X)} \prod_{\beta \in W_\pi^+} \frac{1}{2 \sinh(\frac{\beta}{2}(-X|_s))} = \frac{i}{x_n} \prod_{k=1}^{n-1} \frac{x_k^2}{(x_k^2 - x_n^2)} \prod_{k=1}^{n-1} \frac{i}{2 \sin \frac{x_k}{2}}.$$

Note that the sum $S_{G/H}$ over W_G/W_H of (2.29) contains n similar terms, where x_n above is replaced by x_j for $j = 1, \dots, n$. Because the last two terms in Theorem 2.33 (2) drop out for a symmetric space, we get (1). The proof of (2) is similarly based on the formula for $\eta_{e^X}(B)$ in [1] and Corollary 2.34. \square

2. THE BOTT LOCALISATION DEFECT

The difference between the classical equivariant η -invariant of [7] and the infinitesimal equivariant η -invariant of [10] can be expressed in terms of the Bott localisation defect as in Theorem 1.9, see [10]. In this section, we prove a fibration formula for the Bott localisation defect, which is then applied to homogeneous spaces.

2.1. A fibration formula. Let $\mathcal{N}_{M_X, M}$ denote the normal bundle to the zero set M_X of the Killing field X_M on M , and let $e_X(\mathcal{N}_{M_X, M})$ denote its equivariant Euler class. Bott’s localisation formula in equivariant cohomology in the local version of [2] states that any equivariantly closed form $\alpha_X \in \Omega_{\mathfrak{g}}(M)$ is equivariantly cohomologous to the current

$$\frac{\delta_{M_X} \alpha_X}{e_X(\mathcal{N}_{M_X, M})}$$

for small $X \in \mathfrak{g}$. Here, δ_{M_X} denotes the δ -distribution at the fixpoint set of X . It follows from Berline-Vergne’s proof in [2] that in fact

$$\alpha_X - \frac{\delta_{M_X} \alpha_X}{e_X(\mathcal{N}_{M_X, M})} = d_X \left(\frac{\vartheta_X}{d_X \vartheta_X} \alpha_X \right), \tag{2.1}$$

where $\vartheta_X = \frac{1}{2\pi i} X_M^*$ as in (1.8). The current $\frac{\vartheta_X}{d_X \vartheta_X}$ also appears in a localisation formula for manifolds with boundary in [13]. In this section, we state some properties of $\frac{\vartheta_X}{d_X \vartheta_X}$. In particular, we give a formula for fibre bundles. Let us begin with the following observation, which holds for arbitrary Riemannian manifolds M with an isometric action by a Lie group G .

2.2. Proposition. *If we set $\frac{\vartheta_X}{d_X \vartheta_X}|_{M_X} = 0$, then the current $\frac{\vartheta_X}{d_X \vartheta_X}$ is locally of class L_1 . In particular, if $\alpha \in \Omega^*(M)$ is continuous and compactly supported, then*

$$\int_M \frac{\vartheta_X}{d_X \vartheta_X} \alpha = - \int_0^\infty \left(\int_M \vartheta_X e^{t d_X \vartheta_X} \alpha \right) dt .$$

In particular, both integrals exist.

Proof. Because

$$d_X \vartheta_X = \frac{1}{2\pi i} dX_M^* - \|X_M\|^2 ,$$

it is clear that $\frac{\vartheta_X}{d_X \vartheta_X}$ is integrable on each compact subset of $M \setminus M_X$. Also, the equation above holds for all α with compact support in $M \setminus M_X$.

Because X_M is a Killing field, the zero set M_X is a totally geodesic submanifold of M . Let $\mathcal{N} \rightarrow M_X$ denote the normal bundle, then the Levi-Civita connection on M restricts to a connection $\nabla^{\mathcal{N}}$ on \mathcal{N} , and the infinitesimal action $\mu_X^{\mathcal{N}}$ of X is parallel with respect to $\nabla^{\mathcal{N}}$. The normal exponential map $\exp^\perp : \mathcal{N} \rightarrow M$ is a local diffeomorphism near the zero section in \mathcal{N} , and

$$e^{tX} \exp^\perp V = \exp^\perp (e^{t\mu_X^{\mathcal{N}}} V)$$

for all vectors $V \in \mathcal{N}$.

Let \mathcal{R} denote the vector field near M_X given by

$$\mathcal{R}|_{\exp^\perp V} = \left. \frac{d}{dt} \right|_{t=1} \exp^\perp tV,$$

and set $r = |\mathcal{R}|$ near M_X . To estimate the behaviour of dX_M^* and $\|X_M\|^2$ near M_X , we need some facts.

Because X_M is a Killing field, it satisfies $\|X_M\|^2 > cr^2$ near M_X for some $c > 0$. Cartan's formula for the exterior derivative implies that

$$(dX_M^*)(V, W) = 2 \langle \nabla_V X, W \rangle.$$

Now let V be a vector field on M near M_X that is tangential to M_X and parallel along all radial geodesics emanating from and perpendicular to M_X . Let Y, Z be a vector fields normal to M_X with $\nabla_Y Z = 0$, then

$$\begin{aligned} \langle X_M, V \rangle|_{M_X} &= 0, & (Y \langle X_M, V \rangle)|_{M_X} &= \langle \mu_X^{\mathcal{N}} V, Y \rangle = 0, \\ \text{and } Y(Z \langle X_M, V \rangle)|_{M_X} &= \langle R_{Y,V} X_M - (\nabla_V \mu_X^{\mathcal{N}})(Y), Z \rangle|_{M_X} = 0 \end{aligned} \tag{2.3}$$

implies that

$$\iota_V dX_M^* \leq Cr^2 \quad \text{and} \quad \langle X_M, V \rangle \leq Cr^3$$

for some $C > 0$.

We choose a local orthonormal frame e_1, \dots, e_n near M_X that is parallel along radial geodesics emanating from M_X , such that e_1, \dots, e_{2k} are normal to M_X and e_{2k+1}, \dots, e_n are tangential. Then

$$\frac{\vartheta_X}{d_X \vartheta_X} = -\frac{1}{2\pi i} \frac{X_M^*}{\|X_M\|^2 - \frac{dX_M^*}{2\pi i}} = -\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (2\pi i)^{-j-1} \frac{X_M^* (dX_M^*)^j}{\|X_M\|^{2j+2}} = O(r^{1-2k}) \tag{2.4}$$

near M_X , so the left hand side integral in Proposition 2.2 exists.

It also follows from (2.4) that we may write

$$-\int_0^\infty \left(\int_M \vartheta_X e^{t d_X \vartheta_X} \alpha \right) dt = \int_M \frac{\vartheta_X}{d_X \vartheta_X} \alpha - \lim_{t \rightarrow \infty} \int_M \frac{\vartheta_X}{d_X \vartheta_X} e^{t d_X \vartheta_X} \alpha. \quad (2.5)$$

Let U_t be the tubular neighbourhood of M_X of radius $t^{-\frac{1}{4}}$ for $t \gg 0$. Because $\|X_M\|^2 \geq cr^2$, one has

$$\lim_{t \rightarrow \infty} \int_{M \setminus U_t} \frac{\vartheta_X}{d_X \vartheta_X} e^{t d_X \vartheta_X} \alpha = 0, \quad (2.6)$$

so the rightmost term in (2.5) localises near M_X .

