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Extensions of Truncated Discrete Valuation Rings

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Dedicated to Professor Jean-Pierre Serre on the Occasion of His 80^{th} *Birthday*

Abstract: An equivalence is established between the category of at most a-ramified finite separable extensions of a complete discrete valuation field K and the category of at most a-ramified finite extensions of the "length-a truncation" $\mathcal{O}_K/\mathfrak{m}_K^a$ of the integer ring of K.

1. Introduction

Let K be a complete discrete valuation field (abbr. cdvf in the following), $\mathcal{O}_K$ its valuation ring, and $\mathfrak{m}_K$ its maximal ideal. Let a be an integer ≥ 1. In this paper, we prove that the category $\mathcal{FE}_K^{\leq a}$ of finite étale K-algebras with ramification "bounded by a" (*cf.* Def. 3.1) depends only on $\mathcal{O}_K/\mathfrak{m}_K^a$. More precisely, let m be any rational number such that $0 < m \leq a$ and put $A = \mathcal{O}_K/\mathfrak{m}_K^a$. We give an equivalence of $\mathcal{FE}_K^{\leq m}$ with a category $\mathcal{FFP}_A^{\leq m}$ of finite flat principal A-algebras[2] with ramification "bounded by m" (*cf.* Def. 3.2). The morphisms in $\mathcal{FFP}_A^{\leq m}$ are defined (*cf.* Def. 3.3) by using Hattori's functor ([6]); they are the usual A-algebra homomorphisms modulo a certain equivalence relation.

For each object L in $\mathcal{FE}_K^{\leq m}$, let $\mathcal{O}_L$ be the integral closure of $\mathcal{O}_K$ in L. Then the quotient ring $T(L) := \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L$ is an object of $\mathcal{FFP}_A^{\leq m}$ (Cor. 3.5). This correspondence $L \mapsto T(L)$ is functorial, and thus we obtain a functor

$$T:\ \mathcal{FE}_K^{\leq m}\ \to\ \mathcal{FFP}_A^{\leq m}.$$

Our main result in this paper is:

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[2]We mean by a *principal* A-algebra an A-algebra of which every ideal is generated by one element. All algebras in this paper are commutative.

Theorem 1.1. *The functor T is an equivalence of categories.*

Remarks. (i) The case of $a = 1$ in the Theorem is well-known (*cf.* [12], Chap. III, Sect. 5). Indeed, if $m \leq 1$, the objects of $\mathcal{FE}_K^{\leq m}$ are direct products of finite unramified extensions of K, and the Theorem implies that the objects of $\mathcal{FFP}_A^{\leq m}$ are étale over A. Thus our main interest is in the case $a > 1$.

(ii) Let $G_K = \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group of K, and G_K^a its ath ramification subgroup defined by Abbes and Saito ([2], [3]). The category $\mathcal{FE}_K^{\leq m}$ is, and hence $\mathcal{FFP}_A^{\leq m}$ is also, a Galois category whose fundamental group is G_K/G_K^m by the very definition of the ramification filtration (*cf.* Sect. 3). Note that $\mathcal{FE}_K^{\leq m}$ is equivalent also to the category of coverings of $\mathrm{Spec}(\mathcal{O}_K)$ with ramification bounded by $\mathfrak{m}_K^m$ ([7], Def. 2.3); in the terminology of *op. cit.*, we have $\pi_1(\mathrm{Spec}(\mathcal{O}_K), \mathfrak{m}_K^m) = G_K/G_K^m$.

A finite étale K-algebra is the direct product of a finite number of finite separable extension fields of K. Similarly, a finite flat principal A-algebra is the direct product of a finite number of local objects (*cf.* [9], Th. 1.1, Th. 1.2). Since the boundedness of ramification of direct products of K- and A-algebras may be considered componentwise, the above Theorem is equivalent with the following Corollary, in which $\mathrm{FE}_K^{\leq m}$ (resp. $\mathrm{FFP}_A^{\leq m}$) denotes the full subcategory of $\mathcal{FE}_K^{\leq m}$ (resp. $\mathcal{FFP}_A^{\leq m}$) consisting of local rings.

Corollary 1.2. *The functor T induces an equivalence* $\mathrm{FE}_K^{\leq m} \simeq \mathrm{FFP}_A^{\leq m}$.

