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Stratification of Hurwitz Spaces by Closed Modular Subvarieties

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Abstract: In this paper we explain how an arithmetical invariant for G -cover - we call the base invariant - provides a stratification of Hurwitz moduli spaces of G -covers of the projective line by closed subvarieties. This stratification geometrically encodes arithmetical properties of Hurwitz spaces.

Keywords: moduli spaces, curves with group action, inverse Galois theory.

1. PRELIMINARIES

Let S be a connected scheme. An S -curve of genus g is a smooth, proper, geometrically connected S -scheme of dimension 1 whose geometric fibers have genus g .

Given a finite group G of order prime to the characteristics of S , an S - G -curve with group G is a pair (X, α) , where X is an S -curve and $\alpha : G \hookrightarrow \text{Aut}_S(X)$ is a group monomorphism. Two S - G -curves (X_i, α_i) , $i = 1, 2$ with the same group G are S - G -isomorphic if there exists an S -scheme isomorphism $u : X_1 \rightarrow X_2$ such that $u\alpha_1(g)u^{-1} = \alpha_2(g)$, $g \in G$. An S - G -cover with group G is a pair $(f : X \rightarrow Y, \alpha)$, where $f : X \rightarrow Y$ is a Galois cover of S -curves (that is a finite flat surjective morphism with generically separable geometric fibers over S such that $\text{Aut}_Y(X)$ acts transitively on the fibers) and $\alpha : G \xrightarrow{\sim} \text{Aut}_Y(X)$ is a group isomorphism. Two S - G -covers $(X_i \rightarrow Y, \alpha_i)$, $i = 1, 2$ of a given S -curve

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$Y \rightarrow S$ with the same group G are S - G -isomorphic if there exists a Y -scheme isomorphism $u : X_1 \rightarrow X_2$ such that $u\alpha_1(g)u^{-1} = \alpha_2(g)$, $g \in G$. Two S - G -covers $(X_i \rightarrow Y_i, \alpha_i)$, $i = 1, 2$ with the same group G are weakly S - G -isomorphic if there exists an S -scheme isomorphism $v : Y_1 \rightarrow Y_2$ such that the S - G -covers $(v \circ f_1 : X_1 \rightarrow Y_2, \alpha_1)$ and $(f_2 : X_2 \rightarrow Y_2, \alpha_2)$ are S - G -isomorphic. The groupoid of S - G -curves with group G and S - G -isomorphisms is then equivalent to the groupoid of S - G -covers with group G and weak S - G -isomorphisms. In the following, we will drop the α in our notation though it remains part of the data. In particular, the set of G -covers $X \rightarrow \mathbb{P}_S^1$ of the projective line over S with group G can be equipped with two structures of groupoids: the G -structure, where isomorphisms are S - G -isomorphisms and the G/PGL_2 -structure, where isomorphisms are weak S - G -isomorphisms.

Given a field k , we will write Γ_k for its absolute Galois group and we will always assume that a compatible system $(\zeta_n)_{n \geq 1}$ of primitive roots of unity is given in the algebraic closure \bar{k} of k (that is, $\zeta_{nm}^n = \zeta_m$, $n, m \geq 1$).

Let k be an algebraically closed field of characteristic not dividing $|G|$ and let $Y \rightarrow k$ be a k -curve. Then two classical invariants can be associated to a G -cover $f : X \rightarrow Y$ defined over k with group G : the ramification divisor $\mathfrak{t} \subset Y$ and the inertia canonical invariant $\mathbf{C} = (C_t)_{t \in \mathfrak{t}}$.¹ Writing $\mathcal{C}(G)$ for the set of all conjugacy classes of G and $\mathcal{C}(G)^*$ for the set of all nontrivial conjugacy classes of G , the inertia canonical invariant \mathbf{C} of a G -cover f with group G can be regarded as an element of the submonoid $\mathbf{R}_+(G) := \mathbb{N}^{(\mathcal{C}(G)^*)}$ of the free abelian monoid $\mathbf{R}_+(G) := \mathbb{N}^{(\mathcal{C}(G))}$ of all maps $n : \mathcal{C}(G) \rightarrow \mathbb{N}$. More explicitly, $\mathbf{C} = (C_t)_{t \in \mathfrak{t}}$ is identified with $\sum_{C \in \mathcal{C}(G)} n(C)C$, where $n(C)$ is the number of $t \in \mathfrak{t}$ with $C_t = C$. The monoid $\mathbf{R}_+(G)$ can be endowed with a *degree epimorphism* $\deg : \mathbf{R}_+(G) \rightarrow \mathbb{N}$ sending $\mathbf{C} = \sum_{C \in \mathcal{C}(G)} n(C)C$ to $\sum_{C \in \mathcal{C}(G)} n(C)$. Hence, when \mathbf{C} is the inertia canonical invariant of a G -cover f , $\deg(\mathbf{C})$ is just the degree r of the ramification divisor \mathfrak{t} of f . For the general theory of G -covers over arbitrary schemes, we refer to [1, §3]. In particular, given a connected $\mathbb{Z}[\frac{1}{|G|}]$ -scheme S , the inertia canonical

¹Recall that $\mathbf{C} = (C_t)_{t \in \mathfrak{t}}$ is defined as follows. For each $t \in \mathfrak{t}$, choose a place P_t in $k(X)$ above t and let I_{P_t} be the corresponding inertia group, which is cyclic of order e_t . Any uniformizing parameter u of P_t induces a well-defined (independent of the uniformizing parameter u) group monomorphism $\phi_{P_t} : I_{P_t} \hookrightarrow k^\times$, $\omega \mapsto \frac{\omega(u)}{u} \bmod P_t$. The element $\omega_{P_t} := \alpha(\phi_{P_t}^{-1}(\zeta_{e_t})) \in G$ is called *the distinguished generator of I_{P_t}* . The set of all ω_{P_t} for places P_t above t is a full conjugacy class C_t in G . In [1, §2.2.1] an equivalent definition of the inertia canonical invariant is given in terms of “holonomy”.

invariant \mathbf{C}_s of the geometric fiber of an S - G -cover $f : X \rightarrow Y$ does not depend (up to $\Gamma_{\mathbb{Q}}$ -conjugate) on the geometric point $s \in S$. Hence, we denote it by \mathbf{C} and call it the inertia canonical invariant of $f : X \rightarrow Y$.

Given a finite group G , an integer $r \geq 3$ and a degree r element $\mathbf{C} \in R_+^*(G)$, we will denote the groupoids of G -covers and G/PGL_2 -covers of the projective line over S with group G and inertia canonical invariant \mathbf{C} by $\mathcal{H}_G(\mathbf{C})(S)$ and $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})(S)$ respectively. Given furthermore an integer $g \geq 0$, we will denote the groupoid of genus g S - G -curves with group G whose resulting G -cover has inertia canonical invariant \mathbf{C} by $\mathcal{H}_{g,G,\mathbf{C}}(S)$. $S \mapsto \mathcal{H}_G(\mathbf{C})(S)$ and $S \mapsto \mathcal{H}_{g,G,\mathbf{C}}(S)$ are Deligne-Mumford stacks over $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ and their coarse moduli spaces are schemes that we will denote by $H_G(\mathbf{C})$ and $H_{g,G,\mathbf{C}}$ respectively. (See [16] and [1] for more details). Here, $\mathbb{Z}_{\mathbf{C}}$ is the ring of integers of a certain number field $\mathbb{Q}_{\mathbf{C}}$. (See section 2.1 for the definition of $\mathbb{Q}_{\mathbf{C}}$). On the other hand, $S \mapsto \mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})(S)$ defines a prestack but not a stack (since descent data are not effective in general) but its associated stack is $\mathcal{H}_{g(\mathbf{C}),G,\mathbf{C}}$. (See section 2.1 for the definition of $g(\mathbf{C})$).

In section 2.2, we show that the quotient of $H_G(\mathbf{C})$ under the natural action of PGL_2 exists over $\mathbb{Q}_{\mathbf{C}}$ and yields a coarse moduli scheme $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ for the prestack $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})$ over $\mathbb{Q}_{\mathbf{C}}$; we denote the quotient morphism by $\pi : H_G(\mathbf{C}) \rightarrow H_{G/\mathrm{PGL}_2}(\mathbf{C})$. We also explain that $H_{g(\mathbf{C}),G,\mathbf{C}}$ is, as well, a coarse moduli scheme for the prestack $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})$ over $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ (hence we will denote it by $H_{G/\mathrm{PGL}_2}(\mathbf{C})$). The aim of this paper is to give a stratification of (the generic fiber of) these moduli schemes by closed modular subvarieties.

Fix a field k of characteristic 0. Given a G -cover $f : X \rightarrow \mathbb{P}_k^1$ define the *base group* E_f of f to be the stabilizer of the G -isomorphism class of f under PGL_2^2 and the *base invariant* of f to be the conjugacy class \mathcal{E}_f of E_f in PGL_2 ; the latter is an invariant of the G/PGL_2 -isomorphism class of f . Note that E_f stabilizes the ramification divisor \mathbf{t} of f and hence, provided $r := \deg(\mathbf{t}) \geq 3$, it is contained in the permutation group $\mathfrak{S}(\mathbf{t})$. In particular, for a given $r \geq 3$, there are only finitely many possible values for the base invariant. This results from the following classification result for finite subgroups of $\mathrm{PGL}_2(\bar{k})$ [14], which asserts in particular that, for a given $r \geq 3$, there are only finitely many conjugacy classes of finite subgroups of $\mathrm{PGL}_2(\bar{k})$ contained in a copy of the symmetric group \mathcal{S}_r .

²Note that, by definition, the centralizer of G in $\mathrm{Aut}(X)$ is then an extension of E_f by the center $Z(G)$ of G and that E_f acts trivially on $Z(G)$.

Lemma 1.1. *Any finite subgroup of $\mathrm{PGL}_2(\bar{k})$ is conjugate to one of the following:*

- $C_n = \left\{ \begin{pmatrix} \zeta_n^r & 0 \\ 0 & 1 \end{pmatrix}, r = 0, \dots, n-1 \right\}$, where ζ_n is a primitive n th root of unity, $n \geq 1$;
- $D_{2n} = \left\{ \begin{pmatrix} \zeta_n^r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_n^r \\ 1 & 0 \end{pmatrix}, r = 0, \dots, n-1 \right\}$, where ζ_n is a primitive n th root of unity, $n \geq 2$;
- $\mathcal{A}_4 = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i^\nu & i^\nu \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} i^\nu & -i^\nu \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i^\nu \\ 1 & -i^\nu \end{pmatrix}, \begin{pmatrix} -1 & -i^\nu \\ 1 & -i^\nu \end{pmatrix}, \nu = 1, 3 \right\}$;
- $\mathcal{S}_4 = \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i^\nu \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i^\nu & -i^{\nu+\nu'} \\ 1 & i^{\nu'} \end{pmatrix}, \nu, \nu' = 0, 1, 2, 3 \right\}$;
- $\mathcal{A}_5 = \left\{ \begin{pmatrix} \zeta^r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta^r \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta^r \omega & \zeta^{r-s} \\ 1 & -\zeta^s \omega \end{pmatrix}, \begin{pmatrix} \zeta^r \bar{\omega} & \zeta^{r-s} \\ 1 & -\zeta^{-s} \bar{\omega} \end{pmatrix}, r, s = 0, 1, 2, 3, 4 \right\}$, where $\omega = \frac{-1+\sqrt{5}}{2}$, $\bar{\omega} = \frac{-1-\sqrt{5}}{2}$ and ζ is a primitive 5th root of unity.