We identify U_t with the neighbourhood \mathcal{N}_t of radius $t^{\frac{1}{4}}$ by a rescaled normal exponential map

$$v \mapsto \varphi_t(v) = \exp^\perp(t^{-\frac{1}{2}}v). \quad (2.7)$$

Using (2.3), one finds that if $A = \mu_X^{\mathcal{N}} \in \Gamma(\text{End}\mathcal{N})$, then as $t \rightarrow \infty$,

$$\begin{aligned} t \varphi_t^* X_M^* &\rightarrow \langle A\mathcal{R}, \cdot \rangle, & t \varphi_t^* dX_M^* &\rightarrow \langle 2A \cdot, \cdot \rangle, \\ \text{and } t \varphi_t^* \|X_M\|^2 &\rightarrow \|A\mathcal{R}\|^2 \end{aligned} \quad (2.8)$$

uniformly on \mathcal{N}_t , with respect to the canonical metric on the total space of $\mathcal{N} \rightarrow M_X$. Moreover,

$$\lim_{t \rightarrow \infty} \varphi_t^* \alpha = \pi^* \alpha$$

is the pullback of α by the bundle projection $\pi: \mathcal{N} \rightarrow M_X$, because α is of class C^0 . Using (2.6), this allows us to compute

$$\begin{aligned} &\lim_{t \rightarrow \infty} \int_M \frac{\vartheta_X}{d_X \vartheta_X} e^{t d_X \vartheta_X} \alpha \\ &= - \lim_{t \rightarrow \infty} \int_{\mathcal{N}_t} \sum_{j,k} (2\pi i)^{-j-k-1} \frac{t \varphi_t^* X_M^* (t \varphi_t^* dX_M^*)^{j+k}}{(t \varphi_t^* \|X_M\|^2)^{j+1} k!} e^{-t \varphi_t^* \|X_M\|^2} \varphi_t^* \alpha \\ &= - \lim_{t \rightarrow \infty} \int_{\mathcal{N}} \sum_{j,k} (2\pi i)^{-j-k-1} \frac{\langle A\mathcal{R}, \cdot \rangle \langle 2A \cdot, \cdot \rangle^{j+k}}{\|A\mathcal{R}\|^{2j+2} k!} e^{-\|A\mathcal{R}\|^2} \pi^* \alpha \\ &= 0. \end{aligned} \quad (2.9)$$

This proves the existence of the right hand side integral in Proposition 2.2. Together with (2.5), we obtain the claimed equality. \square

2.10. **Theorem.** *Let $p: M \rightarrow B$ be a proper Riemannian submersion, and let G be a compact Lie group of isometries of M that map fibres to fibres. Then*

$$\frac{\vartheta_X^M}{d_X^M \vartheta_X^M} = p^* \left(\frac{\vartheta_X^B}{d_X^B \vartheta_X^B} \right) + p^* \left(\frac{\delta_{B_X}}{e_X(\mathcal{N}_{B_X, B})} \right) \frac{\vartheta_X^M}{d_X^M \vartheta_X^M}$$

as L_1 -currents for any $X \in \mathfrak{g}$ modulo d_X -exact currents.

The first term on the right hand side gives the horizontal part of the Bott localisation defect. The second term is a localisation of the localisation defect to the fibres over the zero set B_X of X_B on B . Note that the Killing vector field X_M on M is tangential to these fibres. We can rephrase Theorem 2.10 as follows. For all compactly supported continuous forms $\alpha \in \Omega^*(M)$, we have

$$\int_M \frac{\vartheta_X^M}{d_X^M \vartheta_X^M} \alpha = \int_B \frac{\vartheta_X^B}{d_X^B \vartheta_X^B} \int_{M/B} \alpha + \int_B \frac{\delta_{B_X}}{e_X(\mathcal{N}_{B_X, B})} \int_{p^{-1}(B_X)/B_X} \frac{\vartheta_X^M}{d_X^M \vartheta_X^M} \alpha .$$

If X_B is not identically zero, this reduces the computation of the left hand side to the computation of integrals over manifolds of smaller dimension.

Proof. By Proposition 2.2,

$$\frac{\vartheta_X^M}{d_X^M \vartheta_X^M} \alpha_X = - \int_0^\infty \vartheta_X^M e^t d_X^M \vartheta_X^M \alpha_X dt .$$

Using the vertical tangent bundle TF and the equivariant horizontal distribution $\mathcal{H} = (TF)^\perp \subset TM$, we decompose $\vartheta_X = \vartheta_X^h + \vartheta_X^v \in \Gamma(\mathcal{H}^*) \oplus \Gamma(T^*F)$. If G maps fibres to fibres, then clearly

$$\vartheta_X^h = \frac{1}{2\pi i} \langle p_* X_M, p_* \cdot \rangle = \frac{1}{2\pi i} p^* \langle X_B, \cdot \rangle = p^* \vartheta_X^B ,$$

hence $\vartheta_X^M = p^* \vartheta_X^B + \vartheta_X^v$.

Regard the manifold $\overline{M} = M \times (0, \infty)^2$, where the action of the Killing field X is extended trivially to \overline{M} . Then the form

$$\beta_X = e^{d_X^{\overline{M}}(s p^* \vartheta_X^B + t \vartheta_X^v)} = (1 - p^* \vartheta_X^B ds) (1 - \vartheta_X^v dt) e^{s p^* d_X^B \vartheta_X^B + t d_X^M \vartheta_X^v}$$

is closed on \overline{M} . For $0 < T \leq S$ consider the domain

$$\Omega = \{ (s, t) \mid 0 \leq t \leq T \text{ and } t \leq s \leq S \} ,$$

and let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ denote the contour $\partial\Omega$, where $s = t$ on Γ_1 , $t = 0$ on Γ_2 , $s = S$ on Γ_3 , and $t = T$ on Γ_4 . It follows from the equivariant Stokes

theorem that integrating β_X over Γ produces a d_X -exact current on M . We will consider the limits $S \rightarrow \infty$ and $T \rightarrow \infty$ in that order to prove Theorem 2.10.

Let $I_j^0 = \int_{\Gamma_j} \beta_X$ for $j = 1, \dots, 4$, then clearly

$$\begin{aligned} I_1^0 &= \int_{\Gamma_1} \beta_X = \int_0^T \vartheta_X^M e^t d_X^M \vartheta_X^M dt, \\ I_1^1 &= \lim_{S \rightarrow \infty} I_1^0 = I_1^0, \end{aligned} \tag{2.11}$$

$$\text{and } I_1^2 = \lim_{T \rightarrow \infty} I_1^1 = -\frac{\vartheta_X^M}{d_X^M \vartheta_X^M}$$

by Proposition 2.2. Similarly,

$$\begin{aligned} I_2^0 &= \int_{\Gamma_2} \beta_X = -\int_0^S p^* \vartheta_X^B e^s p^* d_X^B \vartheta_X^B ds, \\ I_2^1 &= \lim_{S \rightarrow \infty} I_2^0 = p^* \left(\frac{\vartheta_X^B}{d_X^B \vartheta_X^B} \right), \end{aligned} \tag{2.12}$$

$$\text{and } I_2^2 = \lim_{T \rightarrow \infty} I_2^1 = I_2^1.$$

For the term I_3^0 , we use that $e^{S d_X^B \vartheta_X^B}$ forces localisation on the zero set B_X of X_B in the limit $S \rightarrow \infty$, see [2]. This gives

$$\begin{aligned} I_3^0 &= \int_{\Gamma_3} \beta_X = -\int_0^T \vartheta_X^v e^S p^* d_X^B \vartheta_X^B + t d_M \vartheta_X^v dt, \\ I_3^1 &= \lim_{S \rightarrow \infty} I_3^0 = -p^* \left(\frac{\delta_{B_X}}{e_X(\mathcal{N}_{B_X, B})} \right) \int_0^T \vartheta_X^M e^t d_M \vartheta_X^M dt, \end{aligned} \tag{2.13}$$

$$\text{and } I_3^2 = \lim_{T \rightarrow \infty} I_3^1 = p^* \left(\frac{\delta_{B_X}}{e_X(\mathcal{N}_{B_X, B})} \right) \frac{\vartheta_X^M}{d_X^M \vartheta_X^M}.$$

