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Limits of Stable Pairs

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Abstract: Let (X_0, B_0) be the canonical limit of a one-parameter family of stable pairs, provided by the log Minimal Model Program. We prove that X_0 is S_2 and that $\lfloor B_0 \rfloor$ is S_1, as an application of a general local statement: if $(X, B + \epsilon D)$ is log canonical and D is $\mathbb{Q}$-Cartier then D is S_2 and $\lfloor B \rfloor \cap D$ is S_1, i.e. has no embedded components.
When B has coefficients < 1, examples due to Hacking and Hassett show that B_0 may indeed have embedded primes. We resolve this problem by introducing a category of stable branchpairs. We prove that the corresponding moduli functor is proper for families with normal generic fiber.

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Let $U = S \setminus 0$ be a punctured nonsingular curve, and $f : (X_U, B_U) \to Y \times U$ be a family of stable maps (precise definitions follow). It is well understood, see e.g. [KSB88, Ale96] that log Minimal Model Program leads to a natural completion of this family over S, possibly after a finite ramified base change $S' \to S$. This, in turn, leads to the construction of a proper moduli space of stable maps (called stable pairs if Y is a point) once some standard conjectures, such as log MMP in dimension $\dim X + 1$ and boundedness, and some technical questions have been resolved.

The purpose of this paper is solve two such technical issues. The first one is the Serre's S_2-property for the one-parameter limits, which implies that the limit is semi log canonical:

Theorem 0.1. *Let $(X, B) \to S$ be the stable log canonical completion of a family of log canonical pairs. Then for the central fiber one has:*

(1) *X_0 is S_2,*
(2) *$\lfloor B_0 \rfloor$ is S_1, i.e. this scheme has no embedded components.*

As a corollary, if B is reduced (i.e. all $b_j = 1$) then the central fiber $f : (X_0, B_0) \to Y$ is a stable map.

For surfaces with reduced B, Theorem 0.1 was proved by Hassett [Has01]. Even in that case, our proof is different. Whereas the proof in [Has01] is global, i.e. it requires an actual semistable family of projective surfaces with relatively ample $K_X + B$, our proof is based on the following quite general local statement:

Lemma 0.2. *Let (X, B) be a log canonical pair which has no zerodimensional centers of log canonical singularities. Then for every closed point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is S_3.*

As a consequence, we obtain the following theorem from which (0.1) follows at once.

Theorem 0.3. *Let (X, B) be a log canonical pair and D be an effective Cartier divisor. Assume that for some $\epsilon > 0$ the pair $(X, B + \epsilon D)$ is log canonical. Then D is S_2 and $\lfloor B \rfloor \cap D$ is S_1.*

The second question we consider is the following. When the coefficients b_j in are less than one, Hacking and Hassett gave examples of families of stable surface pairs in which the central fiber B_0 of B indeed does have embedded primes. We resolve this problem by introducing, following ideas of [AK06], a new category, that of *stable branchpairs*, which avoids nonreduced schemes. We define the moduli functor in this category and check the valuative criterion of properness for families with normal generic fiber.

With branchdivisors thus well-motivated, we define, in a straighforward way, branchcycles of other dimensions as well.

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Throughout most of the paper we work over an algebraically closed field of characteristic zero, and relax this condition to an arbitrary field for the last section.

1. Basic definitions

All **varieties** in this paper will be assumed to be connected and reduced but not necessarily irreducible. A **polarized variety** is a projective variety X with an ample invertible sheaf L. A pair (X, B) will always consist of a variety X and a $\mathbb{Q}$-divisor $B = \sum b_j B_j$, where B_j are effective Weil divisors on X, and $0 < b_j \leq 1$.

We use standard definitions and notations of Minimal Model Program for discrepancies $a(X, B, E_i)$, notions of log canonical pairs (abbreviated lc), klt pairs, etc., as in [KM98]. We assume standard definitions from commutative algebra for the Serre's conditions S_n. We now list the slightly less standard definitions.

Definition 1.1. Let (X, B) be an lc pair. A **center of log canonical singularities** of (X, B) (abbreviated to a center of LCS(X, B)) is the image of a divisor $E_i \subset Y$ on a resolution $f : Y \to X$ that has discrepancy $a(X, B, E_i) = -1$.

If $f : Y \to X$ is log a smooth resolution of (X, B) and $E = \sum E_i$ is the union of all divisors with discrepancy -1 (some exceptional, some strict preimages of components of B with $b_j = 1$) then the centers of $\mathrm{LCS}(X,B)$ are the images of the nonempty strata $\cap E_i$.