One can define a partial order \subset on conjugacy classes of finite subgroups of $\mathrm{PGL}_2(\bar{k})$ by $\mathcal{E}_1 \subset \mathcal{E}_2$ if and only if $E_1 \subset E_2$ for some representatives E_1, E_2 of $\mathcal{E}_1, \mathcal{E}_2$ or, equivalently, there exists a monomorphism $i : E_1 \hookrightarrow E_2$ (indeed, by lemma 1.1, two finite subgroups of $\mathrm{PGL}_2(\bar{k})$ are isomorphic if and only if they are conjugate in $\mathrm{PGL}_2(\bar{k})$ and note that giving a monomorphism $E_1 \hookrightarrow E_2$ is equivalent to giving an isomorphism between E_1 and a subgroup of E_2). Now, given a conjugacy class \mathcal{E} of finite subgroups of $\mathrm{PGL}_2(\bar{k})$ denote by $H_G(\mathbf{C}, \supset \mathcal{E})(k)$ (resp. $H_G(\mathbf{C}, \mathcal{E})(k)$) the subset of k -rational points of $H_G(\mathbf{C})$ corresponding to G -covers f with base invariant $\mathcal{E}_f \supset \mathcal{E}$ (resp. with base invariant $\mathcal{E}_f = \mathcal{E}$). We define similarly $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \supset \mathcal{E})(k)$ (resp. $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \mathcal{E})(k)$).

In [3], constructing a cohomological obstruction for the lifting of k -rational points from $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ to $H_G(\mathbf{C})$, one shows that, for any nontrivial conjugacy class \mathcal{E} , there exists an integer $1 \leq d(\mathcal{E}) \leq 6$ depending only on \mathcal{E} and not on r such that the image of the natural map

$$\bigcup_{[l:k] \leq d(\mathcal{E})} H_G(\mathbf{C}, \mathcal{E})(l) \rightarrow H_{G/\mathrm{PGL}_2}(\mathbf{C}, \mathcal{E})(\bar{k})$$

contains $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \mathcal{E})(k)$. So, the subsets $H_G(\mathbf{C}, \supset \mathcal{E})$ (resp. $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$) of points in $H_G(\mathbf{C})$ (resp. in $H_{G/\mathrm{PGL}_2}(\mathbf{C})$) corresponding to G -covers f with base invariant $\mathcal{E}_f \supset \mathcal{E}$ have particularly “good” arithmetical properties. They actually are geometrical objects, as follows.

Theorem 1.2. *For any conjugacy class \mathcal{E} of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{Q}})$, the set $H_G(\mathbf{C}, \supset \mathcal{E})$ is closed in $H_G(\mathbf{C})$. In particular, $H_G(\mathbf{C}, \supset \mathcal{E})$ can be equipped with a unique structure of reduced closed subvariety defined over $\mathbb{Q}_{\mathbf{C}}$.*

To prove theorem 1.2, it is enough to show that the inverse image of $H_G(\mathbf{C}, \supset \mathcal{E})$ is closed in $H_G(\mathbf{C}) \times_{\mathbb{Q}_{\mathbf{C}}} \overline{\mathbb{Q}}$, that is, this inverse image coincides with the underlying set of a closed subvariety $H_G(\mathbf{C}, \supset \mathcal{E})$ of $H_G(\mathbf{C}) \times_{\mathbb{Q}_{\mathbf{C}}} \overline{\mathbb{Q}}$. Indeed, the fact that this subvariety is defined over $\mathbb{Q}_{\mathbf{C}}$ results from (i) $H_G(\mathbf{C})$ is defined over $\mathbb{Q}_{\mathbf{C}}$, (ii) for any $\mathbf{p} \in H_G(\mathbf{C})(\overline{\mathbb{Q}})$ corresponding to the G -isomorphism class of a G -cover f , ${}^\sigma \mathbf{p}$ corresponds to the G -isomorphism class of the G -cover ${}^\sigma f$, $\sigma \in \Gamma_{\mathbb{Q}_{\mathbf{C}}}$ and (iii) for any G -cover f over $\overline{\mathbb{Q}}$, $\mathcal{E}_{\sigma f} = \mathcal{E}_f$, $\sigma \in \Gamma_{\mathbb{Q}_{\mathbf{C}}}$. Properties (ii) and (iii) imply that ${}^\sigma H_G(\mathbf{C}, \supset \mathcal{E}) = H_G(\mathbf{C}, \supset \mathcal{E})$, $\sigma \in \Gamma_{\mathbb{Q}_{\mathbf{C}}}$ and property (i) allows us to apply the Galois criterion for rationality [2, AG, Th. 14.4].

Corollary 1.3. *For any conjugacy class \mathcal{E} of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{Q}})$, the set $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$ is closed in $H_{G/\mathrm{PGL}_2}(\mathbf{C})$. In particular, it can be equipped with a unique structure of reduced closed subvariety of $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ defined over $\mathbb{Q}_{\mathbf{C}}$.*

Proof. As the quotient morphism $\pi : H_G(\mathbf{C}) \rightarrow H_{G/\mathrm{PGL}_2}(\mathbf{C})$ is submersive and $H_G(\mathbf{C}, \supset \mathcal{E}) = \pi^{-1}(H_{G/\mathrm{PGL}_2}(\mathbf{C}, \supset \mathcal{E}))$, theorem 1.2 also implies that the set $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$ is closed in $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ and hence, by the same argument as for theorem 1.2, $H_{G/\mathrm{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$ can be equipped with a unique structure of reduced closed subvariety of $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ defined over $\mathbb{Q}_{\mathbf{C}}$. \square

Theorem 1.2 thus provides a stratification of Hurwitz spaces by closed subvarieties which encodes arithmetical properties of the morphism π .

We present two approaches to theorem 1.2. The first approach (section 3) relies on results from Mumford’s geometric invariant theory and uses that $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ is the universal geometric quotient of $H_G(\mathbf{C})$ under PGL_2 . The second approach (section 4) exploits the functorial properties of the Hurwitz schemes $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ constructed in [1]. This second approach is more explicit and, in particular, it provides a natural method to describe the geometrically irreducible components of our strata (section 4.2.3) and to search for \mathbb{Q} -rational points on $H_G(\mathbf{C})$ (section 5).

Remark 1.4. Theorem 1.2 and corollary 1.3 could be extended to base fields k of positive characteristic $p > 0$ provided p divides neither $|G|$ nor the (common) order $|E|$ of $E \in \mathcal{E}$ (take for instance $p \nmid |G|r!$): under these assumptions, our proofs work without any change. However, for primes $p > 0$ which do not divide $|G|$ but may divide $|E|$, our proofs would not work, at least as they are. More specifically, a key theorem 3.3 of our first approach fails in general. (If $A := \mathrm{PGL}_2$ and $a := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $C(a) = A_p \setminus \{1_A\}$ ($A_p := \{a \in A \mid a^p = 1_A\}$), which is not closed in A .) Also, our second approach no longer works as it is since it resorts to coarse moduli schemes associated with groups of order $|G||E|$. Anyway, to avoid technical details, we only deal with the case of characteristic 0.

2. A DESCRIPTION OF $H_{g(\mathbf{C}), G, \mathbf{C}}$

Let S be a scheme and $\mathcal{X} \rightarrow (\mathrm{Sch}/S)$ a category fibered in groupoids over the category (Sch/S) of S -schemes (which we will shorten by “an S -groupoid”). Any S -scheme $X \rightarrow S$ is naturally identified with an S -groupoid by considering $(\mathrm{Sch}/X) \rightarrow (\mathrm{Sch}/S)$. A *coarse moduli scheme* for $\mathcal{X} \rightarrow (\mathrm{Sch}/S)$ is an S -scheme $X \rightarrow S$ together with an S -groupoid morphism $F : \mathcal{X} \rightarrow X$ such that (i) for any geometric point $k \rightarrow S$, $F(k) : \mathcal{X}(k) \rightarrow X(k)$ induces a bijection between the isomorphism classes of objects in $\mathcal{X}(k)$ and $X(k)$ (which we will shorten by “is a bijection”) and (ii) for any S -scheme $X' \rightarrow S$ and any S -groupoid morphism $F' : \mathcal{X} \rightarrow X'$, there exists a unique morphism $\phi : X \rightarrow X'$ in \mathcal{C} such that $\phi \circ F = F'$.

Note that a coarse moduli scheme - when it exists - is unique (by condition (ii)) and, also, that from the definition of an S -groupoid morphism, for any k -rational point $k \rightarrow S$ (k : a field), the bijection $F(\bar{k}) : \mathcal{X}(\bar{k}) \rightarrow X(\bar{k})$ is compatible with the natural action of Γ_k , that is, for any $x \in \mathcal{X}(\bar{k})$, $\sigma \in \Gamma_k$, one has $F(\bar{k})(x^\sigma) = F(\bar{k})(x)^\sigma$.

2.1. Notation for Hurwitz spaces. Fix a finite group G and an element $\mathbf{C} \in \mathbb{R}_+^*(G)$. Set $\mathbb{Q}_{\mathbf{C}} := \overline{\mathbb{Q}}^{\Delta_{\mathbf{C}}}$, where $\Delta_{\mathbf{C}} \subset \Gamma_{\mathbb{Q}}$ is the closed subgroup of all $s \in \Gamma_{\mathbb{Q}}$ such that $\mathbf{C}^{\chi(s)} = \mathbf{C}$, where $\chi : \Gamma_k \rightarrow \widehat{\mathbb{Z}}^\times$ is the cyclotomic character of k . Finally, let $\mathbb{Z}_{\mathbf{C}}$ denote the ring of integers of $\mathbb{Q}_{\mathbf{C}}$.

Given a conjugacy class C of G we denote by $o(C)$ the common order of the

elements in C . With this notation, we associate to \mathbf{C} the corresponding genus $g(\mathbf{C})$ defined by the Riemann-Hurwitz formula

$$g(\mathbf{C}) = \frac{|G|}{2} \left(-2 + \sum_{C \in \mathcal{C}(G)} n(C) \left(1 - \frac{1}{o(C)}\right)\right) + 1.$$

We retain the notation $\mathcal{H}_G(\mathbf{C})$, $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})$ and $\mathcal{H}_{g,G,\mathbf{C}}$ of section 1. Also, for an integer $r \geq 0$ (resp. integers $g, r \geq 0$ with $2 - 2g - r < 0$), we denote by \mathcal{U}_r (resp. $\mathcal{M}_{g,[r]}$) the \mathbb{Z} -stack of projective lines (resp. genus g curves) with r unordered marked points (or, more precisely, relatively étale divisors of degree r) and by U_r (resp. $M_{g,[r]}$) its coarse moduli scheme. (Thus, $\mathcal{U}_r = U_r$, i.e., \mathcal{U}_r is representable.) Set $r = \mathrm{deg}(\mathbf{C})$. Then there are natural prestack morphisms $\mathcal{H}_G(\mathbf{C}) \rightarrow \mathcal{U}_r$, $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C}) \rightarrow \mathcal{U}_r/\mathrm{PGL}_2$, $\mathcal{H}_{g,G,\mathbf{C}} \rightarrow \mathcal{M}_{g,[a]}$ and $\mathcal{H}_{g,G,\mathbf{C}} \rightarrow \mathcal{M}_{g',[r]}$, where

$$a := |G| \sum_{C \in \mathcal{C}(G)} \frac{n(C)}{o(C)} = |G|(2g' - 2 + r) - (2g - 2),$$

$$g' := 1 + \frac{g-1}{|G|} - \frac{1}{2} \sum_{C \in \mathcal{C}(G)} n(C) \left(1 - \frac{1}{o(C)}\right).$$

(If g' is not an integer ≥ 0 , set $\mathcal{M}_{g',[r]} = \emptyset$.)