The remaining term and its limit as $S \rightarrow \infty$ are easily computed as

$$I_4^0 = \int_{\Gamma_4} \beta_X = e^T d_X^M \vartheta_X^M \int_0^{S-T} p^* \vartheta_X^B e^s p^* d_X^B \vartheta_X^B ds, \tag{2.14}$$

$$\text{and } I_4^1 = \lim_{S \rightarrow \infty} I_4^0 = -p^* \left(\frac{\vartheta_X^B}{d_X^B \vartheta_X^B} \right) e^T d_X^M \vartheta_X^M.$$

Then as in (2.6) above, the current I_4^1 localises near M_X as $T \rightarrow \infty$, and the limit I_4^2 of I_4^1 as $T \rightarrow \infty$ can be computed using a rescaling argument.

Let $\mathcal{N} = \mathcal{N}_{M_X, M}$ denote the normal bundle to M_X in M . We still identify the tubular neighbourhood U_t of M_X of radius $t^{-\frac{1}{4}}$ for $t = T \gg 0$ with the

neighbourhood \mathcal{N}_t of radius $t^{\frac{1}{4}}$ by the rescaled normal exponential map $\varphi_t: \mathcal{N}_t \rightarrow U_t$ of (2.7). Define $A \in \Gamma(\text{End}\mathcal{N})$ as above, then (2.8) holds unchanged. Similarly, let $A_0 \in \Gamma(\text{End}\mathcal{N})$ denote the horizontal lift of the action of $\mu_X^{\mathcal{N}_{B_X/B}}$ on $\mathcal{N}_{B_X/B}$, then the analogue of (2.8) implies that

$$t \varphi_t^* p^* X_B^* \rightarrow \langle A_0 \mathcal{R}, \cdot \rangle, \quad t \varphi_t^* p^* dX_B^* \rightarrow \langle 2A_0 \cdot, \cdot \rangle,$$

$$\text{and} \quad t \varphi_t^* p^* \|X_B\|^2 \rightarrow \|A_0 \mathcal{R}\|^2$$

uniformly on \mathcal{N}_t .

In analogy with (2.9), we calculate for any continuous form α that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \int_M p^* \left(\frac{\vartheta_X^B}{d_X^B \vartheta_X^B} \right) e^{t d_X^M \vartheta_X^M} \alpha \\ &= \lim_{t \rightarrow \infty} \int_{\mathcal{N}_t} \sum_{j,k} \varphi_t^* p^* \left(\frac{t X_B^* (t dX_B^*)^j}{(2\pi i t \|X_B\|^2)^{j+1}} \right) \frac{(t \varphi_t^* dX_M^*)^k}{(2\pi i)^k k!} e^{-t \varphi_t^* \|X_M\|^2} \varphi_t^* \alpha \\ &= \lim_{t \rightarrow \infty} \int_{\mathcal{N}} \sum_{j,k} \frac{\langle A_0 \mathcal{R}, \cdot \rangle \langle 2A_0 \cdot, \cdot \rangle^j \langle 2A \cdot, \cdot \rangle^k}{(2\pi i)^{j+k+1} \|A_0 \mathcal{R}\|^{2j+2} k!} e^{-\|A \mathcal{R}\|^2} \pi^* \alpha \\ &= 0. \end{aligned}$$

With (2.14), this implies that

$$I_4^2 = \lim_{T \rightarrow \infty} I_4^1 = 0.$$

Theorem 2.10 follows from (2.11)–(2.13), because $X_1^2 + X_2^2 + X_3^2 + X_4^2$ is exact. \square

2.2. The Bott localisation defect on G/H . Let $G \supset K \supset S$ be compact Lie groups, then

$$\begin{array}{ccc} P = K/S & \longrightarrow & E = G/S \\ & & \downarrow p \\ & & Q = G/K \end{array} \tag{2.15}$$

is a G -equivariant fibration. The left and the right hand triangle in the following diagram represent two such equivariant fibrations with $K = T$ and $K = H$,

respectively.

$$\begin{array}{ccc}
 & G & \\
 & / \quad \backslash & \\
 T & & H \\
 & \backslash \quad / & \\
 & S &
 \end{array} \tag{2.16}$$

We apply Theorem 2.10 to these fibrations to compute

$$\int_{G/H} \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(T(G/H), \nabla^{\text{LC}}) \text{ch}_X(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) .$$

Together with the formula for $\eta_{e-x}(\tilde{D}) + h_{e-x}(\tilde{D})$ in [8], [9], we obtain a formula for $\eta(D) + h(D)$ up to a possible contribution in $2\mathbb{Z}$ coming from the spectral flow.

Recall the definition of X_E for $X \in \mathfrak{g}$ in (1.3). With the notation of (1.10), we represent X_E by $\hat{X}: G \rightarrow \mathfrak{e} = \mathfrak{s}^\perp$ with

$$\hat{X}_E(g) = -(\text{Ad}_g^{-1} X)_\mathfrak{e} .$$

The Lie derivative \mathcal{L}_X^κ acts on $V^\kappa E$ by

$$\widehat{\mathcal{L}_X^\kappa} s = \left. \frac{d}{dt} \right|_{t=0} \hat{s}(e^{-tX} g) = -(\text{Ad}_g^{-1} X)(\hat{s}) ,$$

so the moment of X with respect to the reductive connection is simply

$$\widehat{\mu_X^{0,\kappa}} s = \widehat{\mathcal{L}_X^\kappa} s - \widehat{\nabla_{X_E}^{0,\kappa}} s = -(\text{Ad}_g^{-1} X)_\mathfrak{h}(\hat{s}) = \kappa_{*(\text{Ad}_g^{-1} X)_\mathfrak{h}} \hat{s} . \tag{2.17}$$

The equivariant curvature of $\nabla^{0,\kappa}$ is then given by

$$\begin{aligned}
 F_X^{0,\kappa} &= (\nabla^{0,\kappa} - 2\pi i \iota_{X_E})^2 + 2\pi i \mathcal{L}_X^\kappa \\
 &= F^{0,\kappa} + 2\pi i \mu_X^{0,\kappa} = \kappa_*(-[\cdot, \cdot]_\mathfrak{h} + 2\pi i (\text{Ad}_g^{-1} X)_\mathfrak{h}) .
 \end{aligned} \tag{2.18}$$