We will use the following important results of Florin Ambro, which were further clarified by Osamu Fujino. The first is Ambro's generalization of Kollár's injectivity theorem [Kol86], and the second describes properties of log centers.

Theorem 1.2 (Injectivity for varieties with normal crossings, simple form)**.** *Let Y be a nonsingular variety, $E+S+\Delta$ be a normal crossing $\mathbb{R}$-divisor on Y, E, S and Δ have no components in common, $E+S$ is reduced, and $\lfloor\Delta\rfloor = 0$.*

Let $f: Y \to X$ be a proper morphism, A a Cartier divisor on E, and assume that the divisor $H \sim_{\mathbb{R}} A - (K_E + S + \Delta)$ on E is f-semiample. Then every nonzero section of $R^i f_* \mathcal{O}_E(A)$ contains in its support the f-image of some strata of $(E, S+\Delta)$.

Here, K_E stands for the dualizing invertible sheaf ω_E, and the strata of $(E, S+\Delta)$ are the intersections of the components of E and S.

Proof. This is a special case of [Amb03, 3.2(i)], see also [Amb07] for another exposition. This theorem was also reproved in [Fuj07a, 5.7,5.15], see also [Fuj07b]. □

Theorem 1.3 (Properties of log centers)**.** (1) *Every irreducible component of the intersection of two centers is a center.*
(2) *For any $x \in X$ the minimal center containing x is normal.*
(3) *A union of any set of centers is seminormal.*

Proof. (1) and (2) are contained in [Kaw97, Kaw98] in the case when there exists a klt pair (X, B') with $B' \le B$. For the general case these are in [Amb03, 4.8]. (3) is [Amb98] and [Amb03, 4.2(ii),4.4(i)].

Also, a very easy, one-page proof of these properties, which uses only the above injectivity theorem, is contained in [Amb07, §4]. □

Definition 1.4. A pair (X,B) is called **semi log canonical** (slc) if

(1) X satisfies Serre's condition S_2,
(2) X has at worst double normal crossing singularities in codimension one, and no divisor B_j contains any component of this double locus,

(3) some multiple of the Weil $\mathbb{Q}$-divisor $K_X + B$, well defined thanks to the previous condition, is $\mathbb{Q}$-Cartier, and
(4) denoting by $\nu : X^\nu \to X$ the normalization, the pair $(X^\nu, \text{(double locus)} + \nu^{-1}B)$ is log canonical.

Definition 1.5. A pair $(X, B = \sum b_j B_j)$ (resp. a map $f : (X, B) \to Y$) is called a **stable map** if the following two conditions are satisfied:

(1) *on singularities:* the pair (X, B) is semi log canonical, and
(2) *numerical:* the divisor $K_X + B$ is ample (resp. f-ample).

A **stable pair** is a stable map to a point.

Definition 1.6. A variety X is **seminormal** if any proper bijection $X' \to X$ is an isomorphism.

It is well-known, see e.g. [Kol96, I.7], that every variety has a unique seminormalization X^{sn} and it has a universal property: any morphism $Y \to X$ with seminormal Y factors through X^{sn}.

2. S_2 and seminormality

We collect some mostly well-known facts about the way the S_2 property and seminormality are related.

Definition 2.1. The **S_2-fication**, or **saturation in codimension 2** of a variety X is defined to be

$$\pi_X^{\text{sat}} : X^{\text{sat}} = \varinjlim \operatorname{Spec}_{\mathcal{O}_X} \mathcal{O}_{X \setminus Z} \;\to\; X$$

in which the limit goes over closed subsets $Z \subset X$ with $\operatorname{codim}_X Z \geq 2$. The morphism π_X^{sat} is finite: indeed, it is dominated by the normalization of Y.

More generally, for any closed subset $D \subset X$ the **saturation in codimension 2 along** D

$$\pi_{X,D}^{\text{sat}} : X_D^{\text{sat}} \to X$$

is defined by taking the limit as above that goes only over $Z \subset D$. Hence, $\pi_X^{\text{sat}} = \pi_{X,X}^{\text{sat}}$.

