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Ramification in Local Galois Groups-the Second Central Step

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Abstract: In this paper we review results concerning the ramification filtration of local Galois groups restricted to the second central step. Such results have direct impact on the problem of determining the conductor of a local Galois representation.

Keywords: Representations of local Galois groups, Ramification and extension theory, local class field theory.

0. Introduction

Inspired by the Langlands program much attention has been paid to Galois representations during the past several decades. Early on, there was the progress made by Langlands, Tunnell, and Buhler toward understanding Artin's conjecture (regarding the holomorphicity of L-functions) for two dimensional representations. Since Galois representations are usually not induced by one-dimensional characters, automorphic induction does not bring about a simple reduction of the Langlands correspondence to class field theory. Instead, as a basic question, it became of interest to understand the so-called *primitive* Galois representations, i.e. those representations which are not induced representations. However besides some numerical results these considerations did not contribute much to the proof of Langlands' conjecture. Nevertheless it is challenging to have a more

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direct view at Galois representations. Replacing primitive by stable representations the irreducible Galois representations are parametrized by conjugacy classes of admissible pairs, and a basic step to better understand the conductor is the conductor problem for Heisenberg representations which is the main topic of this paper.

The Galois group G of a local field F, viewed from a global perspective, is the decomposition group of a fixed prime ideal. The group structure of G implies that the degrees of its irreducible primitive representations are powers of the residue characteristic p. A. Weil [W] studied these representations in the case $p = 2$ and H. Koch [Ko] treated the general case. Next J. Buhler [B] and G. Henniart [He] succeeded in finding formulas for the conductors of primitive representations of degree exactly p. Because the Langlands correspondence commutes with character twists it was natural to study the conductor problem for twist classes, and it was rather obvious that, to deal with the general case, it is enough to know the minimal conductor in each twist class. Moreover, in local Galois theory it is possible to reduce the minimal conductor problem for primitive representations (or, more generally, *stable* representations, see section 4) to the simplest type of nonabelian irreducible representations, namely the Heisenberg representations. Whereas one-dimensional representations of G factor through quotient commutator groups $G/[G,G]$, Heisenberg representations factor through maximal quotients of the form $G/[[G,G],G]$, which are two-step nilpotent groups. The minimal conductor of a twist class of Heisenberg representations is then directly related to the filtration of the second central step $[G,G]/[[G,G],G]$ which is induced by the filtration $\{G^\nu\}_{\nu\in\mathbb{Q}_+}$ of ramification subgroups of G. Thus the minimal conductor problem reduces to the consideration of filtered quotient groups of the form $G^\nu \cap [G,G]/G^\nu \cap [[G,G],G]$ which are indexed by positive rational numbers ν (Proposition 2.2). Systematic study of these filtered groups began in [Zi1],[Zi2] and continued in work of Cram [Cr2],[Cr3] and Kaufhold [Kh2]. Because these results have never been put into journal articles, we shall review them here with some added remarks.

For $G := \mathrm{Gal}(\bar{F}|F)$, the Galois group of a local field F, class field theory implies that a dense subgroup of the abelian quotient $G/[G,G]$ identifies with the multiplicative group $F^\times$. Consequently, the second central step $[G,G]/[[G,G],G]$ identifies with the alternating square $F^\times \wedge F^\times$. It follows that the filtration of the second central step (see above) corresponds to a filtration, denoted UU^ν, of $F^\times \wedge F^\times$ and this becomes the basic object to be studied. In fact, for the conductor problem it would be enough to know this filtration modulo p-powers, i.e. on $F^\times \wedge F^\times/(F^\times)^p \wedge F^\times$, where p is the residue characteristic of F. Actually it is enough to consider the group U^1 of principal units instead of $F^\times$.

Class field theory also implies that the filtration of $G/[G,G]$ by ramification subgroups corresponds to the filtration of $F^\times$ by the principal unit subgroups $U^\nu = 1 + \mathfrak{p}^\nu$. The crux of the matter is that a concrete characterization for the filtration UU^ν of $F^\times \wedge F^\times$ involves a choice of coordinates; in other words, in the spirit of [Se1] it would be necessary to represent $F^\times \wedge F^\times$, or rather $U \wedge U = U^1 \wedge U^1$, as the covering group of some proalgebraic group. But the coordinates which give the filtration

$$\overline{UU}^\nu := UU^\nu \cap (U^1 \wedge U^1)/(U^1)^p \wedge U^1 \tag{0.1}$$

arise naturally by applying the so-called truncated exponential(see section 7) or in the general case the Artin-Hasse exponential (see section 12). The coordinates are naturally indexed by a certain set of triples $(s, \ell, r) \in S$ which is denoted $S_{f,t}$ and $S_{f,I}$ in sections 7 and 13 resp. However, it turns out to be a delicate problem to make the filtration (0.1) explicit by writing down a sequence of coordinate relations. There are two problems involved in passing from the coordinates to the filtration. The first is to specify the jumps of the filtration (0.1), i.e, to specify those rational numbers $\nu = j(s, \ell, r) \in \mathbb{Q}_+$ such that $\overline{UU}^\nu \supset \overline{UU}^{\nu+\varepsilon}$ is a proper inclusion for all $\varepsilon > 0$. Cram solved this problem completely in [Cr3] (see section 13).

The more difficult problem of describing the filtration in terms of coordinates remains without a complete solution at the present time. A basic step toward the solution of this problem is the study of s-extensions, i.e. of the factors $U^1 \wedge U^1/N \wedge U^1$ such that

$$U^s/U^{s+1} \twoheadrightarrow U^1/N \tag{0.2}$$

is surjective for some integer $s \geq 1$. In particular this implies $(U^1)^p \subset N$. Therefore if $e = e_{F|\mathbb{Q}_p}$ is the absolute ramification exponent of F, then s must be prime to p and less than $e^* := ep/(p-1)$. We call N a complement of U^s if (0.2) is an isomorphism. It appeared as an irritating complication that the filtration $\overline{UU}^\nu$ on $U^1 \wedge U^1/N \wedge U^1$ depends on the choice of the complement N. However, the right complement was chosen in [Cr2]. In section 11 we try to explain how the filtration varies with the complement, and in the appendix to that section we discuss the implications for the conductor formula of a primitive representation. The final results on the filtration on

$$U^1 \wedge U^1/(U^{s+1}(U^1)^p \wedge U^1)(N \wedge N), \tag{0.3}$$

where $N = C_s$ is Cram's complement, are Theorems 7.1* and 15.4. At first only the case $s \leq p-1$ had been studied, but the more complicated general result appeared as [Cr3], Proposition 4.1.2. From Theorem 7.1* follows immediately Theorem 7.1 which describes

1) the filtration of $U^1 \wedge U^1/U^{t+1} \wedge U^1$ if $t \leq \min\{e, p-1\}$ or equivalently $t < \min\{e^*, p\}$. In this case the filtration is simply given by the fact that more and more coordinates vanish.

The best result so far is

2) a description of the filtration of $U^1 \wedge U^1/U^{t+1}(U^1)^p \wedge U^1$ for $t < \min\{e^*, p^2\}$ ([Cr3], Theorem 1.5.2). Here it occurs for the first time that we need relations between different coordinates to describe the filtration. The result does not follow directly from results concerning s-extensions; it depends upon a study of extensions with two jumps. This will be sketched in section 16.

Let us briefly summarize the contents of this paper. In sections 1-4 we give basic facts concerning representations of local Galois groups, in particular Heisenberg representations, which serve as motivation for what follows. In sections 5-7 we state the main results in the simpler case 1). In sections 8-10 we give some selected proofs and in section 11 we discuss the filtration for complements other than Cram's complement and implications for the conductor formula of primitive Galois representations. In sections 12-16 we sketch some of the powerful results of [Cr3] and state the main results concerning the filtrations of (0.3) and in case 2).

We will freely use results of local class field theory where our main reference is [Se].

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Notation: Often we consider quotients A/B such that A does not contain B. This means that $A/B = AB/B$. Beginning from section 6 we will write UU^ν instead of $UU^\nu \cap (U^1 \wedge U^1)$.

1. Heisenberg representations

Let ρ be an irreducible representation of a (pro-)finite group G. Then ρ is called a **Heisenberg representation** if it represents commutators by scalar matrices. If $C^1G = G$, $C^{i+1}G = [C^iG, G]$ denotes the descending central series of G, the Heisenberg property means $C^3G \subset \mathrm{Ker}(\rho)$, and therefore ρ determines a character X on the alternating square of $A := G/C^2G$ such that

$$\rho([\hat{a}_1, \hat{a}_2]) = X(a_1, a_2) \cdot E$$

for $a_1,\ a_2 \in A$ with lifts $\hat{a}_1,\ \hat{a}_2 \in G$. The equivalence class of ρ is determined by the projective kernel Z_ρ which has the property that Z_ρ/C^2G is the radical of X and by the character χ_ρ of Z_ρ such that $\rho(g) = \chi_\rho(g) \cdot E$ for all $g \in Z_\rho$.

Proposition 1.1 [ZiRF] **Proposition 4.2** *The map $\rho \mapsto (Z_\rho, \chi_\rho)$ is a bijection between equivalence classes of Heisenberg representations of G and pairs Z, χ such that*

(i) $Z \subseteq G$ is a coabelian normal subgroup,

(ii) χ is a G-invariant character of Z,

(iii) $X(g_1, g_2) := \chi(g_1 g_2 g_1^{-1} g_2^{-1})$ is a nondegenerate alternating character on G/Z.

For pairs (Z, ρ) with properties (i)-(iii) the corresponding Heisenberg representation ρ is determined by the identity:

$$\sqrt{(G : Z)} \cdot \rho = Ind_Z^G(\chi).$$

Two Heisenberg representations $\rho_1,\ \rho_2$ induce the same alternating character $X_1 = X_2$ if and only if $\rho_2 = \chi \otimes \rho_1$ for some character χ of A.

Moreover, assume that every projective representation of A **lifts** to an ordinary representation of G. Then by I. Schur's results:

(i) the commutator map

$$A \wedge_{\mathbb{Z}} A \cong C^2G/C^3G, \quad a_1 \wedge a_2 \mapsto [\hat{a}_1, \hat{a}_2]$$

is an isomorphism;

(ii) the map $\rho \longmapsto X_\rho \in X(A \wedge_{\mathbb{Z}} A)$ from Heisenberg representations to alternating characters on A is surjective.

Now let $F|\mathbb{Q}_p$ be a p-adic number field and $G = \mathrm{Gal}(\bar{F}|F)$ the absolute Galois group. Then the lifting property holds, and via class field theory we turn (i) into an isomorphism:

$$(1) \qquad c : FF^\times \cong C^2G/C^3G,$$

where $FF^\times := \varprojlim(F^\times/N \wedge_{\mathbb{Z}} F^\times/N)$ is the profinite completion of the alternating square of $F^\times$. If $K|F$ is the abelian extension corresponding to the norm subgroup $N \subset F^\times$ and if $W_{K|F}$ denotes the relative Weil group, then the commutator map for $W_{K|F}$ induces an isomorphism:

$$(2) \qquad c : F^\times/N \wedge F^\times/N \longrightarrow K_F^\times/I_FK^\times,$$

where $K_F^\times$ denotes the norm 1 subgroup and $I_FK^\times$ the augmentation with respect to $K|F$. Taking the projective limit over all abelian extensions $K|F$ the isomorphisms (2) induce:

$$(3) \qquad c : FF^\times \cong \varprojlim K_F^\times/I_FK^\times,$$

where the limit on the right side refers to the norm maps. This gives a more explicit description of Heisenberg representations of the Galois group:

Corollary1.2 *The set of Heisenberg representations ρ of $G = G_F$ is in bijective correspondence with the set of all pairs (X, χ) such that:*

(i) X is a character of $FF^\times$,

(ii) χ is a character of $K^\times/I_FK^\times$, where the abelian extension $K|F$ corresponds to the radical $N \subset F^\times$ of X, and

(iii) via (2) the alternating character X corresponds to the restriction of χ to $K_F^\times$.

Given a pair (X, χ), we construct the Heisenberg representation ρ by induction from G_K to G_F:

$$\sqrt{(F^\times : N)} \cdot \rho = \mathrm{Ind}_{K\uparrow F}(\chi),$$

where N, K are as in (ii) and where the induction of χ (to be considered as a character of G_K) produces a multiple of ρ. From $(F^\times : N) = [K : F]$ we obtain the dimension formula:

$$\dim(\rho) = \sqrt{(F^\times : N)},$$

where N is the radical of X.

Let M be any subgroup between N and $F^\times$ and consider the class field $L|F$ which corresponds to M. Then we have an exact commutative diagram:

$$\begin{array}{ccccc}
L^\times\wedge L^\times/N_{K|L}\wedge L^\times & \xrightarrow{N_{L|F}\wedge N_{L|F}} & F^\times\wedge F^\times/N\wedge F^\times & \longrightarrow & F^\times\wedge F^\times/(M\wedge M)(N\wedge F^\times) \\
{\scriptstyle c}\big\downarrow & & {\scriptstyle c}\big\downarrow & & {\scriptstyle c}\big\downarrow \\
K_L^\times/I_LK^\times & \xrightarrow{\iota} & K_F^\times/I_FK^\times & \xrightarrow{N_{K|L}} & L_F^\times/I_FN_{K|L}.
\end{array}$$

Note that $L_F^\times \subseteq N_{K|L}(K^\times)$ because $N_{L|F} : L^\times/N_{K|L}(K^\times) \xrightarrow{\sim} M/N$ is a surjective map between groups of the same order. From the diagram we see a more direct construction of ρ. Choose M maximal isotropic for X and consider the class field $L|F$ which corresponds to M. Since X is trivial on $M \wedge M$ we see that $\chi|_{K_F^\times}$ is trivial on the kernel of $N_{K|L} : K_F^\times/I_FK^\times \to L_F^\times/I_FN_{K|L}$ and therefore there is a character χ_L of $L^\times/I_FN_{K|L}$ such that $\chi = \chi_L \circ N_{K|L}$. Then we obtain $\rho = \mathrm{Ind}_{L\uparrow F}(\chi_L)$ independently of the choice of χ_L. Heisenberg representations of dimension p are obtained from bicyclic extensions $K \supset L \supset F$ of degree p^2 and characters χ_L of $L^\times/I_FN_{K|L}$ which are nontrivial on $L_F^\times = I_FL^\times$.

Finally we remark that for any field extension $E|F$ the diagram

$$\begin{array}{ccc}
EE^\times & \xrightarrow{c} & C^2G_E/C^3G_E \\
{\scriptstyle N_{E|F}\wedge N_{E|F}}\big\downarrow & & \big\downarrow \\
FF^\times & \xrightarrow{c} & C^2G_F/C^3G_F
\end{array}$$

is commutative. Therefore:

Proposition 1.3. *Let $E|F$ be a finite extension.*

(i) The Heisenberg representation $\rho = (X, \chi)$ restricts to an irreducible representation ρ_E if and only if the norm map $N_{E|F} : E^\times \to F^\times/N$ is surjective, where N denotes the radical of X. For the corresponding class field K we have then $K \cap E = F$ and $\rho_E = (X \circ (N_{E|F} \wedge N_{E|F}), \chi \circ N_{EK|K})$.

(ii) The restriction $\rho|_{G_E}$ is isotypic if and only if $N \cap N_{E|F}$ is the radical of $X|_{N_{E|F}\wedge N_{E|F}}$. Then we obtain:

$$\rho|_{G_E} = \sqrt{(F^\times : N \cdot N_{E|F})} \cdot \rho_E.$$

We still mention the following characterization of Heisenberg representations which is useful if we want to identify them under the local Langlands correspondence:

Let ρ be an irreducible representation of $G = G_F$ and identify finite characters χ_F of $F^\times$ with characters of G_F. The torsion number $t_F(\rho)$ is the number of all χ_F such that $\chi_F \otimes \rho \cong \rho$. Then:

Proposition 1.4. *The torsion number $t_F(\rho)$ of an irreducible representation divides $dim^2(\rho)$, and equality holds if and only if ρ is a Heisenberg representation.*

Appendix: Heisenberg representations with symmetric or symplectic structure

Let $\rho = (Z_\rho, \chi_\rho)$ be a Heisenberg representation of G of dimension $n = \sqrt{(G : Z_\rho)}$ in the space $V|\mathbb{C}$. We consider the decomposition

$$(*) \qquad V \otimes V = Sym^2(V) \oplus Alt^2(V), \qquad tr^2_\rho = tr^2_{\rho,\sigma} + tr^2_{\rho,\alpha}$$

into the sum of the symmetric and alternating square of V. If ψ is a 1-dimensional character of G, then $< \psi, \rho \otimes \rho > \neq 0$ is equivalent to the existence of a nondegenerate bilinear form

$$\Phi : V \otimes V \to \mathbb{C}, \quad \text{such that} \quad \Phi(gv_1, gv_2) = \psi(g)\Phi(v_1, v_2).$$

Depending on whether ψ meets $Sym^2(V)$ or $Alt^2(V)$ the form Φ will be symmetric or symplectic.