We will assume that $\mathcal{E} = \mathcal{S}^\kappa E \rightarrow E$ is constructed as in (1.14). If we regard the Dirac operator $D^\kappa = D^{\frac{1}{2},\kappa}$ on $\mathcal{S}^\kappa E$, then the equivariant twisting curvature in (1.5) is just the equivariant curvature of the reductive connection on $V^\kappa E$ (even if this bundle does not exist globally), so formally

$$\begin{aligned}
 \text{ch}_X(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) &= \text{ch}_X(V^\kappa E, \nabla^0) \\
 &= \text{tr}_{V^\kappa} \left(e^{\frac{\kappa_*[\cdot, \cdot]_\mathfrak{h} - \kappa_{*(\text{Ad}_g^{-1} X)_\mathfrak{h}}}{2\pi i}} \right) = \chi_H^\kappa \left(e^{\frac{[\cdot, \cdot]_\mathfrak{h} - (\text{Ad}_g^{-1} X)_\mathfrak{h}}{2\pi i}} \right) .
 \end{aligned} \tag{2.19}$$

Let $P \hookrightarrow E \rightarrow Q$ be an equivariant fibration as in (2.15) with $K = H$, so the fibre $P = H/S$ is a flag manifold. An Ad_G -invariant metric on \mathfrak{g} induces a normal metric on E such that p is an equivariant Riemannian submersion. We write $\mathfrak{p} = \mathfrak{s}^\perp \cap \mathfrak{h}$ and $\mathfrak{q} = \mathfrak{h}^\perp$, so $\mathfrak{e} = \mathfrak{p} \oplus \mathfrak{q}$. By Ad_G -invariance, the isotropy representation of E splits as $\pi = \varphi \oplus \iota^*\psi$, where φ and ψ are the isotropy representations of P and Q , and $\iota: S \rightarrow H$ is the inclusion. Thus the tangent bundle TE of the total space splits naturally as $p^*TQ \oplus TP$ with vertical tangent bundle $TP = G \times_S \mathfrak{p}$ and horizontal complement $p^*TQ = G \times_S \mathfrak{q}$.

We define $\nabla^{\lambda,P}, \nabla^{\mu,Q}$ as in (1.13) above, regarding $\nabla^{\lambda,P}$ as a connection on the vertical tangent bundle $TP \rightarrow E$. Then

$$\widehat{\nabla_U^{\lambda,P} Y} = \hat{U}(\hat{Y}) + \lambda [\hat{U}, \hat{Y}]_{\mathfrak{p}} \quad \text{and} \quad \widehat{\nabla_V^{\mu,Q} W} = \hat{V}(\hat{W}) + \mu [\hat{V}, \hat{W}]_{\mathfrak{q}}$$

for S -equivariant $\hat{U}: G \rightarrow \mathfrak{e}, \hat{Y}: G \rightarrow \mathfrak{p}$ and H -equivariant $\hat{V}, \hat{W}: G \rightarrow \mathfrak{q}$.

Let $\kappa \in \mathfrak{s}$ be a weight of \mathfrak{h} , and consider the bundle $V^{\kappa+\rho_H} P \rightarrow P$ associated to the S -representation with highest weight $\kappa + \rho_H$. Note that this bundle lives on the universal cover of $P = H/S$. However, its equivariant Chern form $\text{ch}_X(V^{\kappa+\rho_H} P, \nabla^0)$ descends to P , even if P is not simply connected. Similarly, the character $\chi_H^\kappa(e^X)$ is well-defined in terms of $X \in \mathfrak{h}$ even if χ_H^κ is maybe not defined as a function on H . Bott's localisation formula and the equivariant index theorem imply that

$$\chi_H^\kappa(e^{-X}) = \int_P \hat{A}_X(TP, \nabla^{\lambda,P}) \text{ch}_X(V^{\kappa+\rho_H} P, \nabla^0), \tag{2.20}$$

see [2], Section 8.2.

The curvature $F^{\lambda,P}$ and the moment $\mu_X^{\lambda,P}$ on E are given by

$$\begin{aligned} \widehat{F_{V,W}^{\lambda,P} Y} &= -\varphi_{*[\hat{V}, \hat{W}]_{\mathfrak{s}}} \hat{Y} - \lambda [[\hat{V}, \hat{W}]_{\mathfrak{p}}, \hat{Y}]_{\mathfrak{p}} \\ &\quad + \lambda^2 [\hat{V}, [\hat{W}, \hat{Y}]_{\mathfrak{p}}]_{\mathfrak{p}} - \lambda^2 [\hat{W}, [\hat{V}, \hat{Y}]_{\mathfrak{p}}]_{\mathfrak{p}} \\ \text{and} \quad \widehat{\mu_X^{\lambda,P} Y} &= \varphi_{*(\text{Ad}_h^{-1} X)_{\mathfrak{s}}} \hat{Y} + \lambda [(\text{Ad}_h^{-1} X)_{\mathfrak{p}}, \hat{y}]_{\mathfrak{p}}. \end{aligned}$$

Using these equations, one checks that the equivariant curvature of the connection $\nabla^{\lambda,P}$ and the Killing field X on E is formally the same as the equivariant curvature of $\nabla^{\lambda,P}$ and the Killing field $-\text{Ad}_h \frac{[\cdot, \mathfrak{q}, \cdot, \mathfrak{q}]_{\mathfrak{h}}}{2\pi i} + X$, considered only on P .

By (2.19) and (2.20), this gives

$$\begin{aligned} \text{ch}_X(V^\kappa Q, \nabla^0) &= \int_P \hat{A}_{-\text{Ad}_h \frac{[\cdot, \mathfrak{q}, \cdot, \mathfrak{q}]_h}{2\pi i} + X}(TP, \nabla^{\lambda, P}) \text{ch}_{-\text{Ad}_h \frac{[\cdot, \mathfrak{q}, \cdot, \mathfrak{q}]_h}{2\pi i} + X}(V^{\kappa + \rho_H} P, \nabla^0) \\ &= \int_{E/Q} \hat{A}_X(TP, \nabla^{\lambda, P}) \text{ch}_X(V^{\kappa + \rho_H} E, \nabla^0), \end{aligned} \quad (2.21)$$

where we integrate over the fibres of $E \rightarrow Q$ in the last line.

Let $p^*\nabla^{\mu, Q} \oplus \nabla^{\lambda, P}$ denote the product connection on $TE = p^*TQ \oplus TF$. By multiplicativity of the \hat{A} -form, we have

$$\hat{A}_X(TE, p^*\nabla^{\mu, Q} \oplus \nabla^{\lambda, P}) = p^*\hat{A}_X(TQ, \nabla^{\mu, Q}) \hat{A}_X(TP, \nabla^{\lambda, P}).$$

Now, Theorem 2.10 and equation (2.21) have the following implication.