This extends a theorem of Deligne ([4], Th. 2.8) to the imperfect residue field case, except that our construction of the category $\mathcal{FFP}_A^{\leq m}$ for $A = \mathcal{O}_K/\mathfrak{m}_K^a$ depends on the cdvf K and hence our result is somewhat weaker than the "true" generalization of Deligne's theorem[3]. We expect, however, the category $\mathcal{FFP}_A^{\leq m}$ depends only on the isomorphism class of A as a ring (such a ring as $A = \mathcal{O}_K/\mathfrak{m}_K^a$ is called a *truncated discrete valuation ring*; see Sect. 2). If this is the case, we may define *the* Galois group G_A of A to be G_K/G_K^a (or equivalently, to be the fundamental group of the Galois category $\mathcal{FFP}_A^{\leq a}$) together with the ramification subgroups $G_A^m := G_K^m/G_K^a$, where K is any cdvf such that $A \simeq \mathcal{O}_K/\mathfrak{m}_K^a$. The filtered group G_A should depend (up to inner automorphisms) only on the isomorphism class of A as a ring. It is natural to ask the converse:

[3]Note also that Deligne uses a category, instead of $\mathcal{FFP}_A^{\leq m}$, of certain triples which have a priori less information than the objects of $\mathcal{FFP}_A^{\leq m}$.

Question. If A and A' are two truncated discrete valuation rings of length a and if there is an isomorphism $\gamma : G_A \to G_{A'}$ of groups such that $\gamma(G_A^m) = G_{A'}^m$ for all $m \leq a$, then is it true that $A \simeq A'$ as a ring?

This problem is a version of the Grothendieck conjecture in anabelian geometry. It will certainly be necessary to assume that the residue fields of A and A' are either finite or of some "anabelian" nature. For the case of local fields (or, the case of "$a = \infty$" and finite residue fields), see [10] and [1].

In Section 2, we study basic properties of truncated discrete valuation rings. After recalling some basics of the ramification theory of Abbes-Saito ([2], [3]) and Hattori ([6]), we construct the category $\mathcal{FFP}_A^{\leq m}$ and prove the Theorem in Section 3.

Throughout this paper, K is a complete discrete valuation field with residual characteristic $p > 0$. We denote by $\mathcal{O}_K$ the valuation ring of K, $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, π_K a uniformizing element of K, and $\overline{K}$ a fixed separable closure of K. For any étale K-algebra L, we denote by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in L. For A-algebras B and B', we denote by $\mathrm{Hom}_A(B, B')$ the set of A-algebra homomorphisms $B \to B'$. We use the following abbreviations:
cdvf := complete discrete valuation field,
cdvr := complete discrete valuation ring,
tdvr := truncated discrete valuation ring.

It is our pleasure to dedicate this paper to Professor Jean-Pierre Serre, whose mathematical influence on us has been enormous. In particular, the Book *Corps Locaux* has ever been our main source of inspiration in ramification theory.

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2. Truncated discrete valuation rings

A *tdvr* is an Artinian local ring whose maximal ideal is generated by one element. The *length* of a tdvr A is the length of A as an A-module. It is known that a tdvr A is principal, and any ideal is of the form $\mathfrak{m}_A^i$ for some $i \geq 0$ if $\mathfrak{m}_A$ is the maximal ideal of A. Any generator π_A of $\mathfrak{m}_A$ is said to be a *uniformizer* of A. Any non-zero element x of A can be written as $x = u\pi_A^i$ with $u \in A^\times$, π_A a uniformizer of A, and $0 \leq i < \mathrm{length}(A)$ (with the convention $0^0 = 1$ if $\mathrm{length}(A) = 1$). If $\mathrm{length}(A) > 1$ (resp. $\mathrm{length}(A) = 1$), we mean by an *extension* B/A of tdvr's a local ring homomorphism $A \to B$ of tdvr's via which B is flat over A (resp. an extension B/A of fields); thus we refrain from calling a homomorphism such as $A \hookrightarrow A[t]/(t^a)$ an extension if A is a field. An extension B/A is said to be *finite* if B is finite as an A-module. If $a > 1$, an A-algebra is a finite extension of A if and only if it is finite, flat, principal and local. In general, the objects of the category $\mathrm{FFP}_A^{\leq m}$ are finite extensions of the tdvr A. The *ramification index* $e_{B/A}$ of a homomorphism $f : A \to B$ of tdvr's is defined to be the integer e such that $f(\mathfrak{m}_A)B = \mathfrak{m}_B^e$ (with the convention $e_{B/A} = 1$ if $\mathrm{length}(A) = 1$). Note that the homomorphism f is an extension of tdvr's if and only if one has the equality $\mathrm{length}(B) = e_{B/A}\mathrm{length}(A)$ (*cf.* [4], Sect. 1.4 and [8], Exer. 22.1).