Proposition1.5. *For a Heisenberg representation $\rho = (Z_\rho, \chi_\rho)$ the following are equivalent:*

(i) $\rho \otimes \rho$ contains a 1-dimensional character ψ,

(ii) $\psi|_{Z_\rho} = \chi^2_\rho$,

(iii) χ^2_ρ is a character of Z_ρ/C^2G,

(iv) $X(g_1, g_2) = \chi_\rho(g_1 g_2 g_1^{-1} g_2^{-1})$ satisfies $X^2 \equiv 1$.

(v) G/Z_ρ is an $\mathbb{F}_2$ vector space.

The proof follows from Frobenius reciprocity because Proposition 1.1 implies

$$\rho \otimes \rho = Ind^G_Z(\chi^2_\rho).$$

As a consequence we see that $n = dim(\rho)$ is a power of 2, and $\rho \otimes \rho$ is the sum of 1-dimensional characters, namely of all ψ such that $\psi|_{Z_\rho} = \chi_\rho^2$. Because of (*) we see that (for $n \neq 1$) we must have characters ψ which meet $Sym^2(V)$ and others which meet $Alt^2(V)$. Therefore if the conditions of Proposition 1.5 are fulfilled, then ρ will admit symmetric structures and symplectic sructures as well (provided that $n \neq 1$). The function $f(g) = \chi_\rho(g^2)$ is then a well defined function on G/C^2G (but not a character because $(df)(g_1, g_2) = X(g_1, g_2) = X(g_2, g_1)$), and all characters ψ must have the properties: $\psi(g) \in \{\pm\chi_\rho(g^2)\}$ if $g \in G - Z_\rho$, and $\psi(g) = \chi_\rho(g^2)$ if $g \in Z_\rho$. They are obtained as $\psi = \varphi f$, where φ is a ± 1function on G/Z_ρ such that $d\varphi = X$.

In the case $G = G_F$ we have Corollary 1.2. Then in the case of Proposition 1.5 the radical N of X must contain $F^{\times 2}$, and therefore $N = F^{\times 2}$ is the only nontrivial possibility if the residue characteristic is $p \neq 2$.

Remark. In a letter to M.F.Vigneras (25.11.1984) I asserted that Heisenberg representations cannot admit symplectic structures. A counterexample by D.Prasad prompted me to write this appendix. One can do more considering primitive or stable representations (see sections 3, 4 below) but I will not go into this.

2. Next we want to know the **Swan conductor** of $\rho = (X, \chi)$. Let $\{G^i\}_{i \geq 0, \in \mathbb{Q}}$ be the ramification subgroups of $G = G_F$ in the upper notation. For irreducible representations ρ of G we have the numerical invariants

$$j(\rho) := \max\{i;\ \rho|_{G^i} \neq 1\}, \quad \mathrm{sw}_F(\rho) = \dim(\rho) \cdot j(\rho),$$

where the second number, the (exponential) Swan conductor of ρ, is always an integer. This follows from [Se],VI,§2, proof of Cor.1, because the restriction of ρ (being irreducible) to ramification subgroups is either trivial which means $\langle \rho|_{G_i}, u_i \rangle = 0$ or disjoint from trivial and then $\langle \rho|_{G_i}, u_i \rangle = \dim(\rho)$.

Definition 2.1. Let $UU^i \subseteq FF^\times$ be the subgroup which under (1) corresponds to $G^i \cap C^2G / G^i \cap C^3G \subseteq C^2G/C^3G$, and for a character X of $FF^\times$ put

$$j(X) := \max\{i;\ X|_{UU^i} \neq 1\}. \tag{4}$$

Proposition 2.2. [Zi1]**section 2** *The Swan conductor of $\rho = (X, \chi)$ satisfies*

$$sw_F(\rho) \geq \widetilde{sw}_F(\rho) = \sqrt{(F^\times : N)} \cdot j(X),$$

where N is the radical of X. The right side is precisely the minimum of all Swan conductors for Heisenberg representations ρ corresponding to X. Let $\rho_0 = (X, \chi_0)$ be a minimal representation corresponding to X. Then in general we will have $\rho = \chi_F \otimes \rho_0 = (X, (\chi_F \circ N_{K|F})\chi_0)$ and

$$sw_F(\chi_F \otimes \rho_0) = \sqrt{(F^\times : N)} \cdot max\{j(\chi_F), j(X)\}, \tag{5}$$

where $j(\chi_F) = max\{i;\ \chi_F|_{U^i} \neq 1\}$ refers to the principal unit filtration in $F^\times$.

It is suggestive to consider the groups UU^i as principal unit subgroups of $FF^\times$. A **jump** of the filtration UU^i is a rational number $\nu \geq 0$ such that $UU^\nu \supset UU^{\nu+\epsilon}$ is a proper inclusion for all $\epsilon > 0$. As we see from (4), we have direct access to the conductors of Heisenberg representations of our Galois group G if we know the filtration UU^i and its jumps explicitly.

3. In connection with the local Langlands conjectures it became of interest to know the irreducible representations of $G = \mathrm{Gal}(\bar{F}|F)$ and in particular those representations which are **primitive**, i.e. those irreducible representations which cannot be constructed as induced representations from a proper subgroup $H \subset G$. A. Weil [W] commenced the study of these representations and complete results were given in H. Koch [Ko]:

Theorem 3.1., [Ko].*An irreducible representation ρ of G is primitive if and only if the following conditions are fulfilled:*

(i) The restriction of ρ to the subgroup of wild ramification $V \subset G$ is irreducible; and

(ii) there exists a finite tame normal extension $K|F$ such that the restriction ρ_K to $G_K \subset G$ is a Heisenberg representation $\rho_K = (X, \chi)$ where the character X of $KK^\times$ is totally anisotropic with respect to the natural action of $G_{K|F}$.

Let N be the radical of X. Since both ρ and $\rho_K = (X, \chi)$ fulfill condition (i), it follows that:

$$U_K^1 \cdot N = K^\times,$$

where U_K^1 denotes the principal units of K. Thus N determines a totally and wildly ramified abelian extension of K. Moreover we must have:

$$(U_K^1)^p \subseteq N.$$

Otherwise the maximal power p^ν, $(\nu \geq 1)$ such that $(U_K^1)^{p^\nu} \not\subseteq N$ would be an isotropic $G_{K|F}$-module. In particular we see that $(K^\times)^p \subset N$.

Proposition 3.2. [He],**Theorems 1.7 and 5.3.** *Let ρ be a primitive irreducible representation of G. Then (with the notations of Theorem 3.1(ii)):*

$$sw_F(\rho) = \frac{1}{e_{K|F}} \cdot sw_K(\rho_K) \geq \widetilde{sw}_F(\rho) = \frac{1}{e_{K|F}} \cdot \sqrt{(K^\times : N)} \cdot j_{KK}(X), \tag{6}$$

where the estimate actually occurs for appropriate character twists of ρ.

The conductor formulas (5) and (6) suggest that one should try to give the filtration UU^i of $FF^\times$ and also the jumps of that filtration explicitly. The remarks following Theorem 3.1 suggest that one should begin by studying the filtration modulo $(F^\times)^p \wedge F^\times$. We are going to review here the results of [Cr3] and [Kh2] concerning this question. To put things into perspective we mention that Proposition 3.2 applies to a wider class of representations, which we will call stable representations.

4. Stable representations and admissible pairs

In order to obtain parameters for the irreducible representations of the Galois group $G = \mathrm{Gal}(\bar{F}|F)$ it turns out to be convenient to work with stable instead of primitive representations. An irreducible representation ρ of G is called **stable** if: $(*)$ The restriction of ρ to ramification subgroups G^ν for $\nu \in \mathbb{Q}$, $\nu \geq 0$, is always isotypic.

Remark. We remark that ρ is primitive if and only if $(*)$ holds for all normal subgroups N of G, because G is pro-solvable (Theorem of T.R.Berger).

As a variant of Theorem 3.1 one can prove [ZiRF], sections 4-6:

Proposition 4.1. *An irreducible representation ρ of G is stable if and only if the following conditions are fulfilled:*

(i) The restriction of ρ to the subgroup of wild ramification $V \subset G$ is still irreducible,

(ii) there exists a finite tame normal extension $K|F$ such that the restriction ρ_K to $G_K \subset G$ is a Heisenberg representation $\rho_K = (X, \chi)$, where the character X of $KK^\times$ is nondegenerate on all principal unit subgroups U_K^i in the following sense:

If $N \subset K^\times$ is the radical of X and if X_i is the restriction of X to $U_K^i \wedge U_K^i$, then

$$N \cap U_K^i = rad(X_i) \quad \textit{for all}\, i \geq 1.$$

Again we have $U_K^1 \cdot N = K^\times$ because condition (i) is the same as in Theorem 3.1. Moreover let $M_i = U_K^i(N \cap U_K^1)/(N \cap U_K^1)$ for all $i \geq 1$, and let $M_i^\perp \subset U_K^1/N \cap U_K^1$ be the orthogonal complement with respect to X. Then we have

$$U_K^1/N \cap U_K^1 = M_1 = \bigoplus_{i \geq 1} M_i \cap M_{i+1}^\perp$$

and $M_i \cap M_{i+1}^\perp \cong M_i/M_{i+1}$ and therefore $K^\times/N \cong M_1$ is again p-elementary.

Now for a fixed base field F and $G = G_F$ we consider **admissible pairs** $(K|F, \rho)$ which means:

(i) ρ is a stable representation of G_K.

(ii) For each ramification subgroup G^ν the induction $\mathrm{Ind}_{G_K \cap G^\nu}^{G^\nu}(\rho_{G_K \cap G^\nu})$ is irreducible. (We remark that $G_K \cap G^\nu = G_K^{\psi_{K|F}(\nu)}$ is a ramification subgroup of G_K and therefore we have up to isomorphism a unique irreducible representation $\rho_{G_K \cap G^\nu} \subseteq \rho|_{G_K \cap G^\nu}$.)

(iii) The extension $K|F$ is minimal in the sense that $\mathrm{Ind}_{K|E}(\rho)$ will not be stable for proper subextensions $K \supset E \supseteq F$. In fact it is enough to check here the cases where $K|E$ is ramified of degree p.

In an obvious way we can form conjugate admissible pairs $(sK|F, s\rho s^{-1})$ for any $s \in G$.

Proposition 4.2.([ZiRF], **2.2 and 6.1**). *The map*

$$(K|F, \rho) \longmapsto \mathit{Ind}_{K|F}(\rho) \in \widehat{G}$$

induces a bijection between G-conjugacy classes of admissible pairs and irreducible representations of G.

It is obvious that the conductor formula (6) also includes the case of stable representations.

Remark. If we consider only representations of dimension prime to p, then admissible pairs $(K|F, \rho)$ consist of $[K : F]$ prime to p, $\dim(\rho) = 1$ and our properties are equivalent to the properties introduced by R. Howe [Ho]. (see also [KoZ] 1.8 and §3.)

Example. If $\sigma \in \widehat{G}$ is irreducible then in general it seems hard to find an admissible pair $(K|F, \rho)$ which defines σ. According to [ZiRF],Theorem 1.4 one has to consider a representation filter with leading term σ. But if $\sigma = (X, \chi)$ is a Heisenberg representation then the construction of ρ is much more direct. Consider $X_i = X|_{U^i \wedge U^i}$ for all $i \geq 0$ and denote $R_i := rad(X_i) \subseteq U^i$. If $rad(X) = N$ then obviously $N \cap U^i \subseteq R_i$, and according to Proposition 1.3(ii) σ is stable if and only if this is an equality for all i, i.e. $R_i \subseteq N$. Otherwise consider $M = \prod_{i=0}^{\infty} N \cdot R_i$.

Then M is isotropic for X and therefore:

$$N \subset M \subset M^{\perp} \subset F^{\times} \quad \text{corresponding to} \quad K \supset E \supset L \supset F,$$

where K is the class field of N. We consider the commutative diagram:

$$\begin{tikzcd}
FF^{\times}/N_{K|F} \wedge F^{\times} \arrow[r, "c_{K|F}"] & K_F^{\times}/I_F K^{\times} \\
LL^{\times}/N_{K|L} \wedge L^{\times} \arrow[r, "c_{K|L}"] \arrow[u, "N_{L|F} \wedge N_{L|F}"'] \arrow[d] & K_L^{\times}/I_L K^{\times} \arrow[u] \arrow[d, "N_{K|E}"] \\
LL^{\times}/(N_{E|L} \wedge L^{\times}) \arrow[r, "c_{E|L}"] & E_L^{\times}/I_L E^{\times}
\end{tikzcd}$$

We have the isomorphism $N_{L|F} : L^{\times}/N_{E|L} \cdot L_F^{\times} \to N_{L|F}/N_{E|F} = M^{\perp}/M$. Since X restricted to $M^{\perp}$ has the radical M we see that the alternating character X_L of $L^{\times}$ such that $X_L(a \wedge b) = X(N_{L|F}(a) \wedge N_{L|F}(b))$ will have the radical $N_{E|L} \cdot L_F^{\times} \supset N_{E|L}$. So it lives on the lowest row of the diagram. In correspondence we find χ_E a character of $E^{\times}/I_L E^{\times}$ such that $\chi_E \circ N_{K|E} = \chi$. Then $\rho = (X_L, \chi_E)$ is a stable Heisenberg representation of G_L and $(L|F, \rho)$ is the admissible pair which defines σ. Note that Heisenberg representations are monomial but nevertheless they can be stable of dimension greater than one, and therefore the admissible pair will not be supported by a character in general.

Finally we remark that stable representations come as tensor products of elementary stable representations.

An irreducible representation ρ of $G = \mathrm{Gal}(\bar{F}|F)$ is called **elementary stable** if there is a positive $\nu = \nu(\rho) \in \mathbb{Q}$ such that:

(a) ρ on G^ν is still irreducible,

(b) ρ on $G^{\nu+\varepsilon}$ is scalar for all rational $\varepsilon > 0$.

This implies that $0 < \nu < \frac{p \cdot e_{F|\mathbb{Q}_p}}{p-1}$ and that there exists a finite tame Galois extension $K|F$ such that

(c) $\rho_K = (X, \chi)$ is a Heisenberg representation of G_K,

(d) $\nu_K = e_{K|F} \cdot \nu$ is an integer,

(e) there is a natural surjection $U_K^{\nu_K}/U_K^{\nu_K+1} \twoheadrightarrow K^\times/N$ where N denotes the radical of X

(f) the restriction of X onto $U_K^{\nu_K} \wedge U_K^{\nu_K}$ has the radical $N \cap U_K^{\nu_K} \supseteq U_K^{\nu_K+1}$.

Proposition 4.3.[Kh1]. *Every stable representation ρ of G decomposes into a tensor product $\rho = \rho_1 \otimes \cdots \otimes \rho_r$ of elementary stable representations with different numbers $\nu_i = \nu(\rho_i)$. The decomposition is unique up to character twists. In particular the rational numbers ν_i are uniquely determined. Conversely every tensor product of elementary stable representations ρ_i with different numbers $\nu(\rho_i)$ is an irreducible stable representation.*

The proof depends on the following facts from [ZiRF]:

(i) A stable representation is a tensor product of irreducible projective representations of factor groups $G/G^{\nu+\varepsilon}$ such that the representation on $G^\nu/G^{\nu+\varepsilon}$ is still irreducible. (By a projective representation we understand here a representation with a multiplier.)

(ii) Stable representations with respect to a descending central series are always Heisenberg.