From [16], the $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ -stack $\mathcal{H}_G(\mathbf{C})$ is an algebraic stack proper, quasi-finite and étale over \mathcal{U}_r ; it admits a coarse moduli scheme $\mathcal{H}_G(\mathbf{C}) \rightarrow H_G(\mathbf{C})$, which is an affine, smooth $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ -scheme finite étale over U_r .

From [1, Th. 6.9 and Prop. 6.14], the $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ -stack $\mathcal{H}_{g,G,\mathbf{C}}$ admits a stable compactification $\overline{\mathcal{H}}_{g,G,\mathbf{C}}$, which is a Deligne-Mumford stack smooth and proper over $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$. The natural extended morphism $\overline{\mathcal{H}}_{g,G,\mathbf{C}} \rightarrow \overline{\mathcal{M}}_{g,[a]}$ is finite (representable) and unramified whereas the natural extended morphism $\overline{\mathcal{H}}_{g,G,\mathbf{C}} \rightarrow \overline{\mathcal{M}}_{g',[r]}$ is proper, quasi-finite and flat (but representable if and only if either G has trivial center or $\overline{\mathcal{H}}_{g,G,\mathbf{C}} = \emptyset$); at the level of coarse moduli spaces, the latter is finite and surjective. We will only consider the coarse moduli scheme $\mathcal{H}_{g,G,\mathbf{C}} \rightarrow H_{g,G,\mathbf{C}}$, which is normal, of finite type over $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ and quasi-finite over $M_{g,[a]}$.³

³In fact, $H_{g,G,\mathbf{C}}$ is finite over $M_{g,[a]}$, which is crucial in our second approach. See the proof of lemma 4.3.

2.2. A description of $H_{g(\mathbf{C}),G,\mathbf{C}}$. Recall that $S \mapsto \mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})(S)$ is a prestack over $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ but not a stack in general; its associated stack [8, Chapter 3] is $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C}) \hookrightarrow \mathcal{H}_{g(\mathbf{C}),G,\mathbf{C}}$. Then, the composition $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C}) \hookrightarrow \mathcal{H}_{g(\mathbf{C}),G,\mathbf{C}} \rightarrow H_{g(\mathbf{C}),G,\mathbf{C}}$ is a coarse moduli scheme for $\mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C})$; condition (i) is straightforward and condition (ii) results from the universal property of the stack associated to a prestack.

Let $\mathcal{H}_G(\mathbf{C})^*$ denote the presheaf of isomorphism classes $S \mapsto \mathcal{H}_G(\mathbf{C})/\simeq$. Then the $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ -group scheme PGL_2 acts naturally on $\mathcal{H}_G(\mathbf{C})^*$ (by translation on the left). By condition (ii) of the definition of coarse moduli scheme, this action induces an algebraic action defined over $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]$ of PGL_2 on $H_G(\mathbf{C})$. We classically denote the corresponding quotient stack by $[H_G(\mathbf{C})/\mathrm{PGL}_2/\mathbb{Z}_{\mathbf{C}}[\frac{1}{|G|}]]$ [8, Ex. 2.4.2].

We explain below that the $\mathbb{Q}_{\mathbf{C}}$ -stack $[H_G(\mathbf{C})/\mathrm{PGL}_2/\mathbb{Q}_{\mathbf{C}}]$ admits a coarse moduli scheme.

Let k be a field of characteristic 0 and let A be a connected reductive k -group acting on an affine k -scheme of finite type X via $\sigma : A \times_k X \rightarrow X$. Consider the following cartesian square

$$(1) \quad \begin{array}{ccc} S & \longrightarrow & X \\ \downarrow & \square & \downarrow \Delta_{X/k} \\ A \times_k X & \xrightarrow{\Phi} & X \times_k X, \end{array}$$

where Φ is the map $A \times_k X \rightarrow X \times_k X$ sending (a, x) to (ax, x) . Then $S \rightarrow X$ is a closed subgroup scheme of $A \times_k X \xrightarrow{p_2} X$ called the *stabilizer group scheme*. Also, given a (k -rational) geometric point $x \in X$, the *orbit* of x is the subset $O(x) = \Phi(A \times_k x) \subset X$; it is a smooth locally closed subvariety of X [2, I.1.8, Prop.].

A *categorical quotient* for σ is a k -morphism $\pi : X \rightarrow X/A$ such that (i) $\pi \circ \sigma = \pi \circ p_2$ and (ii) for any k -morphism $\phi : X \rightarrow Y$ such that $\phi \circ \sigma = \phi \circ p_2$ holds there is a unique k -morphism $\bar{\phi} : X/A \rightarrow Y$ such that $\bar{\phi} \circ \pi = \phi$. A *geometric quotient* for σ is a k -morphism $\pi : X \rightarrow X/A$ such that (i) $\pi \circ \sigma = \pi \circ p_2$, (ii) for any geometric point $x \in X$, the fiber $\pi^{-1}(\pi(x))$ is precisely $O(x)$, (iii) π is submersive (*i.e.* $U \subset X/A$ is open if and only if $\pi^{-1}(U) \subset X$ is open in X .) and (iv) $\mathcal{O}_{X/A} = (\pi_* \mathcal{O}_X)^A$. A geometric quotient is always a categorical quotient [11, Prop. 0.1], and, in particular, is unique up to a unique isomorphism. A universal

geometric quotient⁴ for σ is - when it exists - a coarse moduli scheme for the k -stack $[X/A/k]$ [8, Chap. 19].

Lemma 2.1. *Assume that A acts on X via σ with finite stabilizers, then there exists a universal geometric quotient $\pi : X \rightarrow X/A$ for σ .*

Proof. Lemma 2.1 is a consequence of geometric invariant theory. By [11, Th. 1.1], a universal categorical quotient $\pi : X \rightarrow X/A$ exists for σ with X/A affine and π universally submersive. Furthermore, by [11, Amp. 1.3], this quotient is a (universal) geometric quotient if and only if σ is closed, that is, for any geometric point $x \in X$ the orbit $O(x) \subset X$ is closed in X . So, lemma 2.1 amounts to proving that σ is closed. For each $x \in X(k)$, the natural surjective morphism $A \rightarrow O(x)$, $a \mapsto ax$ is quasi-finite by assumption. Thus, $\dim(O(x)) = \dim(A)$ (see [7, II, ex. 3.22]) is independent of x . Now the assertion follows from the so-called closed orbit lemma [2, I.1.8, Prop.]. \square

We deduce from lemma 2.1 that there exists a universal geometric quotient $\pi : H_G(\mathbf{C}) \rightarrow H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C})$ defined over $\mathbb{Q}_{\mathbf{C}}$. The notation $H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C})$ is explained by the following lemma.

Lemma 2.2. *The $\mathbb{Q}_{\mathbf{C}}$ -scheme $H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C})$ is canonically isomorphic to the generic fiber of the $\mathbb{Z}_{\mathbf{C}}[\frac{1}{|\mathbf{G}|}]$ -scheme $H_{G/\mathrm{PGL}_2}(\mathbf{C})$.*

Proof. When G has trivial center, this is straightforward since $\mathcal{H}_G(\mathbf{C}) = H_G(\mathbf{C})$. Else, it formally results from condition (ii) of the definition of coarse moduli scheme. Indeed, consider the following commutative diagram, which sums up the situation and where the first row has to be considered as restricted to the category of $\mathbb{Q}_{\mathbf{C}}$ -schemes.

$$\begin{array}{ccccccc}
 & & & & a & & \\
 & & & & \curvearrowright & & \\
 \mathcal{H}_G(\mathbf{C}) & \longrightarrow & \mathcal{H}_{G/\mathrm{PGL}_2}(\mathbf{C}) & \longrightarrow & \mathcal{H}_{g(\mathbf{C}),G,\mathbf{C}} & \longrightarrow & H_{G/\mathrm{PGL}_2}(\mathbf{C}) = H_{g(\mathbf{C}),G,\mathbf{C}} \\
 \downarrow b & & & \searrow & \downarrow & & \\
 H_G(\mathbf{C}) & \xrightarrow{c} & [H_G(\mathbf{C})/\mathrm{PGL}_2/\mathbb{Q}_{\mathbf{C}}] & \longrightarrow & H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C}) & & \\
 & & & \curvearrowleft & & & \\
 & & & & d & &
 \end{array}$$

⁴For the definition of universal categorical/geometric quotients, see [11, Def. 0.7].

From condition (ii) of the definition of a coarse moduli scheme applied to $H_{g(\mathbf{C}),G,\mathbf{C}}$, there exists a unique morphism $\alpha : H_{g(\mathbf{C}),G,\mathbf{C}} \rightarrow H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C})$ making the last square commute. Also, the same condition applied to $H_G(\mathbf{C})$ yields a unique morphism $\beta : H_G(\mathbf{C}) \rightarrow H_{g(\mathbf{C}),G,\mathbf{C}}$ such that $\beta \circ b = a$. This morphism, in turn, factors through c and condition (ii) of the definition of a coarse moduli scheme applied to $H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C})$ yields a unique morphism $\gamma : H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C}) \rightarrow H_{g(\mathbf{C}),G,\mathbf{C}}$ such that $\gamma \circ d = \beta$. Then, invoking one more time condition (ii) of the definition of a coarse moduli scheme, one gets $\alpha \circ \gamma = \mathrm{Id}$ and $\gamma \circ \alpha = \mathrm{Id}$. \square

In the following, we will no longer distinguish between $H_{G/\mathrm{PGL}_2}^{gen}(\mathbf{C})$ and the generic fiber of $H_{G/\mathrm{PGL}_2}(\mathbf{C})$, both of which we will denote by $H_{G/\mathrm{PGL}_2}(\mathbf{C})$.

3. FIRST APPROACH

We retain the notation of section 2.2.

Lemma 3.1. *Assume that A acts on X with finite stabilizers, then the map $\Phi : A \times_k X \rightarrow X \times_k X$ is proper and unramified.*

Proof. Since σ is closed (see the proof of Lemma 2.1) and π is affine, the properness of Φ follows from [11, Prop. 0.8]. The unramifiedness of Φ follows from the fact that, for each geometric point (x, y) of $X \times_k X$, the fiber of Φ at (x, y) is either empty or an S_x -torsor, where S_x is the fiber at x of the stabilizer group scheme $S \rightarrow X$. (Note that any finite group scheme over a field of characteristic 0 is étale.) \square

Corollary 3.2. *Fix $a_1, \dots, a_m \in A$ and let E be the subgroup of A generated by a_1, \dots, a_m . Assume that the A -orbit $C(a_1, \dots, a_m)$ of (a_1, \dots, a_m) (with respect to the diagonal, conjugate action) is closed in A^m . Then the set of points $x \in X$ such that S_x contains a conjugate of E is closed in X .*

Proof. By lemma 3.1 applied to A^m acting on X^m by the direct product action induced by $\sigma : A \times_k X \rightarrow X$, the set $\Phi(C(a_1, \dots, a_m) \times_k X^m)$ is closed in $X^m \times_k X^m$. Let $\Delta_{X|k}^m : X \rightarrow X^m$ denote the m th diagonal map over k and set $\Delta_{X|k}^{m,2} = \Delta_{X|k}^m \times_k \Delta_{X|k}^m \circ \Delta_{X|k}^2$. Then the set $(\Delta_{X|k}^{m,2})^{-1}(\Phi(C(a_1, \dots, a_m) \times_k X^m))$

is closed in X , which yields the announced conclusion. \square

Combined with the following result from the theory of algebraic groups, corollary 3.2 yields theorem 1.2.