Lemma 2.2. *$\pi_{X,D}^{\text{sat}}$ is an isomorphism iff for any subvariety $Z \subset D$ the local ring $\mathcal{O}_{X,Z}$ is S_2.*

Proof. Let $Z \subset D$ be a subvariety with $\operatorname{codim}_X Z \geq 2$. By the cohomological characterization of depth (see f.e. [Mat89, Thm. 28] or [Eis95, 18.4]) the local ring $\mathcal{O}_{X,Z}$ has depth ≥ 2 iff any short exact sequence

$$0 \to \mathcal{O}_{X,Z} \to F \to Q \to 0$$

of $\mathcal{O}_{X,Z}$-modules with $\operatorname{Supp} Q = Z$ splits.

If π_D^{sat} is an isomorphism then for every exact sequence as above $F_D^{\mathrm{sat}} = \mathcal{O}_{X,Z}$, and the canonical restriction morphism $F \to F_D^{\mathrm{sat}}$ provides the splitting. If π_D^{sat} is not an isomorphism over some $Z \subset D$ then the localization of

$$0 \to O_X \to \pi_*^{\mathrm{sat}} \mathcal{O}_{X^{\mathrm{sat}}} \to Q \to 0$$

at Z does not split and $Q \neq 0$. □

Lemma 2.3. *Assume that X is seminormal and $\pi_{X,D}^{\mathrm{sat}}$ is a bijection. Then for any subvariety $Z \subset D$ the local ring $\mathcal{O}_{X,Z}$ is S_2.*

Proof. Since X is seminormal, $\pi_{X,D}^{\mathrm{sat}}$ is an isomorphism, so the the previous lemma applies. □

Lemma 2.4. *Assume X is S_2 and is seminormal in codimension 1. Then X is seminormal.*

Proof. We have $(X^{\mathrm{sn}})^{\mathrm{sat}} = X^{\mathrm{sat}}$ and $X^{\mathrm{sat}} = X$, hence $X^{\mathrm{sn}} \to X$ is an isomorphism. □

Corollary 2.5. *Semi log canonical $\implies$ seminormal.*

3. Singularity theorems

Let X be a normal variety, which by Serre's criterion implies that X is S_2. Let $f : Y \to X$ be a resolution of singularities. Then we have:

Lemma 3.1. *Assume* $\dim X > 2$. *Then X is S_3 at every closed point $x \in X$ iff $R^1 f_* \mathcal{O}_Y$ has no associated components of dimension 0, i.e. the support of every section of $R^1 f_* \mathcal{O}_Y$ has dimension > 0.*

Proof. By considering an open affine neighborhood of x and then compactifying, we can assume that X is projective with an ample invertible sheaf L. (Since the property of being S_3 at closed points is open, one can compactify without introducing "worse" points.) Then by the proof of [Har77, Thm.III.7.6], X is S_3 at every closed point iff for all $r \gg 0$ one has $H^2\big(\mathcal{O}_X(-rL)\big) = 0$.

The spectral sequence

$$E_2^{p,q} = H^p\big(R^q f_* \mathcal{O}_Y(-rL)\big) \Rightarrow H^{p+q}\big(\mathcal{O}_Y(-rf^*L)\big)$$

together with the fact that $H^{1 \text{ and } 2}\big(\mathcal{O}_Y(-rf^*L)\big) = 0$ by Generalized Kodaira's vanishing theorem [KM98, 2.70], imply that

$$d_2^{0,1} : H^0\big(R^1 f_* \mathcal{O}_Y(-rL)\big) \to H^2\big(\mathcal{O}_X(-rL)\big)$$

is an isomorphism. Further, $H^0\big(R^1 f_* \mathcal{O}_Y(-rL)\big) = 0$ for $r \gg 0$ precisely when the sheaf $R^1 f_* \mathcal{O}_Y$ has no associated components of dimension 0. □

Log terminal pairs have rational singularities, and hence are Cohen-Macaulay, see [KM98, Thm.5.22] for a simple proof. Log canonical singularities need not be S_3. The easiest example is a cone over an abelian surface S. Indeed, in this case $R^1 f_* \mathcal{O}_Y = H^1\big(\mathcal{O}_S\big)$ is non-zero and supported at one point. However, we will prove the following:

Lemma 3.2. *Let (X, B) be a log canonical pair which has no zerodimensional centers of log canonical singularities. Then for every closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is S_3.*

Proof. As in the previous proof, we can assume that (X, L) is a polarized variety, and we must prove that for $r \gg 0$ one has $H^2\big(\mathcal{O}_X(-rL)\big) = 0$. Let $f : Y \to X$ be a resolution of singularities of (X, B) such that $f^{-1}B \cup \operatorname{Exc}(f)$ is a divisor with global normal crossings. Then we can write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + B) - E + A - \Delta,$$

where

(1) $E = \sum E_j$ is the sum of the divisors B_j with $b_j = 1$ and the exceptional divisors of f with discrepancy -1,
(2) A is effective and integral,
(3) Δ is effective and $\lfloor \Delta \rfloor = 0$.