And moreover

(iii) The full Galois group G has trivial Schur multiplier and therefore if c is any cocycle on a finite factor group G/N we find always a function φ on G such that $c = d\varphi$.

5. A strategy. Our base field is $F|\mathbb{Q}_p$. We want to know more on the filtration UU^i of $FF^\times$ which has been defined in section 2. For an abelian extension $K|F$ let $\psi_{K|F}$ be the corresponding Herbrand function (Serre IV,§3). Then under (3) we must have:

$$c : UU^i \cong \varprojlim_K (U_K^{\psi_{K|F}(i)} \cap K_F^\times)/I_F K^\times, \tag{7}$$

where the principal unit subgroup U_K^ν for $\nu \notin \mathbb{Z}$ is defined through the next integer $\nu+ > \nu$, and the projective limit has to be taken with respect to the norm maps. The idea is to exploit this formula.

For subgroups $A,\ B \subseteq F^\times$ we let $A \wedge B$ denote the product inside of $FF^\times$. Some preliminary results are the following:

Proposition 5.1.[Zi1].

(i) It is basically enough to study the induced filtration $UU^i \cap (U^1 \wedge U^1)$ on the alternating square of principal units. $UU^i = UU^i \cap (U^1 \wedge U^1) \ \times \ U^i \wedge \langle \pi_F \rangle$.

(ii) For all $i \in \mathbb{Q}$, $i \geq 1$ we have $U^i \wedge U^1 \subseteq UU^i \cap (U^1 \wedge U^1)$, and equality holds for the integers $1, \ldots, p$.

(iii) All jumps of the filtration (i) are bigger than 1, not integral, and there exists a power p^ν depending on F such that all jumps are in $\frac{1}{p^\nu} \cdot \mathbb{Z}$.

Restricting to the filtration $UU^i \cap (U^1 \wedge U^1)$ we get the following refinement of (3). We fix a complementary group C_F of U^1 in $F^\times$ and consider only abelian extensions $K|F$ with norm group $N_{K|F}(K^\times) \supseteq C_F$. We will call them **$C_F$-extensions.** Then we have the following diagram ([Zi2], Prop.2.2(iii)):

$$\begin{array}{ccccc} N_1 \wedge U^1/N_1 \wedge N_1 & \xrightarrow{\iota} & U^1 \wedge U^1/N_1 \wedge N_1 & \xrightarrow{\jmath} & U^1 \wedge U^1/N_1 \wedge U^1 \\ \downarrow & & \downarrow & & \downarrow \\ I_F U_K^1/I_F K_F^\times & \xrightarrow{\iota} & C^2 W_{K|F}^1/I_F K_F^\times & \xrightarrow{\jmath} & C^2 W^1/I_F U_K^1, \end{array} \tag{8}$$

in which the maps ι and $\jmath$ are, respectively, injective and surjective. We also note that $N_1 = N_{K|F}(K^\times) \cap U^1$ and that the commutator c in $W_{K|F}$ induces vertical isomorphisms in (8) which connect the filtrations UU^i and $U_K^{\psi_{K|F}(i)}$, respectively. The left vertical map is given explicitly as $c(x \wedge y) = \hat{x}^{w_y - 1}$, where $\hat{x} \in U_K^1$ is a preimage of $x \in N_1$ under the norm map and $y \mapsto w_y \in G_{K|F}$ via class field theory. In particular, we obtain

$$c : UU^i \cap (U^1 \wedge U^1) \cong \varprojlim_K (U_K^{\psi_{K|F}(i)} \cap C^2 W_{K|F}^1)/I_F U_K^1, \tag{9}$$

where the projective limit extends over all C_F-extensions and $C^2W^1_{K|F}$ denotes the commutator subgroup of the first ramification group of the relative Weil group. Since K and F have the same residue field k_F the formula (9) tells us that a quotient $UU^i \cap (U^1 \wedge U^1)/UU^{i+\epsilon} \cap (U^1 \wedge U^1)$ is a subquotient of k_F. Actually the quotient is $\cong k_F$ or in some exceptional cases it is half of that. We will speak of full jumps or half jumps. In accordance with what we have said above we will concentrate on the filtration $UU^i \cap (U^1 \wedge U^1) \bmod ((U^1)^p \wedge U^1)$. In order to simplify notation, beginning from now **we will write UU^i instead of** $UU^i \cap U^1 \wedge U^1$. We begin with two special cases.

6. The filtration on $U^1 \wedge U^1/U^2 \wedge U^1$.

We fix a prime π_F and use the identification

$$(10) \qquad k_F \wedge_{\mathbb{F}_p} k_F \cong U^1 \wedge U^1/U^2 \wedge U^1, \quad a \wedge b \longmapsto (1 + a\pi_F) \wedge (1 + b\pi_F).$$

For a convenient description of our filtration we need a second identification. Let ϕ be the $\mathbb{F}_p$-Frobenius and $k_F\{\phi\}$ the noncommutative polynomial ring such that $\phi \cdot a = \phi(a) \cdot \phi = a^p \cdot \phi$ for $a \in k_F$. Let $f = [k_F : \mathbb{F}_p]$ denote the absolute inertial degree. We will write:

$$tr_\phi = 1 + \phi + \cdots + \phi^{f-1} \in k_F\{\phi\}.$$

Then the second identification is $L : k_F \wedge k_F \mapsto k_F\{\phi\}$:

$$(11) \quad a \wedge b \in k_F \wedge k_F \longmapsto L(a \wedge b) := \sum_{\nu=1}^{f-1} (a\phi^\nu(b) - b\phi^\nu(a))\phi^\nu = a \cdot tr_\phi \cdot b - b \cdot tr_\phi \cdot a.$$

The image are polynomials $\sum c_\nu \phi^\nu$ such that $\phi^\nu(c_{f-\nu}) = -c_\nu$. If f is even we obtain the relation $\phi^{f/2}(c_{f/2}) + c_{f/2} = 0$. We call them alternating polynomials.

Theorem 6.1. *The induced filtration $\overline{UU}^i$ on $U^1 \wedge U^1/U^2 \wedge U^1$ has the jumps $s_\ell = 1 + \frac{1}{p^{f-\ell}}$ for $\ell = f - [f/2], \ldots, f-1$, and via (10), (11) the subgroup $\overline{UU}^{s_\ell}$ corresponds to alternating polynomials with coefficients $c_i = 0$ for $i = f - [f/2], \ldots, \ell - 1$. If f is even then the first jump $s_{f/2}$ is a half jump.*

Instead of reviewing a proof (see [Zi2]) we go immediately to a more general case.

7. The filtration on $U^1 \wedge U^1/U^{t+1} \wedge U^1$.
Let e be the ramification exponent of $F|\mathbb{Q}_p$ and put $t = \mathbf{min}\{e, p-1\}$. Then $(U^1)^p \subset U^{t+1}$, hence the factor group U^1/U^{t+1} is p-elementary, and the truncated exponential $\mathrm{Exp}(x) := 1 + x + \cdots + \frac{x^{p-1}}{(p-1)!}$ induces the isomorphisms

$$E^t : k_F^t \ni (a_1, \ldots, a_t) \longmapsto \prod_{i=1}^{t} \mathrm{Exp}(a_i \pi_F^i) \in U^1/U^{t+1}, \tag{12}$$

$$E^t \wedge E^t : k_F^t \wedge k_F^t \xrightarrow{\sim} U^1 \wedge U^1/U^1 \wedge U^{t+1}, \quad a \wedge b \mapsto E^t(a) \wedge E^t(b), \tag{13}$$

for vectors $a, b \in k_F^t$ where π_F is again a fixed prime. **We will always identify $a_i \in k_F$ with its multiplicative representative in the valuation ring O_F.**

On the other hand we embed $k_F^t \wedge k_F^t$ into the noncommutative ring $k_F^{t\times t}\{\phi\}$ of polynomials with matrix coefficients such that $\phi A = \phi(A)\phi$ for $A \in k_F^{t\times t}$. (It is the endomorphism ring of the additive group G_a^t defined over k_F.) For row vectors $a = (a_1, ..., a_t) \in k_F^t$ the embedding $L : k_F^t \wedge k_F^t \to k_F^{t\times t}\{\phi\}$ is the following generalization of (11):

$$k_F^t \wedge k_F^t \ni a \wedge b \longmapsto L(a \wedge b) = \sum_{\nu=0}^{f-1} L_\nu \phi^\nu = {}^t a \cdot tr_\phi \cdot b - {}^t b \cdot tr_\phi \cdot a \in k_F^{t\times t}\{\phi\} \tag{14}$$

with matrices L_ν, such that $(L_\nu)_{i,j} = a_i \phi^\nu(b_j) - b_i \phi^\nu(a_j)$. Note that the polynomials have now a coefficient L_0 which is a skew symmetric matrix. The image of (14) consists of all polynomials $L = \sum_{\nu=0}^{f-1} L_\nu \phi^\nu$ such that $\phi^\nu(L_{f-\nu}) = -{}^t L_\nu$, for $\nu = 1, \ldots, f-1$ and ${}^t L_0 = -L_0$, where for $p = 2$ one has to add that the diagonal of L_0 is zero. We obtain the relations

$$\phi^{f/2}((L_{f/2})_{i,i}) + (L_{f/2})_{i,i} = 0, \quad \text{for } f \text{ even} \tag{15}$$

which will give us t half jumps in that case.

Remark 7.0 The image $L(k_F^t \wedge k_F^t)$ of the map (14) is an $\mathbb{F}_p$-space acted on by the multiplicative group $(k_F^\times)^t$ if we put:

$$xL = diag(x_1, x_2, \ldots, x_t) \cdot L \cdot diag(x_1, x_2, \ldots, x_t)$$

for $x \in (k_F^\times)^t$ and $L = \sum_\nu L_\nu \phi^\nu$. More explicitely this means $((xL)_\nu)_{i,j} = x_i \phi^\nu(x_j) \cdot (L_\nu)_{i,j}$. We are going to filter $L(k_F^t \wedge k_F^t)$ by $(k_F^\times)^t$-subspaces. In particular we have an action of $k_F^\times$ using $diag(x, x^2, ..., x^t)$ if $x \in k_F^\times$. The more sophisticated filtrations found by Cram [Cr3] (see beginning from section 12 below) are invariant only under this latter action. In fact as one can see from section 12, the action of $k_F^\times$ only uses powers x^i where i is prime to p.

Theorem7.1.[Kh2].*Let be* $t \leq min\{e, p-1\}$. *Then:*

(i) The jumps of the filtration $\overline{UU}^\nu$ *on* $U^1 \wedge U^1/U^1 \wedge U^{t+1}$ *are the numbers*

$$\nu(s, \ell, r) := s + r/p^{f-\ell}$$

for integers $1 \leq r \leq s \leq t$ *and* $\ell = 0, \ldots, f-1$, *where equality* $r = s$ *is allowed only if* $\ell \geq f/2$. *These numbers begin from* $\nu(1, f-[f/2], 1)$ *and increase to* $\nu(t, f-1, t)$, *where* r, ℓ, s *play the role of digits of first, second and third order, respectively.*

(ii) Take the coefficients $(L_\ell)_{s,r}$ *as independent coefficients of our polynomials (14), where* r, ℓ, s *vary as in (i). Then under*

$$(E^t \wedge E^t) \circ L^{-1} : L(k_F^t \wedge_{\mathbb{F}_p} k_F^t) \longrightarrow U^1 \wedge U^1/U^1 \wedge U^{t+1} \tag{16}$$

the filtration $U\bar{U}^\nu$ *corresponds to the filtration* $\{\mathcal{F}^\nu\}_\nu$ *such that* $\mathcal{F}^\nu = L(k_F^t \wedge_{\mathbb{F}_p} k_F^t)$ *if* $\nu = \nu(1, f-[f/2], 1)$, *and for* $\mathcal{F}^{\nu(s,\ell,r)}$ *the next term* $\mathcal{F}^{\nu'}$ *of the filtration is given by adding the relation* $(L_\ell)_{s,r} = 0$. *So we end with* $\mathcal{F}^{\nu'} = 0$ *for* $\nu = \nu(t, f-1, t)$. *And in the case when* f *is even, the jumps* $\nu(i, f/2, i)$ *for* $i = 1, \ldots, t$ *are half jumps.*

Remarks. 1. For $k_F = \mathbb{F}_p$ the example of section 6 gives nothing, whereas for $t \neq 1$ the map (16) will give us the $\frac{t(t-1)}{2}$ full jumps $\nu(s, 0, r) = s + r/p$ for $1 \leq r < s \leq t$.

2. Because of the relation $\phi^\ell((L_{f-\ell})_{r,s}) = -(L_\ell)_{s,r}$ it would be possible to take $(L_{f-\ell})_{r,s}$ as independent coefficients.

3. In the case $t = 2$, $f = 4$ the polynomials L have the form:

$$L = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} + \begin{pmatrix} -\phi(b) & -\phi(g) \\ d & -\phi(h) \end{pmatrix} \phi + \begin{pmatrix} a & -\phi^2(e) \\ e & f \end{pmatrix} \phi^2 + \begin{pmatrix} b & -\phi^3(d) \\ g & h \end{pmatrix} \phi^3$$

with $a, \ldots, h \in k$ where $a = (L_2)_{1,1}$, $f = (L_2)_{2,2}$ satisfy (15), and the jumps are: $1 + 1/p^2 \Rightarrow a = 0$, $1 + 1/p \Rightarrow b = 0$, $2 + 1/p^4 \Rightarrow c = 0$, $2 + 1/p^3 \Rightarrow d = 0$, $2 + 1/p^2 \Rightarrow e = 0$, $2 + 2/p^2 \Rightarrow f = 0$, $2 + 1/p \Rightarrow g = 0$, $2 + 2/p \Rightarrow h = 0$.

The proof uses the fact (see Proposition 5.1(ii)) that for $U^1 \wedge U^1/U^1 \wedge U^{t+1}$ the filtration UU^ν is a refinement of the filtration $U^1 \wedge U^s$ for $s = 1, \ldots, t$. Let k_i for

$i = 1, \dots, t$ denote the different copies of k_F and use the notation $k_r \wedge k_s = k_r \otimes k_s$ for $r < s$. Then instead of (13) (14) we may consider the maps:

$$E^s \wedge E_s : (\oplus_{r=1}^s k_r) \wedge k_s \longrightarrow U^1 \wedge U^s / U^1 \wedge U^{s+1} \tag{17}$$

$$L : (\oplus_{r=1}^s k_r) \wedge k_s \longrightarrow \lrcorner^s\{\phi\} \subset k^{s \times s}\{\phi\}, \tag{18}$$

where the matrices $L_\nu \in k_F^{s \times s}$, $(L_\nu)_{i,j} = a_i \phi^\nu(b_j) - b_i \phi^\nu(a_j)$ are now latches of level s, i.e. $(L_\nu)_{i,j} = 0$ unless ($i \leq s$ and $j = s$) or ($i = s$ and $j \leq s$.)