Theorem 3.3. *With the notation of corollary 3.2 and assuming furthermore that E is finite, the A -orbit $C(a_1, \dots, a_m)$ is closed in A^m .*

Proof. This is a special case of [12, Th. 2.16]. \square

However, it is possible to prove theorem 1.2 in most cases without resorting to theorem 3.3. This relies on a group-theoretic lemma and a semi-continuity argument combined with the elementary case of theorem 1.2 for cyclic groups proved in example 3.4. The rest of this section is devoted to this elementary (partial) proof of theorem 1.2.

Example 3.4. Consider $A := \text{PGL}_2$. Then, as A is separated, the set $A_n := \{a \in A \mid a^n = 1_A\}$ is closed for any integer $n \geq 1$. Furthermore, there are exactly $\phi(n)/2$ conjugacy classes of order n elements in $\text{PGL}_2(\bar{k})$, namely the ones of $a_n^l := \begin{pmatrix} \zeta_n^l & 0 \\ 0 & 1 \end{pmatrix}$, $l \in (\mathbb{Z}/n)^\times / \pm 1$.⁵ Considering for instance the commutative diagram

$$\begin{array}{ccc} \text{GL}_2 & \xrightarrow{q} & \mathbb{P}^1, \\ \downarrow & \nearrow \bar{q} & \\ A & & \end{array}$$

where $q(a) = [\text{Tr}(a)^2 - 2\det(a) : \det(a)]$, one obtains that $C(a_n^l) = \bar{q}^{-1}([\zeta_n^l + \zeta_n^{-l} : 1]) \cap A_n$ is closed. Combined with corollary 3.2 (for $m = 1$) this yields theorem 1.2 for cyclic groups.

Given a finite group E denote by $I(E)_n \geq 0$ the number of elements with order exactly n in E and set $I(E) = (I(E)_n)_{n \geq 1}$. For any two sequences of integers $I_i = (I_{i,n})_{n \geq 1}$, $i = 1, 2$ say that $I_1 \leq I_2$ if $I_{1,n} \leq I_{2,n}$, $n \geq 1$. With these notations

⁵Indeed, if a has order exactly n , then the cyclic subgroup $\langle a \rangle$ it generates is conjugate in A to the group C_n of lemma 1.1, hence a is conjugate in A to an element of the form a_n^l for some $l \in (\mathbb{Z}/n)^\times / \pm 1$. Eventually, two elements a_n^l and $a_n^{l'}$ with $l \neq l' \in (\mathbb{Z}/n)^\times / \pm 1$ are not conjugate in A .

we have:

Lemma 3.5. *Assume that E_1, E_2 are two groups isomorphic to one of the following:*

- *Cyclic group:* $C_n, n \geq 1$;
- *Dihedral group:* $D_{2n} = C_n \rtimes C_2, n \geq 2$;
- *Symmetric group:* S_4 ;
- *Alternating group:* A_4, A_5 .

If $I(E_1) \leq I(E_2)$ then one of the following holds:

- (i) *there exists a monomorphism $E_1 \hookrightarrow E_2$;*
- (ii) *$E_1 \simeq D_4$ and $E_2 \simeq D_{2n}$ with $n \geq 3$ odd; or*
- (iii) *$E_1 \simeq D_6$ and $E_2 \simeq A_4$.*

Proof. Compute first the different possible values of $I(E)$:

$$I(C_n) = (I(C_n)_m)_{m \geq 1} \text{ with } I(C_n)_m = I(C_m)_m \text{ if } m|n;$$

$$I(C_n)_m = 0 \text{ else;}$$

$$I(D_{2n}) = (1, n + I(C_n)_2, (I(C_n)_m)_{m \geq 3});$$

$$I(A_4) = (1, 3, 8, 0, 0, 0, \dots);$$

$$I(S_4) = (1, 9, 8, 6, 0, 0, \dots);$$

$$I(A_5) = (1, 15, 20, 0, 24, 0, \dots).$$

Also observe that $I(C_n)_2 \leq 1, n \geq 2$ and $I(D_{2n})_3 \leq 2, n \geq 2$. We now fix E_1 and consider what the condition $I(E_1) \leq I(E_2)$ imposes on E_2 .

- If $I(C_n) \leq I(E)$ then, in particular, $1 \leq I(C_n)_n \leq I(E)_n$ that is E contains at least one element of order n thus $C_n \hookrightarrow E$.

- If $I(D_{2n}) \leq I(E)$ then, in particular, $2 \leq I(D_{2n})_2 = n + I(C_n)_2 \leq I(E)_2$ thus E is not cyclic and $1 \leq I(D_{2n})_n = n \leq I(E)_n$ thus E contains at least one element of order n . So:

- - either E is a dihedral group D_{2N} with $n|N$ thus $D_{2n} \hookrightarrow E$;

- - or $E \simeq A_4$ and $n + I(C_n)_2 \leq I(E)_2 = 3$ so $n = 2, 3$. If $n = 2$ we are in the situation $D_4 \hookrightarrow A_4$, if $n = 3$ we are in the exceptional case $E_1 \simeq D_6$ and $E_2 \simeq A_4$;

- - or $E \simeq S_4$ and $n + I(C_n)_2 \leq I(E)_2 = 9$ so $2 \leq n \leq 9$. If $n = 2, 3, 4$ we are in the case $D_{2n} \hookrightarrow S_4$ and the case $n \geq 5$ can never occur because $I(S_4)_n = 0, n \geq 5$;

- - or $E \simeq A_5$ and $n + I(C_n)_2 \leq I(E)_2 = 15$ so $2 \leq n \leq 15$. If $n = 2, 3, 5$ we

are in the case $D_{2n} \hookrightarrow \mathcal{A}_5$, the case $n = 4$ can never occur because $I(E)_4 = 0$ and neither can the case $n \geq 6$ because $I(\mathcal{A}_5)_n = 0$, $n \geq 6$.

- If $I(\mathcal{A}_4) \leq I(E)$ then the only possibilities for E are $E \simeq \mathcal{A}_4, \mathcal{S}_4, \mathcal{A}_5$ and, in any case, $\mathcal{A}_4 \hookrightarrow E$.

- If $I(\mathcal{S}_4) \leq I(E)$ then $E \simeq \mathcal{S}_4$.

- If $I(\mathcal{A}_5) \leq I(E)$ then $E \simeq \mathcal{A}_5$. \square

Corollary 3.6. *Theorem 1.2 holds except possibly for the dihedral groups D_4 and D_6 .*

Proof. From now on X/k stands for the coarse moduli scheme $H_G(\mathbf{C}) \times_{\mathbb{Q}\mathbf{C}} \overline{\mathbb{Q}}/\overline{\mathbb{Q}}$ and A stands for PGL_2 . For any $n \geq 1$ set $A_n^0 := \{a \in A \mid | \langle a \rangle | = n\}$. By example 3.4 this is a closed subset of A . (Indeed, A_n^0 is a (disjoint) union of $C(a_n^l)$ for $l \in (\mathbb{Z}/n)^\times / \pm 1$.) As a result, the morphism

$$\Psi_n : A_n^0 \times_k X \hookrightarrow A \times_k X \xrightarrow{\Phi} X \times_k X$$

is a finite morphism by lemma 3.1. In particular, by [7, II, ex. 5.8] the map

$$\begin{aligned} I_n : X \times_k X &\longrightarrow \mathbb{Z}_{\geq 0} \\ \omega &\longmapsto \dim_{k(\omega)} \mathcal{O}_{A_n^0 \times_k X, \omega} \otimes_{\mathcal{O}_{X \times_k X, \omega}} k(\omega) \end{aligned}$$

is upper semi-continuous. Observe that, for each geometric point ω of $X \times_k X$, we have $I_n(\omega) = |\Psi_n^{-1}(\omega)|$, since Ψ_n is unramified by lemma 3.1. So, given any sequence of positive integers $I = (I_n)_{n \geq 1}$ the set

$$\mathcal{L}(I) := \bigcap_{n \geq 1} \{\omega \in X \times_k X \mid I_n(\omega) \geq I_n\} \subset X \times_k X$$

is closed thus so is $\Delta_{X|k}^{-1}(\mathcal{L}(I)) \subset X$. But $\Delta_{X|k}^{-1}(\mathcal{L}(I)) = \{x \in X \mid I(S_x) \geq I\}$. Corollary 3.6 now results from lemma 3.5 and lemma 1.1. \square

Remark 3.7. The above arguments also work with $X \rightarrow k$ standing for $U_r \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. In particular, if we denote by $U_r(\supset \mathcal{E})$ the subset of all points of U_r corresponding to r -tuples \mathbf{t} whose stabilizer contains a representative of \mathcal{E} , then we have also proved: *Let \mathcal{E} be a conjugacy class of finite subgroups in $\mathrm{PGL}_2(\overline{\mathbb{Q}})$ then the set $U_r(\supset \mathcal{E})$ is closed in U_r . In particular, it can be equipped with a unique structure of reduced closed subvariety of U_r defined over \mathbb{Q} .*

4. SECOND APPROACH

4.1. Preliminary remarks. The second approach interprets the base locus $H_G(\mathbf{C}, \supset \mathcal{E})$ as the last element of a series of inverse and direct images of morphisms between moduli schemes. In the following, given an extension of finite groups $1 \rightarrow K \rightarrow F \xrightarrow{p} Q \rightarrow 1$, we will say that Q acts trivially on K if there exists a set-theoretic section $s : Q \rightarrow F$ of p such that $s(q)ks(q)^{-1} = k$, $q \in Q$, $k \in K$. This is equivalent to requiring that the canonical representation $Q \rightarrow \text{Out}(K)$ be trivial. To explain our second approach, start with a G -cover $f : X \rightarrow \mathbb{P}^1$ with group G and with a nontrivial base group E and consider the resulting cover $\tilde{f} : X \xrightarrow{f} \mathbb{P}^1 \xrightarrow{\Pi} \mathbb{P}^1/E$. Fixing an isomorphism $i : \mathbb{P}^1/E \simeq \mathbb{P}^1$ we obtain a cover $i \circ \tilde{f} : X \rightarrow \mathbb{P}^1$ which, by definition of the base group, is a G -cover with group, say, \tilde{E} . We can make the following three remarks:

(1) The group \tilde{E} is an extension $1 \rightarrow G \rightarrow \tilde{E} \xrightarrow{p} E \rightarrow 1$ where E acts trivially on G . Indeed, by definition of the base group, any $e \in E$ admits a lift ϵ to $\text{Aut}(X)$ which is compatible with $\text{Aut}(f) \xrightarrow{\alpha} G$ that is, $\alpha(\epsilon g \epsilon^{-1}) = \alpha(g)$, $g \in \text{Aut}(f)$ and, hence, $\epsilon g \epsilon^{-1} = g$, $g \in G$. In the following, we will denote by $\mathcal{E}(E, G)$ the set of all group extensions $1 \rightarrow G \rightarrow \tilde{E} \xrightarrow{p} E \rightarrow 1$ with E acting trivially on G . This set is always non-empty since it contains the direct product $\tilde{E} = G \times E$. Hence, by classical group cohomology, there is a canonical bijection $(\mathcal{E}(E, G)/\simeq) \xrightarrow{\sim} \text{H}^2(E, Z(G))$ sending the direct product $G \times E$ to the trivial class in $\text{H}^2(E, Z(G))$. In particular, when $Z(G)$ is trivial, the only possible extension is $\tilde{E} = G \times E$.