Lemma 2.1. *Let B and C be extensions of A. Then any A-algebra homomorphism $f : B \to C$ is an extension.*

Proof. We have to show that $\mathrm{length}(C) = e_{C/B}\mathrm{length}(B)$. We may assume that $\mathrm{length}(A) > 1$. Let $\mathfrak{m}_A$, $\mathfrak{m}_B$ and $\mathfrak{m}_C$ be respectively the maximal ideals of A, B and C. By the definition of ramification index, we have $\mathfrak{m}_A B = \mathfrak{m}_B^{e_{B/A}}$, $\mathfrak{m}_A C = \mathfrak{m}_C^{e_{C/A}}$, and $f(\mathfrak{m}_B)C = \mathfrak{m}_C^{e_{C/B}}$. The equality $\mathfrak{m}_C^{e_{C/A}} = f(\mathfrak{m}_B^{e_{B/A}})C$ (= the ideal generated by $\mathfrak{m}_A$) implies that $e_{C/A} = e_{C/B}e_{B/A}$. Since B and C are extensions of A, we have $\mathrm{length}(C) = e_{C/A}\mathrm{length}(A) = e_{C/B}e_{B/A}\mathrm{length}(A) = e_{C/B}\mathrm{length}(B)$. □

If K is a cdvf, then $\mathcal{O}_K/\mathfrak{m}_K^a$ is a tdvr for any integer $a \geq 1$. If L/K is a finite extension of cdvf's, then $B = \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L$ is a finite extension of $A = \mathcal{O}_K/\mathfrak{m}_K^a$. Conversely, it is known that any tdvr is a quotient of a cdvr ([9], Th. 3.3). More precisely, we have:

Proposition 2.2. (i) *Let A be a tdvr with residue field k of characteristic $p \geq 0$, and let a be the length of A. Then there exists a cdvr $\mathcal{O}$ such that A is isomorphic to $\mathcal{O}/\mathfrak{m}^a$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. If $pA = 0$, then this $\mathcal{O}$ can be taken to be the power series ring $k[\![\pi]\!]$; if $pA \neq 0$, then $\mathcal{O}$ as above must be finite over a Cohen p-ring* ([5], $0_{\mathrm{IV}}, 19.8$) *with residue field k. (If $pA = 0$ and $p \neq 0$, then both types of $\mathcal{O}$ are possible.)*

(ii) *Let K be a cdvf and let $A = \mathcal{O}_K/\mathfrak{m}_K^a$ with $a \geq 1$. For any finite extension B/A of tdvr's, there exist a finite separable extension L/K and an isomorphism*

$\psi : \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L \to B$ such that the diagram

$$\begin{array}{ccc} \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L & \xrightarrow{\psi} & B \\ \uparrow & & \uparrow \\ \mathcal{O}_K/\mathfrak{m}_K^a & = & A \end{array} \tag{1}$$

is commutative, where the left vertical arrow is the one induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$.

Proof. (i) Let W be a Cohen p-ring with residue field k. The reduction map $W \to k$ lifts by the formal smoothness of W to a local ring homomorphism $W \to A$ ([5], 0_{IV}, 19.8.6).

If $pA = 0$, the map $W \to A$ factors through the residue field k, which makes A a k-algebra. Then there exists a surjective A-algebra homomorphism $k[\![\pi]\!] \to A$ which maps π to π_A, where π_A is a uniformizer of A. Hence A is isomorphic to $k[\![\pi]\!]/(\pi^a)$ (*cf.* [9], Th. 3.1).