We denote $\lrcorner^s\{\phi\}$ the image of (18) in $k^{s \times s}\{\phi\}$. Then it is possible to decompose (16) into the maps

$$(E^s \wedge E_s) \circ L^{-1} : \lrcorner^s\{\phi\} \longrightarrow U^1 \wedge U^s / U^1 \wedge U^{s+1} \tag{19}$$

for all $s = 1, \dots, t$. We fix s and according to Theorem 7.1 (ii) we try to identify the groups $UU^\nu / U^1 \wedge U^{s+1}$ for $s < \nu < s+1$ in $\lrcorner^s\{\phi\}$. Theorem 7.1 is then a consequence of

Theorem 7.1*. *The filtration* $\{\overline{UU}^\nu\}_{\nu \geq s}$ *of* $U^1 \wedge U^s / U^1 \wedge U^{s+1}$ *has the jumps* $\nu(s, \ell, r)$ *for integers* r, ℓ *as in 7.1(i), and under the isomorphism (19) the filtrations* $\mathcal{F}^\nu \cap \lrcorner^s\{\phi\}$ *and* $\{\overline{UU}^\nu\}_{\nu \geq s}$ *correspond.*

On the other hand we also want to describe the dual filtration $\mathcal{F}^*_{\nu(s,\ell,r)}$ on the space $(k^t \wedge k^t)^* \cong (U^1 \wedge U^1 / U^1 \wedge U^{t+1})^*$ of alternating characters which is given as

$$X \in \mathcal{F}^*_{\nu(s,\ell,r)} \quad \text{if } j(X) < \nu(s, \ell, r)$$

i.e. if X is trivial on $UU^{\nu(s,\ell,r)}$. For this we consider the form $\langle a, b \rangle = \mathrm{tr}(a \cdot {}^t b) \in \mathbb{F}_p$ on k^t and identify $\mathbb{F}_p$ with roots of unity $\mu_p \subset \mathbb{C}$. Then we obtain

$$L(k^t \wedge k^t) \cong (k^t \wedge k^t)^*, \quad P \mapsto X_P$$

which sends the polynomial $P = \sum_{\nu=0}^{f-1} P_\nu \cdot \phi^\nu$ to the alternating form

$$X_P(a, b) = \mathrm{tr}(a \cdot P({}^t b)).$$

Proposition 7.2 *We have* $rad(X_P) = Ker(P)$ *hence* $dim(X_P) = \sqrt{[k^t : Ker(P)]}$ *is the dimension of Heisenberg representations* (X, χ) *where* X *identifies with* X_P *in the sense explained after 5.1. The minimal conductor is*

$$\widetilde{sw}_F(X_P) = \sqrt{[k^t : Ker(P)]} \cdot j(X_P) \in \mathbb{Z},$$

hence $\frac{1}{2} codim_{\mathbb{F}_p}(Ker(P)) - log_p(denom(j(X_P)) \geq 0$.

Now we ask for **the filtration $\mathcal{F}^*_{\nu(s,\ell,r)}$ of $L(k^t \wedge k^t)$ such that**

$$P \in \mathcal{F}^*_{\nu(s,\ell,r)} \quad \text{if and only if} \quad j(X_P) < \nu(s,\ell,r). \tag{19.1}$$

Let $S = S_{f,t}$ be the set of all triples (s,ℓ,r) such as in Theorem 7.1(i) and consider on $L(k^t \wedge k^t)$ the $\mathbb{F}_p$- bilinear pairing

$$< P, L >= \sum_{i=1}^{t} tr_{f/2}((P_{f/2})_{i,i} \cdot (L_{f/2})_{i,i}) + \sum_{(s,\ell,r)\in S'} tr((P_\ell)_{s,r} \cdot (L_\ell)_{s,r}), \tag{19.2}$$

where the first terms only occur if f is even and where S' is S without the triples from the first terms. We note that (15) for the polynomials P and L implies that $(P_{f/2})_{i,i} \cdot (L_{f/2})_{i,i}$ is invariant under $\phi^{f/2}$, and therefore the trace $tr_{f/2} = 1+\phi+...+\phi^{\frac{f}{2}-1}$ is well defined. Concerning the action of $(k_F^\times)^t$ introduced in remark 7.0 we see:

$$< xP, L >=< P, xL > . \tag{19.3}$$

Now an easy computation shows:

Proposition 7.3 *If $P = \sum_{\nu=0}^{f-1} P_\nu \cdot \phi^\nu \in L(k^t \wedge k^t)$, then $X_P(a,b) =< P, L(a \wedge b) >$. Therefore $\mathcal{F}^*_{\nu(s,\ell,r)} = \mathcal{F}^{\nu(s,\ell,r)^\perp}$ with respect to the nondegenerate pairing (19.2).*

Remark. We note here that the pairing (19.2) and the Proposition 7.3 extend to the more general case where we consider $U^1 \wedge U^1$ modulo p-powers. See the remark at the beginning of section 14.

Therefore Theorem 7.1 has the following

Corollary 7.4. *Let be $t \le min\{e, p-1\}$ and consider the isomorphism*

$$(U^1 \wedge U^1 / U^1 \wedge U^{t+1})^* \xrightarrow{\sim} L(k_F^t \wedge k_F^t)$$

which takes X to the polynomial P_X such that $X(E^t(a) \wedge E^t(b)) = X_{P_X}(a,b)$ for all $a,\ b \in k_F^t$. Then:

$j(X) < \nu(s,\ell,r)$ if and only if $P_X \in \mathcal{F}^*_{\nu(s,\ell,r)}$, where $\mathcal{F}^*_{\nu(s,\ell,r)}$ consists of all polynomials P such that $(P_\lambda)_{\sigma,\rho} = 0$ for all $(\sigma,\lambda,\rho) \in S$ such that $\nu(\sigma,\lambda,\rho) \ge \nu(s,\ell,r)$.

Remark. Now the assignment $\nu \mapsto \mathcal{F}^*_\nu$ is covariant in the sense that $\mathcal{F}^*_\nu$ increases together with ν. If we add relations $(L_\ell)_{s,r} = 0$ where we go now in opposite direction then the conductor $j(X_L)$ goes down. In the case $t = 2$, $f = 4$ (see remark 3) we obtain

$$h = 0 \Leftrightarrow j(X_L) < 2 + 2/p,$$
$$h = g = 0 \Leftrightarrow j(X_L) < 2 + 1/p, \dots,$$
$$h = g = \cdots = a = 0 \Leftrightarrow j(X_L) < 1 + 1/p^2,$$

which means that $X_L \equiv 1$.

8. What are the jumps ? [Zi3]

Let $C_s \subseteq U^1$ be a subgroup which is a complement of U^s/U^{s+1}, i.e. $C_s \supseteq U^{s+1}$ and $U^1/U^{s+1} = U^s/U^{s+1} \times C_s/U^{s+1}$ is a direct product. Such a C_s exists for all $s \leq t$ but besides the trivial case $C_1 = U^2$ it is not unique. We obtain natural isomorphisms

$$U^1 \wedge U^s/U^1 \wedge U^{s+1} \xrightarrow{\sim} U^1 \wedge U^1/(U^1 \wedge U^{s+1})(C_s \wedge C_s), \tag{20}$$

and the natural projection

$$U^1 \wedge U^1/U^1 \wedge U^{t+1} \longrightarrow \prod_{s=1}^{t} U^1 \wedge U^1/(U^1 \wedge U^{s+1})(C_s \wedge C_s) \tag{21}$$

is also an isomorphism.

Proposition 8.1 *Assume $s \leq t$. Then the filtrations $UU^\nu/(U^1 \wedge U^{s+1})(C_s \wedge C_s)$ on the right side of (21) have disjoint jumps for the different values of s. Therefore (21) is a direct product of filtered groups and the maps (20) are isomorphisms of filtered groups.*

For the proof we consider s**-extensions** $K|F$, where $s \geq 1$ is an integer. This means that $K|F$ is a Galois extension such that s is the only jump in the filtration of $G_{K|F}$ by ramification subgroups. Equivalently $K|F$ is abelian and the norm residue map induces a surjection:

$$k_F \cong U^s/U^{s+1} \twoheadrightarrow F^\times/N_{K|F} \cong G_{K|F}. \tag{22}$$

Hence $p^n := [K : F] \leq p^f$, and, with $N_i := N_{K|F} \cap U^i = N_{K|F}(U^i_K)$ for $i = 1, \dots, s$, we obtain $U^{s+1} \subseteq N_s$ and $U^s/N_s \xrightarrow{\sim} U^i/N_i$. Because (22) factorizes via $U^s/U^{s+1} \to U^s(F^\times)^p/U^{s+1}(F^\times)^p \to F^\times/N_{K|F}$ we have $1 \leq s \leq \frac{pe}{p-1}$ and $p \nmid s$ (unless s is the upper bound), where e refers to $F|\mathbb{Q}_p$. Maximal s-extensions will correspond to norm subgroups $N_{K|F} = C_s \cdot C_F \subset F^\times$ for complements C_F and

C_s as above. If $s \neq p\frac{e}{p-1}$, then every s-extension is extendable to such a maximal s-extension. From (8) we obtain:

Lemma 8.2 *If $K|F$ is an s-extension with norm subgroup $N_1 = N_{K|F}(K^\times) \cap U^1$ then (3) induces an isomorphism*

$$(23) \qquad c : U^1 \wedge U^1/(U^1 \wedge U^{s+1})(N_1 \wedge N_1) \cong C^2W^1_{K|F}/I_F(K_F^\times \cdot U_K^{s+1}),$$

such that UU^ν corresponds to $U_K^{\psi_{K|F}(\nu)} \cap C^2W^1_{K|F}$. Moreover we have $C^2W^1_{K|F} = U_K^{s+1} \cap K_F^\times$, $K_F^\times \subseteq U_K^s$, and for $a \wedge b \in N_1 \wedge U^1$ we explicitly have $c(a \wedge b) = \hat{a}^{w_b - 1} \in I_F U_K^1/\sim$, where $\hat{a} \in U_K^1$ is a preimage of a under the norm map and $b \mapsto w_b \in G_{K|F}$ via class field theory.

To realize the values of all possible jumps it is convenient to study the bigger quotients $C^2W^1_{K|F}/I_F U_K^{s+1} = \widehat{H}^{-1}(G_{K|F}, U_K^{s+1})$.

Proposition 8.3 *Let $f = f_{F|\mathbb{Q}_p}$ and let r be the position of s in the sequence of all numbers ≥ 1 which are prime to p. Then for any s-extension of degree $[K:F] = p^n$ the group $\widehat{H}^{-1}(G_{K|F}, U_K^{s+1})$ is p-elementary of $\mathbb{F}_p$-dimension $f \cdot r \cdot n$ and the filtration $U_K^i \cap K_F^\times/I_F U_K^{s+1} = U_K^i \cap C^2W^1_{K|F}/I_F U_K^{s+1}$ (for $i > s$) has exactly $r \cdot n$ jumps of full size, namely $i = s + jp^\ell$ where $\ell = 0, \ldots, n-1$, $j = 1, \ldots, s$ and $p \nmid j$. In particular $U_K^{s+p^n} \cap C^2W^1_{K|F}/I_F U_K^{s+1} = \{1\}$ if $s < p$ because then: $s + sp^{n-1} < s + p^n$.*

Proof. The proof is by induction on n. We mention that

$$C^2W^1_{K|F}/I_F U_K^1 \xrightarrow{\sim} K_F^\times/I_F K^\times = \widehat{H}^{-1}(G_{K|F}, K^\times).$$

In the cyclic case $n = 1$ this vanishes and therefore

$$\widehat{H}^{-1}(G_{K|F}, U_K^{s+1}) = I_F U_K^1/I_F U_K^{s+1}$$

in that case. We may use that the groups of our filtration occur as images of the maps $\widehat{H}^{-1}(G_{K|F}, U_K^i) \longrightarrow \widehat{H}^{-1}(G_{K|F}, U_K^{s+1})$ for $i > s$. In the induction step we go from K to $E|K$ such that $[E:K] = p$ and consider the exact sequence

$$\widehat{H}^{-1}(G_{E|K}, U_E^i) \xrightarrow{cor} \widehat{H}^{-1}(G_{E|F}, U_E^i) \xrightarrow{N_*} \widehat{H}^{-1}(G_{K|F}, U_K^{\varphi_{E|K}(i)}) \to \{1\}.$$

Then we can apply the induction hypothesis for $E|K$ and $K|F$. $\square$

Now we may use Lemma 8.2 in order to see that the jumps of Proposition 8.3 will correspond to the jumps $s + j/p^{n-\ell}$ on the left side of (23). Since in

Proposition 8.1 the assumption is $s \leq t < p$ we obtain disjoint sets of jumps for different values of s. This ends the proof.

In the next step we will see the positions of the jumps more precisely. We preserve the assumptions of Proposition 8.3 and for the commutator subgroups of the ramification groups $W^j_{K|F}$ (upper numeration) we simply write: $CW^j := C^2W^j_{K|F}$. Then:

Proposition 8.4. *Let be $1 \leq j < s, p \nmid j$. The filtration $U^i_K \cap CW^j/CW^{j+1}$ has exactly the jumps $i = s + jp^\ell, \quad \ell = 0, \cdots, n-1$ which are full size.*

We will need a series of Lemmas:

Lemma 8.5. $CW^j \subseteq U^{s+j}_K$ for all $j = 1, \ldots, s$.

Proof. Because of $CW^j = CW^s \cdot I_F U^j_K$ and $I_F U^j_K \subseteq U^{j+s}_K$ we can restrict to the case $j = s$. We consider the division algebra D of invariant $1/p^n$, ($p^n = [K : F]$). The relative Weil group $W = W_{K|F}$ is imbedded into D and from [ST] we conclude:

$$W^{\varphi(v)} = W \cap U^v_D \ \ \textit{for all} \ \ v \geq 0, \in \mathbb{Z},$$

where $\varphi = \varphi_{K|F}$ and where U^v_D are the principal units in D. This is because $K|F$ is fully unramified. From $\varphi(s) = s$ we get $W^s = W \cap U^s_D$, hence $CW^s \subseteq W \cap U^{2s}_D$ because $CU^s_D \subseteq U^{2s}_D$. From the exactness of $U^v_K \hookrightarrow W^{\varphi(v)} \longrightarrow G^{\varphi(v)}$ and $G^v = \{1\}$ for $v > s$ we see $W \cap U^{2s}_D = K^\times \cap U^{2s}_D = U^{2s}_K$, because D/K is unramified. $\square$

We fix $\sigma \neq 1 \in G$ and consider

$$x \in U^i_K \longmapsto x^{1-\sigma} \in I_F U^i_K \subseteq CW^i \subseteq U^{i+1}_K.$$

This induces

$$U^i_K/U^{i+1}_K \xrightarrow{1-\sigma} CW^i/CW^{i+1} \longrightarrow U^{i+s}_K/U^{i+s+1}_K \quad ,$$

and in case $p \nmid i$ the combined map is an isomorphism ([Se], p. 79, Ex. 3a.). Thus we obtain:

Lemma 8.6. *Let be $1 \leq j < s$ and $p \nmid j$. The filtration $U^i_K \cap CW^j/CW^{j+1}$ has the first jump $i = s + j$ which is of full size.* $\square$

Now we consider an intermediate field $K \supset E \supset F$ with $[K : E] = p, \quad [E : F] = p^{n-1}$, for instance take E to be the fixed-point field of σ. Then:

$$1 \to (CW^j/CW^{j+1})_{K/E} \to CW^j/CW^{j+1} \xrightarrow{N_{K/E}} (CW^j/CW^{j+1})_{E/F} \to 1$$

is exact and $I_E U^j_K / I_E U^{j+1}_K = (CW^j/CW^{j+1})_{K/E} \xrightarrow{\sim} U^{j+s}_K/U^{j+s+1}_K$. We have an isomorphism

$$(*) \qquad U^{j+s+1}_K \cap CW^j/CW^{j+1} \xrightarrow{N_{K/E}} (CW^j/CW^{j+1})_{E/F}$$

because the left side intersects the kernel of the norm map trivially and it has the same order as the right side.

We must prove that $N_{K/E}$ maps the quotient $U_K^i \cap CW^j/CW^{j+1}$ onto $U_E^{\varphi(i)} \cap (CW^j/CW^{j+1})_{E/F}$ with $\varphi = \varphi_{K/E}$, if $i \geq s+j$. This follows from:

Lemma 8.7. $U_K^i \cap CW^j \xrightarrow{N_{K/E}} U_E^{\varphi(i)} \cap CW^j_{E/F}, \quad \varphi = \varphi_{K/E},$ *is surjective for* $j = 1, \ldots, s$ *and* $i \geq s+1$.

Proof. We proceed by induction on j. The case $j = 1$ follows because $CW^1 = U_K^{s+1} \cap K_F^\times$ and $N_{K/E}(U_K^v \cap K_F^\times) = U_E^{\varphi(v)} \cap E_F^\times$ for all $v \geq s+1$. Now we assume the Lemma to be true for a certain j and consider the commutative diagram

$$\begin{array}{ccccc} U_K^i \cap CW^{j+1} & \xrightarrow{\iota} & U_K^i \cap CW^j & \xrightarrow{\jmath} & U_K^i \cap CW^j/CW^{j+1} \\ \downarrow & & \downarrow & & \downarrow \\ U_E^{\varphi(i)} \cap CW^{j+1}_{E/F} & \xrightarrow{\iota} & U_E^{\varphi(i)} \cap CW^j_{E/F} & \xrightarrow{\jmath} & U_E^{\varphi(i)} \cap (CW^j/CW^{j+1})_{E/F} \end{array}$$

in which the maps ι are injective, the maps $\jmath$ are surjective, and the vertical maps are the norm maps $N_{K/E}$. By the induction hypothesis the middle vertical maps are surjective for $i \geq s+j$. Therefore, by $(*)$, the right and left vertical maps are also isomorphisms for $i \geq s+j+1$. $\square$

Proposition 8.4 now follows by applying $(*)$, Lemma 8.7, and induction on n ($p^n = [K : F]$).