(2) The cover $i \circ \Pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is ramified over two or three points. Indeed, first, recall that there is no nontrivial cover of \mathbb{P}^1 with 0 or 1 ramification point (and by assumption, E is nontrivial). Also, $i \circ \Pi$ is a G -cover with group E . Thus, if e_1, \dots, e_s denote the list of the ramification indices > 1 , the Riemann-Hurwitz formula becomes

$$0 > -2 = |E|(-2 + \sum_{1 \leq i \leq s} (1 - \frac{1}{e_i})) \geq |E|(-2 + \frac{s}{2})$$

This implies $s < 4$.

(3) If we assume furthermore that $f : X \rightarrow \mathbb{P}^1$ is ramified over $r \geq 3$ points then X has genus ≥ 1 . Else, \tilde{E} would necessarily be one of the groups listed in lemma 1.1. In particular, the extension $1 \rightarrow G \rightarrow \tilde{E} \xrightarrow{p} E \rightarrow 1$ would be one of the following

$$\begin{aligned} 1 \rightarrow C_n \rightarrow C_{nm} \xrightarrow{p} C_m \rightarrow 1, n, m \geq 2; \\ 1 \rightarrow C_n \rightarrow D_{2nm} \xrightarrow{p} D_{2m} \rightarrow 1, n, m \geq 2; \\ 1 \rightarrow C_n \rightarrow D_{2n} \xrightarrow{p} C_2 \rightarrow 1, n \geq 2; \\ 1 \rightarrow D_4 \rightarrow \mathcal{A}_4 \xrightarrow{p} C_3 \rightarrow 1; \\ 1 \rightarrow D_4 \rightarrow \mathcal{S}_4 \xrightarrow{p} D_6 \rightarrow 1; \\ 1 \rightarrow \mathcal{A}_4 \rightarrow \mathcal{S}_4 \xrightarrow{p} C_2 \rightarrow 1. \end{aligned}$$

Then, recalling that E acts trivially on G , the only remaining cases are $1 \rightarrow C_2 \rightarrow D_{4m} \rightarrow D_{2m} \rightarrow 1, m \geq 2, 1 \rightarrow C_n \rightarrow C_{nm} \xrightarrow{p} C_m \rightarrow 1, n, m \geq 2$ and $1 \rightarrow C_2 \rightarrow D_4 \xrightarrow{p} C_2 \rightarrow 1$. In particular, they all correspond to a cyclic group G . But, by the Riemann-Hurwitz formula, a nontrivial cyclic G -cover $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ ramifies above exactly two points, which contradicts $r \geq 3$.

Also, $i \circ \tilde{f}$ is only defined after fixing an isomorphism $i : \mathbb{P}^1/E \simeq \mathbb{P}^1$ but, for two such choices i_1, i_2 , the resulting covers $i_1 \circ \tilde{f}, i_2 \circ \tilde{f}$ are G/PGL_2 -isomorphic and, thus, the G/PGL_2 -isomorphism class of $i \circ \tilde{f}$ no longer depends on i ; we denote it by \tilde{f} . Similarly, we denote by Π the G/PGL_2 -isomorphism class of $i \circ \Pi$. The idea is to reverse this construction and recover f from Π and \tilde{f} .

4.2. Second proof of theorem 1.2.

4.2.1. *Notation.* First, we introduce some notations. We write $\mathbf{R}_+(G) \rightarrow \mathbf{R}_+^*(G)$, $\mathbf{C} = \sum_{C \in \mathcal{C}} n(C)C \mapsto \mathbf{C}^* = \sum_{C \in \mathcal{C}^*} n(C)C$ for the canonical projection. Observe that $g(\mathbf{C}^*) = g(\mathbf{C})$ and $\deg(\mathbf{C}^*) = \deg(\mathbf{C}) - n(\{1\})$ hold.

Then any group epimorphism $p : \tilde{E} \twoheadrightarrow E$ defines a morphism of monoids $\mu : \mathbf{R}_+(\tilde{E}) \rightarrow \mathbf{R}_+(E)$, sending $\tilde{C} \in \mathcal{C}(\tilde{E})$ to $p(\tilde{C}) \in \mathcal{C}(E)$. Geometrically, if $\tilde{f} : X \rightarrow \mathbb{P}^1$ is a G -cover with group \tilde{E} and inertia canonical invariant $\tilde{\mathbf{C}}$ then $\mu^*(\tilde{\mathbf{C}}) := \mu(\tilde{\mathbf{C}})^*$ is the inertia canonical invariant of the G -cover $X/G \rightarrow \mathbb{P}^1$ with group E .

Similarly, any group monomorphism $i : G \hookrightarrow \tilde{E}$ defines a morphism of monoids $\nu : \mathbf{R}_+(\tilde{E}) \rightarrow \mathbf{R}_+(G)$ as follows. Consider the canonical map $C_G : G \rightarrow \mathcal{C}(G)$,

sending $g \in G$ to its conjugacy class $C_G(g)$ in G and let $s : \mathcal{C}(G) \hookrightarrow G, C \rightarrow s(C)$ a section of it. Then ν sends $\tilde{C} \in \mathcal{C}(\tilde{E})$ to

$$\sum_{\tilde{e} \in G \backslash \tilde{E} / \langle s(\tilde{C}) \rangle} C_G(\tilde{e}s(\tilde{C})^{\frac{o(\tilde{C})}{|G \cap \tilde{e} \langle s(\tilde{C}) \rangle \tilde{e}^{-1}|}} \tilde{e}^{-1}) \in R_+(G).$$

Geometrically, if $\tilde{f} : X \rightarrow \mathbb{P}^1$ is a G -cover with group \tilde{E} and inertia canonical invariant $\tilde{\mathbf{C}}$ then $\nu^*(\tilde{\mathbf{C}}) := \nu(\tilde{\mathbf{C}})^*$ is the inertia canonical invariant of the G -cover $X \rightarrow X/G$ with group G . Note that for an extension of finite groups $1 \rightarrow G \xrightarrow{i} \tilde{E} \xrightarrow{p} E \rightarrow 1$ where E acts trivially on G and for any $\tilde{C} \in \mathcal{C}(\tilde{E})$, the set $\{g^{o(p(\tilde{C}))} \mid g \in \tilde{C}\}$ generates exactly one conjugacy class in G ; we denote it by $\tilde{C}^{o(p(\tilde{C}))}$. Then $\nu(\tilde{C})$ is simply $\frac{|E|}{o(p(\tilde{C}))} \tilde{C}^{o(p(\tilde{C}))} \in R_+(G)$.

4.2.2. *Second proof of theorem 1.2.* From now on, let $1 \rightarrow G \xrightarrow{i} \tilde{E} \xrightarrow{p} E \rightarrow 1$ be an extension of finite groups where E acts trivially on G .

Lemma 4.1. *For each $\mathbf{C} \in R_+^*(G)$ the set of all $\tilde{\mathbf{C}} \in (\nu^*)^{-1}(\mathbf{C}) \cap R_+^*(\tilde{E})$ such that $g(\mu^*(\tilde{\mathbf{C}})) = 0$ is finite.*

Proof. Fix $\mathbf{C} \in R_+^*(G)$ and $\tilde{\mathbf{C}} \in (\nu^*)^{-1}(\mathbf{C}) \cap R_+^*(\tilde{E})$ such that $g(\mu^*(\tilde{\mathbf{C}})) = 0$. Then we have

$$\deg(\tilde{\mathbf{C}}) \leq \frac{\deg(\mathbf{C})}{|E|} + \deg(\mu^*(\tilde{\mathbf{C}})) \leq \frac{\deg(\mathbf{C})}{|E|} + 3,$$

where the last inequality is implied by $g(\mu^*(\tilde{\mathbf{C}})) = 0$. \square

Now, fix an integer $r \geq 3$, a finite nontrivial subgroup E of $\text{PGL}_2(\overline{\mathbb{Q}})$ and a degree r element $\mathbf{C} \in R_+^*(G)$. Define

$$H_{G/\text{PGL}_2}^0(\mathbf{C}, E) := \coprod_{(\tilde{E}, p) \in \mathcal{E}(E, G) / \simeq} \coprod_{\substack{\tilde{\mathbf{C}} \in (\nu^*)^{-1}(\mathbf{C}) \cap R_+^*(\tilde{E}) \\ g(\mu^*(\tilde{\mathbf{C}})) = 0}} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}})$$

Note that, by point (3) above, the defining conditions $r \geq 3$, $\tilde{\mathbf{C}} \in (\nu^*)^{-1}(\mathbf{C}) \cap R_+^*(\tilde{E})$ and $g(\mu^*(\tilde{\mathbf{C}})) = 0$ force $g(\mathbf{C}) = g(\tilde{\mathbf{C}}) \geq 1$ (hence, in particular, $\deg(\tilde{\mathbf{C}}) \geq 3$). Accordingly, $H_{G/\text{PGL}_2}^0(\mathbf{C}, E) = \emptyset$ if $g(\mathbf{C}) = 0$.

By functoriality of reduced Hurwitz spaces [1, §6.2.2], we have - corresponding to ν^* - a canonical morphism

$$\nu : H_{G/\text{PGL}_2}^0(\mathbf{C}, E) \rightarrow H_{G/\text{PGL}_2}(\mathbf{C})$$

Geometrically, ν corresponds to sending the G/PGL_2 -isomorphism class of a G -cover $\tilde{f} : X \rightarrow \mathbb{P}^1$ with group \tilde{E} to the G/PGL_2 -isomorphism class of the G -cover $f : X \rightarrow X/G$ with group G . The defining condition $g(\mu^*(\tilde{\mathbf{C}})) = 0$ in $H_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E)$ ensures that X/G is isomorphic to \mathbb{P}^1 .

Remark 4.2. There is no canonical choice for the isomorphism $X/G \simeq \mathbb{P}^1$ but the G/PGL_2 -isomorphism class of f is well and canonically defined from \tilde{f} . In other words, we have a diagram

$$\begin{array}{ccc} H_G^0(\mathbf{C}, E) & \xrightarrow{\nu^0} & H_G(\mathbf{C}) \\ \downarrow & & \downarrow \pi \\ H_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E) & \xrightarrow{\nu} & H_{G/\mathrm{PGL}_2}(\mathbf{C}) \end{array}$$

where $H_G^0(\mathbf{C}, E)$ is defined by replacing $H_{G/\mathrm{PGL}_2}(\tilde{\mathbf{C}})$ with $H_G(\tilde{\mathbf{C}})$ in the definition of $H_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E)$. However, the lifting ν^0 of ν does not exist, a priori.