In the general case, we can write A as a quotient of the polynomial ring $W[X]$ by sending X to π_A. Then we obtain a surjection onto A from a cdvr $\mathcal{O}$ which is finite over W by the same procedure as in the proof of (ii) below.

(ii) Since B is finite over $A = \mathcal{O}_K/\mathfrak{m}_K^a$, there exists a surjective $\mathcal{O}_K$-algebra homomorphism $\phi : R \to B$ from a polynomial ring $R = \mathcal{O}_K[X_1, ..., X_n]$ onto B. Let $\mathfrak{m} = \phi^{-1}(\mathfrak{m}_B)$ and $R_\mathfrak{m}$ the localization of R at the maximal ideal $\mathfrak{m}$. Then $R_\mathfrak{m}$ is a regular local ring of Krull dimension $n+1$ ([5], 0_{IV}, 17.3.7), and ϕ extends to a surjective $\mathcal{O}_K$-algebra homomorphism $\varphi : R_\mathfrak{m} \to B$. By abuse of notation, we denote also by $\mathfrak{m}$ the maximal ideal of $R_\mathfrak{m}$. Put $\mathfrak{n} = \mathrm{Ker}\,(\varphi)$. We identify the residue field k' of $R_\mathfrak{m}$ with that of B via φ. Since $\varphi(\mathfrak{m}^2) = \mathfrak{m}_B^2$, the map φ induces a surjective k'-linear map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ and its kernel is $(\mathfrak{n} + \mathfrak{m}^2)/\mathfrak{m}^2 \simeq \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$. Thus we have an exact sequence

$$0 \to \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2) \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2 \to 0.$$

Assume $a \geq 2$, as the case $a = 1$ can be treated similarly and more easily. Then $\dim_{k'}(\mathfrak{m}_B/\mathfrak{m}_B^2) = 1$ and $\dim_{k'}(\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)) = n$. Choose a regular system of parameters $(w, f_1, ..., f_n)$ of $R_\mathfrak{m}$ such that $\varphi(w)$ gives a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$ and $f_1, ..., f_n \in \mathfrak{n}$ give a basis of $\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$. Let $\mathfrak{p}$ be the ideal of $R_\mathfrak{m}$ generated by $f_1, ..., f_n$. Then by [5], 0_{IV}, 17.1.7, the quotient ring $\mathcal{O} = R_\mathfrak{m}/\mathfrak{p}$ is a regular local ring of dimension 1 and hence a discrete valuation ring. It contains $\mathcal{O}_K$ since φ maps π_K to a non-zero non-unit in B, and is finite over $\mathcal{O}_K$. Hence it is a cdvr.

Since $\mathfrak{n} \supset \mathfrak{p}$, the map φ factors through $\mathcal{O}$. Thus we see the diagram (1) commutes (with $\mathcal{O}$ in place of $\mathcal{O}_L$). Since B is flat over A, the induced homomorphism ψ is bijective.

To make the fraction field L of $\mathcal{O}$ separable over K, we "deform" the prime ideal $\mathfrak{p}$ if necessary. By multiplying the f_i with some $u \in R \smallsetminus \mathfrak{m}$, we may assume that all f_i are in the polynomial ring R. Note that the composite map $R \hookrightarrow R_\mathfrak{m} \to R_\mathfrak{m}/\mathfrak{p} = \mathcal{O}$ is surjective by Nakayama's lemma, since its image generates $B = \mathcal{O}/\mathfrak{m}_K^a\mathcal{O}$. Let $\mathfrak{q}$ be its kernel, so that $\mathcal{O} = R/\mathfrak{q}$. We have $\mathfrak{q}R_\mathfrak{m} = \mathfrak{p}$, *i.e.*, $\mathfrak{q}$ is generated by $f_1, ..., f_n$ locally at $\mathfrak{m}$. By the Jacobian criterion ([11], V, Sect. 2, Th. 5), the K-algebra L is separable (*i.e.*, the $\mathcal{O}_K$-algebra $\mathcal{O}$ is étale at the generic point of Spec($\mathcal{O}$)) if and only if the Jacobian $\det\left(\dfrac{\partial f_i}{\partial X_j}\right)_{1\le i,j\le n} \not\equiv 0 \pmod{\mathfrak{q}}$. Let $g_i := f_i + xX_i$ with $x \in \mathfrak{m}_K^a$. Then, since $g_i \in \mathfrak{n}$ and $g_i \equiv f_i \pmod{\mathfrak{n} \cap \mathfrak{m}^2}$, the ideal $\mathfrak{p}' = (g_1, ..., g_n)$ of $R_\mathfrak{m}$ has similar properties as $\mathfrak{p}$ so that the quotient ring $\mathcal{O}' := R_\mathfrak{m}/\mathfrak{p}'$ is a cdvr which contains $\mathcal{O}_K$ and surjects onto B. Moreover, if $J := \left(\dfrac{\partial f_i}{\partial X_j}\right)_{1\le i,j\le n,}$ we have