Corollary 8.8. *Let* $K|F, G, s, p^n, W = W_{K|F}$ *be as above. Then* $CW^s/I_F U_K^{s+1}$ *is p-elementary of* $\mathbb{F}_p$*-dimension* fn $(f = f_{F|\mathbb{Q}_p})$ *and the principal unit filtration* $U_K^i \cap CW^s/I_F U_K^{s+1}$ *has the jumps* $i = s + sp^\ell$ *which are of full size* ($\ell = 0, \ldots, n-1$).

For the last result see [Zi1](8.5).

9. Cram's complements

We restrict again to $s \leq t$, and we consider the different factors on the right side of (21). Each factor has its filtration

$$U^1 \wedge U^1 \supset (U^s \wedge U^s)(C_s \wedge C_s) \supset (U^1 \wedge U^{s+1})(C_s \wedge C_s).$$

The terms we denote $\Gamma \supset \Gamma_s \supset \Delta_s$ resp. This filtration has a splitting, namely:

$$\Gamma/\Delta_s = C_s \wedge U^1/\Delta_s \quad \times \quad \Gamma_s/\Delta_s, \tag{24.1}$$

$$C^2W^1_{L|F}/I_F U_L^s = I_F U_L^1/I_F U_L^s \quad \times \quad C^2W^s_{L|F}/I_F U_L^s, \tag{24.2}$$

where $L|F$ is the maximal s-extension corresponding to the norm subgroup $C_s{\cdot}C_F$, hence $L_F^\times U_L^{s+1} = U_L^s$. The maps from (24.1) down to (24.2) are given through the commutator in $W_{L|F}$ and connect the filtrations as described in Lemma 8.2. Because of Propositions 8.3, 8.4 and Corollary 8.8 we have some information on the jumps on $C^2W^1_{L|F}/I_FU^s_L$, $C^2W^1_{L|F}/C^2W^s_{L|F}$ and $C^2W^s_{L|F}/I_FU^s_L$, and we can transport this to get information on the jumps on Γ/Δ_s, Γ/Γ_s, Γ_s/Δ_s resp..

So far everything is independent of the complement C_s which we have chosen. But **in general it is not true that the splitting (24) is a direct product of filtered groups,** a phenomenon which gave rise to many irritations. To ensure that (24) is in fact such a product one has to choose an appropriate complement C_s of U^s/U^{s+1}, namely as a variation of (12) take

$$C'_s := E^s(k_1 \oplus \cdots \oplus k_{s-1} \oplus \{0\}) \subset U^1/U^{s+1} \tag{25}$$

The C'_s were considered first by G.-M.Cram [Cr2], and we call them Cram's complements. We will also use the notation $C'_s = E^{s-1}(k_F^{s-1}) \subset U^1/U^{s+1}$. Then we obtain

Proposition 9.1. *For $C_s = C'_s$ the filtrations induced by $\{UU^\nu\}_\nu$ on the factors of the right side of (24) have disjoint jumps, namely $\nu = \nu(s,\ell,r)$ for $r < s$ on the first factor and $\nu = \nu(s,\ell,s)$ for $\ell \geq f/2$ on the second factor.*

Remark. The jumps on the second factor are the same for any choice of C_s but in general these jumps may occur also on the first factor.

Proof. The crux of the proof is to identify the filtration of

$$c : C_s \wedge U^1/\Delta_s \xrightarrow{\sim} I_FU^1_L/I_FU^s_L, \quad a \wedge b \mapsto \widehat{a}^{w_b-1} \tag{26}$$

where the map refers to the end of Lemma 8.2 in the case when $N_1 = C_s = C'_s$ is Cram's complement. In particular one wants to show that the filtration $U^i_L \cap I_FU^1_L/I_FU^s_L$ does not admit jumps of the type $i = s + sp^\ell$. **In a first step one has to express the norm map in terms of coordinates:**

Fix a prime π_L such that $N_{L|F}(\pi_L) = \pi_F$ and let $\pi_K = N_{L|K}(\pi_L)$ for any subextension $K|F$ of L. For the pair $(K,\ \pi_K)$ let $C_s(K)$ be Cram's complement on the K-level. Then it is possible to describe the norm map in terms of coordinates, namely:

$$\begin{array}{ccc} k_1 \oplus \cdots \oplus k_{s-1} & \xrightarrow{\phi^d \oplus \cdots \oplus \phi^d} & k_1 \oplus \cdots \oplus k_{s-1} \\ {\scriptstyle E^{s-1}}\big\downarrow & & \big\downarrow{\scriptstyle E^{s-1}} \\ C_s(K)/U_K^{s+1} & \xrightarrow{N_{K|F}} & C_s/U^{s+1} \end{array} \tag{27}$$

is commutative if $[K : F] = p^d$. In particular the norm map for Cram's complements is surjective. This gives us a modified version of (26). Put $\Delta_s(K) := (U^1 \wedge U^{s+1})(C_s \wedge N_1(K|F))$, i.e. $\Delta_s(K) \supseteq \Delta_s(L) = \Delta_s$. Then (26) turns into

$$C_s \wedge U^1/\Delta_s(K) \xrightarrow{\sim} I_FC_s(K)/I_FU_K^{s+1} \tag{28}$$

for all subextensions $K|F$ of L.

Lemma 9.2. *We have natural isomorphisms*

$$I_F C_s(K)/I_F U_K^{s+1} \xrightarrow{\sim} I_F U_K^1/I_F U_K^s \xrightarrow{\sim} C^2W^1_{K|F}/C^2W^s_{K|F}, \tag{29}$$

which respect the principal unit filtration, i.e.

$$U_K^i \cap I_F C_s(K)/I_F U_K^{s+1} \xrightarrow{\sim} U_K^i \cap C^2W^1_{K|F}/C^2W^s_{K|F}$$

for all i. Equivalently the filtration $U_K^i \cap I_F C_s(K)/I_F U_K^{s+1}$ has no jumps of type $i = s + sp^\ell$.

Proof. The maps (29) are isomorphisms because the norm map for Cram's complements is surjective. Next we want to exclude the jump $i = 2s$. Via the left vertical of (27) we obtain a map

$$(k_1 \oplus \cdots \oplus k_{s-1}) \otimes G_{K|F} \xrightarrow{\sim} I_F C_s(K)/I_F U_K^{s+1} \longrightarrow U_K^{s+1}/U_K^{2s+1}. \tag{30}$$

To make this explicit a careful computation is necessary.

Lemma 9.3, [Cr2], [Kh2], 4.3 *Let $x_r \in k_r = k$, $\sigma \in G_{K|F}$, and let $b(\sigma) \in k$ such that $\pi_K^{\sigma-1} \equiv 1 + b(\sigma) \cdot \pi_K^s$ mod U_K^{s+1}. Then:*

$$Exp(x_r \pi_K^r)^{\sigma-1} \equiv 1 + r x_r b(\sigma)\varepsilon \cdot \pi_K^{r+s} \mod U_K^{2s+1},$$

for all $r = 1, \ldots, s-1$ where $\varepsilon \in U_K^1$ is a principal unit which does not depend on σ and r. ☐

Now we use the isomorphism

$$\mathfrak{p}_K^{s+1}/\mathfrak{p}_K^{2s+1} \cong U_K^{s+1}/U_K^{2s+1}$$

and observe that $k_F = W(k_F)/pW(k_F)$ (where $W(k_F)$ denotes the Witt ring) acts on $\mathfrak{p}_K^{s+1}/\mathfrak{p}_K^{2s+1}$, because $\nu_K(p) = [K : F]e \geq t \geq s$. So we may consider $\mathfrak{p}_K^{s+1}/\mathfrak{p}_K^{2s+1}$ as a k_F-space with basis $\{\varepsilon \cdot \pi_K^{r+s};\ r = 1, \ldots, s\}$. From Lemma 12 we see that via (29) $I_F C_s(K)$ is contained in the subspace generated by $\{\varepsilon \cdot \pi_K^{r+s};\ r = 1, \ldots, s-1\}$. and therefore $I_F C_s(K) \cap U_K^{2s} \subseteq U_K^{2s+1}$, which means that $i = 2s$ cannot be a jump for the filtration $U_K^i \cap I_F C_s(K)/I_F U_K^{s+1}$.

For the induction step we consider the direct product

$$C^2W^1_{K|F}/I_F U_K^{s+1} = I_F C_s(K)/I_F U_K^{s+1} \times C^2W^s_{K|F}/I_F U_K^{s+1}. \tag{31}$$

We denote the three terms A, B, C, respectively.We use induction on ℓ to show that:

(i) $U_K^i \cap B$ has no jumps of type $i = s + sp^\ell$.

(ii) $U_K^i \cap A = U_K^i \cap B \times U_K^i \cap C$ for all $i \leq s + sp^{\ell+1}$.

Proof. We know the possible jumps of A and C. For $\ell = 0$ we can use $U_K^{s+1} \cap A = A$, $U_K^{2s} \cap C = C$ and $U_K^{2s} \cap B = U_K^{2s+1} \cap B$ to obtain (ii) for all $i \leq 2s+1$. Then we apply Corollary 8.8: $U_K^{2s+1} \cap C = U_K^{s+sp} \cap C$ which gives us (ii) for all $i \leq s+sp$.

We assume (i), (ii) for some $\ell \geq 0$. Thus we know (ii) for $i_0 = s + sp^{\ell+1}$. This implies

$$U_K^{i_0+1} \cap A \subseteq \quad U_K^{i_0} \cap B \quad \times \quad U_K^{i_0} \cap C, \quad x = y \cdot z.$$

Now we consider a tower $K \supset E \supseteq F$ such that $[K : E] = p$. Then by Lemma 8.7 we obtain an isomorphism:

$$N_{K|E} : U_K^{s+sp^{\ell+1}} \cap A \xrightarrow{\sim} U_E^{s+sp^{\ell}} \cap A_{E|F}.$$

Now we use $N_{K|E}(x) = N_{K|E}(y) \cdot N_{K|E}(z)$ and $N_{K|E}(x) \in U_E^{s+sp^{\ell}+1} \cap A_{E|F}$ because $x \in U_K^{i_0+1}$. Moreover $N_{K|E}(y) \in U_E^{s+sp^{\ell}} \cap B_{E|F} = U_E^{s+sp^{\ell}+1} \cap B_{E|F}$ by induction hypothesis. Therefore:

$$N_{K|E}(z) \in U_E^{s+sp^{\ell}+1} \cap C_{E|F}.$$

We know that C has only jumps of type $s + sp^{\ell}$ (Corollary 8.8) and again by Lemma 8.7 we have the isomorphism:

$$N_{K|E} : U_K^{i_0} \cap C / U_K^{i_0+1} \cap C \xrightarrow{\sim} U_E^{s+sp^{\ell}} \cap C_{E|F} / U_E^{s+sp^{\ell}+1} \cap C_{E|F}.$$

Then we conclude

$$z \in U_K^{i_0+1} \cap C = U_K^{s+sp^{\ell+2}} \cap C.$$

Therefore our original equation $x = y \cdot z$ implies now $y \in U_K^{i_0+1} \cap B$ which gives (ii) for $i = i_0 + 1$. Since i_0 is a full jump for C we must have $U_K^{i_0} \cap B = U_K^{i_0+1} \cap B$ which is our assertion (i) for $\ell + 1$. Finally using once more Corollary 8.8 we see that (ii) for $i = i_0 + 1$ will imply (ii) for all $i \leq s + sp^{\ell+2}$.

We have seen that (31) is a direct product of filtered groups. But then (24.2) and (24.1) are direct products of filtered groups too and Proposition 9.1 is proved. □

Corollary 9.4. *Let $s \leq t$ and X be a character of $U^1 \wedge U^1 / (U^1 \wedge U^{s+1})(N_1 \wedge N_1)$ as in Lemma 8.2. Moreover assume that N_1 is in the radical of X.*

(i) If Cram's complement C'_s is contained in N_1, then

$$j(X) = j(X|_{(U^s \wedge U^s)(N_1 \wedge N_1)}) \in \{s + \frac{s}{p^{n-\ell}};\ \ell = n - [\frac{n}{2}], \ldots, n-1\}.$$

(ii) If $C'_s \nsubseteq N_1$, then it may happen that $j(X) = s + \frac{j}{p^{n-\ell}}$ for some $j < s$.

For 9.4(ii) see Proposition 11.4 and remark.

10. Proof of Theorem 7.1

Now in particular we know that for Cram's complements Lemma 9.2 holds. We consider the diagram:

$$
\begin{array}{lccc}
 & (k_1 \oplus \cdots \oplus k_{s-1}) \wedge (k_s/N) & \xrightarrow{\mathrm{Norm}^{-1}\otimes w} & (k_1 \oplus \cdots \oplus k_{s-1}) \otimes G_{K|F} \\
(32) & \Big\downarrow{\scriptstyle E^{s-1}\wedge E_s} & & \Big\downarrow{\scriptstyle (30)} \\
 & C_s \wedge U^1/\Delta_s(K) & \xrightarrow{(28)} & I_F C_s(K)/I_F U_K^{s+1},
\end{array}
$$

where $N \subset k_s$ corresponds through (22) to the s-extension $K|F$, and Norm^{-1} means reversing the first row of (27). We denote $\{F_N^\nu\}_\nu$ the filtration of $(k_1 \oplus \cdots \oplus k_{s-1}) \wedge (k_s/N)$ which via the main diagonal of (32) corresponds to the filtration $U_K^{\psi_{K|F}(\nu)} \cap I_F C_s(K)/I_F U_K^{s+1}$. Then we have seen the following properties:

(i) The filtration $F^\nu := F^\nu_{\{0\}}$ has the jumps $\nu = \nu(s, \ell, r)$ where s is fixed and $\ell \in \{0, \ldots, f-1\}$, $r \in \{1, \ldots, s-1\}$ may vary.

(ii) F_N^ν is a quotient of F^ν, and the filtration F_N^ν has the jumps $\nu(s, \ell, r)$ for $\ell \geq \dim_{\mathbb{F}_p}(N)$, whereas the induced filtration $F^\nu \cap ((k_1 \oplus \cdots \oplus k_{s-1}) \wedge N)$ has the complementary jumps $\nu(s, \ell, r)$ for $\ell < \dim_{\mathbb{F}_p}(N)$.

(iii) If we combine the main diagonal of (32) with the second map of (30) which is given by Lemma 9.3 and if we use the basis $\{\varepsilon\pi_K^{r+s};\ r = 1, \ldots, s-1\}$ resulting from Lemma 12, then we obtain a coordinate map

$$(k_1 \oplus \cdots \oplus k_{s-1}) \wedge (k_s/N) \longrightarrow k_{s+1} \oplus \cdots \oplus k_{2s},$$

which takes $F_N^{\nu(s,\ell,r)}$ for $\ell = \dim_{\mathbb{F}_p}(N)$ to coordinate vectors $(x_{s+1}, \ldots, x_{2s-1}, 0)$ such that $x_{s+1} = \cdots = x_{s+r-1} = 0$.

With these three properties the filtration on the first factors of the factorizations (24.1), (24.2) is identified if we assume that $C_s = C_s'$ is Cram's complement. We still have to deal with the filtrations on the second factors. In the case $s = 1$ this comes down to Theorem 6.1 and has been dealt with in [Zi2]sections 3, 5 and 6. Also in the case $s > 1$ the result is independent of the choice of C_s and the proof is very similar as for $s = 1$. ([Cr3],4.3 and [Kh2],5. resp.) Altogether this proves Theorem 7.1*.