As a result, we obtain a diagram of morphisms

$$H_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E) \xrightarrow{\nu} H_{G/\mathrm{PGL}_2}(\mathbf{C}) \xleftarrow{\pi} H_G(\mathbf{C})$$

Furthermore, we have

Lemma 4.3. *The morphism $\nu : H_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E) \rightarrow H_{G/\mathrm{PGL}_2}(\mathbf{C})$ is proper.*

Proof. Given a finite group G , an integer $r \geq 3$ and a degree r element $\mathbf{C} \in \mathbb{R}_+^*(G)$, set $g = g(\mathbf{C})$, $d = |G|$ and $a = d(-2 + r) - (2g - 2)$. Recall from section 2.2 that $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ is the Hurwitz scheme $H_{g,G,\mathbf{C}}$ of [1].

Denote by $\phi : H_{g,G,\mathbf{C}} \rightarrow M_{g,[a]}$ the morphism corresponding to the stack morphism Φ sending a G -cover $f : X \rightarrow Y$ to the curve X with the reduced ramification divisor. (Observe that the degree of this divisor is a .) Then, since ϕ is separated (as both $H_{g,G,\mathbf{C}}$ and $M_{g,[a]}$ are separated), it is enough to prove that $\psi = \phi \circ \nu : H_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E) \rightarrow M_{g,[a]}$ is proper. This, in turn, amounts to showing the valuative criterion of properness [8, Rem. 7.5.2] for the corresponding stack morphism $\Psi : \mathcal{H}_{G/\mathrm{PGL}_2}^0(\mathbf{C}, E) \rightarrow \mathcal{M}_{g,[a]}$.⁶ So let R be a discrete valuation ring

⁶Indeed, this follows from the fact in general algebraic stack theory [5] that the natural morphism from a Deligne-Mumford stack to its coarse space is dominant and proper, together with the fact that ψ is separated. (See [7, II, Ex. 4.4].)

with field of fractions K and consider any commutative diagram

$$\begin{array}{ccccc}
 X_R & \longleftarrow & X_K & & \\
 \downarrow & & \downarrow & \searrow \tilde{E} & \\
 R & \longleftarrow & K & \longleftarrow & Y_K
 \end{array}$$

in which X_R is a proper smooth curve of genus g over R and is equipped with a relatively étale divisor D_R of degree a , such that D_K is \tilde{E} -stable. Then we have to show that the action of \tilde{E} on (X_K, D_K) extends uniquely to (X_R, D_R) and that the resulting monomorphism $\alpha : G \hookrightarrow \text{Aut}(X_R)$ defines a G -cover $(X_R \rightarrow X_R/G, \alpha)$. But the former follows from the uniqueness of the stable model (X_R, D_R) of the marked curve (X_K, D_K) (cf. [1, Prop. 5.2]) and the latter is a consequence of [1, §3.1]. (Precisely speaking, [1, §3.1] put the assumption $g \geq 2$, but this is superfluous. See also [15, Lem. 5.5]) \square

Now, fix a finite group G , an integer $r \geq 3$ and a degree r element $\mathbf{C} \in \mathbb{R}_+^*(G)$. Then, for any nontrivial conjugacy class \mathcal{E} and any representative E of \mathcal{E} , by construction, we have $H_G(\mathbf{C}, \supset \mathcal{E}) = \varphi^{-1}(\nu(H_{G/\text{PGL}_2}^0(\mathbf{C}, E)))$, which is a closed subset of $H_G(\mathbf{C})$ since ν is proper. This yields theorem 1.2 as announced.

4.2.3. *About the morphism ν .* By lemma 4.3, the morphism $\nu : H_{G/\text{PGL}_2}^0(\mathbf{C}, E) \rightarrow H_{G/\text{PGL}_2}(\mathbf{C})$ is proper. But as $H_{G/\text{PGL}_2}^0(\mathbf{C}, E)$ and $H_{G/\text{PGL}_2}(\mathbf{C})$ are both affine schemes ($H_{G/\text{PGL}_2}(\mathbf{C})$ is a finite cover of the affine scheme $M_{0,[r]}$), it results from [9, Lem. 3.3.17] that ν is finite. By [9, Ex. 5.1.25], to compute its degree, it is enough to compute the cardinality of its geometric fibers.

The aim of this section is to describe the geometrically irreducible components of $H_{G/\text{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$ for finite groups G with trivial center, where \mathcal{E} is the conjugacy class of $E = C_n$ ($n \geq 6$), D_{2n} ($2|n, n \geq 6$), $\mathcal{S}_4, \mathcal{A}_5$. It will appear in the course of the proof (lemma 4.7) why our method does not seem to work for other groups E .

More precisely, since G has trivial center, it follows from remark (1) of section 4.1 that the only possible extensions $1 \rightarrow G \rightarrow \tilde{E} \rightarrow E \rightarrow 1$ are the trivial split extensions $\tilde{E} = G \times E$. Let us write $p_G : G \times E \twoheadrightarrow G$ and $p_E : G \times E \twoheadrightarrow E$ for the canonical projections, $\mu_G^* : \mathbb{R}_+(G \times E) \rightarrow \mathbb{R}_+^*(G)$ and $\mu_E^* : \mathbb{R}_+(G \times E) \rightarrow \mathbb{R}_+^*(E)$ for the corresponding morphisms of monoids defined in section 4.2.1. Similarly,

write $i_G : G \hookrightarrow G \times E$ and $i_E : E \hookrightarrow G \times E$ for the canonical embeddings, $\nu_G^* : \mathbb{R}_+(G \times E) \rightarrow \mathbb{R}_+^*(G)$ and $\nu_E^* : \mathbb{R}_+(G \times E) \rightarrow \mathbb{R}_+^*(E)$ for the corresponding morphisms of monoids.

To proceed, set

$$\text{Aut}^G(G \times E) := \{\tilde{\beta} \in \text{Aut}(G \times E) \mid \tilde{\beta} \circ i_G = i_G\}.$$

Then we have two natural homomorphisms

$$a : \text{Aut}(E) \rightarrow \text{Aut}^G(G \times E), \beta \mapsto \text{Id}_G \times \beta$$

and

$$b : \text{Aut}^G(G \times E) \rightarrow \text{Aut}(E), \tilde{\beta} \mapsto \bar{\beta},$$

where $\bar{\beta}$ denotes the automorphism that $\tilde{\beta}$ induces on the quotient $E = (G \times E)/(G \times \{1\})$, or, equivalently, $\bar{\beta}$ is characterized by: $\bar{\beta} \circ p_E = p_E \circ \tilde{\beta}$.

Lemma 4.4. (Assuming that G has trivial center.) The homomorphisms a and b are inverse to each other. In particular, $\text{Aut}(E) \simeq \text{Aut}^G(G \times E)$.

Proof. It is trivial that $b \circ a = \text{Id}$. To see that $a \circ b = \text{Id}$, take any $\tilde{\beta} \in \text{Aut}^G(G \times E)$ and compare $\tilde{\beta}$ and $(b \circ a)(\tilde{\beta}) = \text{Id}_G \times \bar{\beta}$. Since G has trivial center, the subgroup $\{1\} \times E$ coincides with the centralizer of $G \times \{1\}$, hence is preserved by $\tilde{\beta}$. Now, since the subgroup $\{1\} \times E$ is isomorphically mapped onto the quotient $E = (G \times E)/(G \times \{1\})$, $\tilde{\beta}$ must induce $\bar{\beta}$ on the subgroup $\{1\} \times E$. Thus, $\tilde{\beta}$ and $\text{Id}_G \times \bar{\beta}$ coincide with each other both on $G \times \{1\}$ and on $\{1\} \times E$. This completes the proof, since $G \times E$ is generated by $G \times \{1\}$ and $\{1\} \times E$. \square

Note that a and b induce an isomorphism $\text{Out}(E) \simeq \text{Out}^G(G \times E)$, where $\text{Out}(E) := \text{Aut}(E)/\text{Inn}(E)$ and $\text{Out}^G(G \times E) := \text{Aut}^G(G \times E)/\text{Inn}(\{1\} \times E)$.

Next, let $\mathcal{X}(\mathbf{C}, E)$ denote the set of all $\tilde{\mathbf{C}} \in (\nu_G^*)^{-1}(\mathbf{C}) \cap \mathbb{R}_+^*(G \times E)$ such that $g(\mu_E^*(\tilde{\mathbf{C}})) = 0$. Then $\text{Aut}^G(G \times E)$ acts naturally on $\mathcal{X}(\mathbf{C}, E) \subset \mathbb{R}_+^*(G \times E)$ (by pulling-back). Moreover, compatibly with this action, $\text{Aut}^G(G \times E)$ also acts naturally on $\coprod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}})$, by sending, for each $\tilde{\beta} \in \text{Aut}^G(G \times E)$, a

G -curve $(\tilde{X}, j : G \times E \hookrightarrow \text{Aut}(\tilde{X}))$ to $(\tilde{X}, j \circ \tilde{\beta} : G \times E \hookrightarrow \text{Aut}(\tilde{X}))$.

Lemma 4.5. The actions of $\text{Aut}^G(G \times E)$ on $\mathcal{X}(\mathbf{C}, E)$ and on $\coprod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}})$

factor through $\text{Aut}^G(G \times E) \twoheadrightarrow \text{Out}^G(G \times E)$.

Proof. As $\text{Inn}(G \times E)$ acts trivially on $\mathbb{R}_+^*(G \times E)$, $\text{Inn}(\{1\} \times E)$ acts trivially on $\mathcal{X}(\mathbf{C}, E)$. Next, let $(\tilde{X}, j : G \times E \hookrightarrow \text{Aut}(\tilde{X}))$ be a G -curve corresponding to a point of $\prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}})$, and $\tilde{\beta} = \text{Inn}((1, e)) \in \text{Inn}(\{1\} \times E)$, where $e \in E$. Then, $j((1, e))$ tautologically gives a G -curve isomorphism from (\tilde{X}, j) to $(\tilde{X}, j \circ \tilde{\beta})$. Thus, the action of $\tilde{\beta}$ on $\prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}})$ is trivial, as desired. \square

Denote by \sim the equivalence relation on $\mathcal{X}(\mathbf{C}, E)$ induced by the action of $\text{Out}^G(G \times E)$. Also, for each $\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)$, denote by $\text{Out}^G(G \times E)_{\tilde{\mathbf{C}}}$ the stabilizer subgroup of $\text{Out}^G(G \times E)$ at $\tilde{\mathbf{C}}$.

Proposition 4.6. *Let \mathcal{E} be the conjugacy class in $\text{PGL}_2(\bar{k})$ of one of the following groups: C_n , $n \geq 6$; D_{2n} , $2|n$, $n \geq 6$; S_4 ; A_5 (cf. lemma 1.1), and E a representative of \mathcal{E} . Let G be a finite group with trivial center. Then the finite morphism $\nu_G : \prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) \rightarrow H_{G/\text{PGL}_2}(\mathbf{C})$ induces a finite, radicial (i.e., universally injective) morphism*

$$\left(\prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) \right) / \text{Out}^G(G \times E) \rightarrow H_{G/\text{PGL}_2}(\mathbf{C})$$

with image $H_{G/\text{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$. Moreover, the left-hand side is naturally identified with $\prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E) / \sim} (H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) / \text{Out}^G(G \times E)_{\tilde{\mathbf{C}}})$. In particular, the geomet-

rically irreducible components of $H_{G/\text{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$ can be identified with those of $\prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E) / \sim} (H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) / \text{Out}^G(G \times E)_{\tilde{\mathbf{C}}})$.