2.22. Proposition. *Let $G \supset H$ be compact Lie groups, and let $S \subset H$ be a maximal torus. If the Killing field X_Q has no zeros, then for any λ and μ ,*

$$\begin{aligned} \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TQ, \nabla^{\mu, Q}) \text{ch}_X(V^\kappa Q, \nabla^0) \\ = \int_{E/Q} \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TE, p^*\nabla^{\mu, Q} \oplus \nabla^{\lambda, P}) \text{ch}_X(V^{\kappa + \rho_H} E, \nabla^0). \quad \square \end{aligned}$$

To evaluate the right hand side of Proposition 2.22, we now consider the left triangle in (2.16). In particular, $T \subset G$ is a maximal torus containing S . We assume in addition that X is regular. Then the only fixed points of X on G/T are the isolated fixpoints wT/T . In particular, the normal bundle to $N_G(T)/T$ in G/T is just the tangent bundle $T(G/T)$, and its equivariant Euler form is invertible near $N_G(T)/T$. We find by Proposition 2.22 that

$$\begin{aligned} \int_Q \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TQ, \nabla^{\mu, Q}) \text{ch}_X(V^\kappa Q, \nabla^0) \\ = \int_E \tilde{\hat{A}}_X(TE, \nabla^0, p^*\nabla^{\mu, Q} \oplus \nabla^{\lambda, P}) \text{ch}_X(V^{\kappa + \rho_H} E, \nabla^0) \\ + \int_E \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TE, \nabla^0) \text{ch}_X(V^{\kappa + \rho_H} E, \nabla^0). \end{aligned}$$

Applying Theorem 2.10 once more gives

$$\begin{aligned}
 & \int_Q \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TQ, \nabla^{\mu, Q}) \operatorname{ch}_X(V^\kappa Q, \nabla^0) \\
 &= \int_E \tilde{\hat{A}}_X(TE, \nabla^0, p^* \nabla^{\mu, Q} \oplus \nabla^{\lambda, P}) \operatorname{ch}_X(V^{\kappa + \rho_H} E, \nabla^0) \\
 & \quad + \int_{G/T} \frac{\vartheta_X}{d_X \vartheta_X} \int_{E/(G/T)} \hat{A}_X(TE, \nabla^0) \operatorname{ch}_X(V^{\kappa + \rho_H} E, \nabla^0) \\
 & \quad + \int_{G/T} \frac{\delta_{N_G(T)/T}}{e_X(T(G/T))} \int_{N_G(T)/S} \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TE, \nabla^0) \operatorname{ch}_X(V^{\kappa + \rho_H} E, \nabla^0)
 \end{aligned} \tag{2.23}$$

We can still simplify this expression. First of all, the reductive connection respects the splitting $TE \cong p^*TQ \oplus TP$. We can thus fix $\lambda = 0$ and rewrite the first term of the right hand side of (2.23) as

$$\begin{aligned}
 & \int_E \tilde{\hat{A}}_X(TE, \nabla^0, p^* \nabla^{\mu, Q} \oplus \nabla^{0, P}) \operatorname{ch}_X(V^{\kappa + \rho_H} E, \nabla^0) \\
 &= \int_E \tilde{\hat{A}}_X(p^*TQ, \nabla^0, p^* \nabla^{\mu, Q}) \hat{A}(TP, \nabla^{0, P}) \operatorname{ch}_X(V^{\kappa + \rho_H} E, \nabla^0) .
 \end{aligned}$$

We want to show that for $\mu = 0$, the expression above vanishes.

Note that the curvature of the reductive connection on an equivariant vector bundle over E depends only on the $(G \times_H \mathfrak{s})$ -valued two form $[\cdot, \cdot]_{\mathfrak{s}}$ by (1.12). However, this form vanishes on $\mathfrak{g}/\mathfrak{h} \otimes \mathfrak{h}/\mathfrak{s}$, so we find that

$$\hat{A}_X(TP, \nabla^{0, P}) \operatorname{ch}_X(V^{\kappa + \rho_H} E, \nabla^0) \in \Gamma(p^* \Lambda^{\text{even}} T^*Q \otimes \Lambda^{\text{even}} T^*P) . \tag{2.24}$$

It remains to analyse the Chern-Simons class $\tilde{\hat{A}}_X(p^*TQ, \nabla^0, p^* \nabla^{0, Q})$. Once again, let $\mathfrak{p} = \mathfrak{s}^\perp \cap \mathfrak{h}$ and $\mathfrak{q} = \mathfrak{h}^\perp \subset \mathfrak{g}$. Note that a vector field V on Q is given by an H -equivariant function $\hat{V}: G \rightarrow \mathfrak{q}$. This function also describes the horizontal lift of V to $p^*TQ \subset TE$. Therefore, the pull-back connection $p^* \nabla^{0, Q}$ on the horizontal tangent bundle $p^*TQ \subset TE$ is given by

$$p^* \widehat{\nabla_V^{\mu, Q}} W = \hat{V}(\hat{W}) + [\hat{V}_{\mathfrak{p}}, \hat{W}] \tag{2.25}$$

for S -equivariant functions $V: G \rightarrow \mathfrak{e}$ and $W: G \rightarrow \mathfrak{q}$, since then

$$p^* \widehat{\nabla_V^{\mu, Q}} W = \hat{V}_{\mathfrak{q}}(\hat{W})$$

for an H -equivariant \hat{W} .

We fix a family of connections $\nabla^{0,\lambda}$ on $p^*TQ \subset TE$ by

$$\widehat{\nabla_V^{0,\lambda} W} = \hat{V}(\hat{W}) + \lambda[\hat{V}_p, \hat{W}] .$$

By (1.11) and (2.25), we have $\nabla^{0,0} = \nabla^0$ and $\nabla^{0,1} = p^*\nabla^{0,Q}$. A straightforward computation gives the differential and the curvature of this family of connections as

$$\frac{\partial}{\partial \lambda} \widehat{\nabla_V^{0,\lambda}} = [\hat{V}_p, \cdot] \in \Lambda^1 \mathfrak{p} \otimes \text{End} \mathfrak{q} ,$$

and
$$\begin{aligned} \widehat{F_{V,W}^{0,\lambda}} &= -\psi_{*[\hat{V}, \hat{W}]_{\mathfrak{s}}} - \lambda [[\hat{V}, \hat{W}]_{\mathfrak{p}}, \cdot] \\ &\quad + \lambda^2 [\hat{V}_p, [\hat{W}_p, \cdot]] - \lambda^2 [\hat{W}_p, [\hat{V}_p, \cdot]] \in (\Lambda^2 \mathfrak{p} \oplus \Lambda^2 \mathfrak{q}) \otimes \text{End} \mathfrak{q} , \end{aligned}$$

where we have used that $[\mathfrak{p}, \mathfrak{q}] \subset \mathfrak{q}$ and $\mathfrak{q} \cap \mathfrak{s} = 0$. This implies that

$$\tilde{\hat{A}}_X(p^*TQ, \nabla^0, p^*\nabla^{0,Q}) \in \Gamma(p^*\Lambda^{\text{even}}T^*Q \otimes \Lambda^{\text{odd}}T^*P) . \tag{2.26}$$

Combining (2.24) and (2.26) with the fact that $\dim Q$ is odd while $\dim P$ is even, we see that

$$\begin{aligned} \int_E \tilde{\hat{A}}_X(TE, \nabla^0, p^*\nabla^0 \oplus \nabla^0) \text{ch}_X(V^{\kappa+\rho_H} E, \nabla^0) \\ = \int_E \tilde{\hat{A}}_X(p^*TQ, \nabla^0, p^*\nabla^0) \hat{A}_X(TP, \nabla^0) \text{ch}_X(V^{\kappa+\rho_H} E, \nabla^0) = 0 . \end{aligned}$$

Of course, this conclusion would fail for most other possible choices of connections.

The analogue of (2.24) for the left hand side of (2.16) is

$$\hat{A}_X(TE, \nabla^0) \text{ch}_X(V^{\kappa+\rho_H} E, \nabla^0) \in \Gamma(p^*\Lambda^{\text{even}}T^*(G/T) \otimes \Lambda^{\text{even}}T^*(T/S)) .$$

It implies that the middle term of the right hand side of (2.23) vanishes. Note that this conclusion would also fail for many other possible choices of connections.