Since the pair (X, B) is log canonical and the coefficients of B satisfy $0 < b_j \le 1$, it follows that A is f-exceptional, E has no components in common with

$\operatorname{Supp}\Delta$ and with A, and the union $E\cup\operatorname{Supp}A\cup\operatorname{Supp}\Delta$ is a divisor with global normal crossings.

Then $-E+A\sim_{\mathbb{Q}}K_Y+\Delta-f^*(K_X+B)$. The Generalized Kodaira Theorem (Kawamata-Viehweg theorem) gives $R^qf_*\mathcal{O}_Y(-E+A)=0$ for $q>0$. Therefore, by pushing forward the exact sequence

$$0\to\mathcal{O}_Y(-E+A)\to\mathcal{O}_Y(A)\to\mathcal{O}_E(A)\to 0$$

we obtain $R^1f_*\mathcal{O}_Y(A)\simeq R^1f_*\mathcal{O}_E(A)$. Now, on E one has

$$A\sim_{\mathbb{Q}}K_E+\Delta-f^*(K_X+B),$$

where K_E stands for the (invertible) dualizing sheaf ω_E. Therefore, by Ambro's injectivity theorem 1.2, applied here with $H=-f^*(K_X+B)$ and $S=0$, the support of every nonzero section of the sheaf $R^1f_*\mathcal{O}_E(A)$ contains a center of $\operatorname{LCS}(X,B)$, hence has dimension >0.

Now consider the following commutative diagram

$$\begin{array}{ccc}
H^2\big(\mathcal{O}_Y(-rf^*L)\big) & \longrightarrow & H^2\big(\mathcal{O}_Y(A-rf^*L)\big)\\
\uparrow & & \uparrow\phi\\
H^2\big(f_*\mathcal{O}_Y(-rf^*L)\big) & & H^2\big(f_*\mathcal{O}_Y(A-rf^*L)\big)\\
\| & & \|\\
H^2\big(\mathcal{O}_X(-rL)\big) & = & H^2\big(\mathcal{O}_X(-rL)\big)
\end{array}$$

Since by Generalized Kodaira's vanishing theorem $H^2\big(\mathcal{O}_Y(-rf^*L)\big)=0$, this implies $H^2\big(\mathcal{O}_X(-rL)\big)=0$ if we could prove that ϕ is injective. Finally, the spectral sequence

$$E_2^{p,q}=H^p\big(R^qf_*\mathcal{O}_Y(A)(-rL)\big)\Rightarrow E^{p+q}=H^{p+q}\big(\mathcal{O}_Y(A-rf^*L)\big)$$

in a standard way produces the exact sequence

$$E_2^{0,1}\xrightarrow{d_2^{0,1}}E_2^{2,0}\longrightarrow E^2$$

In our case, $E_2^{0,1}=H^0\big(R^1f_*\mathcal{O}_Y(A)(-rL)\big)=0$ by what we proved above (the sheaf $R^1f_*\mathcal{O}_Y(A)$ has no associated components of dimension 0), and the second homomorphism is ϕ. Hence, ϕ is injective. This completes the proof. $\square$

Remark 3.3. One has to be careful that (3.2) does *not* imply that X is S_3. Indeed, let X' be a variety which is S_2 but not S_3, for example a cone over an abelian surface, and let X be the cartesian product of X' with a curve C. Then X is S_3 at every closed point but not at the scheme point corresponding to (vertex)$\times C$.

Theorem 3.4. *Let (X,B) be an lc pair and D be an effective Cartier divisor. Assume that for some $\epsilon > 0$ the pair $(X, B+\epsilon D)$ is lc. Then D is S_2.*

Proof. Suppose that for some subvariety $Z \subset D$ the local ring $\mathcal{O}_{D,Z}$ is not S_2, then $\mathcal{O}_{X,Z}$ is not S_3. Let

$$(X^{(d)}, B^{(d)}) = (X,B) \cap H_1 \cap \cdots \cap H_d$$

be the intersection with $d = \dim Z$ general hyperplanes such that $Z^{(d)} = Z \cap H_1 \cap \cdots \cap H_d \neq \emptyset$. Then

(1) the pair $(X^{(d)}, B^{(d)})$ is lc by the general properties of lc (apply Bertini theorem to a resolution), and
(2) $X^{(d)}$ is not S_3 at a closed point $P \in Z^{(d)}$ (by the semicontinuity of depth along Z on fibers a morphism; applied to a generic projection $X \to \mathbb{P}^d$).