11. Other complements

Each linear map $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k_1 \oplus \cdots \oplus k_{s-1}, k_s)$ induces

$$i \oplus \tau : k_1 \oplus \cdots \oplus k_{s-1} \to k_1 \oplus \cdots \oplus k_s,\quad x \mapsto x \oplus \tau(x).$$

Therefore

$$\mathrm{Hom}_{\mathbb{F}_p}(k_1 \oplus \cdots \oplus k_{s-1}, k_s) \ni \tau \longmapsto C_s(\tau) := E^s(\mathrm{Im}(i \oplus \tau)) \subset U^1/U^{s+1}$$

turns $\mathrm{Hom}_{\mathbb{F}_p}(k_1\oplus\cdots\oplus k_{s-1}, k_s)$ into the space of parameters for the complements C_s of U^s/U^{s+1} in U^1/U^{s+1}. Cram's complement $C_s' = C_s(0)$ is obtained for the trivial map $\tau \equiv 0$.

We make some remarks on the filtration of (26):

$$c : C_s(\tau) \wedge U^1/\Delta_s(\tau) \xrightarrow{\sim} I_F U_L^1/I_F U_L^s$$

if $C_s = C_s(\tau)$ is an arbitrary complement of U^s/U^{s+1} in U^1. We restrict to the case where $s \le t \le \min\{e, p-1\}$, which implies $U^1 \wedge U^s = UU^s$. Then the natural isomorphism

$$C_s(\tau) \wedge U^s/U^1 \wedge U^{s+1} \xrightarrow{\sim} C_s(\tau) \wedge U^1/\Delta_s(\tau)$$

is compatible with the filtration because for $\nu \ge s$:

$$(C_s(\tau) \wedge U^1) \cap UU^\nu = (C_s(\tau) \wedge U^1) \cap (U^1 \wedge U^s) \cap UU^\nu = (C_s(\tau) \wedge U^s) \cap UU^\nu.$$

We consider the isomorphism

$$U^1 \wedge U^s/U^1 \wedge U^{s+1} \xleftarrow{E^s\wedge E_s} k^s \wedge k_s \xrightarrow{L} \lrcorner^s\{\phi\} \tag{33.1}$$

which is compatible with the filtrations UU^ν mod $U^1 \wedge U^{s+1}$ and $\mathcal{F}^\nu \cap \lrcorner^s\{\phi\}$ resp.. Put:

$$I_s(\tau) := \mathrm{Im}(i \oplus \tau) \subset k^s.$$

Then under (33.1) we have:

$$C_s(\tau) \wedge U^s/U^1 \wedge U^{s+1} \xleftarrow{\sim} I_s(\tau) \wedge k_s \xrightarrow{\sim} L(I_s(\tau) \wedge k_s)$$

which is compatible with the induced filtrations. The map $i \oplus \tau : k^{s-1} \to k^s$ induces $k^{s-1} \wedge k_s \xrightarrow{\sim} I_s(\tau) \wedge k_s \subset k^s \wedge k_s$ and therefore

$$(i \oplus \tau)^L : L(k^{s-1} \wedge k_s) \xrightarrow{\sim} L(I_s(\tau) \wedge k_s) \subset \lrcorner^s\{\phi\}.$$

This map is a section for the natural projection map

$$\lrcorner^s\{\phi\} = L(k^s \wedge k_s) = L(k^{s-1} \wedge k_s) \oplus L(k_s \wedge k_s) \to L(k^{s-1} \wedge k_s) \tag{33.2}$$

which forgets the diagonal entries $(L_\ell)_{s,s}$ of the polynomials $L \in \lrcorner^s\{\phi\}$ for $\ell = 1, \ldots, f-1$.

Lemma 11.1 *The projection (33.2) restricted to $L(I_s(\tau) \wedge k_s)$ is an isomorphism*

$$L(I_s(\tau) \wedge k_s) \xrightarrow{\sim} L(k^{s-1} \wedge k_s)$$

which maps $L(I_s(\tau) \wedge k_s) \cap \mathcal{F}^\nu$ into $L(k^{s-1} \wedge k_s) \cap \mathcal{F}^\nu$.

Proof. According to Theorem 7.1 a polynomial $L \in L(I_s(\tau) \wedge k_s)$ is contained in $L(I_s(\tau) \wedge k_s) \cap \mathcal{F}^\nu$ for $\nu = \nu(s, \ell, r)$ if a certain subset of the coefficients $(L_\lambda)_{s,\rho}$ vanish. But the projection map (33.2) replaces the diagonal coefficients $(L_\lambda)_{s,s}$ (where λ varies) by zero, hence the condition to be in $\mathcal{F}^\nu$ is preserved. □We note that, according to Theorem 7.1, the jumps of $\mathcal{F}^\nu \cap L(k^{s-1} \wedge k_s)$ are precisely the numbers $\nu(s, \ell, r) = s + r/p^{f-\ell}$ for $r < s$ and $\ell = 0, \ldots, f-1$, and these are full jumps. **We ask whether it is possible for the filtration** $\mathcal{F}^\nu \cap L(I_s(\tau) \wedge k_s)$

to have other jumps, i.e. jumps $\nu = \nu(s, \ell, s)$ for $\ell \geq f/2$. For this we interpret the map $(i \oplus \tau)^L : L(k^{s-1} \wedge k_s) \xrightarrow{\sim} L(I_s(\tau) \wedge k_s)$ as $(i \oplus \tau)^L = i \oplus \tau^L$, where $\tau^L : L(k^{s-1} \wedge k_s) \to L(k_s \wedge k_s)$ is induced by τ. That means:

(33.3) $$\tau^L[L((a_1, ..., a_{s-1}) \wedge b_s)] = L(\tau(a_1, ..., a_{s-1}) \wedge b_s).$$

In other words, polynomials $L = \sum_\nu L_\nu \phi^\nu \in L(I_s(\tau) \wedge k_s)$ have the property that the diagonal coefficients $(L_\ell)_{s,s}$ for $\ell = f - [f/2], \ldots, f-1$ are linear functions of the other coefficients $(L_\ell)_{s,r}$ where $r < s$. Assume that $(L_\ell)_{s,s}$ for a fixed ℓ is a linear combination of coefficients $\{(L_\lambda)_{s,\rho};\ \lambda = 0, \ldots, f-1, \quad \rho < s\}$ with nontrivial contributions from coefficients $(L_\lambda)_{s,\rho}$ such that $s + \rho/p^{f-\lambda} > s + s/p^{f-\ell}$. Then $(L_\lambda)_{s,\rho} = 0$ for $s + \rho/p^{f-\lambda} < s + s/p^{f-\ell}$ does not imply $(L_\ell)_{s,s} = 0$, and therefore $L(I_s(\tau) \wedge k_s) \cap \mathcal{F}^\nu$ has a jump for $\nu = \nu(s, \ell, s)$.

If $a = (a_1, \ldots, a_s) \in k^s$ and $b = (0, \ldots, 0, b_s) \in k_s$, then $L(a \wedge b) \in \lrcorner^s\{\phi\}$ and according to (14) we obtain:

$$\begin{aligned}
&(L_\nu)_{s,j} = -b_s \phi^\nu(a_j) \quad \text{if } j \neq s \\
&(L_\nu)_{i,s} = a_i \phi^\nu(b_s) = -\phi^\nu((L_{f-\nu})_{s,i}) \quad \text{if } i \neq s \\
&(L_\nu)_{s,s} = a_s \phi^\nu(b_s) - b_s \phi^\nu(a_s).
\end{aligned}$$

For $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k^{s-1}, k_s)$ we obtain $a_s = \tau(a_1, ..., a_{s-1})$ and therefore:

$$(L_\nu)_{s,s} = \tau(a_1, \ldots, a_{s-1}) \phi^\nu(b_s) - b_s \phi^\nu(\tau(a_1, \ldots, a_{s-1})).$$

In (33.3) we have defined τ^L for homogeneous arguments. But we need now to give a general expression for the linear extension $\tau^L : L(k^{s-1} \wedge k_s) \to L(k_s \wedge k_s)$. Our result will be Proposition 11.3 below. Since we can write:

(34) $$L(k^{s-1} \wedge k_s) = L(k_1 \wedge k_s) \oplus \cdots \oplus L(k_{s-1} \wedge k_s)$$

it is basically enough to study the case $s = 2$. Then $\tau \in Hom_{\mathbb{F}_p}(k_1, k_2)$ and in order to describe τ^L we have to express $(L_\nu)_{2,2}$ in terms of $(L_\ell)_{2,1}$ for $\ell = 0, \ldots, f-1$. We put $\lambda_\nu := (L_\nu)_{2,1}$ and $\lambda := (\lambda_0, \ldots, \lambda_{f-1}) \in k^f$. We have now only two copies of k and consider $a \in k_1$, $b \in k_2$. Then $a \wedge b \mapsto \lambda(a \wedge b) = -(ab, \phi(a)b, \ldots, \phi^{f-1}(a)b)$ is an isomorphism

(*) $$\lambda : k \otimes k \xrightarrow{\sim} L(k_1 \wedge k_2) = k^f.$$

Now let $\alpha = (\alpha_0, \ldots, \alpha_{f-1})$ be a basis for $k|\mathbb{F}_p$ and form the matrix

$$A = \begin{pmatrix} \alpha_0 & \ldots & \phi^{f-1}(\alpha_0) \\ \vdots & \ddots & \vdots \\ \alpha_{f-1} & \ldots & \phi^{f-1}(\alpha_{f-1}) \end{pmatrix}.$$

The matrix A is invertible because under (*) the rows of A correspond to the k-basis $\alpha_0 \otimes 1, \ldots, \alpha_{f-1} \otimes 1$ of $k \otimes k$.

Lemma 11.2. *Identifying $L(k_1 \wedge k_2) = k^f$ the map $\tau^L : k^f \to L(k_2 \wedge k_2)$ is given explicitly as $\tau^L(\lambda) = \sum_{\nu=1}^{f-1} L_\nu \phi^\nu$, where*

$$L_\nu = (L_\nu)_{2,2} = \langle \lambda A^{-1}, \phi^\nu(\tau(\alpha))\rangle - \langle \phi^\nu(\lambda A^{-1}), \tau(\alpha)\rangle. \tag{35.1}$$

If $\tau \in Hom_{\mathbb{F}_p}(k_1, k_2)$ is even k-linear, then:

$$(L_\nu)_{2,2} = \phi^\nu(\tau(1))\lambda_\nu - \tau(1)\phi^\nu(\lambda_{f-\nu}). \tag{35.2}$$

Proof. Obviously the map $L_\nu = L_\nu(\lambda)$ is $\mathbb{F}_p$-linear. Therefore it is enough to prove that in the case $\lambda = \lambda(a \wedge b) = -(ab, \phi(a)b, \dots, \phi^{f-1}(a)b)$ we obtain $(L_\nu)_{2,2} = \tau(a)\phi^\nu(b) - b\phi^\nu(\tau(a))$.

Let $[a]_\alpha = (a_0, \dots, a_{f-1}) \in \mathbb{F}_p^f$ be the coordinate vector of a with respect to the base α. Then obviously $[a]_\alpha A = (a, \dots, \phi^{f-1}(a))$, hence $\lambda = -b \cdot [a]_\alpha A$. If we substitute this on the right side of (35.1) and use that $[a]_\alpha$ is ϕ-invariant then (35.1) turns into:

$$(L_\nu)_{2,2} = -b\langle [a]_\alpha, \phi^\nu(\tau(\alpha))\rangle + \phi^\nu(b)\langle [a]_\alpha, \tau(\alpha)\rangle = -b\phi^\nu(\tau(a)) + \phi^\nu(b)\tau(a),$$

as asserted.

If τ is k-linear we get $\tau(\alpha) = \tau(1) \cdot \alpha$ and therefore:

$$L_\nu = \phi^\nu(\tau(1))\langle \lambda A^{-1}, \phi^\nu(\alpha)\rangle - \tau(1)\phi^\nu(\langle \lambda A^{-1}, \phi^{f-\nu}(\alpha)\rangle).$$

Now (35.2) follows because ${}^t\phi^\nu(\alpha)$ is the $\nu + 1$-st column of A. $\square$

Remarks. 1. The formula (35.1) does not depend on the choice of the base α. If $\beta = \alpha M$, $M \in GL_f(\mathbb{F}_p)$ is another base then A is replaced by $B = {}^tMA$ and $\tau(\beta) = \tau(\alpha)M$.

2. If we write $\tau(\alpha) = \alpha[\tau]_\alpha$ where $[\tau]_\alpha \in \mathbb{F}_p^{f\times f}$ is the coordinate matrix of τ then in (35.1) A^{-1}, $\tau(\alpha)$ are replaced by $A^{-1}\,{}^t[\tau]_\alpha$, α resp.

Using (34) we state the result in the general case:

Proposition 11.3. *Consider $\tau \in Hom_{\mathbb{F}_p}(k^{s-1}, k)$ and let τ_i be the restriction of τ to the i-th copy of k. Then for $L \in L(k^{s-1} \wedge k_s)$ with coefficient vectors $(L_*)_{s,i} = ((L_0)_{s,i}, \dots, (L_{f-1})_{s,i})$ we obtain $\tau^L(L) = \sum_{\nu=1}^{f-1}(L_\nu)_{s,s}\phi^\nu$, where*

$$(L_\nu)_{s,s} = \sum_{i=1}^{s-1} \langle (L_*)_{s,i}A^{-1}, \phi^\nu(\tau_i(\alpha))\rangle - \langle \phi^\nu((L_*)_{s,i}A^{-1}), \tau_i(\alpha)\rangle.$$

In particular if τ is k-linear then

$$(L_\nu)_{s,s} = \sum_{i=1}^{s-1} \phi^\nu(\tau_i(1))(L_\nu)_{s,i} - \tau_i(1)\phi^\nu((L_{f-\nu})_{s,i}).$$

Therefore in this case the complement $C_s(\tau)$ is as good as Cram's complement $C_s(0)$ because the filtration $\overline{UU}^{\nu}$ on $C_s(\tau)\wedge U^1/\Delta_s(\tau)$ has only the jumps $\nu(s,\ell,r)$ for $r<s$.

Proof. We are left only with the last statement. We have to prove that the filtration $\mathcal{F}^{\nu}\cap L(I_s(\tau)\wedge k_s)$ has no jumps $\nu=\nu(s,\ell,s)$ for $\ell\geq f/2$. But according to our second equation $(L_\ell)_{s,s}$ linearly depends on $(L_\ell)_{s,i}$ and $(L_{f-\ell})_{s,i}$ for $i<s$. We have $s+i/p^{f-\ell}<s+s/p^{f-\ell}$ and $s+i/p^{\ell}<s+s/p^{f-\ell}$ because $\ell\geq f/2$. Therefore if we come to $\nu=\nu(s,\ell,s)$ the coefficients $(L_\ell)_{s,i}$ and $(L_{f-\ell})_{s,i}$ are already zero, hence the condition $(L_\ell)_{s,s}=0$ is empty. □

Now we consider the decomposition $\lrcorner^s\{\phi\}=L(k^{s-1}\wedge k_s)\oplus L(k_s\wedge k_s)$ with projection maps pr_1 and pr_2 resp. and the map

$$d_\tau:\lrcorner^s\{\phi\}\to L(k_s\wedge k_s),\quad d_\tau(L):=\mathrm{pr}_2(L)-\tau^L(\mathrm{pr}_1(L)).$$

Proposition 11.4. *Let $\overline{UU}^{\nu}$ be the induced filtration on*

$U^1\wedge U^s/C_s(\tau)\wedge U^s=U^1\wedge U^1/C_s(\tau)\wedge U^1$. *Then under*

$$U^1\wedge U^s/C_s(\tau)\wedge U^s\leftarrow U^s\wedge U^s/U^{s+1}\wedge U^s\to L(k_s\wedge k_s)$$

the filtration $\overline{UU}^{\nu}$ corresponds to the filtration $d_\tau(\mathcal{F}^{\nu}\cap\lrcorner^s\{\phi\})$.