Finally, over $W_G = N_G(T)/T \subset G/T$, the Killing field X_E becomes tangential to the fibres. Let $w = nT \in N_G(T)/T = W_G$, then $X_E|_{nT}$ is diffeomorphic to the Killing field $wX_{T/S}$ on T/S . Let us summarise our computations so far.

2.27. Proposition. *Let $G \supset H \supset S$ and $G \supset T \supset S$ be as above, and assume that $X \in \mathfrak{t}$ is regular and that the Killing field $X_{G/H}$ has no zeros. Then*

$$\begin{aligned} & \int_{G/H} \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(T(G/H), \nabla^0) \operatorname{ch}_X(V^\kappa(G/H), \nabla^0) \\ &= \sum_{w \in W_G} \frac{1}{e_{wX}(T(G/T))} \\ & \int_{T/S} \frac{\vartheta_{wX}}{d_{wX} \vartheta_{wX}} \hat{A}_{wX}(T(G/S), \nabla^0) \operatorname{ch}_{wX}(V^{\kappa+\rho_H}(G/S), \nabla^0) . \end{aligned}$$

2.3. Evaluation of the Bott localisation defect. In the previous subsection, we have reduced the Bott localisation defect to the quotient of the maximal tori. We will now give two formulas in representation theoretic terms.

We start with a Lemma concerning the Weyl groups of G and H .

2.28. Lemma. *Let $H \subset G$ be a pair of compact Lie groups with maximal tori $S \subset T$, and let W_G, W_H be the corresponding Weyl groups. Then W_H is a subgroup of*

$$\{ w|_{\mathfrak{s}} \mid w \in W_G \text{ and } w(\mathfrak{s}) = \mathfrak{s} \} \subset \operatorname{Aut}(S) .$$

Proof. Let $w' = n'S \in W_H = N_H(S)/S$, then w' acts on S by $s \mapsto n'sn'^{-1}$. Clearly, $n'Tn'^{-1} \subset Z_G(S)$ is a maximal torus in the centraliser of S in G . In particular, we find $z \in Z_G(S)$ such that $zn'T(zn')^{-1} = T$, and $zn's(zn')^{-1} = w'(s)$ for all $s \in S$. Thus, $w = zn'T \in W_G = N_G(T)/T$ acts on the subset $S \subset T$ as w' . □

To state our main result, let $\delta \in i\mathfrak{t}^*$ be defined as in (1.17). Let $\prod_{\beta \in W_\pi^+}$ denote the product over all positive weights of the isotropy representation π , counted with multiplicity by abuse of notation. Let $\hat{A}: z \rightarrow \frac{z/2}{\sinh(z/2)}$ denote the \hat{A} -function. We will denote by $X|_{\mathfrak{s}}$ the orthogonal projection of $X \in \mathfrak{t}$ onto \mathfrak{s} . If $f: \mathfrak{t} \rightarrow \mathbb{C}$ satisfies $f(X) = f(wX)$ for all $w \in W_G$ that map \mathfrak{s} to itself, then we set

$$S_{G/H}(f)(X) = \frac{1}{\#W_H} \sum_{w \in W_G} \operatorname{sign}(w) f(wX) . \tag{2.29}$$

If W_H can be identified with a subgroup of W_G , this amounts to summing over W_G/W_H .

2.30. **Theorem.** *Let $M = G/H$ be a quotient of compact Lie groups. If $\text{rk } G = \text{rk } H + 1$, then*

$$\begin{aligned} \int_M \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TM, \nabla^0) \text{ch}_X(V^\kappa M, \nabla^0) &= A_G \left(\frac{e^{(\kappa + \rho_H)(\cdot|_s)}}{\delta(\cdot)} \prod_{\beta \in \Delta_G^+} \hat{A}(\beta(\cdot|_s)) \right) (-X) \prod_{\beta \in \Delta_{\mathfrak{g}}^+} \frac{-1}{\beta(X)} \quad (1) \\ &= S_{G/H} \left(\frac{\chi_H^\kappa(e^{\cdot|_s})}{\delta(\cdot)} \prod_{\beta \in \Delta_G^+} \frac{\beta(\cdot|_s)}{\beta(\cdot)} \prod_{\beta \in W_\pi^+} \frac{1}{2 \sinh(\frac{\beta}{2}(\cdot|_s))} \right) (-X). \quad (2) \end{aligned}$$

Otherwise,

$$\int_M \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TM, \nabla^0) \text{ch}_X(V^\kappa M, \nabla^0) = 0. \quad (3)$$

Formula (2) gives an advantage for explicit computations if the subgroup H has a large Weyl group and the character χ_H^κ is known, e.g., if $D^\kappa = D$ is the untwisted Dirac operator. We have used this formula in Section 1.3 when dealing with spheres and odd Grassmannians. In some cases, one can even improve on (2). Suppose that there exists another subgroup $K \subset G$, such that $H \subset K$ and S is a maximal torus of K . Then one obtains a similar formula where K replaces H , π becomes the isotropy representation of G/K , and κ gets replaced by $\kappa + \rho_H - \rho_K$.

Proof. We evaluate the right hand side of Proposition 2.27 term by term. The tangent bundle $T(G/T)$ is oriented by the choice of Δ_G^+ as in (1.15), so for $w = nT$,

$$\frac{\delta_{nT/T}}{e_X(T(G/T))} = \prod_{\beta \in \Delta_G^+} \frac{-1}{\beta(wX)} = \text{sign}(w) \prod_{\beta \in \Delta_G^+} \frac{-1}{\beta(X)},$$

independent of any connection on $T(G/T)$, since we evaluate at isolated fixpoints.

Because the vertical tangent bundle of $G/S \rightarrow G/T$ is trivial, G -invariant and parallel with respect to ∇^0 , the equivariant \hat{A} -form of $T(G/S)$ is given as

$$\hat{A}_{wX}(T(G/S), \nabla^0)|_{T/S} = \hat{A}_{wX}(p^*T(G/T), \nabla^0)|_{T/S} = \prod_{\beta \in \Delta_G^+} \hat{A}(\beta(-wX|_s))$$

by (2.17); because T/S is one-dimensional, only the moment μ^0 of ∇^0 enters. Similarly, the equivariant Chern character form equals

$$\text{ch}_{wX}(V^{\kappa + \rho_H}(G/S), \nabla^0)|_{T/S} = e^{(\kappa + \rho_H)(-wX|_s)}.$$

Assume that $\text{rk } G = \text{rk } H + 1$. The expression $\int_{T/S} \frac{\vartheta_{wX}}{d_{wX} \vartheta_{wX}}$ is clearly independent of the metric chosen, so we may assume that $\text{vol}(T/S) = 2\pi$. Then for a positively oriented unit vector E , we have $\delta(E) = i$ and $\delta(X) = i\langle X, E \rangle$. Then

$$\int_{T/S} \frac{\vartheta_{wX}}{d_{wX} \vartheta_{wX}} = - \int_{T/S} \frac{wX_{T/S}^*}{2\pi i \|wX_{T/S}\|^2} = - \frac{\text{vol}(T/S)}{2\pi i \langle wX, E \rangle} = \frac{1}{\delta(-wX)} .$$

This proves (1).