Let W be a center of $\mathrm{LCS}(X,B)$. Since $(X, B+\epsilon D)$ is lc, W is not contained in D. Then the corresponding centers, irreducible components of $W^{(d)} = W \cap H_1 \cap \cdots \cap H_d$ are not contained in $D^{(d)}$. Hence, by shrinking a neighborhood of $D^{(d)}$ in $X^{(d)}$ we can assume that $(X^{(d)}, B^{(d)})$ has no zerodimensional centers of LCS. But then $X^{(d)}$ is S_3 at P by (3.2), a contradiction. □

Theorem 3.5. *Under the assumptions of* (3.4)*, the scheme $\lfloor B \rfloor \cap D$ is S_1.*

Proof. Note that $\lfloor B \rfloor$ is a union of several centers of $\mathrm{LCS}(X,B)$. As such, it is seminormal by Theorem 1.3.

We claim that the saturation $\pi = \pi^{\mathrm{sat}}_{\lfloor B \rfloor, \lfloor B \rfloor \cap D}$ of $\lfloor B \rfloor$ in codimension 2 along $\lfloor B \rfloor \cap D$ is a bijection. Otherwise, there exists a subvariety $Z \subset \lfloor B \rfloor$ intersecting D such that $\pi : \pi^{-1}(Z) \to Z$ is several-to-one along Z. Then cutting by generic hyperplanes, as above, we obtain a pair $(X^{(d)}, B^{(d)})$ such that $Z^{(d)}$ is a point P and $\lfloor B \rfloor^{(d)}$ has several analytic branches intersecting at P.

After going to an étale cover, which does not change the lc condition, we can assume that P is a component of the intersection of two irreducible component of the locus of $\mathrm{LCS}(X^{(d)}, B^{(d)})$.

But then P is a center of LCS itself, by Theorem 1.3(1). This is not possible, again because $(X^{(d)}, B^{(d)} + \epsilon D)$ is lc; contradiction.

The saturation morphism $\pi : \lfloor B \rfloor^{\mathrm{sat}}_{\lfloor B \rfloor, \lfloor B \rfloor \cap D} \to \lfloor B \rfloor$ is a bijection, and $\lfloor B \rfloor$ is seminormal. By Lemma 2.3 this implies that $\lfloor B \rfloor$ is S_2 along any subvariety $Z \subset D$. Therefore $\lfloor B \rfloor \cap D$ is S_1. □

4. One-parameter limits of stable pairs

Let $U = (S, 0)$ be a punctured nonsingular curve and let $f_U : (X_U, B_U) \to Y \times U$ be a family of stable maps with normal X_U, so that (X_U, B_U) is lc.

The stable limit of this family is constructed as follows. Pick some extension family $f : (X, B) \to Y \times S$. Take a resolution of singularities, which introduces some exceptional divisors E_i. Apply the Semistable Reduction Theorem to this resolution together with the divisors, as in [KM98, Thm.7.17]. The result is that after a ramified base change $(S', 0) \to (S, 0)$ we now have an extended family $\tilde{f}' : (\widetilde{X}', \widetilde{B}')$ such that $\widetilde{X}'$ is smooth, the central fiber $\widetilde{X}'_0$ is a reduced normal crossing divisor, and, moreover, $\widetilde{X}'_0 \cup \operatorname{Supp} \widetilde{B}' \cup \widetilde{E}'_i$ is a normal crossing divisor. Let us drop the primes in this notation for simplicity, and write X, S, etc. instead of X', S' etc.

It follows that the pair $(\widetilde{X}, \widetilde{B} + \widetilde{X}_0 + \sum \widetilde{E}_i)$ has log canonical singularities and is relatively of general type over $Y \times S$. Now let $f : (X, B + X_0) \to Y \times S$ be its log canonical model, guaranteed by the log Minimal Model Program. The divisor $K_X + B + X_0$ is f-ample and the pair $(X, B + X_0)$ has canonical singularities.