$$\det\left(\frac{\partial g_i}{\partial X_j}\right)_{1\le i,j\le n} = \det(xI_n + J) = x^n + \mathrm{Tr}(J)x^{n-1} + \cdots + \det(J).$$

Considering this modulo $\mathfrak{q}$ and noticing that $\mathcal{O}_K \subset \mathcal{O} = R/\mathfrak{q}$, we find an $x \in \mathfrak{m}_K^a$ such that $\det\left(\dfrac{\partial g_i}{\partial X_j}\right) \not\equiv 0 \pmod{\mathfrak{q}}$. Then the fraction field of $\mathcal{O}'$ is separable over K. □

3. Ramification

Let G_K be the absolute Galois group of K. A. Abbes and T. Saito ([2], [3]) defined a decreasing filtration $(G_K^m)_{m\ge 0}$ by closed normal subgroups G_K^m of G_K indexed with rational numbers $m \ge 0$, in such a way that $\cap_{m\ge 0} G_K^m = 1$, $G_K^0 = G_K$ and G_K^1 is the inertia subgroup of G_K. The filtration coincides with the classical upper numbering ramification filtration shifted by one if the residue field of K is perfect (see [12], Chap. IV, Sect. 3, for the classical case). It is defined by using certain functors F and F^m from the category $\mathcal{FE}_K$ of finite étale K-algebras to the category $\mathcal{S}_K$ of finite G_K-sets. We recall here the definition of F and F^m assuming for simplicity that m is a positive integer. Let L be a finite étale K-algebra, and let $\mathcal{O}_L$ be the integral closure of $\mathcal{O}_K$ in L. We define $F(L) := \mathrm{Hom}_K(L, \overline{K}) = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{\overline{K}})$. The functor F gives an anti-equivalence of $\mathcal{FE}_K$ with $\mathcal{S}_K$, thereby making $\mathcal{FE}_K$ a Galois category. To define

F^m, we proceed as follows: An *embedding* of $\mathcal{O}_L$ is a pair $(\mathbb{B}, \mathbb{B} \to \mathcal{O}_L)$ consisting of an $\mathcal{O}_K$-algebra $\mathbb{B}$ which is formally of finite type and formally smooth over $\mathcal{O}_K$ and a surjection $\mathbb{B} \to \mathcal{O}_L$ of $\mathcal{O}_K$-algebras which induces an isomorphism $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \to \mathcal{O}_L/\mathfrak{m}_L$, where $\mathfrak{m}_{\mathbb{B}}$ and $\mathfrak{m}_L$ are respectively the radicals of $\mathbb{B}$ and $\mathcal{O}_L$ (*cf.* [3], Def. 1.1). Let I be the kernel of the surjection $\mathbb{B} \to \mathcal{O}_L$. Define an affinoid algebra $\boldsymbol{B}^m$ over K by $\boldsymbol{B}^m = \mathbb{B}[I/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$, where $\wedge$ means the π_K-adic completion. Let $X^m(\mathbb{B} \to \mathcal{O}_L)$ be the affinoid variety $\mathrm{Sp}(\boldsymbol{B}^m)$ associated with $\boldsymbol{B}^m$. For any affinoid variety X over K, let $\pi_0(X_{\overline{K}})$ denote the set $\varprojlim_{K'} \pi_0(X \otimes_K K')$ of geometric connected components, where K' runs through the finite separable extensions of K. Then we define the functor F^m by

$$F^m(L) := \varprojlim_{(\mathbb{B} \to \mathcal{O}_L)} \pi_0(X^m(\mathbb{B} \to \mathcal{O}_L)_{\overline{K}}),$$

where $(\mathbb{B} \to \mathcal{O}_L)$ runs through the category of embeddings of $\mathcal{O}_L$. The projective system in the right-hand side is constant. The finite set $F(L)$ can be identified with a subset of $X^m(\mathbb{B} \to \mathcal{O}_L)(\overline{K})$, and this causes a natural surjective map $F(L) \to F^m(L)$. The mth ramification subgroup G_K^m is characterized by the property that $F(L)/G_K^m = F^m(L)$ for all L.