To prove (2), we use Lemma 2.28 to simplify (1). If we assume that $\text{rk } G = \text{rk } H + 1$, there are two possibilities. Either, each element of W_H corresponds to precisely one element of W_G ; then these elements form a subgroup of W_G that we identify with W_H . Or each element of W_H corresponds to precisely two elements of W_G which differ by the reflexion on S ; then we can identify W_H with a subgroup of W_G^{even} , or with those elements that preserve the normal orientation of S in T . We fix one of these identifications. Let $\text{sign}_G(w) = \det(w|_{\mathfrak{t}})$ and $\text{sign}_H(w) = \det(w|_{\mathfrak{s}})$ denote the sign of an element $w \in W_H \subset W_G$ as an element of the Weyl groups W_G and W_H , respectively. These signs are related by

$$\text{sign}_G(w) \delta(wX) = \text{sign}_H(w) \delta(X) . \tag{2.31}$$

As representations of $\mathfrak{s}, \mathfrak{g}/\mathfrak{t} \oplus \mathbb{R} E$ is isomorphic to $\mathfrak{h}/\mathfrak{s} \oplus \pi$. If we have chosen Δ_G^+ and Δ_H^+ carefully, the restriction to $\mathfrak{s} \subset \mathfrak{t}$ maps the positive roots Δ_G^+ bijectively onto $\Delta_H^+ \dot{\cup} W_\pi^+$, where the positive weights of π are counted with multiplicity. We assume that this is the case, even though our final formula does not depend on this choice, only on the compatibility of orientations in (1.16). Recall the Weyl formulas,

$$\prod_{\beta \in \Delta_H^+} 2 \sinh\left(\frac{\beta}{2}(X)\right) = A_H(e^{\rho_H})(X) \quad \text{and} \quad \frac{A_H(e^{\kappa+\rho_H})}{A_H(e^{\rho_H})}(X) = \chi_H^\kappa(e^X) , \tag{2.32}$$

where A_H denotes the alternating sum over the Weyl group of H . Regarding W_H as a subgroup of W_G and using (2.31) and (2.32), we find that

$$\begin{aligned} & A_H \left(\frac{e^{(\kappa+\rho_H)(\cdot|_{\mathfrak{s}})}}{\delta(\cdot)} \prod_{\beta \in \Delta_G^+} \hat{A}(\beta(\cdot|_{\mathfrak{s}})) \right) (-X) \prod_{\beta \in \Delta_G^+} \frac{1}{\beta(-X)} \\ &= \frac{1}{\delta(-X)} \frac{A_H(e^{\kappa+\rho_H})}{A_H(e^{\rho_H})} (-X|_{\mathfrak{s}}) \prod_{\beta \in \Delta_G^+} \frac{\hat{A}(\beta(X|_{\mathfrak{s}}))}{\beta(-X)} \prod_{\beta \in \Delta_H^+} 2 \sinh\left(\frac{\beta}{2}(X|_{\mathfrak{s}})\right) \\ &= \frac{\chi_H^\kappa(e^{-X|_{\mathfrak{s}}})}{\delta(-X)} \prod_{\beta \in \Delta_G^+} \frac{\beta(-X|_{\mathfrak{s}})}{\beta(-X)} \prod_{\beta \in W_\pi^+} \frac{1}{2 \sinh(\frac{\beta}{2}(-X|_{\mathfrak{s}}))} . \end{aligned}$$

Replacing X by wX and summing over wX for $w \in W_G/W_H$, we obtain (2).

Finally, assume that $\text{rk } G \neq \text{rk } H + 1$. First of all, if $\text{rk } G - \text{rk } H$ is even, then so is $\dim M$, and the integral in Theorem 2.30 vanishes for parity reasons. Otherwise, $\text{rk } G - \text{rk } H = \dim T/S \geq 3$. Because T is abelian, the two-form part of the equivariant curvature $F_{wX}^{0,\kappa}$ vanishes on T/S by (2.18). This implies that $\hat{A}_{wX}(T(G/S), \nabla^0)$ and $\text{ch}_{wX}(V^{\kappa+\rho_H}(G/S), \nabla^0)$ have no components of non-zero exterior degree. Also, the Killing vector field on T/S generated by $wX \in \mathfrak{t}$ is parallel, so $\frac{\vartheta_{wX}}{d_{wX}\vartheta_{wX}} \in \Omega^1(T/S)$. Then

$$\frac{\vartheta_{wX}}{d_{wX}\vartheta_{wX}} \hat{A}_{wX}(T(G/S), \nabla^0) \text{ch}_{wX}(V^{\kappa+\rho_H}(G/S), \nabla^0) \in \Omega^1(T/S) ,$$

and the integral over T/S vanishes, which proves (3). □

2.4. Evaluation of η -invariants. We combine Theorem 2.30 with Theorem 1.9 and Theorem 1.19 to establish our final formula for the infinitesimally equivariant η -invariant of the Dirac operator D^κ on $M = G/H$.

Let \hat{G} denote the set of equivalence classes of irreducible unitary representations of G . Because G is compact, all these representations are finite dimensional. Recall that by Frobenius reciprocity and the Peter-Weyl theorem,

$$\Gamma(S^\kappa M) = \overline{\bigoplus_{\gamma \in \hat{G}} V^\gamma \oplus \text{Hom}_H(V^\gamma, S \otimes V^\kappa)} .$$

Because D^κ and \tilde{D}^κ are G -equivariant, they respect this decomposition, and we have

$$D^\kappa = \overline{\bigoplus_{\gamma \in \hat{G}} \text{id}_{V^\gamma} \otimes \gamma D^\kappa},$$

and similarly for \tilde{D}^κ . Let $\eta(\gamma D^\kappa) \in \mathbb{Z}$ and $h(\gamma D^\kappa) \in \mathbb{Z}$ denote the η -invariant and the dimension of the kernel of γD^κ acting on $\text{Hom}_H(V^\gamma, S \otimes V^\kappa)$. We also recall the definitions of δ and $\alpha \in \mathfrak{t}^*$ in (1.17), (1.18), and the definition of W_π^+ and $S_{G/H}$ in section 2.3.

2.33. Theorem. *Assume that $\text{rk } G = \text{rk } H + 1$, then*

$$\begin{aligned} \eta_X(D^\kappa) &= 2 \sum_{w \in W_G} \frac{\text{sign}(w)}{\delta(wX)} \left(\prod_{\beta \in \Delta_G^+} \hat{A}(\beta(wX)) \hat{A}(\delta(wX)) e^{-(\alpha - \frac{\delta}{2})(wX)} \right. \\ &\quad \left. - \prod_{\beta \in \Delta_G^+} \hat{A}(\beta(wX|_{\mathfrak{s}})) e^{-(\kappa + \rho_H)(wX|_{\mathfrak{s}})} \right) \prod_{\beta \in \Delta_G^+} \frac{-1}{\beta(X)} \tag{1} \\ &+ 2 \int_M \tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) \text{ch}_X(V^\kappa M, \nabla^\kappa) \\ &+ \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right). \end{aligned}$$

Equivalently,

$$\begin{aligned} \eta_X(D^\kappa) &= \frac{A_G\left(\frac{1}{\sinh(\frac{\delta}{2}(\cdot))} e^{(\alpha - \frac{\delta}{2})(\cdot)}\right)}{A_G(\rho_G)}(-X) \\ &+ 2 S_{G/H} \left(\frac{\chi_H^\kappa(e^{\cdot|_{\mathfrak{s}}})}{\delta(\cdot)} \prod_{\beta \in \Delta_G^+} \frac{\beta(\cdot|_{\mathfrak{s}})}{\beta(\cdot)} \prod_{\beta \in W_\pi^+} \frac{1}{2 \sinh(\frac{\beta}{2}(\cdot|_{\mathfrak{s}}))} \right) (-X) \tag{2} \\ &+ 2 \int_M \tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) \text{ch}_X(V^\kappa M, \nabla^\kappa) \\ &+ \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right). \end{aligned}$$

If $\text{rk } G \neq \text{rk } H + 1$, then

$$\begin{aligned} \eta_X(D^\kappa) &= 2 \int_M \tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) \text{ch}_X(V^\kappa M, \nabla^\kappa) \\ &\quad + \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right). \end{aligned} \tag{3}$$

The classical η -invariant $\eta(D^\kappa)$ is attained at $X = 0$ in (1)–(3).