Theorem 4.1. *The central fiber X_0 is S_2, and the scheme $\lfloor B \rfloor \cap X_0$ is S_1.*

Proof. Immediate from (3.4) and (3.5) by taking $D = X_0$ and $\epsilon = 1$. □

5. Branchpairs

In [Has03] Hassett constructed moduli spaces of weighted stable curves, i.e. one-dimensional pairs $(X, \sum b_i B_i)$ with $0 < b_i \le 1$. It is natural to try to extend this construction to higher dimensions.

However, in the case of surfaces Hacking and Hassett gave examples of one-parameter families of pairs $(X, bB) \to S$ with irreducible B such that B_0 has an embedded point. Such examples are constructed by looking at families $(X, B) \to S$ in which B is *not* $\mathbb{Q}$-Cartier. Recall that by the definition of a log canonical pair $K_X + B$ must be $\mathbb{Q}$-Cartier but neither K_X nor B are required to be such. The following explicit example was communicated to me by Brendan Hassett, included here with his gracious permission.

Example 5.1. Let $\mathbb{F}_n$ denote the Hirzebruch ruled surface with exceptional section s_n ($s_n^2 = -n$) and fiber f_n; in particular $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with two rulings denoted by f_0 and s_0.

Let $l \sim s_0 + 2f_0$ be a smooth curve in $\mathbb{F}_0$ and let $\widetilde{X}$ be the blowup of $\mathbb{F}_0 \times S$ along $l \times 0$ in the central fiber. Then $\widetilde{X}_0$ is the union of two irreducible components $\widetilde{X}_0^{(1)} = \mathbb{F}_0$ and $\widetilde{X}_0^{(2)} = \mathbb{F}_4$ intersecting along l, and $l \sim s_4$ in $\mathbb{F}_4$.

Let $\widetilde{B}_0 = \widetilde{B}_0^{(1)} \cup \widetilde{B}_0^{(2)}$ be a curve in the central fiber such that $\widetilde{B}_0^{(1)} \sim 2s_0$ is the union of two generic lines and $\widetilde{B}_0^{(2)} \sim 4(s_4 + 4f_4) + 4f_4$, intersecting at 4 points P_1, P_2, P_3, P_4. Then $\widetilde{B}_0$ is a nodal curve of genus 35. Let $\widetilde{B}$ be a family of curves obtained by smoothing $\widetilde{B}_0$.

Denote by $f : \widetilde{X} \to X$ the morphism blowing down the divisor $\widetilde{X}_0^{(1)}$, and $B = f(\widetilde{B})$. One easily computes that K_X is not $\mathbb{Q}$-Cartier (because $s_0 + 2f_0$ is not proportional to $K_{\widetilde{X}_0^{(1)}}$) but $K_X + 1/2B$ is, and that $(X, 1/2B)$ has canonical singularities.

After the blowdown, the curve $(B_0)_{\mathrm{red}}$ in the central fiber is obtained from $\widetilde{B}_0^{(2)}$ by gluing the four points P_i together. The curve $\widetilde{B}_0$ and its smoothings have arithmetic genus 35. The curve $(B_0)_{\mathrm{red}}$ has genus 36. Hence, B_0 has an embedded point.

So, if one wants to work with arbitrary coefficients, which is very natural, one must enlarge the category of pairs in some way, or use some other trick to solve the problem. There are at least two ways to proceed:

(1) One can work with *floating coefficients.* This means that we must require the divisors B_j to be $\mathbb{Q}$-Cartier, and the pairs $(X, \sum(b_j + \epsilon_j)B_j)$ to be semi log canonical and ample for all $0 < \epsilon_j \ll 1$. Hacking did just that in [Hac04] for planar pairs $(\mathbb{P}^2, (3/d + \epsilon)D)$. And the moduli of stable toric, resp. abelian pairs in [Ale02] can be interpreted as moduli of semi log canonical stable pairs $(X, \Delta + \epsilon B)$, resp. $(X, \epsilon B)$.

However, it is very desirable to work with constant coefficients, and the coefficients appearing in the above-mentioned examples are fairly simple, such as $b_1 = 1/2$.

(2) One can work with the pairs $(X, \sum b_j B_j)$, where B_j are codimension-one subschemes of X, possibly with embedded components. This can be done in two ways:

(a) *Natural.* One should define (semi) log canonical pairs (X, Y) of a variety X with a subscheme Y. This was done for pairs with *smooth* variety X (see, e.g. [Mus02]) and more generally when X is $\mathbb{Q}$-Gorenstein. But: this is insufficiently general for our purposes, especially if we consider the case of pairs of dimension ≥ 3.