Proof. For $L\in\lrcorner^s\{\phi\}$ we have $d_\tau(L)\equiv L \mod L(I_s(\tau)\wedge k_s)$ because

$$\mathrm{pr}_1(L)+\tau^L(\mathrm{pr}_1(L))\in L(I_s(\tau)\wedge k_s).$$

Therefore the diagram

$$\begin{array}{ccc}\lrcorner^s\{\phi\} & \xrightarrow{d_\tau} & L(k_s\wedge k_s)\\ {\scriptstyle =}\big\downarrow & & \big\downarrow\\ \lrcorner^s\{\phi\} & \longrightarrow & \lrcorner^s\{\phi\}/L(I_s(\tau)\wedge k_s)\end{array}$$

is commutative and $d_\tau(\mathcal{F}^{\nu}\cap\lrcorner^s\{\phi\})$ can be identified with

$$(\mathcal{F}^{\nu}\cap\lrcorner^s\{\phi\}) \mod L(I_s(\tau)\wedge k_s).\quad\square$$

Remark. We note that $L(k_s\wedge k_s)\subset\mathcal{F}^{\nu}\cap\lrcorner^s\{\phi\}$ if $\nu\leq\nu(s,f-[f/2],s)$ such that $\nu(s,f-[f/2],s)$ is the first jump of the filtration $d_\tau(\mathcal{F}^{\nu}\cap\lrcorner^s\{\phi\})$. But depending on τ this can now be a much more subtle filtration of $L(k_s\wedge k_s)$ than the standard filtration (for $\tau=0$), and in particular jumps $\nu(s,\ell,r)$ with $r<s$ may occur.

Appendix: Application to primitive representations

If ρ is a primitive irreducible representation of G of dimension p^d, then according to Proposition 3.2 we have

$$\widetilde{sw}(\rho) = \frac{1}{e_{K|F}} p^d j_{KK}(X),$$

where $K|F$ is a tame extension and X is an alternating character defined on $K^\times \wedge K^\times / N \wedge K^\times$ such that for $N_1 := N \cap U_K^1$:

$$K^\times / N \cong U_K^1 / N_1 \cong U_K^1/\Delta_1 \times \cdots \times U_K^1/\Delta_r$$

where $\Delta_1 \cap \cdots \cap \Delta_r = N_1$ and $U_K^{s_i}/U_K^{s_i+1} \twoheadrightarrow U^1/\Delta_i$ for different $s_1, ..., s_r$. This corresponds to the fact that ρ is the tensor product of r elementary stable representations (see section 4).

We restrict to the case that ρ is elementary stable which means $r = 1$, $N_1 = \Delta_1$ and $s_1 = s$. Then we obtain the conductor formulas

$$j_{KK}(X) = s + s/p^{f-\ell}, \quad \widetilde{sw}(\rho) = p^{d+\ell-f} \cdot \frac{s(p^{f-\ell}+1)}{e_{K|F}}$$

if N_1 fits with a complement C_s which is as good as Cram's. We must have $\ell \geq f - d \geq f/2$. On the other hand if N_1 does not fit with such a good complement then we could have:

$$j_{KK}(X) = s + j/p^{f-\ell} \geq s + s/p^{f-[f/2]}, \quad \widetilde{sw}(\rho) = p^{d+\ell-f} \frac{sp^{f-\ell}+j}{e_{K|F}}$$

for some $j = 1, ..., s$, where again $\ell \geq f - d \geq f/2$. For a particular example see [Cr1].

12. The Artin-Hasse exponential and a model for $\bar{U}^1 \wedge \bar{U}^1$

So far the filtration UU^i has not been described modulo $(U^1)^p \wedge U^1$ but modulo appropriate larger subgroups of $U^1 \wedge U^1$. In his attempt to come close to this final aim G. M. Cram [Cr3] makes use of the Artin-Hasse exponential (see H. Hasse [H] and I. R. Shafarevich [Sh]).

Notation: Let e, f be the ramification exponent and inertial degree of $F|\mathbb{Q}_p$ and let $O_0 = W(k_F)$ be the Witt ring in O_F. Let I be the set of integers prime to p which are less than $e^* = p\frac{e}{p-1}$. Note that $|I| = e$ and that $\overline{U}^1 := U^1/R := U^1/(U^1)^p U^{e^*}$ is filtered by the subgroups $\overline{U}^\nu = U^\nu/R$ for $\nu \in I$.

We want to identify the e-dimensional k_F-spaces $k_F^I = \{(a_i)_{i \in I}\}$ and U^1/R.

The Artin-Hasse exponentials are power series

$$\mathrm{Exp}(a,x)=\exp(\sum_{n=0}^{\infty} a^{\phi^n}\cdot x^{p^n}\cdot p^{-n})\in O_0[[x]]$$

which can be defined for all $a\in O_0$. One has $\mathrm{Exp}(a+b,x)=\mathrm{Exp}(a,x)\mathrm{Exp}(b,x)$, and $\mathrm{Exp}(a,x)\equiv \mathrm{Exp}(ax) \bmod x^p$ is the truncated exponential. For a fixed prime element π_F we will use the identification

$$\mathrm{Exp}^I : k^I \ni (a_i)_{i\in I}\longmapsto \prod_{i\in I}\mathrm{Exp}(a_i,\pi_F^i)\in U^1/R. \tag{36}$$

which maps k_i into $\overline{U}^i=U^iR/R$. This induces

$$\mathrm{Exp}^I\wedge\mathrm{Exp}^I : k^I\wedge_{\mathbb{F}_p} k^I \xrightarrow{\sim} (U^1\wedge U^1)/U^1\wedge R.$$

Again we want to use the injection (14)

$$L: k^I\wedge k^I\hookrightarrow k_F^{I\times I}\{\phi\}$$

in order to identify the filtration UU^i.

13. The jumps in the general case

Let $S=S_{f,I}$ be the set of all triples (s,ℓ,r) such that $\ell\in\{0,\dots,f-1\}$, $s,r\in I$, $r\le s$ and $r=s$ only if $\ell\ge f/2$. The image $L(k^I\wedge k^I)$ consists again of polynomials $\sum_{\nu=0}^{f-1}L_\nu\phi^\nu$ with $L_\nu\in k_F^{I\times I}$ where the coefficients $(L_\ell)_{s,r}$ for $(s,\ell,r)\in S$ may serve as independent coefficients. A problem is posed now by the fact that the map

$$(s,\ell,r)\in S\longmapsto \nu(s,\ell,r)=s+\frac{r}{p^{f-\ell}} \tag{*}$$

need not be injective. It may happen that s is much larger than p. As a consequence the values $\nu(s,\ell,r)$ from Theorem 7.1 are no longer distinguished by s. For instance for $s=p^\nu+1$ we could have: $p^\nu+1+\frac{p^\nu+1}{p^\nu}=p^\nu+2+\frac{1}{p^\nu}$ and $2^\nu+1+\frac{2^\nu+1}{2^{\nu-1}}=2^\nu+3+\frac{1}{2^{\nu-1}}$ resp. Injectivity of (*) is preserved only if s is fixed.

We write $(s,\ell,r)\sim(\sigma,\lambda,\rho)$ to mean that $\nu(s,\ell,r)=\nu(\sigma,\lambda,\rho)$ and we let $[s,\ell,r]$ denote the equivalence class of (s,ℓ,r). Equivalence implies $\lambda=\ell$ and $\rho=r-(\sigma-s)p^{f-\ell}$. In the present case the filtration UU^ν of $\overline{U}^1\wedge\overline{U}^1$ will have other jumps $\nu=j(s,\ell,r)$. These jumps separate the elements of $[s,\ell,r]$.

Definition 13.1. *For $s\in I$ consider the inverse Herbrand functions*

$$\varphi_s(x)=x \textit{ if } x\le s \quad \textit{and } =s+\frac{x-s}{p^f} \textit{ if } x\ge s,$$

$$\varphi^{(s)}(x)=\cdots\circ\varphi_{s''}\circ\varphi_{s'},$$

where $s < s' < s'' < \cdots \subset I$ is the set of successors of s which are in I. In particular put $\varphi^{(s)}(x) = x$ if $s \in I$ is maximal.

Theorem 13.2 ([Cr3], **Th.5.2.1(a)).** *The jumps of the filtration UU^ν of $\overline{U}^1 \wedge \overline{U}^1$ are the numbers*

$$\nu = j(s,\ell,r) := \varphi^{(s)}(s + \frac{r}{p^{f-\ell}}) \quad \text{for } (s,\ell,r) \in S_{f,I}.$$

Remarks.

1. The numbers $j(s,\ell,r)$ for $(s,\ell,r) \in S_{f,I}$ are all different.

2. $\nu(s,\ell,r)$ is the maximum of all numbers $j(\sigma,\ell,\rho)$ such that $(\sigma,\ell,\rho) \sim (s,\ell,r)$.

3. Instead of $\overline{U}^1 \wedge \overline{U}^1$ we may also consider $U^1 \wedge U^1/U^1 \wedge U^{t+1}(U^1)^p$ for any $t \in I$. Then the Theorem holds if we replace I by $I_t = \{s \in I;\ s \leq t\}$ and also change the definition of $j(s,\ell,r)$ by using I_t.

Ad 1. The denominator of $j(s,\ell,r)$ has to be $p^{af-\ell}$ for some $a \geq 1$. Therefore $j(s,\ell,r) = j(\sigma,\lambda,\rho)$ implies $p^{af-\ell} = p^{\alpha f-\lambda}$, hence $\ell = \lambda$, $a = \alpha$ because $\ell,\lambda \in \{0,\ldots,f-1\}$. Furthermore $j(s,\ell,r) = s_1 + k_1/p^{af-\ell}$ such that $k_1 > 0$ and $j(s,\ell,r) < s_1'$, or $s_1 \in I$ is maximal. Now assume $s_1 + k_1/p^{af-\ell} = \sigma_1 + \kappa_1/p^{af-\ell}$ and $s_1 < \sigma_1$. Then $j(s,\ell,r) \geq \sigma_1 \geq s_1'$, hence $s_1 \in I$ must be maximal, which is a contradiction.

Ad 2. The elements $(\sigma,\lambda,\rho) \in [s,\ell,r]$ all have different values σ. If σ is the maximal of these values then $\sigma + \rho/p^{f-\lambda} < \sigma'$ or $\sigma \in I$ is maximal, and therefore $j(\sigma,\lambda,\rho) = \nu(\sigma,\lambda,\rho) = \nu$. On the other hand $j(s,\ell,r) = \varphi^{(s)}(\nu) < \varphi^{(\sigma)}(\nu) = \nu$ if $s < \sigma$.

We will quote now the main result leading to the proof of 13.2. Let $s \in I$ and $C_s \subseteq U^1$ be a complement such that $U^s/U^{s+1} \xrightarrow{\sim} U^1/C_s$. Consider the maximal s-extension $K|F$ such that $N_{K|F}(U_K^1) = C_s$. Then we obtain an exact sequence :

$$(37)\quad \overline{U}_K^1 \wedge \overline{U}_K^1/\overline{U}_K^1 \wedge \overline{U}_K^s \xrightarrow{N_{K|F}} \overline{U}^1 \wedge \overline{U}^1/\overline{U}^1 \wedge \overline{U}^{s+1} \to \overline{U}^1 \wedge \overline{U}^1/(\overline{U}^1 \wedge \overline{U}^{s+1})(C_s \wedge C_s),$$

where the arrows are injective and surjective resp. and $N_{K|F}(a \wedge b) = N_{K|F}(a) \wedge N_{K|F}(b)$.

Proposition 13.3 ([Cr3]**Th.5.2.1(b)).** *Concerning the first arrow of (37) we have*

$$N_{K|F}(UU_K^i) = UU^{\varphi(i)} \cap Im(N_{K|F}),$$

where $\varphi = \varphi_{K|F}$ is the inverse Herbrand function for $K|F$.

The Theorem is easily deduced from the proposition. We have $[K:F] = p^f$ and $K|F$ is an s-extension. Therefore $\varphi_{K|F} = \varphi_s$ in the notation of definition

13.1. By induction we may assume that $\overline{U}_K^1 \wedge \overline{U}_K^1 / \overline{U}_K^1 \wedge \overline{U}_K^s$ has the jumps $j(\sigma, \lambda, \rho)$ for $(\sigma, \lambda, \rho) \in S_{f,I^{<s}}$. They are transferred into the jumps $\varphi_s \circ j(\sigma, \lambda, \rho)$. And from the right side of (37) we get the additional jumps $\nu(s, \ell, r)$.

14. First remarks on the filtration in the general case

As in section 7 we have a bijection between the set $(L_\ell)_{s,r}$ of independent coefficients in $L(k^I \wedge k^I)$ and the set $j(s, \ell, r)$ of possible jumps of our filtration. In both cases the parameters are the triples $(s, \ell, r) \in S_{f,I}$. The arrangement of the numbers $j(s, \ell, r)$ in terms of the parameters (s, ℓ, r) is now more involved and the same holds for the filtration $\mathcal{F}^{j(s,\ell,r)}$ of $L(k^I \wedge k^I)$. Nevertheless Proposition 7.3 holds and therefore $\mathcal{F}^*_{j(s,\ell,r)} = \mathcal{F}^{j(s,\ell,r)^\perp}$.

If we replace F by an unramified extension $F'|F$ then I is left unchanged whereas f increases. But for $(s, \ell, r) \in S_{f,I}$ the numbers $\nu(s, \ell, r) \leq s + s/p < e^* + e^*/p$ have an upper bound which depends only on e, not on f. Therefore replacing F by F' which means to take f sufficiently large, we can always move to a situation where

$$j(s, \ell, r) = \varphi_{s'}(s + r/p^{f-\ell}) \quad \text{for all} \quad (s, \ell, r) \in S_{f,I},$$

i.e. either $j(s, \ell, r) = \nu(s, \ell, r) < s'$, or

$$j(s, \ell, r) = \varphi_{s'}(\nu(s, \ell, r)) = s' + \frac{s + r/p^{f-\ell} - s'}{p^f} < s''.$$

We note that the last inequality certainly holds if $s < p(1 + p^f)$ because then:

$$r \leq s < p(1 + p^f) \leq p^{f-\ell}(1 + p^f) \leq (s' - s)p^{f-\ell} + (s'' - s')p^{2f-\ell}. \tag{38}$$

Proposition 14.1. *For an unramified extension $F'|F$ we have*

(i) $N_{F'|F}(U^i_{F'}) = U^i_F$ *for all* $i \geq 1$

(ii) $N_{F'|F}(UU^\nu_{F'}) = UU^\nu_F$ *for all* $\nu > 1$. □

We describe now the norm map in terms of our models. We have to find the map T such that the diagram

$$\begin{array}{ccccc}
\overline{U}^1_{F'} \wedge \overline{U}^1_{F'} & \longleftarrow & k'^I \wedge k'^I & \longrightarrow & L(k'^I \wedge k'^I) \\
{\scriptstyle N_{F'|F} \wedge N_{F'|F}} \big\downarrow & & {\scriptstyle tr^I \wedge tr^I} \big\downarrow & & {\scriptstyle T} \big\downarrow \\
\overline{U}^1_F \wedge \overline{U}^1_F & \longleftarrow & k^I \wedge k^I & \longrightarrow & L(k^I \wedge k^I)
\end{array}$$

becomes commutative. One easily checks that

$$T(\sum_{\mu \in \mathbb{Z}/f'} L_\mu \phi^\mu) = \sum_{\nu \in \mathbb{Z}/f} P_\nu \phi^\nu$$

where $P_\nu = tr_{k'|k}(\sum_{\mu\in pr^{-1}(\nu)} L_\mu)$. The inner sum extends over the preimage of ν under the projection map $pr : \mathbb{Z}/f'\mathbb{Z} \to \mathbb{Z}/f\mathbb{Z}$. [Cr3](Proposition1.4.3.) Proposition 14.1(ii) means that the filtrations $\mathcal{F}'^\nu$ of $L(k'^I \wedge k'^I)$ and $\mathcal{F}^\nu$ of $L(k^I \wedge k^I)$ are related through $T(\mathcal{F}'^\nu) = \mathcal{F}^\nu = \mathcal{F}^\mu$, where $\mu \geq \nu$ is the next jump in the filtration of $L(k^I \wedge k^I)$.

15. The norm map for s-extensions - a refinement and a generalization of Theorem 7.1* We consider s-extensions $[K : F] = p^d$ as in section 8, i.e. (22) holds. We use (36) to define Cram's complement $C_s \subset U^1/R$ as the image of the hyperplane of k^I given by $a_s = 0$. Obviously $\overline{U}^{s+1} \subset C_s$, and often we will consider C_s in $U^1/U^{s+1}R$. Again let $L|F$ be the maximal s-extension corresponding to the norm subgroup $C_s \cdot C_F$ and assume $K \subseteq L$. In K we consider $C_s(K)$ relative to a prime element π_K such that $N_{K|F}(\pi_K) = \pi_F$.