For any $\tilde{\mathbf{C}}$ with $H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) \neq \emptyset$, the group $\text{Out}^G(G \times E)_{\tilde{\mathbf{C}}}$ is trivial for $\mathcal{E} \sim S_4, A_5$ and of order at most 2 (depending on $\tilde{\mathbf{C}}$) for $\mathcal{E} \sim C_n$ ($n \geq 6$), D_{2n} ($2|n$, $n \geq 6$). In particular, if $\mathcal{E} \sim S_4, A_5$, then the geometrically irreducible components of $H_{G/\text{PGL}_2}(\mathbf{C}, \supset \mathcal{E})$ can be identified with those of $\prod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E) / \sim} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}})$ ⁷.

Proof. Let us start from $\tilde{\mathbf{C}}_i \in \mathcal{X}(\mathbf{C}, E)$, $i = 1, 2$. For each $i = 1, 2$, let $(\tilde{X}_i, j_i : G \times E \hookrightarrow \text{Aut}(\tilde{X}_i))$ be a G -curve with group $G \times E$ corresponding to an element

⁷which can be explicitly computed by algorithms as in [10].

of $H_{G/\mathrm{PGL}_2}(\tilde{\mathbf{C}}_i)(\bar{k})$ and define a G-curve (X_i, i_i) with group E to be $\mu_E((\tilde{X}_i, j_i))$, or, more precisely, $X_i = \tilde{X}_i/(j_i \circ i_G)(G)$ and $i_i : E \hookrightarrow \mathrm{Aut}(X_i)$ is defined by $f_i \circ j_i(\tilde{e}) = (i_i \circ p_E)(\tilde{e}) \circ f_i$, $\tilde{e} \in G \times E$, where $f_i : \tilde{X}_i \rightarrow X_i$ denotes the G-cover corresponding to the G-curve $(\tilde{X}_i, j_i \circ i_G) = \nu_G((\tilde{X}_i, j_i))$ with group G . We have to show that, for E as in proposition 4.6, $(\tilde{X}_1, j_1 \circ i_G) \simeq (\tilde{X}_2, j_2 \circ i_G)$ implies $(X_1, j_1 \circ \tilde{\beta}) \simeq (X_2, j_2)$ for some $\tilde{\beta} \in \mathrm{Aut}^G(G \times E)$. So, fix a G-curve isomorphism $\tilde{v} : (\tilde{X}_1, j_1 \circ i_G) \xrightarrow{\sim} (\tilde{X}_2, j_2 \circ i_G)$, or, equivalently, a weak G-isomorphism:

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{v}} & \tilde{X}_2 \\ f_1 \downarrow & & \downarrow f_2 \\ X_1 & \xrightarrow{v} & X_2 \end{array}$$

Eventually, write $\Pi_i : X_i \rightarrow B_i$ for the G-cover corresponding to the G-curve (X_i, i_i) with group E . The following commutative diagram sums up the notation we will use:

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{v}} & \tilde{X}_2 \\ f_1 \downarrow & & \downarrow f_2 \\ \tilde{f}_1 \curvearrowright X_1 & \xrightarrow{v} & X_2 \curvearrowleft \tilde{f}_2 \\ \Pi_1 \downarrow & & \downarrow \Pi_2 \\ B_1 & & B_2 \end{array}$$

Note that $X_i \simeq B_i \simeq \mathbb{P}_k^1$, $i = 1, 2$.

Before going on, we need to introduce one more notion. We say that a finite subgroup E of $\mathrm{PGL}_2(\bar{k})$ has property $(*)$ if there is no finite subgroup E' of $\mathrm{PGL}_2(\bar{k})$ which contains two distinct subgroups isomorphic to E (or, equivalently, two distinct conjugates of E). We leave it as an easy exercise to the reader to check the following:

Lemma 4.7. *Let E be a nontrivial finite subgroup of $\mathrm{PGL}_2(\bar{k})$. Then E has property $(*)$ if and only if it is of the form C_n , $n \geq 6$, D_{2n} , $2|n$, $n \geq 6$, \mathcal{S}_4 , A_5 .*

From now on, we assume that E is a group of the form of those listed in lemma 4.7. Let $E_i \subset \mathrm{Aut}(X_i)$ denote the base group of f_i , $i = 1, 2$. Then, on the one hand, $i_i(E) \subset E_i$, $i = 1, 2$, and, on the other hand, $vE_1v^{-1} = E_2$. Hence, according to property $(*)$, we get $vi_1(E)v^{-1} = i_2(E)$. It follows from this

last equality that the isomorphism $v : X_1 \rightarrow X_2$ induces a unique isomorphism $b : B_1 \rightarrow B_2$ that makes the following diagram commutative:

$$\begin{array}{ccc} X_1 & \xrightarrow{v} & X_2 \\ \Pi_1 \downarrow & & \downarrow \Pi_2 \\ B_1 & \xrightarrow{b} & B_2 \end{array}$$

Now, the whole diagram becomes

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{v}} & \tilde{X}_2 \\ \downarrow f_1 & & \downarrow f_2 \\ X_1 & \xrightarrow{v} & X_2 \\ \downarrow \Pi_1 & & \downarrow \Pi_2 \\ B_1 & \xrightarrow{b} & B_2 \end{array} \begin{array}{l} \tilde{f}_1 \curvearrowright \\ \tilde{f}_2 \curvearrowleft \end{array}$$

Here, note that (\tilde{v}, b) does not necessarily give a weak G-isomorphism between $\tilde{f}_1 : \tilde{X}_1 \rightarrow B_1$ and $\tilde{f}_2 : \tilde{X}_2 \rightarrow B_2$, as (v, b) does not necessarily give a weak G-isomorphism between $\Pi_1 : X_1 \rightarrow B_1$ and $\Pi_2 : X_2 \rightarrow B_2$. To adjust this, note that we have two isomorphisms $j_2, j'_2 : G \times E \rightarrow \text{Aut}(\tilde{X}_2/B_2)$, where $j'_2(\tilde{e}) := \tilde{v}j_1(\tilde{e})\tilde{v}^{-1}$, $\tilde{e} \in G \times E$. So, set $\tilde{\beta} := (j'_2)^{-1} \circ j_2 \in \text{Aut}(G \times E)$. The fact that (\tilde{v}, v) is a weak G-isomorphism implies that $\tilde{\beta} \in \text{Aut}^G(G \times E)$. Now, the G-curve $(\tilde{X}_1, j_1 \circ \tilde{\beta})$ is isomorphic (via \tilde{v}) to $(\tilde{X}_2, j'_2 \circ \tilde{\beta}) = (\tilde{X}_2, j_2)$. This completes the proof of the first assertion.

Next, the identification

$$\left(\coprod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E)} H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) \right) / \text{Out}^G(G \times E) = \coprod_{\tilde{\mathbf{C}} \in \mathcal{X}(\mathbf{C}, E) / \sim} (H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) / \text{Out}^G(G \times E)_{\tilde{\mathbf{C}}})$$

just follows from elementary theory of group action.

Eventually, we shall analyze the group $\text{Out}^G(G \times E)_{\tilde{\mathbf{C}}}$, under the assumption that $H_{G/\text{PGL}_2}(\tilde{\mathbf{C}}) \neq \emptyset$. Set $\mathbf{\Gamma} := \mu_E^*(\tilde{\mathbf{C}})$, which must be realized as the inertia canonical invariant of a G-cover $\Pi : X \rightarrow B$ with group E and with $X \simeq B \simeq \mathbb{P}^1/k$. Now, the Riemann-Hurwitz formula gives the possible ramification indices of Π and the possible values for the inertia canonical invariant $\mathbf{\Gamma}$ of Π . With the

notation of [4], we have:

(3)

E	Ramification indices	Inertia canonical invariant
\mathcal{A}_5	(2, 3, 5)	(2A, 3A, 5A) or (2A, 3A, 5B)
\mathcal{S}_4	(2, 3, 4)	(2A, 3A, 4A)
\mathcal{A}_4	(2, 3, 3)	(2A, 3A, 3B)
D_4	(2, 2, 2)	(2A, 2B, 2C)
$D_{2n}, n \geq 3$	(2, 2, n)	(2A, 2A, $(nA)^k$), $(k, n) = 1$ if $2 \nmid n$ (2A, 2B, $(nA)^k$), $(k, n) = 1$ if $2 \mid n$
$C_n, n \geq 2$	(n, n)	$((nA)^k, (nA)^{n-k})$, $(k, n) = 1$

(For the sake of completeness, we have included all the possible groups in lemma 1.1, not only the ones appearing in proposition 4.6.)

A straightforward computation [13, Chap. 7] shows that the tuples in table 3 are all “rigid”, that is, given such E , $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_s)$ ($s := \text{deg}(\mathbf{\Gamma})$), and a ramification divisor $\mathbf{t} = t_1 + \dots + t_s \subset B (\simeq \mathbb{P}^1_k)$ (ordered) of degree s , there is a unique G -isomorphism class of G -cover $X \rightarrow B$ with group E , ramification divisor \mathbf{t} and inertia canonical invariant $\mathbf{\Gamma}$ (with Γ_i corresponding to t_i , $i = 1, \dots, s$). In particular, if $\beta \in \text{Aut}(E)$ satisfies $\beta^*(\Gamma_i) = \Gamma_i$, $i = 1, \dots, s$, then β is an inner automorphism.

Now, take any $\tilde{\beta}$ such that $\tilde{\beta}^*(\tilde{\mathbf{C}}) = \tilde{\mathbf{C}}$ and set $\beta := \overline{\tilde{\beta}} \in \text{Aut}(E)$. Then we have $\beta^*(\mathbf{\Gamma}) = \mathbf{\Gamma}$. This, together with the above rigidity, implies that the natural homomorphism from $\text{Out}^G(G \times E)_{\tilde{\mathbf{C}}}$ to the symmetric group on the conjugacy classes $\{\Gamma_1, \dots, \Gamma_s\}$ is injective. Further, by observing $o(\beta^*(\Gamma_i)) = o(\Gamma_i)$, $i = 1, \dots, s$, we see that the image of this last injective homomorphism is trivial for $E \sim \mathcal{S}_4, \mathcal{A}_5$ and is contained in a group of order 2 for $E \sim C_n$ ($n \geq 6$), D_{2n} ($2 \mid n, n \geq 6$). This completes the proof. \square

5. APPLICATION TO THE REGULAR INVERSE GALOIS PROBLEM OVER \mathbb{Q}

We retain the notation of section 4.2. The regular inverse Galois problem over \mathbb{Q} amounts to finding \mathbb{Q} -rational points on non-reduced Hurwitz spaces [6]. One way to do this is to search for \mathbb{Q} -rational points on reduced Hurwitz spaces and, then, try and lift these \mathbb{Q} -rational points from reduced to non-reduced Hurwitz spaces. One result of [3] is the following. Let \mathcal{E} be the conjugacy class of a

nontrivial finite subgroup of $\mathrm{PGL}_2(\overline{\mathbb{Q}})$ and set $d(\mathcal{E}) = 1$ if $\mathcal{E} \sim \mathcal{A}_5, \mathcal{S}_4$ or D_{2n} , $n \geq 3$, $d(\mathcal{E}) = 2$ if $\mathcal{E} \sim \mathcal{A}_4$ or C_n , $n \geq 2$ and $d(\mathcal{E}) = 6$ if $\mathcal{E} \sim D_4$.