(b) *Unnatural.* One can work with subschemes B_j that possibly have embedded components but then ignore them, by saturating in codimension 2. For example, one should define the sheaf $\mathcal{O}_X(N(K_X + B))$ as

$$\mathcal{O}_X(N(K_X + B)) = \varinjlim_U \, j_{U*}\mathcal{O}_U(NK_U + B),$$

where the limit goes over open dense subsets $j_U : U \to X$ with $\operatorname{codim}(X \setminus U) \geq 2$ such that $B \cap U$ has no embedded components and such that U is Gorenstein. But this does feel quite artificial.

Building on [AK06], I now propose a different solution which avoids nonreduced schemes altogether.

Definition 5.2. Let X be a variety of pure dimension d. A **prime branchdivisor** of X is a variety B_j of pure dimension $d - 1$ together with a *finite* (so, in particular proper) morphism $\varphi_j : B_j \to X$.

Let us emphasize again that by our definition of variety, B_j is connected, possibly reducible and, most importantly, *reduced*. Hence, a prime branchdivisor is simply a connected branchvariety, as defined in [AK06], of pure codimension 1.

Definition 5.3. A **branchdivisor** is an element of a free abelian group $bZ_{d-1}(X)$ with prime branchdivisors B_j as generators. If A is an abelian group (such as $\mathbb{Q}$, $\mathbb{R}$, etc.) then an A-branchdivisor is an element of the group $bZ_{d-1}(X) \otimes A$.

The **shadow** of a branchdivisor $\sum b_j B_j$ is the ordinary divisor $\sum b_j \varphi_{j*}(B_j)$ on X. We will use the shortcut $\varphi_* B$ for the shadow of B.

We will be concerned with $\mathbb{Q}$-branchdivisors in this paper, although $\mathbb{R}$-coefficients are frequently useful in other contexts.

Definition 5.4. A **branchpair** is a pair $(X, \sum b_j B_j)$ of a variety and a $\mathbb{Q}$-branchdivisor on it, where B_j are prime branchdivisors and $0 < b_j \leq 1$. This pair is called **(semi) log canonical** (resp. terminal, log terminal, klt) if so is its shadow $(X, \sum b_j \varphi_{j*}(B_j))$.

Definition 5.5. A **family of branchpairs** over a scheme S is a morphism $\pi : X \to S$ and finite morphisms $\varphi_j : B_j \to X$ such that

(1) $\pi : X \to S$ and all $\pi \circ \varphi_j : B_j \to S$ are flat, and
(2) every geometric fiber $(X_{\bar{s}}, \sum b_j (B_j)_{\bar{s}})$ is a branchpair.

Discussion 5.6. It takes perhaps a moment to realize that anything happened at all, that we defined something new here. But consider the following example: $X = \mathbb{P}^2$, B is a rational cubic curve with a node, and $B' \simeq \mathbb{P}^1$ is the normalization of B, and $f : B' \to X$ is a branchdivisor whose shadow is B. Then the pairs (X, B) and (X, B') can never appear as fibers in a proper family with connected base S. Indeed, $p_a(B) = 1$ and $p_a(B') = 0$, and the arithmetic genus is locally constant in proper flat families.

Definition 5.7. A branchpair (X, B) together with a morphism $f : X \to Y$ is **stable** if $(X, \varphi_* B)$ has semi log canonical singularities and $K_X + \varphi_* B$ is ample over Y.

Finally, we define the moduli functor of stable branchpairs over a projective scheme Y.

Definition 5.8. We choose a triple of positive integers numbers $C = (C_1, C_2, C_3)$ and a positive integer N. We also fix a very ample sheaf $\mathcal{O}_Y(1)$ on Y. Then the basic moduli functor $M_{C,N}$ associates to every Noetherian scheme S over a base scheme the set $M_{C,N}(S)$ of morphisms $f : X \to Y \times S$ and $\varphi_j : B_j \to X$ with the following properties:

(1) X and B_j are flat schemes over S.
(2) Every geometric fiber $(X, B)_s$ is a branchpair,
(3) The double dual $\mathcal{L}_N(X/S) = \big(\omega_{X/S}^{\otimes N} \otimes \mathcal{O}_X(N\varphi_* B)\big)^{**}$ is an invertible sheaf on X, relatively ample over $Y \times S$.
(4) For every geometric fiber, $(L_N)_s^2 = C_1$, $(L_N)_s H_s = C_2$, and $H_s^2 = C_3$, where $\mathcal{O}_X(H) = f^*\mathcal{O}_Y(1)$.