Lemma 15.1. : *Assume $s \leq e = e_{F|\mathbb{Q}_p}$. Then the norm map $N_{K|F} : U^1_K/U^{s+1}_K R_K \to U^1_F/U^{s+1}_F R_F$ respects the splitting $U^1/U^{s+1}R = C_s \times (U^s R/U^{s+1}R)$. In particular it induces an isomorphism $N_{K|F} : C_s(K) \xrightarrow{\sim} C_s$ which in terms of (27) – with the exponential replaced by the Artin-Hasse exponential – is given now by a lower triangular matrix with main diagonal: $diag = (\phi^d, \ldots, \phi^d)$.*

More precisely let $Q = Q(\phi) \in k_F\{\phi\}$ be the additive polynomial which according to [Se], V, §3,6 corresponds to $N_{K|F} : U^s_K/U^{s+1}_K \to U^s_F/U^{s+1}_F$. We note that Q is uniquely determined by the properties:

(i) Q has highest coefficient 1 and divides $\phi^f - 1$.

(ii) Under $a \in k_F \mapsto 1 + a\pi^s_K \in U^s_K/U^{s+1}_K$ the subspaces $\mathrm{Ker}(Q)$ and $\mathrm{Ker}(N_{K|F})$ correspond.

Furthermore let $I_s = \{r \in I;\ r \leq s\}$ and let N_s be the matrix of type $I_s \times I_s$ with entries 1 at the positions $(s, s), (s\{-1\}, s - p), \ldots, (s\{-r\}, s - rp), \ldots$ as long as $s - rp \geq 1$ and with the convention that $s\{-r\}$ means r steps backwards in the sequence I_s of numbers prime to p. All other entries of N_s are 0. Note that for $s < p$ the only entry 1 is at position (s, s).

Lemma 15.2 ([Cr3], Theorem 2.4.2). *Keep the assumptions of the previous Lemma and replace*

$$Q = \phi^d + \eta_{d-1}\phi^{d-1} + \cdots + \eta_0 \in k_F\{\phi\}$$

by

$$Q^{\{s\}} = E\phi^d + \eta_{d-1}N_s\phi^{d-1} + \cdots + \eta_0 N^d_s \in k_F^{I_s\times I_s}\{\phi\}.$$

Then the diagram

$$\begin{array}{ccc} k_F^{I_s} & \xrightarrow{Q^{\{s\}}} & k_F^{I_s} \\ {\scriptstyle Exp_K^{I_s}}\Big\downarrow & & \Big\downarrow{\scriptstyle Exp_F^{I_s}} \\ C_s(K) \times U_K^s/U_K^{s+1}R & \xrightarrow{N_{K|F}} & C_s \times U_F^s/U_F^{s+1}R \end{array}$$

is commutative.

Whereas (27) was valid only for $s \le t < p$, Cram's arguments work for all $s \le e$. Note that in (27) the norm map has been considered only on C_s. For $Q^{\{s\}}$ this means that the last row and column have to be removed. For $s < p$ this turns N_s into the zero matrix.

The proof is by iteration, beginning from loc.cit. Proposition 2.1.7, a2):

$$N_{K|F}(\mathrm{Exp}(a, \pi_K^i)) \equiv \mathrm{Exp}(a^p, \pi_F^i)(1 + a \cdot \mathrm{Tr}_{K|F}(\pi_K^i)) \mod U_F^{s+1}$$

if

$$[K : F] = p \text{ and } 1 \le i < s.$$

Since $K|F$ is in L it is possible to obtain more information on the trace of the powers of π_K. The correcting factor on the right can be replaced by $\mathrm{Exp}(a\eta_0, \pi_F^{s-\frac{s-i}{p}})$ if $i \equiv s \; mod \; p$ and by 1 otherwise. This implies the Lemma in case $[K : F] = p$ (loc.cit. Lemma 2.4.1), and then one can argue by induction.

We want the results of section 9 for the less restrictive assumption $s \le e$. Therefore we also need a more general version of Lemma 9.3.

Lemma 15.3 ([Cr3],3.3.1). *Assume $s \le e$. Then it is possible to choose π_K such that:*

$$\pi_K^{\sigma-1} \equiv 1 + \vartheta_s(\sigma)\pi_K^s \;\; mod \; U_K^{2s} \quad \textit{for all } \sigma \in G_{K|F},$$

where $\vartheta_s(\sigma) \in k_F^{\times}$, and this implies:

$$Exp(a, \pi_K^r)^{\sigma-1} \equiv 1 + r\vartheta_s(\sigma)\pi_K^s(a\pi_K^r + \phi(a)\pi_K^{rp} + \phi^2(a)\pi_K^{rp^2} + \cdots) \mod U_K^{2s+1},$$

for all $1 \le r < s$.

As we did after Lemma 9.3 we may again consider U_K^{s+1}/U_K^{2s+1} as a k_F-space, and the Lemma shows that the $2s$-coordinate of $\mathrm{Exp}(a, \pi_L^r)^{\sigma-1}$ is always zero, if $r < s$. Therefore $I_F C_s(K) \cap U_K^{2s} \subseteq U_K^{2s+1}$, and this proves Proposition 9.1 in our more general context. The induction argument is basically the same as in section 9.

Using (16), (19) and (20) we consider now the commutative diagram

$$\begin{array}{ccc} L(k^I \wedge k^I) & \xrightarrow{\sim} & \overline{U}^1 \wedge \overline{U}^1 \\ \downarrow & & \downarrow \\ \lrcorner^s\{\phi\} & \xrightarrow{\sim} & \overline{U}^1 \wedge \overline{U}^1/(\overline{U}^1 \wedge \overline{U}^{s+1})(C_s \wedge C_s), \end{array} \tag{39}$$

where the left vertical is the natural projection which forgets all entries not in $\lrcorner^s\{\phi\}$. Then as a generalization of Theorem 7.1* we obtain:

Theorem 15.4([Cr3]**Proposition 4.1.2).** *Assume that $1 < s \leq e$. Then:*

(i) The jumps of the filtration $\overline{UU}^\nu$ on $\overline{U}^1 \wedge \overline{U}^1/(\overline{U}^1 \wedge \overline{U}^{s+1})(C_s \wedge C_s)$ are the numbers

$$\nu(s, \ell, r) := s + r/p^{f-\ell}$$

where s is fixed, $1 \leq r \leq s$ and $\ell = 0, \dots, f-1$, and where the equality $r = s$ is allowed only if $\ell \geq f/2$. These numbers begin from $\nu(s, 0, 1)$ and increase to $\nu(s, f-1, s)$, where the order can now be different than in Theorem 7.1.

(ii) Take the coefficients $(L_\ell)_{s,r}$ as independent coefficients of our polynomials $L \in \lrcorner^s\{\phi\}$, where r, ℓ vary as in (i). Then under the lower horizontal map of the diagram the filtration $\overline{UU}^\nu$ corresponds to the filtration $\{\bar{\mathcal{F}}^\nu\}_\nu$ such that $\bar{\mathcal{F}}^\nu = \lrcorner^s\{\phi\}$ if $\nu = \nu(s, 0, 1)$, and for $\bar{\mathcal{F}}^{\nu(s,\ell,r)}$ the next term $\bar{\mathcal{F}}^{\nu'}$ of the filtration is given by adding the relation

$$(L_\ell)_{s,r} + \phi^\ell((L_{f-\ell})_{s,s-p^\ell(s-r)}) = 0 \quad \text{if } r < s,\ \ell \neq 0,\ \text{and } s - p^\ell(s-r) > 0$$
$$(L_\ell)_{s,r} = 0 \quad \text{otherwise.}$$

So we end with $\bar{\mathcal{F}}^{\nu'} = 0$ for $\nu = \nu(s, f-1, s)$. And in the case when f is even, the jump $\nu(s, f/2, s)$ is a half jump.

Due to Lemmas 15.2 and 15.3 the proof is basically the same as in section 10 where we used (27) and Lemma 9.3 resp..

16. More on the filtration in the general case

For $s \in I$ we consider Cram's complement $C_s \subset \overline{U}^1$ as in the last section. We will use the notation

$$CC_s := (C_s \wedge C_s)(\overline{U}^1 \wedge \overline{U}^{s+1}) \subset \overline{U}^1 \wedge \overline{U}^1.$$

Then we have $\cap_{s \leq t} CC_s = \overline{U}^1 \wedge \overline{U}^{t+1}$ and

$$\overline{U}^1 \wedge \overline{U}^1/\overline{U}^1 \wedge \overline{U}^{t+1} \xrightarrow{\sim} \prod_{s \in I, s \leq t} \overline{U}^1 \wedge \overline{U}^1/CC_s \tag{40}$$

as in (21). But contrary to Proposition 8.1 this is no longer a direct product of filtered groups because different factors can have jumps in common. Therefore Theorem 15.4 on the factors $\overline{U}^1 \wedge \overline{U}^1/CC_s$ is not enough to fix the filtration on the left side of (40). Instead we consider

$$CC_{s,s'} := CC_s \cap CC_{s'}$$

$$\overline{U}^1 \wedge \overline{U}^1/CC_{s,s'} \xrightarrow{\sim} \overline{U}^1 \wedge \overline{U}^1/CC_s \quad \times \quad \overline{U}^1 \wedge \overline{U}^1/CC_{s'}, \tag{41}$$

where $s,\ s' \in I$ are consecutive numbers. On the other hand we consider:

$$\lrcorner^{s}\lrcorner\{\phi\} := \lrcorner^{s}\{\phi\} + \lrcorner^{s'}\{\phi\} \subseteq L(k^I \wedge k^I).$$

Then similar to (39) we have the commutative diagram:

$$\begin{array}{ccc} L(k^I \wedge k^I) & \xrightarrow{\ \sim\ } & \overline{U}^1 \wedge \overline{U}^1 \\ \downarrow & & \downarrow \\ \lrcorner^{s}\lrcorner\{\phi\} & \xrightarrow{\ \sim\ } & \overline{U}^1 \wedge \overline{U}^1/CC_{s,s'} \end{array} \tag{42}$$

where the left vertical projection forgets all entries of $L \in L(k^I \wedge k^I)$ which are not in $\lrcorner^{s}\lrcorner\{\phi\}$. As a variation of (37) we have the exact sequence:

$$\overline{U}^1_K \wedge \overline{U}^1_K/CC_s(K) \xrightarrow{N_{K|F}} \overline{U}^1 \wedge \overline{U}^1/CC_{s,s'} \to \overline{U}^1 \wedge \overline{U}^1/CC_{s'}$$

and by an argument which is similar to what we did at the end of section 13 we see:

Lemma 16.1.

(i) The jumps of the filtration $\overline{UU}^{\nu}$ on $\overline{U}^1 \wedge \overline{U}^1/CC_{s,s'}$ are the numbers $\varphi_{s'}(s,\ell,r)$ and $\nu(s',\lambda,\rho)$ where $\ell,\ r,\ \lambda,\ \rho$ may vary as allowed by the definition of $S_{f,I}$.

(ii) The numbers from (i) are all different, and for f large enough we always have: $\varphi_{s'}(s,\ell,r) = j(s,\ell,r)$.

For the very last statement see (38).

Corollary 16.2. *If f is large enough the numbers $j(s,\ell,r)$ where s is fixed occur as jumps of $\overline{U}^1 \wedge \overline{U}^1/CC_{s,s'}$ but never occur as jumps of $\overline{U}^1 \wedge \overline{U}^1/CC_{s_1,s_1'}$ for $s_1 > s$. And the numbers $j(s,\ell,r)$ such that $j(s,\ell,r) \neq \nu(s,\ell,r)$ cannot occur as jumps also in the case when $s_1 < s$.*

Proof. Assume the contrary of the first statement. Then we must have $j(s,\ell,r) = \nu(s_1',\lambda,\rho)$, hence the denominator of $j(s,\ell,r)$ is small. Thus $j(s,\ell,r) = \nu(s,\ell,r)$ and $\nu(s,\ell,r) = \nu(s_1',\lambda,\rho) > s_1'$ which is a contradiction.

It may happen that $j(s,\ell,r)$ occurs as a jump of $\overline{U}^1 \wedge \overline{U}^1/CC_{s_1,s_1'}$ for $s_1 < s$. For that it is necessary that $j(s,\ell,r) = \nu(s,\ell,r) = \nu(s_1',\lambda,\rho)$. But then we must have

$s'_1 = s$. If $s'_1 < s$ then $\nu(s'_1, \lambda, \rho)$ stands for a different jump of $\overline{U}^1 \wedge \overline{U}^1 / \overline{U}^1 \wedge \overline{U}^{s'+1}$ namely for the jump $j(s'_1, \lambda, \rho) \neq \nu(s'_1, \lambda, \rho) = j(s, \ell, r)$. $\square$

Therefore if f is large it is enough to study the filtrations $\overline{UU}^\nu / CC_{s,s'}$ for all pairs of consecutive numbers $s,\ s' \in I$ in order to make the filtration $\{\overline{UU}^\nu\}_\nu$ explicit. This is done in [Cr3] sections 7 and 8. Similar as the study of $\overline{UU}^\nu / CC_s$ needs to consider s-extensions $K|F$ which are related to Cram's complements, it is now necessary to study abelian extensions $E|F$ of the following type:

$E = L \cdot L'$ where $L|F$ is a C_s-extension of degree p^n, $n \geq 1$ and $L'|F$ is of maximal degree p^f corresponding to Cram's complement $C_{s'}$.

Then a careful analysis of the isomorphisms (23) for $E,\ s'$ instead of $K,\ s$ leads to the following

Theorem 16.3 ([Cr3]Theorem 1.5.2). *Assume $t < min\{e^*, p^2\}$ and f large. Then:*

(i) The jumps of the filtration $\overline{UU}^\nu$ on $\overline{U}^1 \wedge \overline{U}^1 / \overline{U}^1 \wedge \overline{U}^{t+1}$ are the numbers $j(s, \ell, r) = \varphi_{s'}(\nu(s, \ell, r))$ for triples $(s, \ell, r) \in S_{f, I_t}$.

(ii) Take the coefficients $(L_\ell)_{s,r}$ as independent coefficients of our polynomials $L \in L(k_F^{I_t} \wedge k_F^{I_t})$, where $r,\ \ell,\ s$ vary as in (i). Then under

$$\overline{U}^1 \wedge \overline{U}^1 / \overline{U}^1 \wedge \overline{U}^{t+1} \cong L(k_F^{I_t} \wedge k_F^{I_t})$$

the filtration $\overline{UU}^\nu$ corresponds to the filtration $\{\mathcal{F}^\nu\}_\nu$ such that for $\mathcal{F}^{\nu(s,\ell,r)}$ the next term $\mathcal{F}^{\nu'}$ of the filtration is given by adding the relation

$$\begin{aligned} (L_\ell)_{s,r} + \phi^\ell((L_{f-\ell})_{s,s-p^\ell(s-r)}) &= 0 \quad \textit{if } r < s,\ \ell \neq 0,\ \textit{and } s - p^\ell(s-r) > 0 \\ (L_\ell)_{s,r} - (L_\ell)_{s',r-(s'-s)p^{f-\ell}} &= 0 \quad \textit{if } \ell = f-1 \textit{ and } r - (s'-s)p^{f-\ell} > 0 \\ (L_\ell)_{s,r} &= 0 \quad \textit{otherwise.} \end{aligned}$$

Remarks.

1. In accordance to Corollary 16.2 the relation which specifies the jump $j(s, \ell, r)$ only includes coefficients $(L_\lambda)_{\sigma,\rho}$ for $\sigma \in \{s,\ s'\}$. Moreover the relation always begins with the term which is prescribed by Theorem 15.4.

2. The assumption $t < p^2$ implies that relations $\nu(s, \ell, r) = \nu(\sigma, \ell, \rho)$ are possible only for $\ell = f-1$, that means $denom(\nu(s, \ell, r)) = p$. For higher denominators we will always have $\nu(s, \ell, r) = s + r/p^{f-\ell} < s'$ and therefore $\nu(s, \ell, r) = j(s, \ell, r)$. This is the reason why the second relation in the Theorem only occurs for $\ell = f-1$.

3. Because of Proposition 7.3 it is easy also to specify the dual filtration $\mathcal{F}^*_{\nu(s,\ell,r)}$. We leave this to the reader.

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