Proposition 5.1. *Any closed point $p \in H_{G/\mathrm{PGL}_2}(\mathbf{C}, \mathcal{E})$ admits a lift $\mathbf{p} \in H_G(\mathbf{C}, \mathcal{E})$ such that $[\kappa(\mathbf{p}) : \kappa(p)] \leq d(\mathcal{E})$.*

In other words, the most interesting strata on reduced Hurwitz spaces are those corresponding to $\mathcal{E} \sim \mathcal{A}_5, \mathcal{S}_4$ or D_{2n} , $n \geq 3$. But the main problem remains: finding \mathbb{Q} -rational points on these strata; except for $r = 3, 4$ [3, §5], we currently have no method to do this.

The idea of the method presented here is to reverse this process, that is, starting from a reduced Hurwitz space $H_{G/\mathrm{PGL}_2}(\mathbf{C})$ with a \mathbb{Q} -rational point (which does not lift, *a priori*, to a \mathbb{Q} -rational point on $H_G(\mathbf{C})$), send this \mathbb{Q} -rational point to a \mathbb{Q} -rational point on a reduced Hurwitz space in a stratum $H_{G/\mathrm{PGL}_2}(\mathbf{C}(\mathcal{E}), \mathcal{E})$ (where $\mathbf{C}(\mathcal{E})$ is an inertia canonical invariant to be explicitly defined), with $\mathcal{E} \sim \mathcal{A}_5, \mathcal{S}_4, D_{2n}$, $n \geq 3$.

Now, let $\mathbf{\Gamma} \in \mathbb{R}_+^*(E)$ be one of the tuples in table 3 and choose $\tilde{\mathbf{C}} \in \mu_G^{-1}(\mathbf{C}) \cap (\mu_E^*)^{-1}(\mathbf{\Gamma}) \cap \mathbb{R}_+^*(G \times E)$.⁸ Then the rigidity of $\mathbf{\Gamma}$ forces a certain relative rigidity of $\tilde{\mathbf{C}}$ over \mathbf{C} . In terms of Hurwitz spaces, this latter rigidity shows that the natural morphism $\mu_G : H_{G/\mathrm{PGL}_2}(\tilde{\mathbf{C}}) \rightarrow H_{G/\mathrm{PGL}_2}(\mathbf{C})$ (defined over $\mathbb{Q}_{\tilde{\mathbf{C}}}$) is radicial. We thus obtain the following diagram defined over $\mathbb{Q}_{\tilde{\mathbf{C}}}$:

$$\begin{array}{ccc} H_{G/\mathrm{PGL}_2}(\mathbf{C}) & \xleftarrow{\mu_G} & H_{G/\mathrm{PGL}_2}(\tilde{\mathbf{C}}) \\ & & \downarrow \nu \\ & & H_{G/\mathrm{PGL}_2}(\nu_G^*(\tilde{\mathbf{C}}), \supset \mathcal{E}) \xleftarrow{\varphi} H_G(\nu_G^*(\tilde{\mathbf{C}}), \supset \mathcal{E}) \end{array}$$

For $\mathcal{E} \sim \mathcal{A}_5$ or \mathcal{S}_4 , one has $H_G(\nu_G^*(\tilde{\mathbf{C}}), \supset \mathcal{E}) = H_G(\nu_G^*(\tilde{\mathbf{C}}), \mathcal{E})$, and, for $\mathcal{E} \sim D_{2n}$, $n \geq 3$, one can write $H_G(\nu_G^*(\tilde{\mathbf{C}}), \supset \mathcal{E})$ as a disjoint union of strata $H_G(\nu_G^*(\tilde{\mathbf{C}}), \mathcal{E})$ with $\mathcal{E} \sim \mathcal{S}_4$ (if $n = 3, 4$) or D_{2m} , $n|m$. Thus, we get:

Corollary 5.2. *Assume that $\mathcal{E} \sim \mathcal{A}_5, \mathcal{S}_4$ or D_{2n} , $n \geq 3$ and that $\mu_G : H_{G/\mathrm{PGL}_2}(\tilde{\mathbf{C}}) \rightarrow H_{G/\mathrm{PGL}_2}(\mathbf{C})$ is surjective (hence universally bijective). If $H_{G/\mathrm{PGL}_2}(\mathbf{C})(\mathbb{Q}_{\mathbf{C}}) \neq \emptyset$, then $H_G(\nu_G^*(\tilde{\mathbf{C}}))(\mathbb{Q}_{\tilde{\mathbf{C}}}) \neq \emptyset$.*

⁸Any orderings $\mathbf{C} = (C_1, \dots, C_r)$, $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_s)$ (with $r \geq 3$, $s = 2, 3$) define such an element by setting $\tilde{\mathbf{C}} = \sum_{1 \leq i \leq r} C_i \times \Gamma_i \in \mathbb{R}_+^*(G \times E)$, and conversely. (Here, we set $\Gamma_i = \{1\}$ for $s < i \leq r$.) In particular, the degree of $\tilde{\mathbf{C}}$ is r .

In the table below, we give an example of tuples $\nu_G^*(\tilde{\mathbf{C}})$ for the orderings $\mathbf{C} = (C_1, \dots, C_r)$, $\mathbf{\Gamma} = (2A, 3A, 5A)$ or $(2A, 3A, 5B)$, $\mathbf{\Gamma} = (2A, 3A, 4A)$ and $\mathbf{\Gamma} = (2A, 2A, (nA)^k)$ ($2 \nmid n, (k, n) = 1$) / $(2A, 2B, (nA)^k)$ ($2 \mid n, (k, n) = 1$) if $E = \mathcal{A}_5, \mathcal{S}_4$ and $D_{2n}, n \geq 3$ respectively.

(4)

E	inertia canonical invariant $\nu_G^*(\tilde{\mathbf{C}})$
\mathcal{A}_5	$30C_1^2 + 20C_2^3 + 12C_3^5 + 60(C_4 + \dots + C_r)$
\mathcal{S}_4	$12C_1^2 + 8C_2^3 + 6C_3^4 + 24(C_4 + \dots + C_r)$
$D_{2n}, n \geq 3$	$n(C_1^2 + C_2^2) + 2C_3^n + 2n(C_4 + \dots + C_r)$

Note that $\mathbb{Q}_{\mathbf{\Gamma}} = \mathbb{Q}(\sqrt{5}), \mathbb{Q}, \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ for $\mathcal{E} \sim \mathcal{A}_5, \mathcal{S}_4$ or $D_{2n}, n \geq 3$ respectively, hence the cases $\mathcal{E} \sim \mathcal{S}_4, D_{2n}, n = 3, 4, 6$, are only worth considering when searching for \mathbb{Q} -rational points.

Remark 5.3. Checking the surjectivity of μ_G amounts to the following purely group-theoretic problem. Given a finite group G and an r -tuple $\mathbf{C} = (C_1, \dots, C_r)$ of nontrivial conjugacy classes in \mathbf{C} , let $\text{ni}_G(\mathbf{C})$ denote the corresponding *Nielsen class*, that is, the set of all r -tuples $\mathbf{g} = (g_1, \dots, g_r) \in G^r$ modulo diagonal componentwise inner conjugation by elements of G such that (i) g_1, \dots, g_r generate G , (ii) $g_i \in C_{\sigma(i)}, i = 1, \dots, r$ for some permutation $\sigma \in \mathcal{S}_r$ and (iii) $g_1 \cdots g_r = 1$. Write $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_s)$ and $\tilde{\mathbf{C}} = (C_1 \times \Gamma_1, \dots, C_r \times \Gamma_r)$ (where we set $\Gamma_i = \{1\}$ for $s < i \leq r$). Then μ_G is surjective if and only if any element $\mathbf{g} = (g_1, \dots, g_r) \in \text{ni}_G(\mathbf{C})$ lifts to some element $\tilde{\mathbf{e}} \in \text{ni}_{\tilde{E}}(\tilde{\mathbf{C}})$. If such a lift exists, by rigidity, it is $\tilde{\mathbf{e}} = ((g_1, \gamma_1), (g_2, \gamma_2), (g_3, \gamma_3), (g_4, 1), \dots, (g_r, 1))$, where $(\gamma_1, \gamma_2, \gamma_3)$ denotes the unique element of $\text{ni}_E(\mathbf{\Gamma})$ with $\gamma_i \in \Gamma_i$. The only difficulty is, thus, to ensure that $\tilde{e}_1, \dots, \tilde{e}_r$ generate \tilde{E} . This does not hold in general. (Take for instance $G = E$ and $\mathbf{C} = \mathbf{\Gamma}$. Then $\tilde{e}_1, \dots, \tilde{e}_s$ only generate (a conjugate of) the diagonal subgroup of $\tilde{E} = E \times E$.) But one can give several sufficient conditions for this to hold. An easy-to-check one (see example 5.4 below) is:

Criterion. If $\text{gcd}(o(C_i), o(\Gamma_i)) = 1$ for at least $s-1$ of the s indices $i = 1, \dots, s$, then μ_G is surjective.

Indeed, assume, for instance, that $s = 3$ and that $\text{gcd}(o(C_i), o(\Gamma_i)) = 1$ for $i = 2, 3$. Then $\tilde{e}_2^{o(\Gamma_2)}, \tilde{e}_3^{o(\Gamma_3)}, \tilde{e}_4, \dots, \tilde{e}_r$ generate $G \times \{1\} \subset \tilde{E} = G \times E$, since $g_2^{o(\Gamma_2)}, g_3^{o(\Gamma_3)}, g_4, \dots, g_r$ generate G . (Recall that $g_1 = (g_2 \cdots g_r)^{-1}$.)

Example 5.4. [3, Ex.5.8] Consider $G = L_2(25)$, $\mathbf{C} = 4(3A)$ and note that $3A$ is a \mathbb{Q} -rational conjugacy class. Regarding the natural compactification $\overline{H}_{G/\mathrm{PGL}_2}(\mathbf{C})$ as a cover $\overline{H}_{G/\mathrm{PGL}_2}(\mathbf{C}) \rightarrow \mathbb{P}^1$ ramified above $0, 1, \infty$, we obtain [10] the following list for the lengths of the usual/reduced orbits and the corresponding reduced genera

$$(1200/300, 7), (936/468, 17), (304/304, 5), (120/30, 0).$$

The unique reduced compactified orbit $\overline{O}_{G/\mathrm{PGL}_2}$ of length 30 and reduced genus 0 has ramification type:

$$\begin{aligned} \text{Above } 0 & : [(3)^{10}] \\ \text{Above } 1 & : [(2)^{15}] \\ \text{Above } \infty & : [(2)^2, (3)^2, (6)^2, (8)^1] \end{aligned}$$

In particular $\overline{O}_{G/\mathrm{PGL}_2}$ carries a \mathbb{Q} -rational ramified point (the one with ramification index 8) above ∞ so O_{G/PGL_2} is birational to $\mathbb{P}_{\mathbb{Q}}^1$ over \mathbb{Q} and, hence, $H_{G/\mathrm{PGL}_2}(\mathbf{C})(\mathbb{Q}) \neq \emptyset$. However, the G -covers classified by O_{G/PGL_2} have trivial base invariant [3, §5] so proposition 5.1 cannot be applied to assert that $O_G(\mathbb{Q}) \neq \emptyset$. But since $3A$ is a \mathbb{Q} -rational conjugacy class, corollary 5.2 can be applied with $\mathcal{E} \sim \mathcal{S}_4$ yields a regular realization of $L_2(25)$ over \mathbb{Q} with inertia canonical invariant $42(3A)$.

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