It is easy to see that the singularities within the parentheses of the first term on the right hand side of (1) cancel, so that we are left with the alternating Weyl sum of a power series in X divided by the linearised Weyl denominator $\prod_{\beta \in \Delta_G^+} (-i\beta)(X)$. The result will then be a W_G -invariant power series representing a modified infinitesimal equivariant ξ -invariant of G/H . Note also that the last sum in (1)–(3) is finite because only finitely many eigenvalues change sign when one passes from D^κ to \tilde{D}^κ .

Proof. Assume that e^{-X} acts freely on M . Then by [8], [9],

$$(\eta + h)_{e^{-X}}(D^\kappa) - (\eta + h)_{e^{-X}}(\tilde{D}^\kappa) = \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left((\eta + h)(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right)$$

is the equivariant spectral flow from \tilde{D}^κ to D^κ . This fits with Theorem 1.6, applied to the cylinder $M \times [0, 1]$, such that the induced operators on the ends $M \times \{0\}$ and $M \times \{1\}$ are precisely \tilde{D}^κ to D^κ . Thus by Theorem 1.9, we have

$$\begin{aligned} \eta_X(D^\kappa) &= \eta_{e^{-X}}(D^\kappa) + 2 \int_M \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TM, \nabla^{\text{LC}}) \text{ch}_X(V^\kappa M, \nabla^0) \\ &= (\eta + h)_{e^{-X}}(\tilde{D}^\kappa) + 2 \int_M \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TM, \nabla^{\text{LC}}) \text{ch}_X(V^\kappa M, \nabla^0) \\ &\quad + \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right). \end{aligned}$$

We still assume that X acts freely on M . Then clearly

$$\begin{aligned} \frac{\vartheta_X}{d_X \vartheta_X} \left(\hat{A}_X(TM, \nabla^{\text{LC}}) - \hat{A}_X(TM, \nabla^0) \right) &= \frac{\vartheta_X}{d_X \vartheta_X} d_X \tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) \\ &= \tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) - d_X \left(\frac{\vartheta_X}{d_X \vartheta_X} \tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) \right), \end{aligned}$$

so we get

$$\begin{aligned} \eta_X(D^\kappa) &= (\eta + h)_{e^{-X}}(\tilde{D}^\kappa) + 2 \int_M \frac{\vartheta_X}{d_X \vartheta_X} \hat{A}_X(TM, \nabla^0) \operatorname{ch}_X(V^\kappa M, \nabla^0) \\ &\quad + 2 \int_M \tilde{\hat{A}}_X(TM, \nabla^0, \nabla^{\text{LC}}) \operatorname{ch}_X(V^\kappa M, \nabla^\kappa) \\ &\quad + \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma D^\kappa) - (\eta + h)(\gamma \tilde{D}^\kappa) \right). \end{aligned}$$

Theorem 1.19 and Theorem 2.30 (2) give (2) if $\operatorname{rk} G = \operatorname{rk} H + 1$, and (3) otherwise.

To obtain (1), we use Theorem 2.30 (1), and we rewrite the result of Theorem 1.19 using the Weyl denominator formula (2.32). \square

Because the odd signature operator $B = D$ often comes up in topological applications, we want to state formula (2) for this special case. Therefore, let $\tilde{\pi} = \kappa_1 \oplus \dots \oplus \kappa_l$ be the decomposition of $\tilde{\pi}$ into \mathfrak{h} -irreducible components, and let $\alpha_1, \dots, \alpha_l$ be the corresponding weights of \mathfrak{g} as in (1.18). Let

$$\begin{aligned} \hat{L}_X(TM, \nabla) &= \hat{A}_X(TM, \nabla) \wedge \operatorname{ch}_X(\mathcal{S}, \nabla) \\ &= \det^{\frac{1}{2}} \left(R_X^{TM} \coth \frac{R_X^{TM}}{2} \right) \end{aligned}$$

denote a rescaled version of Hirzebruch’s L -genus, and let $\tilde{\hat{L}}_X$ denote the corresponding equivariant Chern-Simons class.

2.34. Corollary. *Assume that $\operatorname{rk} G = \operatorname{rk} H + 1$. Then the infinitesimally equivariant η -invariant of the odd signature operator B on M is given by*

$$\begin{aligned} \eta_X(B) &= \sum_l \frac{A_G \left(\frac{1}{\sinh(\frac{\delta}{2}(\cdot))} e^{(\alpha_l - \frac{\delta}{2})(\cdot)} \right)}{A_G(\rho_G)} (-X) \\ &\quad + 2 S_{G/H} \left(\frac{1}{\delta(\cdot)} \prod_{\beta \in \Delta_G^+} \frac{\beta(\cdot | \mathfrak{s})}{\beta(\cdot)} \prod_{\beta \in W_\pi^+} \coth \left(\frac{\beta(\cdot | \mathfrak{s})}{2} \right) \right) (-X) \quad (1) \\ &\quad + 2 \int_M \tilde{\hat{L}}_X(TM, \nabla^0, \nabla^{\text{LC}}) + \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma B) - (\eta + h)(\gamma \tilde{B}) \right). \end{aligned}$$

If $\operatorname{rk} G \neq \operatorname{rk} H + 1$, then

$$\eta_X(B) = 2 \int_M \tilde{\hat{L}}_X(TM, \nabla^0, \nabla^{\text{LC}}) + \sum_{\gamma \in \hat{G}} \chi_G^\gamma(e^{-X}) \left(\eta(\gamma B) - (\eta + h)(\gamma \tilde{B}) \right). \quad (2)$$

The classical η -invariant $\eta(B)$ is attained at $X = 0$ in both cases.

Proof. Let β_1, \dots, β_r denote the positive weights of π , counted with the right multiplicity. If $\text{rk } G = \text{rk } H + 1$, then the weights of $\tilde{\pi}$ take the form $(\pm \frac{\beta_1}{2} \pm \dots \pm \frac{\beta_r}{2})|_{\mathfrak{s}}$, each with multiplicity one [8], [9]. Therefore,

$$\chi_{\tilde{H}}^{\tilde{\pi}}(e^{-X|_{\mathfrak{s}}}) = \prod_{\beta \in W_{\beta}^{+}} 2 \cosh\left(\frac{\beta}{2}(-X|_{\mathfrak{s}})\right).$$

Note also that for the odd signature operator B , we need the Levi-Civita connection on the twist bundle instead of the reductive connection. This is why the correct Chern-Simons contribution is given by $\tilde{L}_X(TM, \nabla^0, \nabla^{\text{LC}})$, not by $\tilde{A}_X(TM, \nabla^0, \nabla^{\text{LC}}) \text{ch}_X(\mathcal{S}, \nabla^0)$. The Corollary now follows easily from Theorem 2.33 (2) and (3). \square

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