Definition 3.1 ([2], Def. 6.3)**.** Let L be a finite étale K-algebra. We say that the *ramification of* L *is bounded by* m if $F(L) \to F^m(L)$ is bijective.

Thus the category $\mathcal{FE}_K^{\leq m}$ of finite étale K-algebras with ramification bounded by m forms a Galois full-subcategory of $\mathcal{FE}_K$ whose fundamental group is G_K/G_K^m ([2], Prop. 2.1) as noted in the Introduction. Note that the above definition of "ramification bounded by m" coincides with Deligne's one in [4] when L is a field and $\mathcal{O}_L$ is monogenic over $\mathcal{O}_K$ (*cf.* [2], Prop. 6.7).

Let a be an integer ≥ 1, and put $A = \mathcal{O}_K/\mathfrak{m}_K^a$. For each rational number $0 < m \leq a$, Hattori ([6]) defined another functor $\mathcal{F}^m$ from the category of finite flat A-algebras to the category $\mathcal{S}_K$ of finite G_K-sets. We next recall the definition of $\mathcal{F}^m$ assuming for simplicity that m is a positive integer. Let B be a finite flat A-algebra. An *embedding* of B is a pair $(\mathbb{B}, \mathbb{B} \to B)$ consisting of an $\mathcal{O}_K$-algebra $\mathbb{B}$ which is formally of finite type and formally smooth over $\mathcal{O}_K$ and a surjection $\mathbb{B} \to B$ of $\mathcal{O}_K$-algebras which induces an isomorphism $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \to B/\mathfrak{m}_B$, where $\mathfrak{m}_{\mathbb{B}}$ and $\mathfrak{m}_B$ are respectively the radicals of $\mathbb{B}$ and B. Let $\mathcal{I}$ be the kernel of the surjection $\mathbb{B} \to B$. Define an affinoid algebra $\mathcal{B}^m$ over K by $\mathcal{B}^m = \mathbb{B}[\mathcal{I}/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$. Let $\mathcal{X}^m(\mathbb{B} \to B)$ be the affinoid variety $\mathrm{Sp}(\mathcal{B}^m)$ associated with $\mathcal{B}^m$. Then we define the functor $\mathcal{F}^m$ by

$$\mathcal{F}^m(B) := \varprojlim_{(\mathbb{B} \to B)} \pi_0(\mathcal{X}^m(\mathbb{B} \to B)_{\overline{K}}),$$

where $(\mathbb{B} \to B)$ runs through the category of embeddings of B. In general, we have $\sharp\mathcal{F}^m(B) \leq \mathrm{rank}_A(B)$. Two key definitions in this paper are the following:

Definition 3.2. Let B be a finite flat A-algebra. We say that the *ramification of B is bounded by m* if $\sharp\mathcal{F}^m(B) = \mathrm{rank}_A(B)$.

Definition 3.3. For any rational number m with $0 < m \le a$, we define $\mathcal{FFP}_A^{\le m}$ to be the category whose objects are finite flat principal A-algebras with ramification bounded by m and whose morphisms are defined as follows: For any B and B' in $\mathcal{FFP}_A^{\le m}$, set

$$\mathrm{Hom}_{\mathcal{FFP}_A^{\le m}}(B, B') \;:=\; \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)). \tag{2}$$

We also define $\mathrm{FFP}_A^{\le m}$ to be the full-subcategory of $\mathcal{FFP}_A^{\le m}$ consisting of local objects.