Theorem 5.9 (Properness with normal generic fiber)**.** *Every family in $M_{C,N}$ over a punctured smooth curve $S \setminus 0$ with normal X_η has at most one extension, and the extension does exist after a ramified base change $S' \to S$.*

Proof. Existence. The construction of the previous section gives an extension for the family of shadows $(X_U, \varphi_* B_U)$. The properness of the functor of branchvarieties [AK06] applied over X/S gives extensions $\varphi_j : B_j \to X$.

The shadow pair $(X, \varphi_* B + X_0)$ has log canonical singularities. We have established that X_0 is S_2. Now by the easy direction of the Inversion of Adjunction (see, e.g. [Fli92, 17.3]) the central fiber $(X_0, (\varphi_* B)_0)$ has semi log canonical singularities, and so we have the required extension.

Uniqueness. We apply Inversion of Adjunction [Kaw06] to the shadow pair. The conclusion is that $(X, \varphi_* B)$ is lc. But then it is the log canonical model of any resolution of singularities of any extension of $(X_U, \varphi_* B_U)$. Since the log canonical model is unique, the extension of the shadow is unique. And by the properness of the functor of branchvarieties [AK06] again, the extensions of the branchdivisors $\varphi_j : B_j \to X$ are unique as well. $\square$

Remark 5.10. One can easily see what happens when we apply our procedure in Example 5.1. The limit branchdivisor, call it B_0', is the curve obtained from $\widetilde{B}_0^{(2)}$ by identifying two pairs of the points P_i separately. The morphism $B_0' \to X_0$ is 2-to-1 above the point $P \in X_0$ and a closed embedding away from P.

Indeed, the surface $\widetilde{B}$ has rational double points, and each of the lines in $B_0^{(1)} \sim 2s_0$ is a (-2)-curve on the resolution of this surface. This implies that these curves are contractible. Let B' be the projective surface obtained by contracting them. Thus, the central fiber B'_0 is obtained from $B_0^{(1)}$ by identifying separately P_1 with P_2 and P_3 with P_4.

Then the curve $(B'_0)_{\mathrm{red}}$ is nodal and has arithmetic genus 35, the same as the generic fiber. Therefore, $B'_0 = (B'_0)_{\mathrm{red}}$. Hence, $B' \to C$ is a family of branchcurves.

Remark 5.11. Examples given by J. Kollár in [Kol07] show that the case of nonnormal generic fiber requires extreme care. One key insight from [Kol07] is that on a properly defined non-normal stable pair, for every component of the double locus, the two ways of applying adjunction should match.

More precisely, let (X, B) be a non-normal stable pair and let C be a component of the double locus. Let $\nu : X^\nu \to X$ be the normalization, $C^\nu = \nu^{-1}(C)$, and $C^\nu \to C$ be the corresponding double cover. Then the divisor Diff, computed from $\nu^*(K_X + B)|_{C^\nu} = K_{C^\nu} + \mathrm{Diff}$, should be invariant under the involution.

We also note that M. A. van Opstall considered the case of nonnormal surfaces with $B = \emptyset$ in [vO06].

6. Branchcycles

Once we have defined the branchdivisors, it is straightforward to define branchcycles as well: the prime k-branchcycles of X are simply k-dimensional branchvarieties over X, and they are free generators of an abelian group $bZ_k(X)$, resp. a free A-module $bZ_k(X, A) = bZ_k(X) \otimes A$.

Definition 6.1. The linear (resp. algebraic) equivalence between k-branchcycles is generated by the following: for any family of k-dimensional branchvarieties $\phi : B \to X \times C$, where $C = \mathbb{P}^1$ (resp. a smooth curve with two points $0, \infty$) the fibers $\phi_0 : B_0 \to X$ and $\phi_\infty : B_\infty \to X$ are equivalent.

We denote the quotient modulo the linear (resp. algebraic) equivalence by $bA_k(X)$ (resp. $bB_k(X)$).

Clearly, any function which is constant in flat families of branchvarieties descends to an invariant of $bB_k(X)$. For example, there exists a natural homomorphism $bB_k(X) \to K_\bullet(X)$ to the K-group of X given by associating to a branchvariety $\phi : B \to X$ the class of the coherent sheaf $\phi_* \mathcal{O}_B$. If we fix a very ample sheaf $\mathcal{O}_X(1)$ then we can compose this homomorphism with taking the Hilbert polynomial to obtain a homomorphism $h : bB_k(X) \to \mathbb{Z}[t]$.

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