To prove Theorem 1.1, we recall the following lemma due to Hattori ([6], Lem. 1):

Lemma 3.4. *Let L be a finite étale K-algebra, and a an integer ≥ 1. If $B = \mathcal{O}_L/\mathfrak{m}_K^a\mathcal{O}_L$, then we have $\mathcal{F}^m(B) = F^m(L)$ as an object of $\mathcal{S}_K$ for any rational number $0 < m \le a$.*

This is because one may choose a common $\mathbb{B}$ in the embeddings $(\mathbb{B}, \mathbb{B} \to \mathcal{O}_L)$ and $(\mathbb{B}, \mathbb{B} \to B)$, so that, if $m \le a$, we have $X^m(\mathbb{B} \to \mathcal{O}_L) = \mathcal{X}^m(\mathbb{B} \to B)$.

By Definitions 3.1 and 3.2, we have:

Corollary 3.5. *For any rational number $0 < m \le a$, the ramification of B is bounded by m if and only if the ramification of L is bounded by m.*

Now we can prove Theorem 1.1. The essential surjectivity of the functor $T : \mathcal{FE}_K^{\le m} \to \mathcal{FFP}_A^{\le m}$ follows from (ii) of Proposition 2.2 and Corollary 3.5, since any object of $\mathcal{FFP}_A^{\le m}$ is a direct product of finite extensions of A. To prove the full-faithfullness of T, let L and L' be two objects in $\mathcal{FE}_K^{\le m}$, and let $B = T(L)$ and $B' = T(L')$. Since the functor F^m gives an anti-equivalence of the Galois category $\mathcal{FE}_K^{\le m}$ with a full-subcategory of $\mathcal{S}_K$, we have

$$\mathrm{Hom}_{\mathcal{FE}_K^{\le m}}(L, L') \;\simeq\; \mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)).$$

By Lemma 3.4, we have

$$\mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) \;=\; \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)).$$

It follows from our definition (2) of Hom in $\mathcal{FFP}_A^{\le m}$ that

$$\mathrm{Hom}_{\mathcal{FE}_K^{\le m}}(L, L') \;=\; \mathrm{Hom}_{\mathcal{FFP}_A^{\le m}}(B, B').$$

This completes the proof of the Theorem.

Remark. The relation of $\mathrm{Hom}_A(B, B')$ to the Hom sets appearing in the above proof is summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_K(L, L') & \xrightarrow[F^m]{\simeq} & \mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) \\ \downarrow & & \| \\ \mathrm{Hom}_A(B, B') & \xrightarrow[\mathcal{F}^m]{} & \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)), \end{array}$$

where the left vertical arrow is the reduction mod $\mathfrak{m}_K^a$ of $\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{L'})$. This shows that the map $\mathcal{F}^m : \mathrm{Hom}_A(B, B') \to \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B')$ is surjective and compatible with the composition of morphisms. It can be shown that this map identifies the set $\mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B')$ with the quotient of $\mathrm{Hom}_A(B, B')$ by an equivalence relation $\overset{m}{\approx}$ defined as follows: Put $\overline{A} = \mathcal{O}_{\overline{K}}/\mathfrak{m}_K^a\mathcal{O}_{\overline{K}}$ and let $\mathcal{X}^m$ be the affinoid variety associated with an embedding of B. Recall that there exists a natural surjective map $\mathcal{X}^m(\overline{K}) \to \mathrm{Hom}_A(B, \overline{A})$ with connected fibers ([2], Lem. 3.2), so that its inverse yields a well-defined map $\xi : \mathrm{Hom}_A(B, \overline{A}) \to \pi_0(\mathcal{X}^m_{\overline{K}})$. Then we have a map

$$\mathrm{Hom}_A(B, B') \times \mathrm{Hom}_A(B', \overline{A}) \ \to \ \pi_0(\mathcal{X}^m_{\overline{K}})$$

which maps (f, α) to $\xi(\alpha \circ f)$. For f and f' in $\mathrm{Hom}_A(B, B')$, define

$$f \overset{m}{\approx} f' \iff \xi(\alpha \circ f) = \xi(\alpha \circ f') \quad \text{for all } \alpha \in \mathrm{Hom}_A(B', \overline{A}).$$

It can also be shown that, if B' is local, then for given f and f', the equality $\xi(\alpha \circ f) = \xi(\alpha \circ f')$ holds for all $\alpha \in \mathrm{Hom}_A(B', \overline{A})$ if it holds for some α.

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